Numerical Methods for Solution of Multibody Kinematics

- The multibody system is described by coordinates

$$\mathbf{s} = [s_1, s_2, \ldots]^T$$

- They can be divided into independent

$$\mathbf{q} = [q_1, q_2, \ldots, q_n]^T$$

  and dependent ones

$$\mathbf{z} = [z_1, z_2, \ldots, z_m]^T$$

- The constraints $\mathbf{f}(\mathbf{s}) = \mathbf{0}$ must be solved for determination of dependent coordinates from independent ones

$$\mathbf{f}(\mathbf{z}, \mathbf{q}) = \mathbf{0}$$

Position problem

- 1. problem – nonlinearity
- In general a system of nonlinear transcendental equations
  - Analytical solution possible only in special cases
  - Numerical solution necessary – in general by modified Newton method

- If $\mathbf{z}^{(k)} = \left[z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)}\right]$ the k-th iteration
- The constraints are expanded into Taylor series around the k-th iteration

- where $$\mathbf{f}(\mathbf{z}, \mathbf{q}) = \mathbf{f}(\mathbf{z}^{(k)}, \mathbf{q}) + \frac{\partial \mathbf{f}}{\partial \mathbf{z}^T} \Delta \mathbf{z}^{(k)} = \mathbf{0}$$

$$\begin{aligned} \mathbf{\Phi}_z(\mathbf{z}, \mathbf{q}) &= \frac{\partial \mathbf{f}(\mathbf{z}, \mathbf{q})}{\partial \mathbf{z}^T} \\ \mathbf{\Phi}_z &= \mathbf{\Phi}_z(\mathbf{z}^{(k)}, \mathbf{q}) \end{aligned}$$

- is the Jacobi
- We obtain a system of linear algebraic equations for unknowns $\Delta \mathbf{z}^{(k)}$

$$\mathbf{\Phi}_z \Delta \mathbf{z}^{(k)} = -\mathbf{f}(\mathbf{z}^{(k)}, \mathbf{q})$$

- Solving it $\Delta \mathbf{z}^{(k)} = -\Phi_z^{-1}\mathbf{f}(\mathbf{z}^{(k)}, \mathbf{q})$
- The iteration of classical Newton method is

$$\mathbf{z}^{(k+1)} = \mathbf{z}^{(k)} + \Delta \mathbf{z}^{(k)}$$

- The great disadvantage of classical Newton method is that it converges only in the vicinity of the solution.
- Therefore the modified Newton method is used

$$\mathbf{z}^{(k+1)} = \mathbf{z}^{(k)} + \varkappa \Delta \mathbf{z}^{(k)}$$

- where scalar parameter $\varkappa$ is determined from the minimisation of

$$|\mathbf{f}(\mathbf{z}^{(k+1)}, \mathbf{q})| = \sqrt{\sum_{i=1}^{n} f_i^2(\mathbf{z}^{(k+1)}, \mathbf{q})}$$

- Algorithm

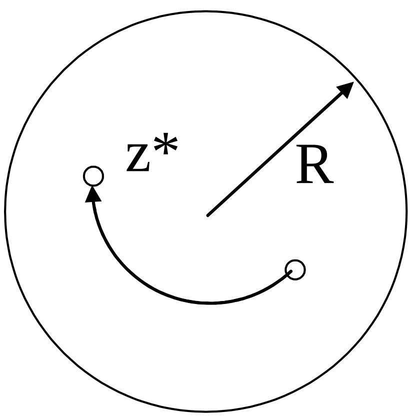
  $\varkappa = 1$
- While $|\mathbf{f}(\mathbf{z}^{\mathbf{k+1}}), \mathbf{q})| \geq |\mathbf{f}(\mathbf{z}^{(k)}, \mathbf{q})|$ do

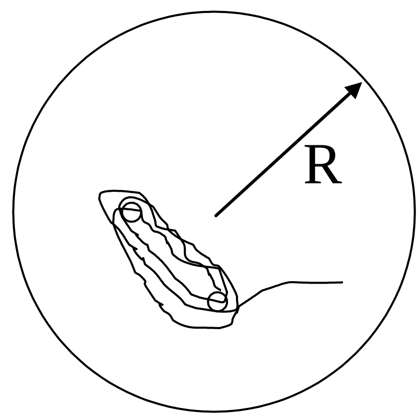
  $\varkappa = \varkappa/2$

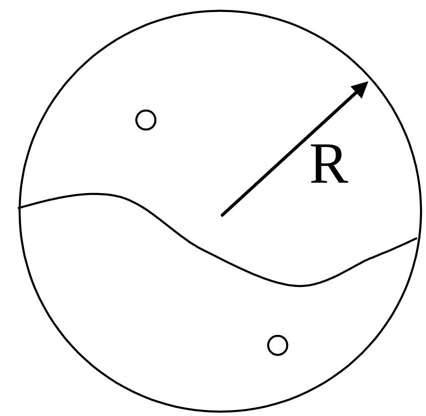
  $\mathbf{z}^{(k+1)} = \mathbf{z}^{(k)} + \varkappa \Delta \mathbf{z}^{(k)}$
- end
- Iteration stop $|\Delta \mathbf{z}^{(k)}| \leq \varepsilon$

