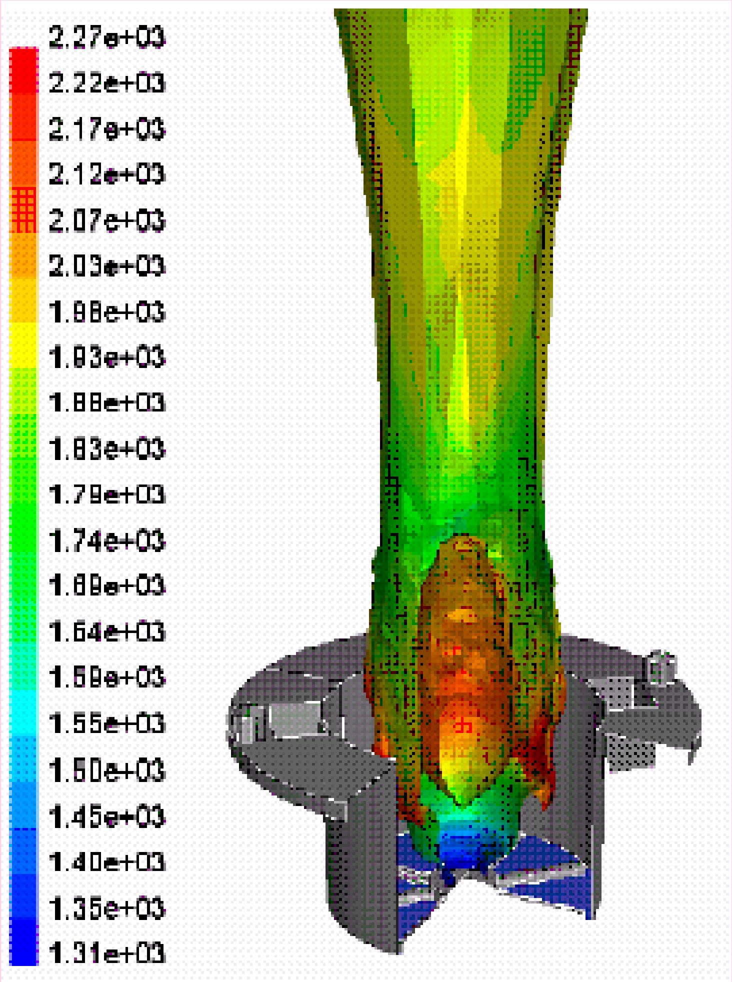
CFD1 Computer Fluid Dynamics E181107



General information about CFD course, prerequisities (tenzor calculus)

Remark: foils with „black background“ could be skipped, they are aimed to the more advanced courses

Rudolf Žitný, Ústav procesní a zpracovatelské techniky ČVUT FS 2013

CFD1

Comp.Fluid Dynamics **181107** 2+2

(Lectures+Tutorials), Exam, 4 credits

- Lectures Prof.Ing.Rudolf Žitný, CSc. Room 366, first lecture 5.10.2018, 12:30-14:00
- Tutorials ing.Karel Petera, PhD.
- Evaluation

A	B	C	D	E	F
90+	80+	70+	60+	50+	..49
excellent	very good	good	satisfactory	sufficient	failed

Summary: Lectures are oriented upon fundamentals of CFD and first of all to control volume methods (application using Fluent)

1. Applications. Aerodynamics. Drag coefficient. Hydraulic systems, Turbomachinery. Chemical engineering reactors, combustion.
2. Implementation CFD in standard software packages Fluent Ansys Gambit. Problem classification: compressible/incompressible. Types of PDE (hyperbolic, eliptic, parabolic) - examples.
3.Weighted residual Methods (steady state methods, transport equations). Finite differences, finite element, control volume and meshless methods.
4. Mathematical and physical requirements of good numerical methods: stability, boundedness, transportiveness. Order of accuracy. Stability analysis of selected schemes.
5. Balancing (mass, momentum, energy). Fluid element and fluid particle. Transport equations.
6. Navier Stokes equations. Turbulence. Transition laminar-turbulent. RANS models: gradient diffusion (Boussinesque). Prandtl, Spalart Alamaras, k-epsilon, RNG, RSM. LES, DNS.
7. Navier Stokes equations solvers. Problems: checkerboard pattern. Control volume methods: SIMPLE, and related techniques for solution of pressure linked equations. Approximation of convective terms (upwind, QUICK). Techniques implemented in Fluent.
8. Applications: Combustion (PDF models), multiphase flows.

