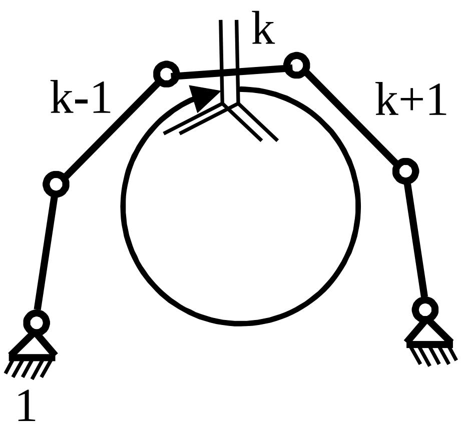
Solution of Kinematical Loops

- The open kinematical chain can be described by independent (relative) coordinates.
- A kinematical loop contains dependent (relative) coordinates.
- The ways for assembling the equations for determination of dependent coordinates.
  - Method of closed loop
  - Method of disconnected loop
  - Method of removed body
  - Method of natural coordinates
  - Method of cartesian coordinates – method of compartments

1. Method of closed loop

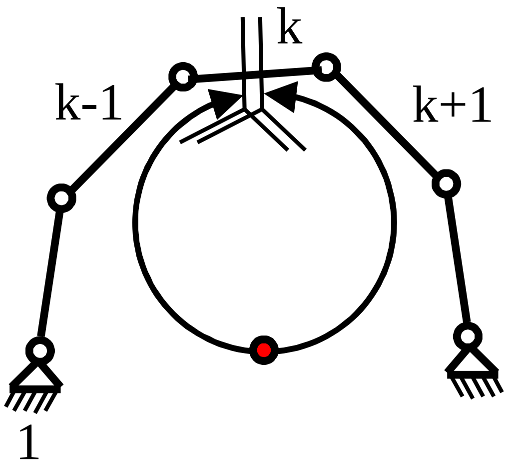


- On some body of the loop (it does not need to be the frame) it is chosen a coordinate system. The kinematical transformation from this coordinate system through the kinematical loop again into the same coordinate system is described. The result is the identity

$$\mathbf{T}_{12}\mathbf{T}_{23}\ldots\mathbf{T}_{k-1,k}\mathbf{T}_{k,k+1}\ldots\mathbf{T}_{n-1,n}\mathbf{T}_{n,1}=\mathbf{E}_4$$

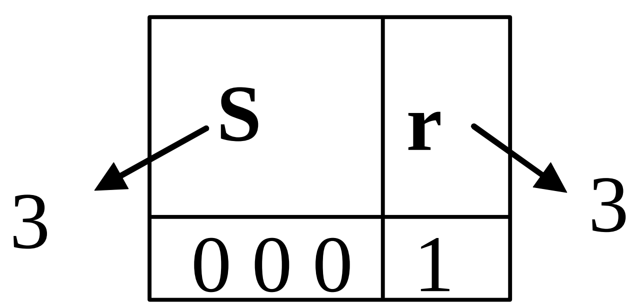
- The other variant appears after multiplication of both sides by

$$\mathbf{T}_{1k}=\mathbf{T}_{k1}^{-1}$$



$$\mathbf{T}_{12}\mathbf{T}_{23}\ldots\mathbf{T}_{k-1,k}=\mathbf{T}_{1n}\mathbf{T}_{n,n-1}\ldots\mathbf{T}_{k+1,k}$$

- In both cases there are assembled 12 nonlinear equations and 4 identitied. Among them there are only 6 independent equations according to the identities in the matrix of directional cosines.



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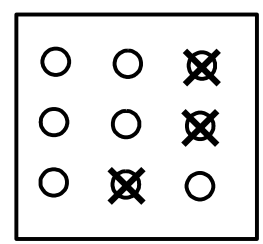
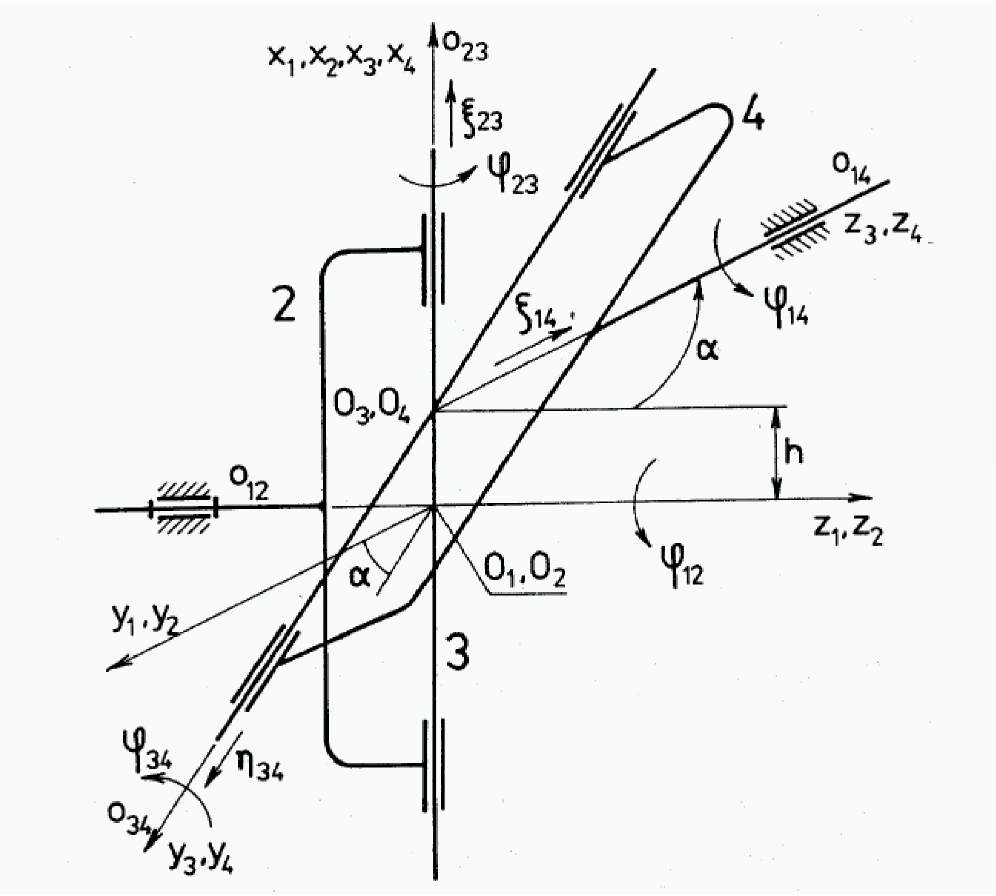