- There is valid very strong theorem

Theorem: In some bounded area let $|\mathbf{f}(\mathbf{z})| \leq R$; further, let function $\mathbf{f}(\mathbf{z})$ have uniformly bounded first and second partial derivatives. In that area let $\mathbf{x} = \mathbf{f}(\mathbf{z})$ be a one-to-one mapping and $|\mathbf{\Phi}_z^{-1}(\mathbf{z})| \leq C$. Further, in that area, let a point $\mathbf{z}^*$ exist (according to the above assumptions only one point), so that $\mathbf{f}(\mathbf{z}^*) = \mathbf{0}$. The modified Newton method then converges from any estimate $|\mathbf{f}(\mathbf{z}^{(0)}| \leq R$, i.e. $\mathbf{z}^{(k)} \longrightarrow \mathbf{z}^*$ for $k \longrightarrow \infty$.







- Two solutions Φ singular
- Problem of oscilation Problem of convergence

- Another modification

$$f_i(\mathbf{z}) \longrightarrow \eta_i f_i(\mathbf{z}) \qquad \frac{1}{\eta_i} = |\mathbf{\Phi}_{iz}(\mathbf{z})| = \sqrt{\sum_{k=1}^{n}\left(\frac{\partial f_i(z_k)}{\partial z_k}\right)^2}$$

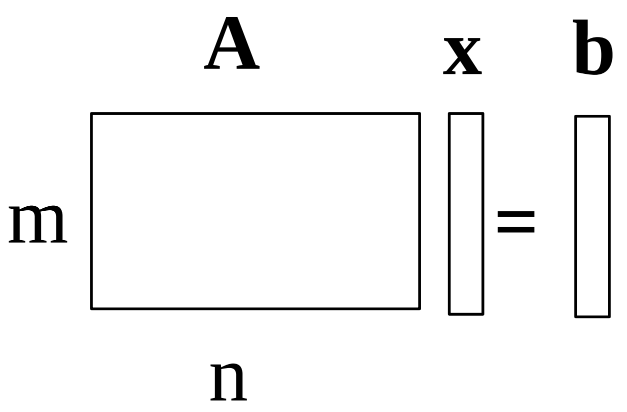
- The computation of $\Delta\mathbf{z} = \mathbf{\Phi}_z^{-1}\mathbf{f}(\mathbf{z})$ is invariant
- The computation of $|\mathbf{f}(\mathbf{z} + \varkappa\Delta\mathbf{z})| < |\mathbf{f}(\mathbf{z})|$ is not invariant
- The convergence is controlled by $|\mathbf{F}(\mathbf{z} + \varkappa\Delta\mathbf{z})| < |\mathbf{F}(\mathbf{z})|$
- where $$\mathbf{F}(\mathbf{z}) = \sqrt{\sum_{i=1}^{N}(\eta_i f_i(\mathbf{z}))^2}$$
- However, then the convergence theorem cańnot be proven.

- 2. problem – overdetermined equations

$$\mathbf{\Phi}_z \Delta \mathbf{z}^{(k)} = -\mathbf{f}(\mathbf{z}^{(k)}, \mathbf{q})$$

9	3
0 0 0	1

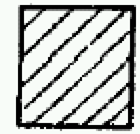
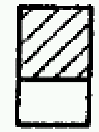
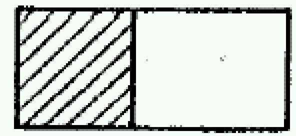
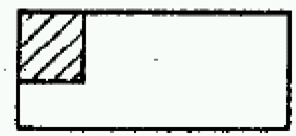
- $\mathbf{\Phi}_z$ generally rectangular matrix
- Solvability and solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$



r_A=rank($\mathbf{A}$)
r_B=rank([$\mathbf{A}$,$\mathbf{b}$])

- <u>Frobenius theorem</u>
- The system of LAE is solvable iff $r_A=r_B$.
- The unique solution exists iff $r_A=r_B=n$.