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CFD KOS

(E181107) Computational Fluid Dynamics / FS

For more information about the CFD course look at my web pages http://users.fs.cvut.cz/rudolf.zitny/

Příjmení	**Jméno**
Kohout	Petr
Mysliveček	Matěj
Skoupý	Pavel
Královec	Václav
Dobos	Botond
Kim	Yujin
Gungor	Ogul Can
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Pawar	Pawan Dasharath
Sekar	Sivaprasadh
Koduru	Vinay Kiran
Akköse	Samed Ali
Balachandran	Praveen Kumar

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LITERATURE

- **Books:**

Versteeg H.K., Malalasekera W.:An introduction to CFD, Prentice Hall,1995

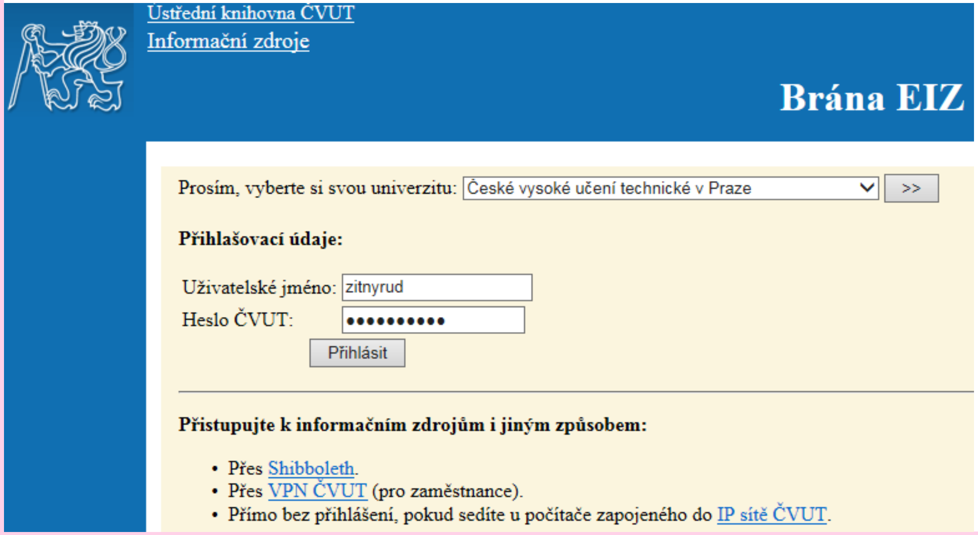
Date A.W.:Introduction to CFD. Cambridge Univ.Press, 2005

Anderson J.: CFD the basics and applications, McGraw Hill 1995

Tu J.et al: CFD a practical approach. Butterworth Heinemann 2ndEd. 2013

Database of scientific articles:

Students of CTU have direct access to full texts of thousands of papers, at knihovny.cvut.cz or directly as https://dialog.cvut.cz