Example – universal joint



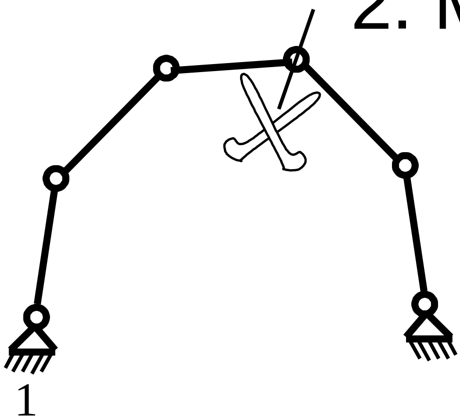
$$\mathbf{T}_{12}\mathbf{T}_{23}\mathbf{T}_{34} = \mathbf{T}_{14}$$

$$\begin{aligned}
\mathbf{T}_{12} &= \mathbf{T}_{Z6}(\varphi_{12}) \\
\mathbf{T}_{23} &= \mathbf{T}_{Z1}(\xi_{23})\mathbf{T}_{Z4}(\varphi_{23}) \\
\mathbf{T}_{34} &= \mathbf{T}_{Z2}(\eta_{34})\mathbf{T}_{Z5}(\varphi_{34}) \\
\mathbf{T}_{14} &= \mathbf{T}_{Z1}(h)\mathbf{T}_{Z4}(\alpha)\mathbf{T}_{Z3}(\xi_{14})\mathbf{T}_{Z6}(\varphi_{14})
\end{aligned}$$

$$\mathbf{A} = \mathbf{T}_{12}\mathbf{T}_{23}\mathbf{T}_{34} - \mathbf{T}_{14} = [a_{ij}]$$

$$
\begin{aligned}
a_{11} &\equiv \mathrm{c}\varphi_{34}\mathrm{c}\varphi_{12} - \mathrm{s}\varphi_{34}\mathrm{s}\varphi_{23}\mathrm{s}\varphi_{12} - \mathrm{c}\varphi_{14} = 0 \\
a_{12} &\equiv -\mathrm{c}\varphi_{23}\mathrm{s}\varphi_{12} + \mathrm{s}\varphi_{14} = 0 \\
a_{13} &\equiv \mathrm{s}\varphi_{34}\mathrm{c}\varphi_{12} + \mathrm{c}\varphi_{34}\mathrm{s}\varphi_{23}\mathrm{s}\varphi_{12} = 0 \\
a_{14} &\equiv \xi_{23}\mathrm{c}\varphi_{12} - \eta_{34}\mathrm{c}\varphi_{23}\mathrm{s}\varphi_{12} - h = 0 \\
a_{21} &\equiv \mathrm{c}\varphi_{34}\mathrm{s}\varphi_{12} + \mathrm{s}\varphi_{34}\mathrm{s}\varphi_{23}\mathrm{c}\varphi_{12} - \mathrm{s}\varphi_{14}\mathrm{c}\alpha = 0 \\
a_{22} &\equiv \mathrm{c}\varphi_{23}\mathrm{c}\varphi_{12} - \mathrm{c}\varphi_{14}\mathrm{c}\alpha = 0 \\
a_{23} &\equiv \mathrm{s}\varphi_{34}\mathrm{s}\varphi_{12} - \mathrm{c}\varphi_{34}\mathrm{s}\varphi_{23}\mathrm{c}\varphi_{12} + \mathrm{s}\alpha = 0 \\
a_{24} &\equiv \xi_{23}\mathrm{s}\varphi_{12} + \eta_{34}\mathrm{c}\varphi_{23}\mathrm{c}\varphi_{12} + \zeta_{14}\mathrm{s}\alpha = 0 \\
a_{31} &\equiv -\mathrm{s}\varphi_{34}\mathrm{c}\varphi_{23} - \mathrm{s}\varphi_{14}\mathrm{s}\alpha = 0 \\
a_{32} &\equiv \mathrm{s}\varphi_{23} - \mathrm{c}\varphi_{14}\mathrm{s}\alpha = 0 \\
a_{33} &\equiv \mathrm{c}\varphi_{34}\mathrm{c}\varphi_{23} - \mathrm{c}\alpha = 0 \\
a_{34} &\equiv \eta_{34}\mathrm{s}\varphi_{23} - \zeta_{14}\mathrm{c}\alpha = 0
\end{aligned}
$$

$$
\begin{aligned}
\xi_{23} &= \frac{\mathrm{c}\varphi_{12}}{1 - \mathrm{s}^2\alpha\mathrm{s}^2\varphi_{12}} h \\
\varphi_{14} &= \mathrm{atan2}(\pm\mathrm{s}\varphi_{12}\mathrm{c}\alpha, \pm\mathrm{c}\varphi_{12}) \\
\eta_{34} &= -\frac{\mathrm{c}\alpha\mathrm{s}\varphi_{12}}{\sqrt{1 - \mathrm{s}^2\alpha\mathrm{s}^2\varphi_{12}}} h \\
\varphi_{23} &= \mathrm{atan2}\left(\mathrm{s}\alpha\mathrm{c}\varphi_{14}, \frac{\mathrm{c}\alpha\mathrm{c}\varphi_{14}}{\mathrm{c}\varphi_{12}}\right) \\
\zeta_{14} &= -\frac{\mathrm{s}\alpha\mathrm{s}2\varphi_{12}}{1 - \mathrm{s}^2\alpha\mathrm{s}^2\varphi_{12}} h \\
\varphi_{34} &= \mathrm{atan2}(-\mathrm{s}\alpha\mathrm{s}\varphi_{14}\mathrm{c}\varphi_{23}, \mathrm{c}\alpha\mathrm{c}\varphi_{23})
\end{aligned}
$$



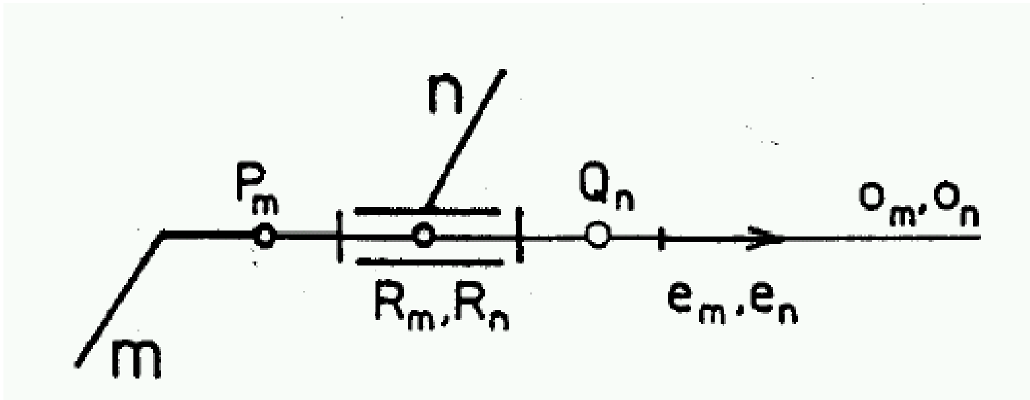
2. Method of disconnected loop

- The number of unknowns and the number of equations can be decreased. The kinematical loop is disconnected by a cut in some kinematical joint and the conditions of its closure are assembled. The resulting equations do not include the relative coordinates of cut loop.
- <u>Spherical joint</u>
- The condition is the equality of radiusvector of the centre of spherical KJ on the body k and k+1.