- Physical interpretation

MBS		solution	type of **A**
regular $r_A = n$	safe $m = n$	1	
	dangerous $m > n$	1 0	
irregular $r_A < n$	safe $r_A = m$	∞	
	dangerous $r_A > m$	∞ 0	

- Number of unknowns equal number of equaqtions
- Superfluous equations
- Superfluous unknowns – some of them cannot be determined – deficiency of equations
- Superfluous equations and superfluous unknowns

Velocity problem

- $\mathrm{f}(\mathrm{z}, \mathrm{q}) = 0$ for solution of position
- By time differentiation

$$\mathbf{\Phi}_z(\mathbf{z}, \mathbf{q})\dot{\mathbf{z}} + \mathbf{\Phi}_q(\mathbf{z}, \mathbf{q})\dot{\mathbf{q}} \quad = \quad \mathbf{0}$$

$$\mathbf{\Phi}_q(\mathbf{z}, \mathbf{q}) = \frac{\partial \mathbf{f}(\mathbf{z}, \mathbf{q})}{\partial \mathbf{q}^T}$$

$$\mathbf{\Phi}_z(\mathbf{z}, \mathbf{q})\dot{\mathbf{z}} \quad = \quad -\mathbf{\Phi}_q(\mathbf{z}, \mathbf{q})\dot{\mathbf{q}}$$

Acceleration problem

- By further time differentiation

$$\mathbf{\Phi}_z(\mathbf{z},\mathbf{q})\ddot{\mathbf{z}} + \dot{\mathbf{\Phi}}_z\dot{\mathbf{z}} \;=\; -\mathbf{\Phi}_q(\mathbf{z},\mathbf{q})\ddot{\mathbf{q}} - \dot{\mathbf{\Phi}}_q\dot{\mathbf{q}}$$

$$\dot{\mathbf{\Phi}}_z = \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{\Phi}_z(\mathbf{z},\mathbf{q}), \qquad \dot{\mathbf{\Phi}}_q = \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{\Phi}_q(\mathbf{z},\mathbf{q})$$

$$\mathbf{\Phi}_z(\mathbf{z},\mathbf{q})\ddot{\mathbf{z}} \;=\; -\mathbf{\Phi}_q(\mathbf{z},\mathbf{q})\ddot{\mathbf{q}} - \dot{\mathbf{\Phi}}_q\dot{\mathbf{q}} - \dot{\mathbf{\Phi}}_z\dot{\mathbf{z}}$$

- In all problems the system matrix, the system of LAE is the same, the same inversion $\mathbf{\Phi}_z^{-1}(\mathbf{z})$

General solution method - SVD

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

$\mathbf{U}$ is (m, m) $\quad \mathbf{U}^T\mathbf{U} = \mathbf{E}_m$

$\mathbf{V}$ is (n, n) $\quad \mathbf{V}^T\mathbf{V} = \mathbf{E}_n$

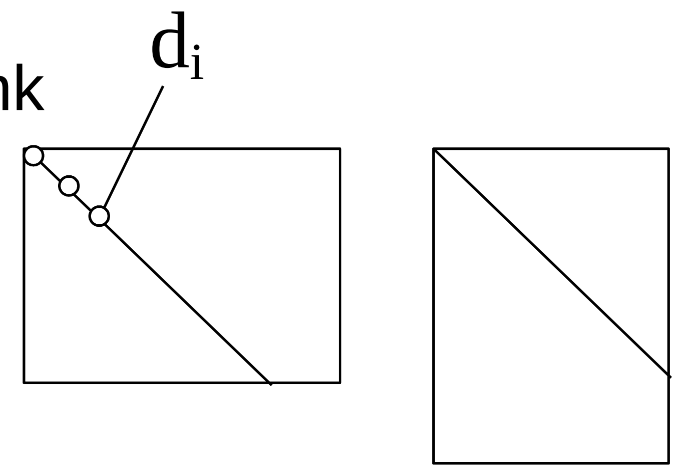
- Orthonormal matrices describing the rotation of coordinate systems
- Diagonal matrix representing the rank

$\mathbf{S}$ is an (m, n)

$$d_1, \ldots, d_k, k = \min(m, n)$$

$$d_1 \geq d_2 \geq \ldots \geq d_k \geq 0$$

S=



- rank(**A**)=r

$$d_r \leq \text{tolerance}, \quad d_{r-1} > \text{tolerance}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

$$\mathbf{x} = \mathbf{V}\mathbf{y}$$

$$\mathbf{U}^T\mathbf{A}\mathbf{V}\mathbf{y} = \mathbf{U}^T\mathbf{b}$$

$$\begin{aligned} \mathbf{y} &= \lceil \mathbf{y}_D, \mathbf{y}_I \rceil \\ \mathbf{c} &= \lceil \mathbf{c}_D, \mathbf{c}_I \rceil \end{aligned}$$

$$\mathbf{c} = \mathbf{U}^T\mathbf{b}$$

$$\mathbf{S}\mathbf{y} = \mathbf{c}$$

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{bmatrix} \mathbf{y}_D = \mathbf{c}_D$$

$$\mathbf{c}_I = 0$$

$$\mathbf{x} = \mathbf{V} \begin{bmatrix} \mathbf{y}_D \\ \mathbf{y}_I \end{bmatrix}$$

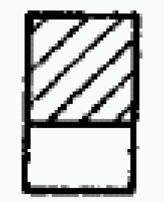
Approximated solution

$$\mathbf{Ax} = \mathbf{b}$$

- 1) $\|\mathrm{Ax} - \mathrm{b}\| \to min$

  $r_A < r_B, r_A = n$

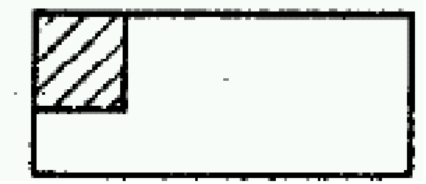
  $$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$