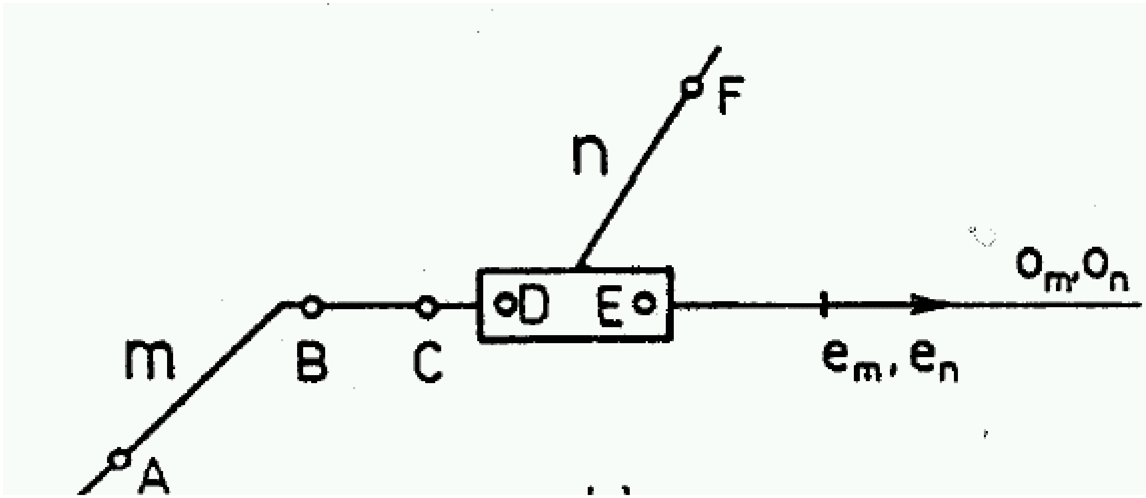
$$ {}^{1}\mathbf{r}_{1S_k} = {}^{1}\mathbf{r}_{1S_{k+1}} $$

$$ \mathbf{T}_{12}\mathbf{T}_{23}\ldots\mathbf{T}_{k-1,k}\mathbf{r}_{kS_k} = \mathbf{T}_{1n}\mathbf{T}_{n,n-1}\ldots\mathbf{T}_{k+2,k+1}\mathbf{r}_{k+1S_{k+1}} $$

- The result is just 3 equations. The 3 relative coordinates of spherical KJ are not used.

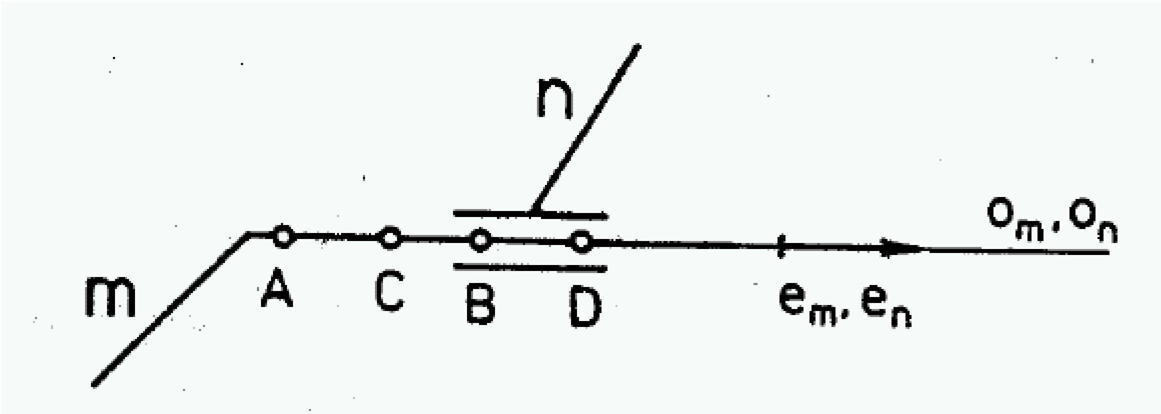


- <u>Revolute joint</u>
- The condition is the equality of radiusvector of the centre of revolute KJ on the body k=m and k+1=n and the equality of unit vector of the axis.
- $\mathbf{u}_{1Rm} = \mathbf{u}_{1Rn}$ 3 equations (independent)
- $\mathbf{e}_m = \mathbf{e}_n$ 3 equations (2 independent)
- The result is 6 equations, 5 of which are independent. The relative coordinate of revolute KJ is not used.



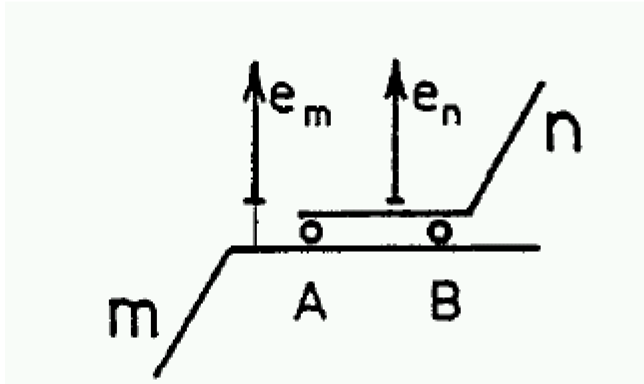
- <u>Translational joint</u>
- The condition is the equality of unit vector of the axis, colinearity of BD and unit vector of the axis and the constant angle of AB, DF
- $\mathbf{e}_m = \mathbf{e}_n$ — 3 equations (2 independent)
- $(\mathbf{u}_{1Bm} - \mathbf{u}_{1Dn}) \times \mathbf{e}_m = \mathbf{0}$ — 3 equations (2 independent)
- $(\mathbf{u}_{1Am} - \mathbf{u}_{1Bm}).(\mathbf{u}_{1Dn} - \mathbf{u}_{1Fn}) = \text{const}$ — 1 equation (independent)

- The result is 7 equations, 5 of which are independent. The relative coordinate of translational KJ is not used.