- 2) $\|\mathrm{x}\| \to min$

  $r_A < r_B, r_A < n = min(m, n)$

  $$\mathbf{x} = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{b}$$



Physical interpretation – number of DOFs

$$\mathbf{f}(\mathbf{s}) = \mathbf{0} \qquad \mathbf{s} = [\mathbf{q}, \mathbf{z}]^T$$

$$n = r - rank(\Phi_{\mathbf{s}})$$

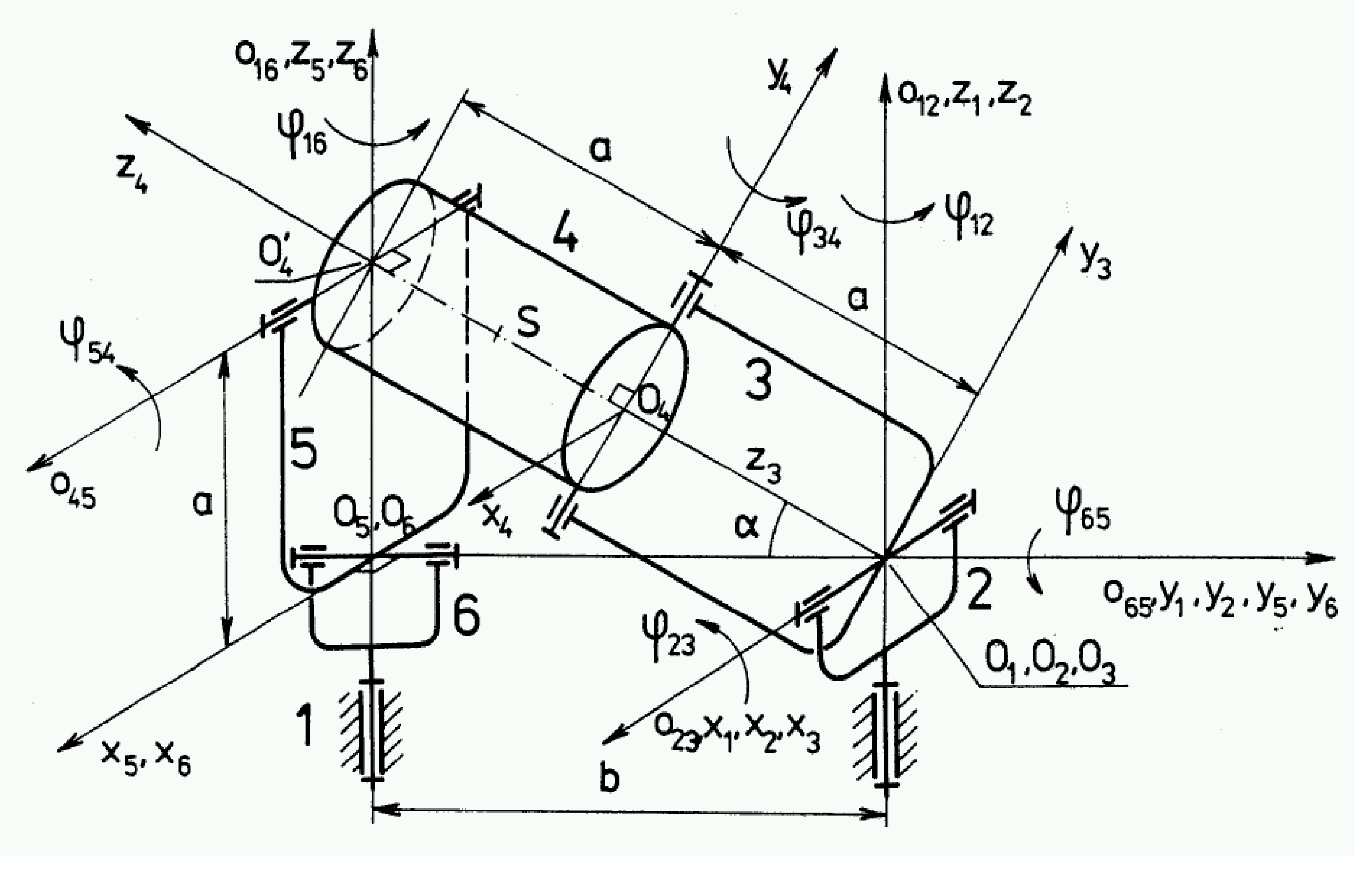
- Number of constraints Rank of Jacobian
- $$n = 6(n_b - 1) - \sum_{j=0}^{6} j w_j$$ sometimes it fails
- There exist singular positions where there is sudden increase of number of DOFs.
- Mechanisms – regular
  – singular in some position
  – steadily singular

- Matrix method gives 12 equations, only 6 of them are independent

$$\left[\begin{array}{c|c} 9 & 3 \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

- Which elements (equations) are really independent, it is determined numerically – the regular kernel is being determined

Example - Turbula



$$\mathbf{T}_{12}\mathbf{T}_{23}\mathbf{T}_{34} = \mathbf{T}_{16}\mathbf{T}_{65}\mathbf{T}_{54}$$

$$\begin{aligned}
\mathbf{T}_{12} &= \mathbf{T}_{Z6}(\varphi_{12}) \\
\mathbf{T}_{23} &= \mathbf{T}_{Z4}(\varphi_{23}) \\
\mathbf{T}_{34} &= \mathbf{T}_{Z3}(a)\mathbf{T}_{Z5}(\varphi_{34}) \\
\mathbf{T}_{54} &= \mathbf{T}_{54}\mathbf{T}_{44} = \mathbf{T}_{Z3}(a)\mathbf{T}_{Z4}(\varphi_{54})\mathbf{T}_{Z3}(a)\mathbf{T}_{Z4}(\pi) \\
\mathbf{T}_{65} &= \mathbf{T}_{Z5}(\varphi_{65}) \\
\mathbf{T}_{16} &= \mathbf{T}_{Z2}(-b)\mathbf{T}_{Z6}(\varphi_{16})
\end{aligned}$$

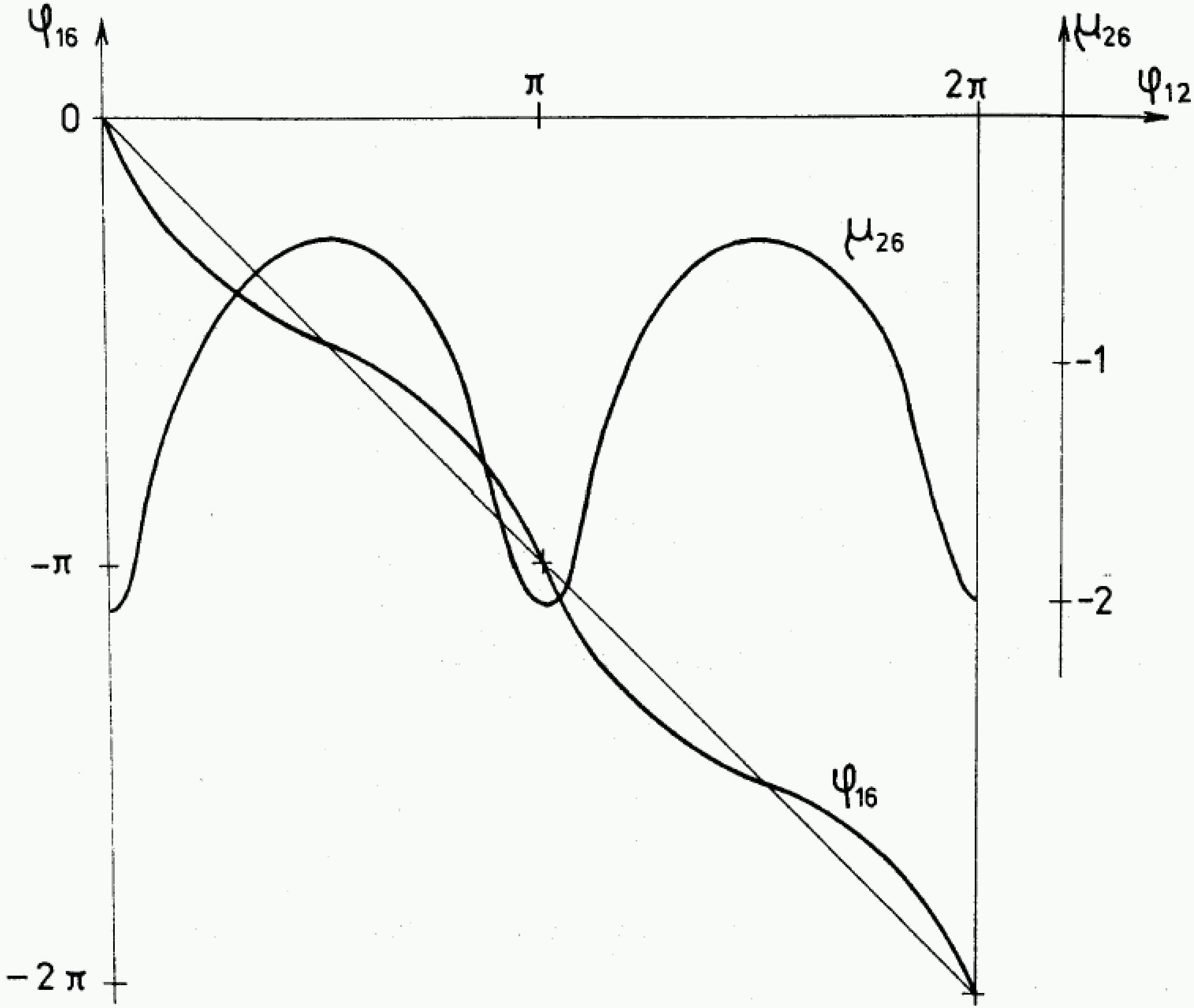
$$\mathbf{r}_{1S} = \mathbf{T}_{12}\mathbf{T}_{23}\mathbf{T}_{34}\mathbf{r}_{4S}$$