- <u>Cylindrical joint</u>
- The condition is the equality of unit vector of the axis and colinearity of AB and unit vector of the axis.
- $\mathbf{e}_m = \mathbf{e}_n$ 3 equations (2 independent)
- $(\mathbf{u}_{1Am} - \mathbf{u}_{1Bn}) \times \mathbf{e}_m = \mathbf{0}$ 3 equations (2 independent)

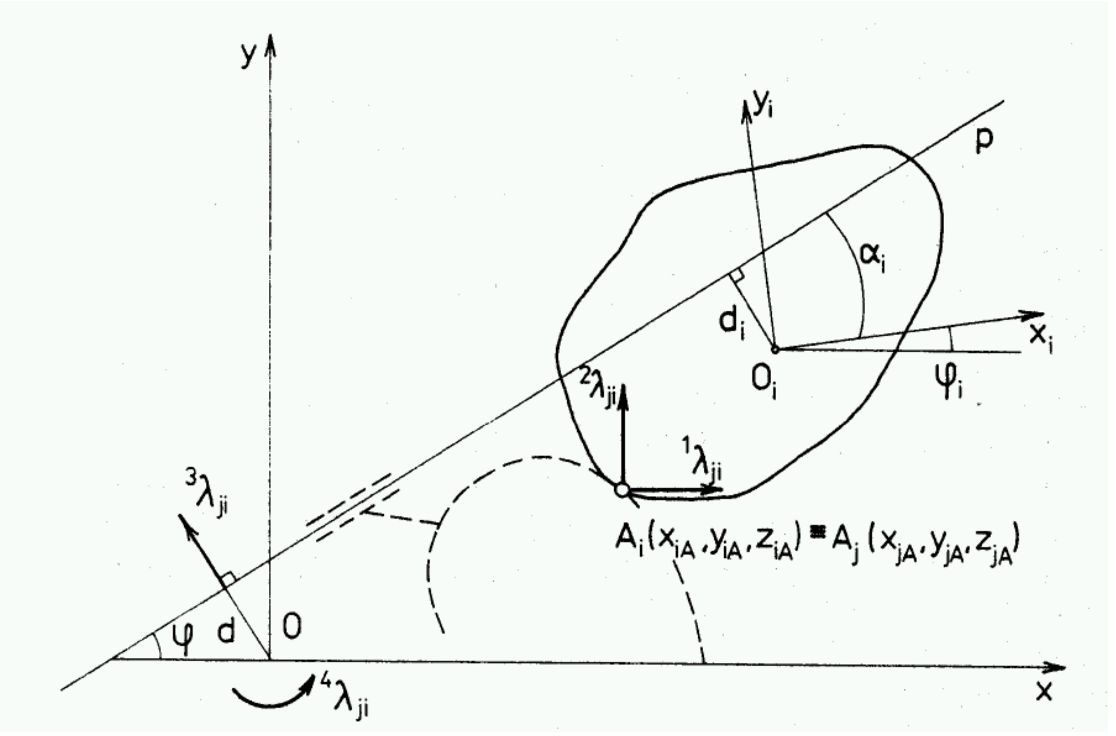
- The result is 6 equations, 4 of which are independent. The relative coordinates of cylindrical KJ are not used.



- <u>Flat joint</u>
- The condition is the equality of unit vector orthogonal to the plane and orthogonality of AB and unit vector orthogonal to the plane.
- $\mathbf{e}_m = \mathbf{e}_n$ 3 equations (2 independent)
- $(\mathbf{u}_{1Am} - \mathbf{u}_{1Bn}).\mathbf{e}_m \; = \; 0$ 1 equation (independent)

- The result is 4 equations, 3 of which are independent. The relative coordinates of flat KJ are not used.

Planar KJ

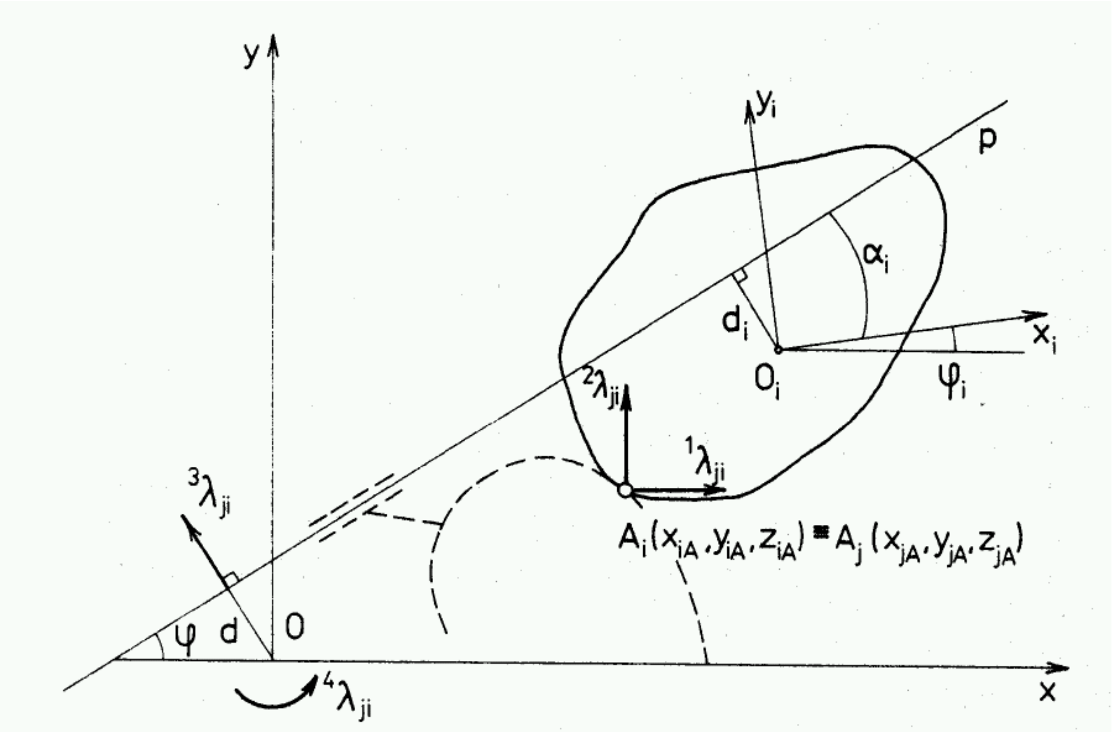


- Revolute KJ
- The condition is the equality of radius vector of the centre of rotation.

$$
\begin{aligned}
x_i + x_{iA} c\varphi_i - y_{iA} s\varphi_i &= x_j + x_{jA} c\varphi_j - y_{jA} s\varphi_j \\
y_i + x_{iA} s\varphi_i + y_{iA} c\varphi_i &= y_j + x_{jA} s\varphi_j + y_{jA} c\varphi_j
\end{aligned}
$$

- The result is 2 equations. The relative coordinate of revolute KJ is not used.

Planar KJ

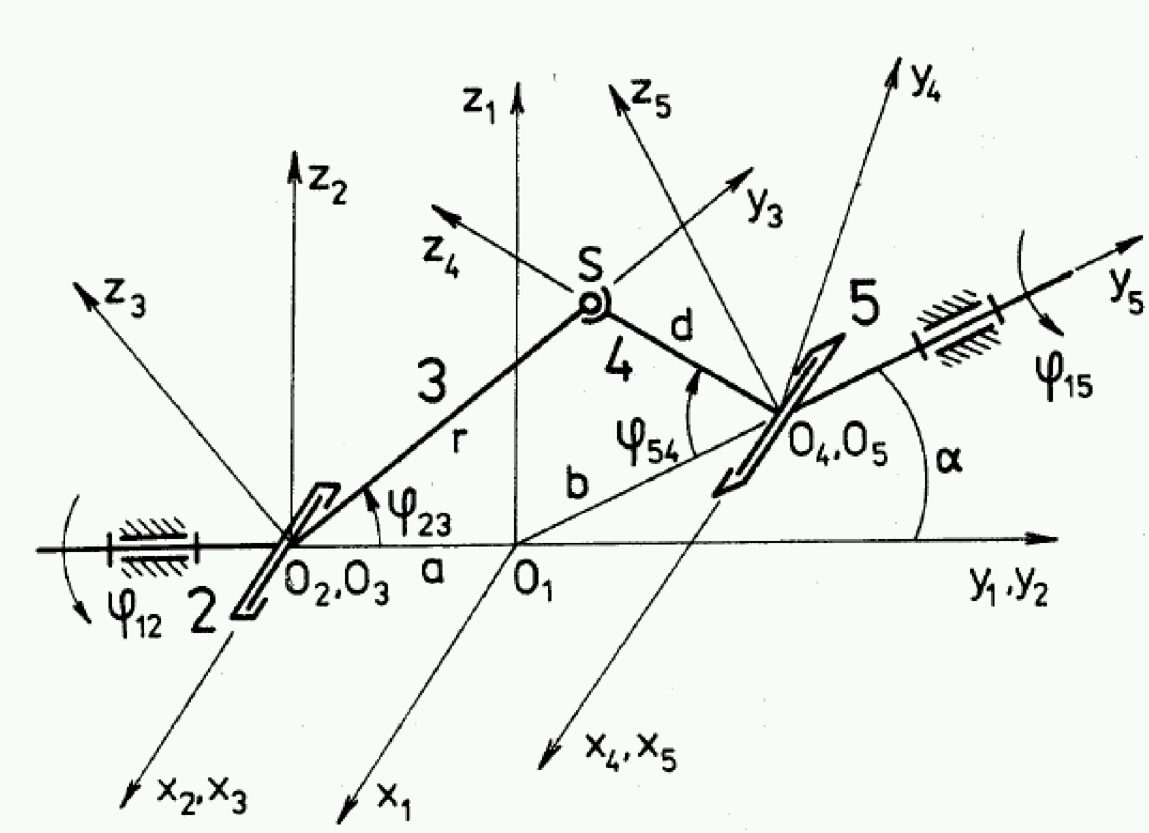


- Translational KJ
- The condition is the equality of unit vector of the axis and the equality of the distance to the axis.

$$d_i + y_i \mathrm{c}(\alpha_i + \varphi_i) - x_i \mathrm{s}(\alpha_i + \varphi_i) =$$
$$= d_j + y_j \mathrm{c}(\alpha_j + \varphi_j) - x_j \mathrm{s}(\alpha_j + \varphi_j)$$
$$\alpha_i + \varphi_i = \alpha_j + \varphi_j$$

- The result is 2 equations. The relative coordinate of translational KJ is not used.

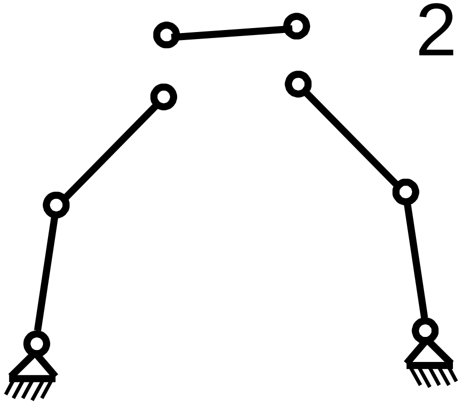
Example – RRSRR mechanism



$$\mathbf{T}_{12}\mathbf{T}_{23}\mathbf{r}_{3S} = \mathbf{T}_{15}\mathbf{T}_{54}\mathbf{r}_{4S}$$

$$\begin{aligned}
\mathbf{T}_{12} &= \mathbf{T}_{Z2}(-a)\mathbf{T}_{Z5}(\varphi_{12}) \\
\mathbf{T}_{23} &= \mathbf{T}_{Z4}(\varphi_{23}) \\
\mathbf{T}_{15} &= \mathbf{T}_{Z4}(\alpha)\mathbf{T}_{Z2}(b)\mathbf{T}_{Z5}(\varphi_{15}) \\
\mathbf{T}_{54} &= \mathbf{T}_{Z4}((\pi/2) - \varphi_{54}) \\
\mathbf{r}_{3S} &= [0, r, 0, 1] \\
\mathbf{r}_{4S} &= [0, 0, d, 1]
\end{aligned}$$

$$\begin{aligned}
r\mathrm{s}\varphi_{23}\mathrm{s}\varphi_{12} &= d\mathrm{s}\varphi_{54}\mathrm{s}\varphi_{15} \\
r\mathrm{c}\varphi_{23} - a &= (b - d\mathrm{c}\varphi_{54})\mathrm{c}\alpha - d\mathrm{s}\varphi_{54}\mathrm{c}\varphi_{15}\mathrm{s}\alpha \\
r\mathrm{s}\varphi_{23}\mathrm{c}\varphi_{12} &= (b - d\mathrm{c}\varphi_{54})\mathrm{s}\alpha + d\mathrm{s}\varphi_{54}\mathrm{c}\varphi_{15}\mathrm{c}\alpha
\end{aligned}$$



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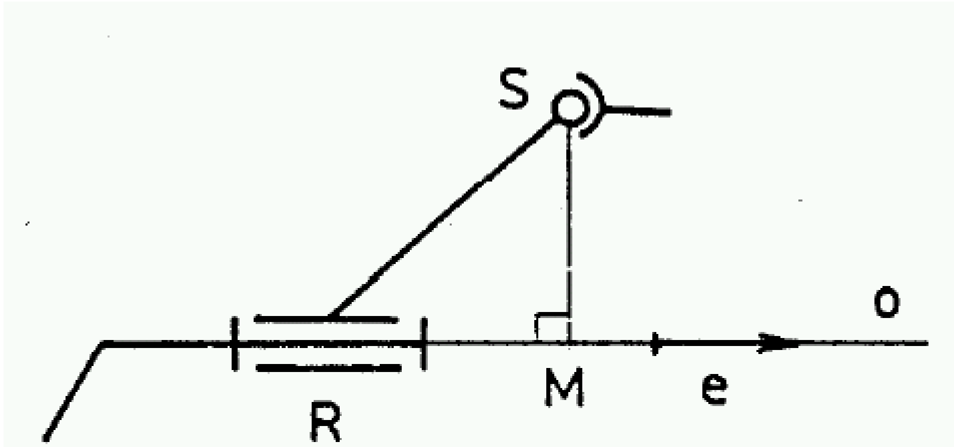
2. Method of removed body