$$
\begin{aligned}
c\varphi_{34}c\varphi_{12} - s\varphi_{34}s\varphi_{23}s\varphi_{12} &= c\varphi_{65}c\varphi_{16} \\
c\varphi_{34}s\varphi_{12} + s\varphi_{34}s\varphi_{23}c\varphi_{12} &= c\varphi_{65}s\varphi_{16} \\
-s\varphi_{34}c\varphi_{23} &= -s\varphi_{65} \\
-c\varphi_{23}s\varphi_{12} &= c\varphi_{54}s\varphi_{16} - s\varphi_{54}s\varphi_{65}c\varphi_{16} \\
c\varphi_{23}c\varphi_{12} &= -c\varphi_{54}c\varphi_{16} - s\varphi_{54}s\varphi_{65}s\varphi_{16} \\
s\varphi_{23} &= -s\varphi_{54}c\varphi_{65} \\
s\varphi_{34}c\varphi_{12} + c\varphi_{34}s\varphi_{23}s\varphi_{12} &= -s\varphi_{54}s\varphi_{16} - c\varphi_{54}s\varphi_{65}c\varphi_{16} \\
s\varphi_{34}s\varphi_{12} - c\varphi_{34}s\varphi_{23}c\varphi_{12} &= s\varphi_{54}c\varphi_{16} - c\varphi_{54}s\varphi_{65}s\varphi_{16} \\
c\varphi_{34}c\varphi_{23} &= -c\varphi_{54}c\varphi_{65}
\end{aligned}
$$

$$
\begin{aligned}
a s\varphi_{23}s\varphi_{12} &= a\left(s\varphi_{54}s\varphi_{16} + c\varphi_{54}s\varphi_{65}c\varphi_{16} + s\varphi_{65}c\varphi_{16}\right) \\
-a s\varphi_{23}c\varphi_{12} &= a\left(-s\varphi_{54}c\varphi_{16} + c\varphi_{54}s\varphi_{65}s\varphi_{16} + s\varphi_{65}s\varphi_{16}\right) - b \\
a c\varphi_{23} &= a\left(c\varphi_{54}c\varphi_{65} + c\varphi_{65}\right)
\end{aligned}
$$

$$
\begin{aligned}
s\varphi_{23} &= \tfrac{\sqrt{3}}{2}c\varphi_{12}, & c\varphi_{23} &= \tfrac{1}{2}\sqrt{4-3c^2\varphi_{12}} \\
s\varphi_{34} &= \frac{2\sqrt{3}s\varphi_{12}}{4-3c^2\varphi_{12}}, & c\varphi_{34} &= \frac{3c^2\varphi_{12}-2}{4-3c^2\varphi_{12}} \\
s\varphi_{54} &= -\tfrac{\sqrt{3}}{2}\sqrt{4-3c^2\varphi_{12}}\,c\varphi_{12}, & c\varphi_{54} &= \tfrac{1}{2}(2-3c^2\varphi_{12}) \\
s\varphi_{65} &= \frac{-\sqrt{3}s\varphi_{12}}{\sqrt{4-3c^2\varphi_{12}}}, & c\varphi_{65} &= \frac{1}{\sqrt{4-3c^2\varphi_{12}}} \\
s\varphi_{16} &= \frac{-2s\varphi_{12}}{\sqrt{4-3c^2\varphi_{12}}}, & c\varphi_{16} &= \frac{c\varphi_{12}}{\sqrt{4-3c^2\varphi_{12}}}
\end{aligned}
$$

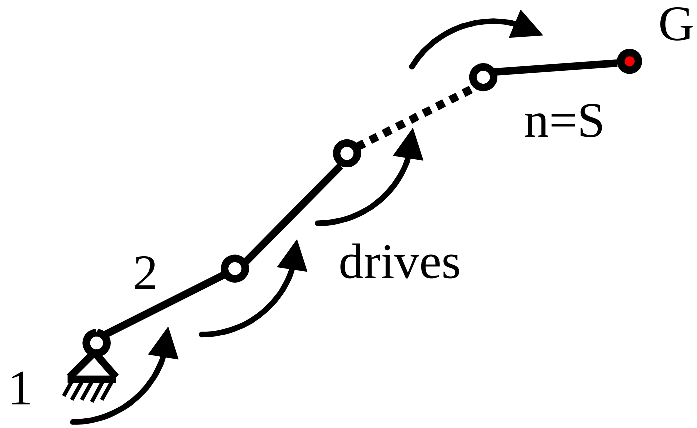
$$\frac{b}{a} = \sqrt{3} \qquad \alpha = \frac{\pi}{6}$$