- The number of unknowns and the number of equations can be further decreased. The kinematical loop is disconnected by two cuts in some kinematical joints of a body and the body is removed and the conditions of its closure are assembled. The resulting equations do not include the relative coordinates of two cut KJs.
- <u>Spherical – spherical joints</u>
- The condition is the equality of the distance of the centres of spherical KJs on the body.

$$\left|\mathbf{u}_{1S_1} - \mathbf{u}_{1S_2}\right| = l$$

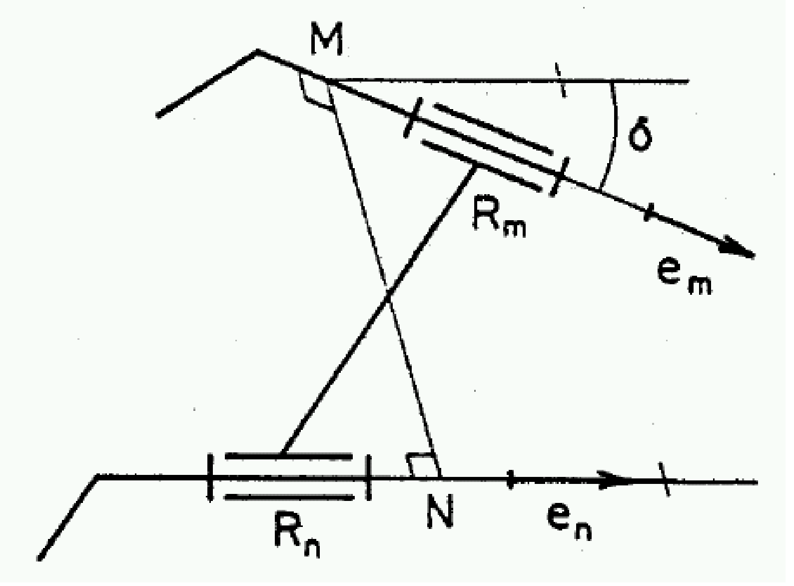
- The result is just 1 equation. The 6 relative coordinates of spherical KJs are not used. ATTENTION: The body can freely rotate around S1S2.



- <u>Spherical – revolute KJs</u>
- The condition is the distance from the centre of spherical KJ to the axis of rotational KJ.

$$
\begin{aligned}
|\mathbf{u}_{1S} - \mathbf{u}_{1M}| &= h \\
(\mathbf{u}_{1S} - \mathbf{u}_{1M}) \cdot \mathbf{e} &= 0
\end{aligned}
$$

- The result is just 2 equations. The 4 relative coordinates of spherical and rotational KJs are not used.

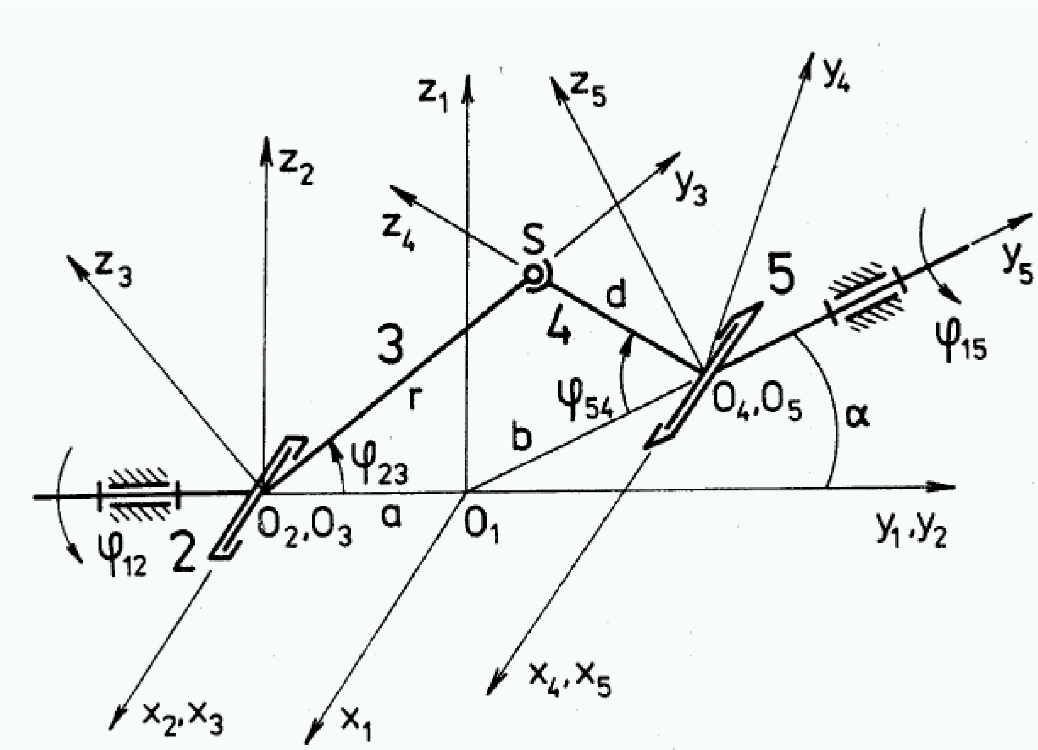


- <u>Revolute – revolute KJs</u>
- The condition is the distance of skew axes of revolute KJs.

$$\begin{aligned} \mathbf{u}_{1M} - \mathbf{u}_{1N} &= \frac{h}{\sin\delta}(\mathbf{e}_m \times \mathbf{e}_n) \\ \mathbf{e}_m . \mathbf{e}_n &= \cos\delta \end{aligned}$$

- The result is just 4 equations. The 2 relative coordinates of rotational KJs are not used.

Example – RRSRR mechanism



$$
\begin{aligned}
[\mathbf{u}_{1S}, 1] &= \mathbf{T}_{15}\mathbf{T}_{54}\mathbf{r}_{4S} = [d\mathrm{s}\varphi_{54}\mathrm{s}\varphi_{15}, (b - d\mathrm{c}\varphi_{54})\mathrm{c}\alpha - \\
&\quad -d\mathrm{s}\varphi_{54}\mathrm{c}\varphi_{15}\mathrm{s}\alpha, (b - d\mathrm{c}\varphi_{54})\mathrm{s}\alpha + d\mathrm{s}\varphi_{54}\mathrm{c}\varphi_{15}\mathrm{c}\alpha, 1] \\
[\mathbf{u}_{1M}, 1] &= \mathbf{T}_{12}\mathbf{r}_{4M} = \mathbf{T}_{12}\mathbf{r}_{4O_2} = [0, -a, 0, 1] \\
[\mathbf{e}, 0] &= \mathbf{T}_{12}[1, 0, 0, 0] = [\mathrm{c}\varphi_{12}, 0, -\mathrm{s}\varphi_{12}, 0]
\end{aligned}
$$

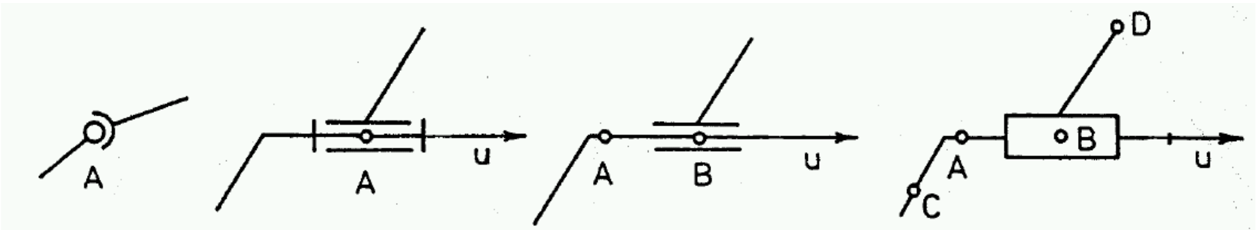
$$
\begin{aligned}
(d\mathrm{s}\varphi_{54}\mathrm{s}\varphi_{15})^2 + ((b - d\mathrm{c}\varphi_{54})\mathrm{c}\alpha - d\mathrm{s}\varphi_{54}\mathrm{c}\varphi_{15}\mathrm{s}\alpha + a)^2 + \\
+((b - d\mathrm{c}\varphi_{54})\mathrm{s}\alpha + d\mathrm{s}\varphi_{54}\mathrm{c}\varphi_{15}\mathrm{c}\alpha)^2 = r^2 \\
d\mathrm{s}\varphi_{54}\mathrm{s}\varphi_{15}\mathrm{c}\varphi_{12} - ((b - d\mathrm{c}\varphi_{54})\mathrm{s}\alpha + d\mathrm{s}\varphi_{54}\mathrm{c}\varphi_{15}\mathrm{c}\alpha)\mathrm{s}\varphi_{12} = 0
\end{aligned}
$$

4. Method of natural coordinates

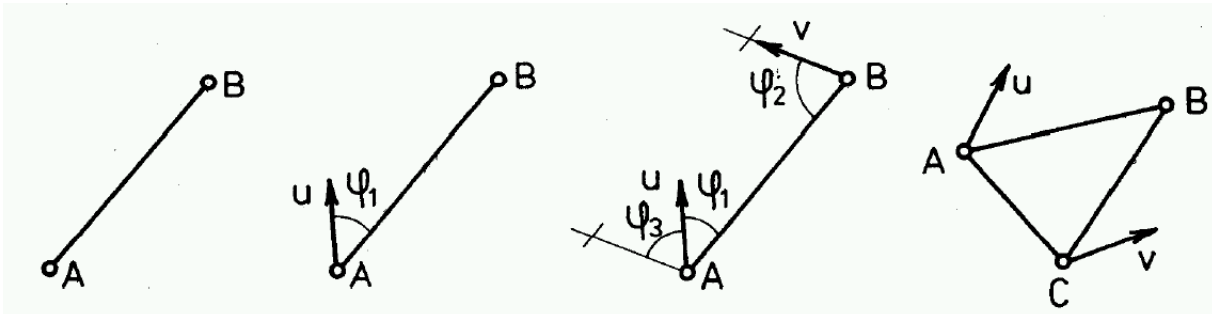
- Natural coordinates – cartesian coordinates of important points and unit vectors of bodies. Important points are usually the centres of particular KJs. Important unit vectors are usually the directions of KJ’s axes

  The description of kinematic loop by natural sharing of important points and unit vectors by connected bodies

  The constraint conditions are only for natural coordinates on the same body.



- Spherical joint is described by sharing the coordinates of the center
- Revolute joint is described by sharing a point of axis and its unit vector
- Cylindrical joint is described by sharing the unit vector and AB//u.
- Translational joint is described by sharing the unit vector, AB//u and constant angle of AC, BD



$$(x_{1B}-x_{1A})^2+(y_{1B}-y_{1A})^2+(z_{1B}-z_{1A})^2=l_{AB}^2$$

$$\begin{aligned}(x_{1B}-x_{1A})u_1+(y_{1B}-y_{1A})u_2+(z_{1B}-z_{1A})u_3 &= l_{AB}\cos\varphi_1\\ (x_{1B}-x_{1A})v_1+(y_{1B}-y_{1A})v_2+(z_{1B}-z_{1A})v_3 &= l_{AB}\cos\varphi_2\\ u_1v_1+u_2v_2+u_3v_3 &= \cos\varphi_3\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{3}u_i^2 &= 1\\ \sum_{i=1}^{3}v_i^2 &= 1\end{aligned}$$

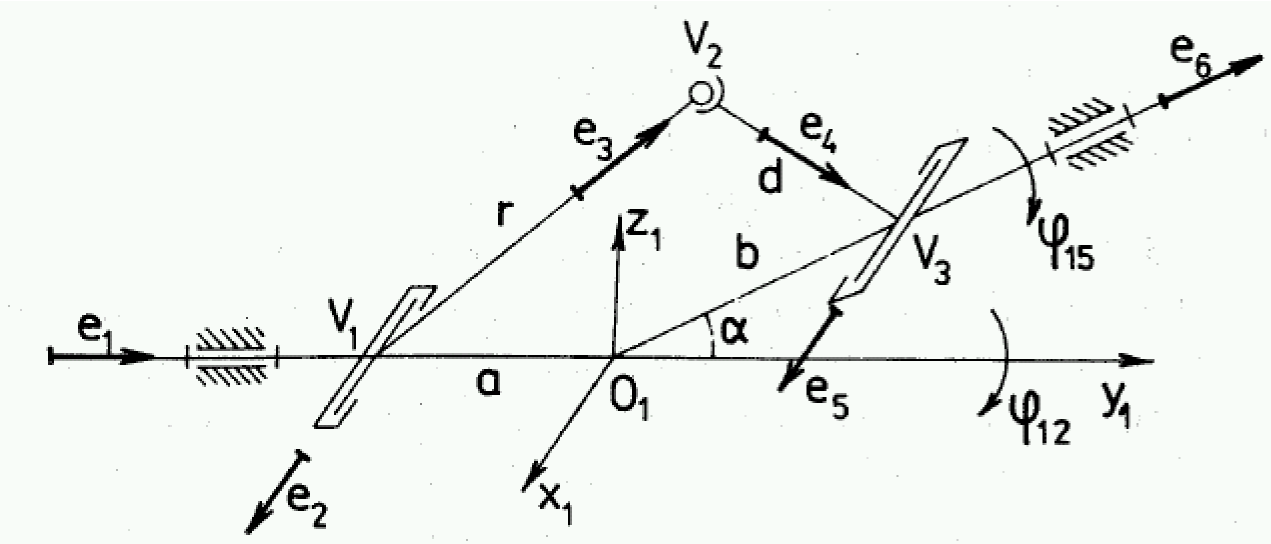
$$\begin{aligned}l_{AB}u_1-(x_{1B}-x_{1A}) &= 0\\ l_{AB}u_2-(y_{1B}-y_{1A}) &= 0\\ l_{AB}u_3-(z_{1B}-z_{1A}) &= 0\end{aligned}$$

$$\overrightarrow{AB}\times\vec{u}=\vec{0}$$

$$\begin{aligned}k_1u_1+k_2v_1-(x_{1B}-x_{1A}) &= 0\\ k_1u_2+k_2v_2-(y_{1B}-y_{1A}) &= 0\\ k_1u_3+k_2v_3-(z_{1B}-z_{1A}) &= 0\end{aligned}$$

$$\begin{aligned}(x_{1C}-x_{1A})(x_{1D}-x_{1B}) &+ (y_{1C}-y_{1A})(y_{1D}-y_{1B})+\\ +(z_{1C}-z_{1A})(z_{1D}-z_{1B}) &= l_{AC}l_{BD}\cos(\overrightarrow{AC},\overrightarrow{BD})\end{aligned}$$