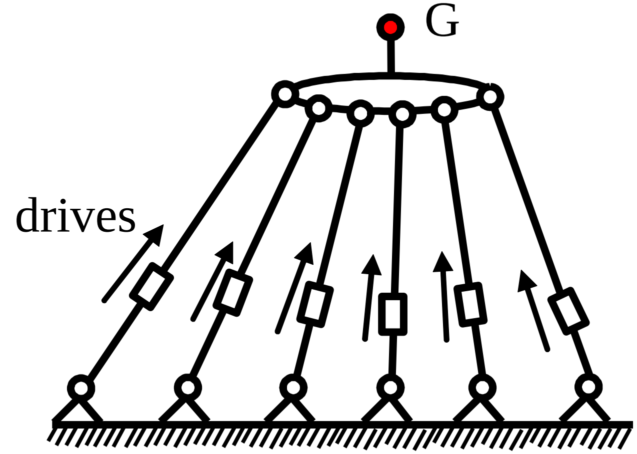
φ16
0
π
2π
φ12
μ26
−1
−2
−π
−2π
μ26
φ16

Kinematic problems of mechanisms

- Mechanism (multibody system) is described by coordinates **s**
- Some of them are always drives or just given – the independent coordinates **q**
- Kinematic problems of mechanisms consist in determination of dependent coordinates **z**
- Robotics introduces the terminology:
  - Robot has drives with driven coordinates and the end effector G that is described by coordinates describing its motion (similarly mechanism has end effecting member and drives that are driving it)
  - <u>Forward kinematic problem</u> – determine from known coordinates of drives the unknown coordinates of position of end effector
  - <u>Inverse kinematic problem</u> – determine from known position of end effector the unknown coordinates of drives



Effector G



Solution of kinematic problems

- Forward kinematic problem for serial chain – no loops, no contraints – usage of relative coordinates and direct computation

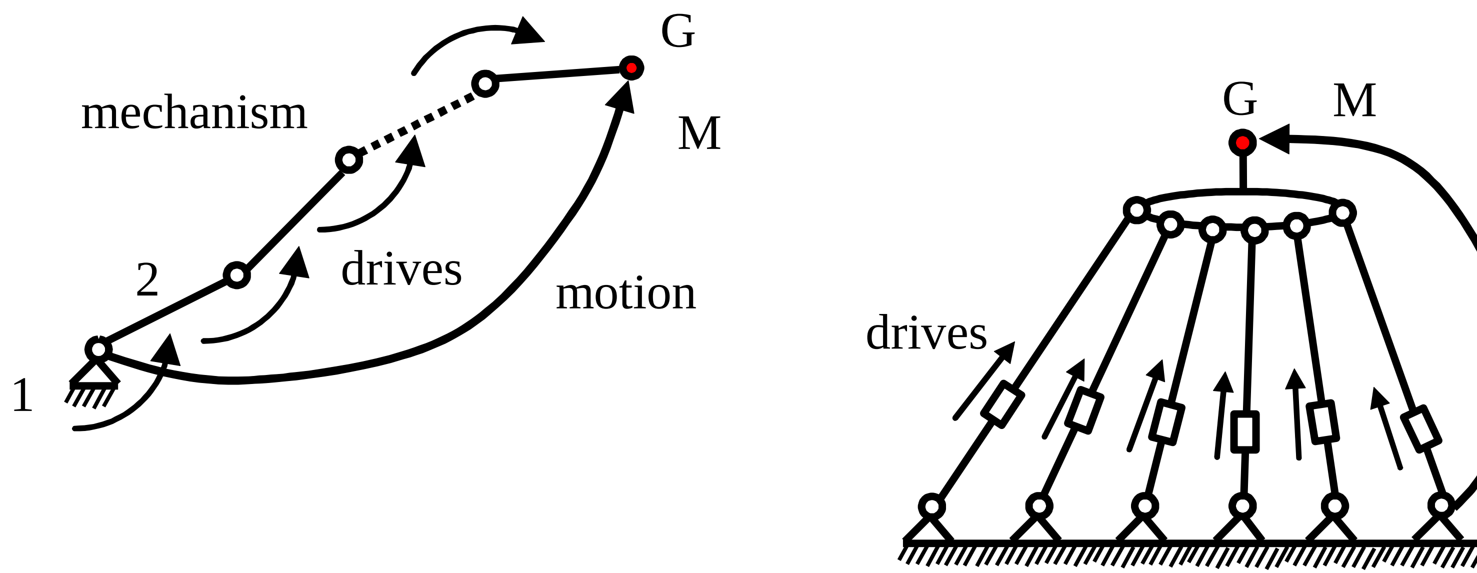
$$\mathbf{T}_{1G} = \mathbf{T}_{1G}(\mathbf{s}), \qquad \mathbf{s} = q$$

- Forward kinematic problem for structures with loops – the constraints solved first $\mathbf{f}(s) = 0$ and subsequently the direct computation

$$\mathbf{T}_{1G} = \mathbf{T}_{1G}(\mathbf{s})$$

- Inverse kinematic problem – the condition of prescribed motion is treated as constraint (they are combined with other contraints)

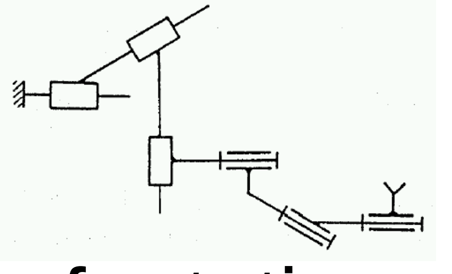
$$\mathbf{T}_{1G}(\mathbf{s}) = \mathbf{T}_{1M}(t) \qquad \mathbf{T}_{1G}(\mathbf{s}) - \mathbf{T}_{1M}(t) = \mathbf{0} \qquad \mathbf{f}_{GM}(\mathbf{s}) = \mathbf{0}$$



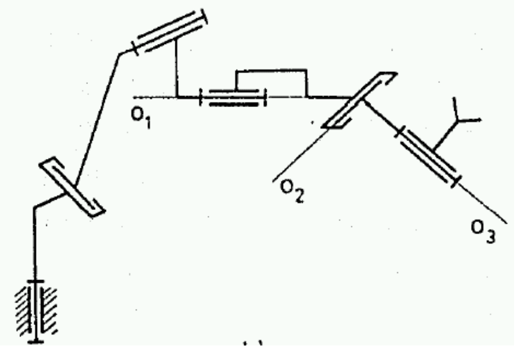
Some kinematical problems can be solved analytically.
Forward kinematical problem for serial robots easy.
Inverse kinematical problem for serial robots difficult.
Forward kinematical problem for parallel robots difficult.
Inverse kinematical problem for parallel robots easy.

Analytical solution of kinematics

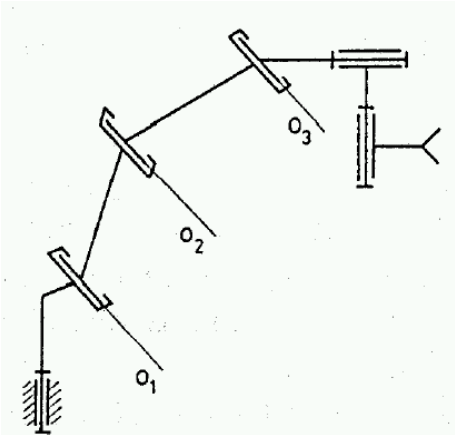
- Analytical solution possible if the equations can be split into two equation systems with three unknowns
- It is possible for the cases:
- 1) three translations

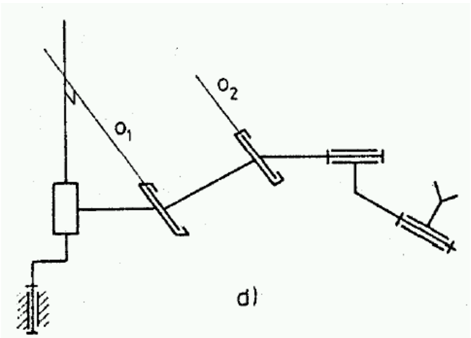


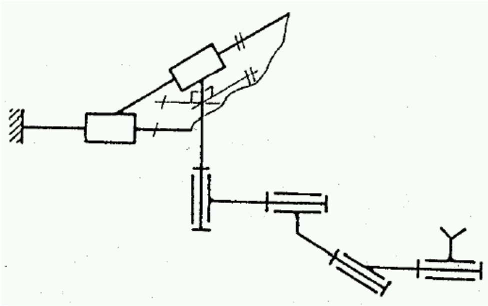
- 2) three intersecting axes of rotations (spherical joint)



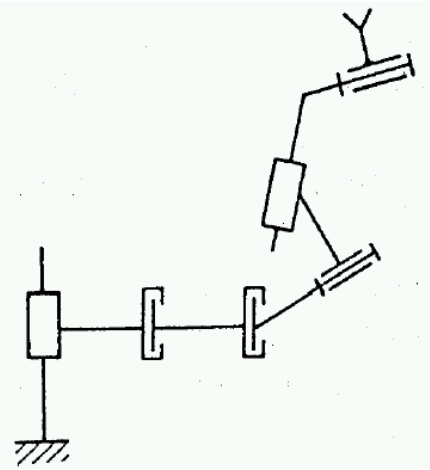
- 3) equivalent to plane joint (three parallel axes of rotation, two parallel axes of rotation and orthogonal translational axis, two translational axes and orthogonal rotational axis)







- 4) two translational axes and two parallel rotational axes



Example – robot with loops