Example – RRSRR mechanism



$$\mathbf{u}_{1V_1} = [0, -a, 0], \qquad \mathbf{u}_{1V_3} = [0, bc\alpha, bs\alpha]$$
$$\mathbf{e}_1 = [0, 1, 0], \qquad \mathbf{e}_6 = [0, c\alpha, s\alpha]$$

$$\begin{aligned} \mathbf{e}_1.\mathbf{e}_5 &= y_5 = 0 \\ \mathbf{e}_5.\mathbf{e}_6 &= y_5 c\alpha + z_5 s\alpha = 0 \end{aligned}$$

$$\begin{aligned} x_{1V_2}^2 + (y_{1V_2} + a)^2 + z_{1V_2}^2 &= r^2 \\ x_{1V_2} x_2 + (y_{1V_2} + a) y_2 + z_{1V_2} z_2 &= 0 \\ x_2^2 + y_2^2 + z_2^2 &= 1 \end{aligned}$$

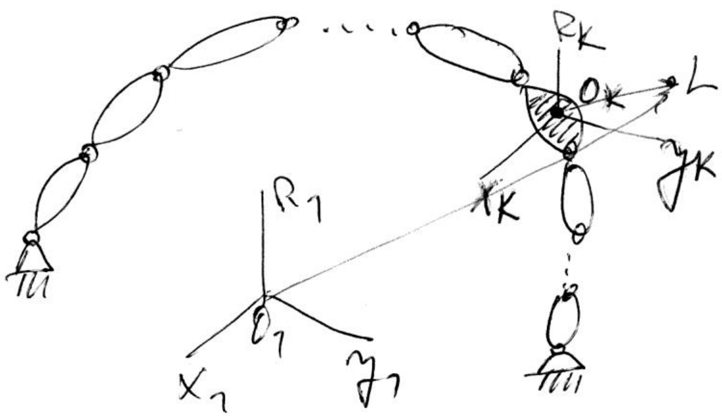
$$\mathbf{e}_2 = [c\varphi_{12}, 0, -s\varphi_{12}]$$

$$\begin{aligned} x_{1V_2}^2 + (y_{1V_2} - bc\alpha)^2 + (z_{1V_2} - bs\alpha)^2 &= d^2 \\ x_{1V_2} x_5 + (y_{1V_2} - bc\alpha) y_5 + (z_{1V_2} - bs\alpha) z_5 &= 0 \\ x_5^2 + y_5^2 + z_5^2 &= 1 \end{aligned}$$

$$\mathbf{e}_5 = [c\varphi_{15}, s\varphi_{15} s\alpha, -s\varphi_{15} c\alpha]$$

4. Method of cartesian coordinates

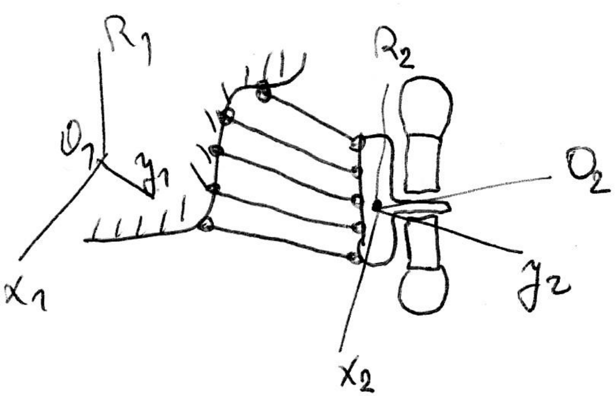
- Cartesian (physical) coordinates – every body is described by cartesian coordinates (coordinates of origin of local coordinate system) and its orientation (e.g. Cardan angles)
- Constraint conditions are described for connection of each body with its neighbours – like method of disconnected loop



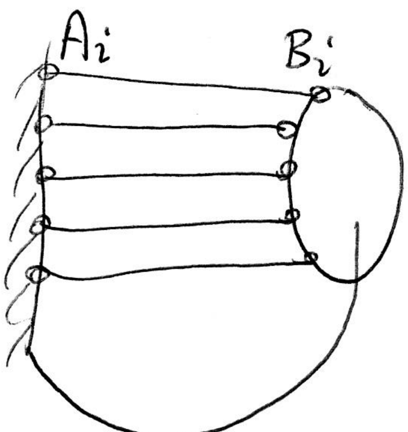
$$\mathbf{T_{1k}} = \mathbf{T_x}(x_k)\mathbf{T_y}(y_k)\,\mathbf{T_z}(z_k)\mathbf{T_{\varphi x}}(\varphi_{xk})\mathbf{T_{\varphi y}}(\varphi_{yk})\,\mathbf{T_{\varphi z}}(\varphi_{yk})$$

$$^{1}\mathbf{r_{1L}} = \mathbf{T_{1k}}\,^{k}\mathbf{r_{kL}}$$

Example – Five point wheel suspension



$[x\ y\ z]^T$, φ_x, φ_y φ_z … Cardan angles



virtual loop

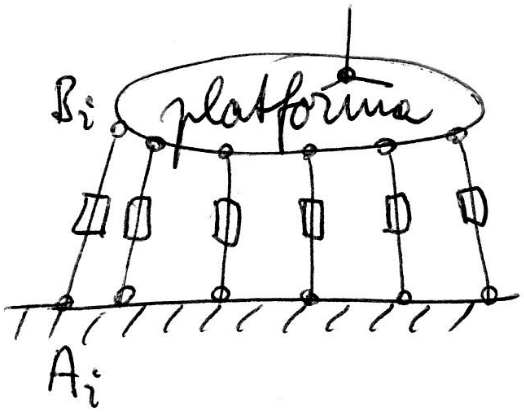
$A_iB_i=l_i$

6 coordinates

5 scalar constraint equations

6-5=1 DOF

Example – Hexapod



$x, y, z, \varphi_x, \varphi_y, \varphi_z, l_i = l_i(t), i=1,2,\ldots,6$

$A_iB_i = l_i$

12 coordinates

6 scalar constraint equations

12-6=6 DOF