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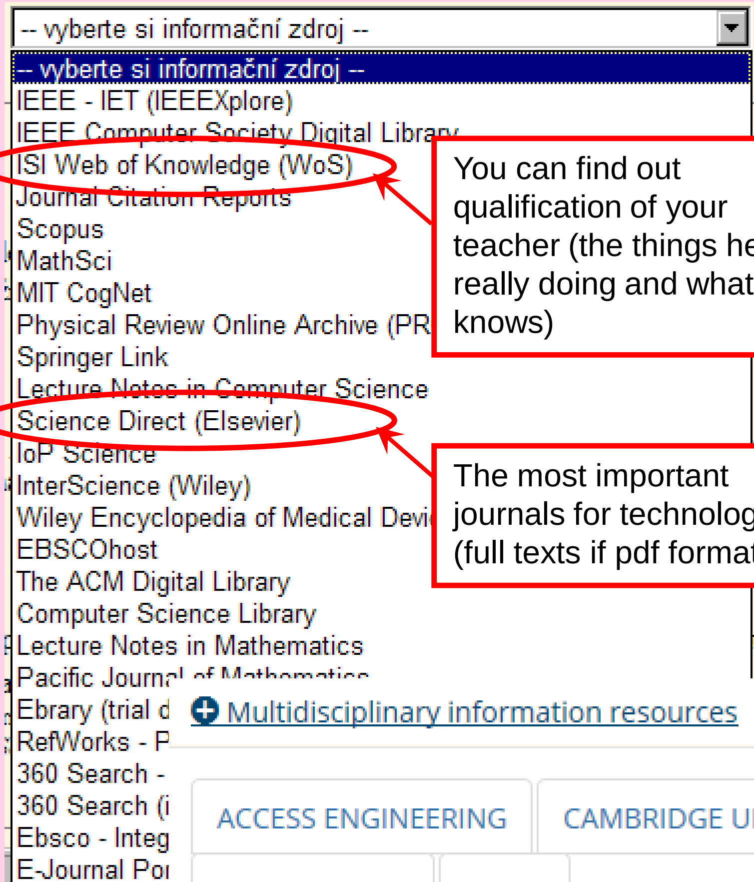
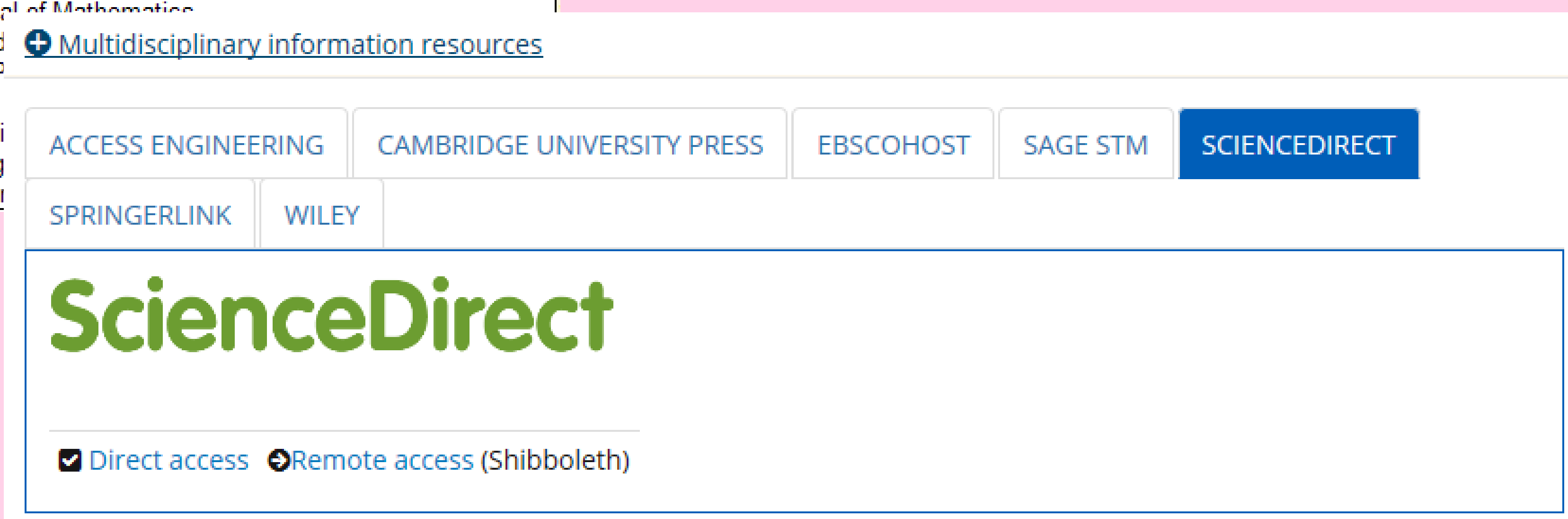
DATABASE selection



You can find out qualification of your teacher (the things he is really doing and what he knows)

The most important journals for technology (full texts if pdf format)



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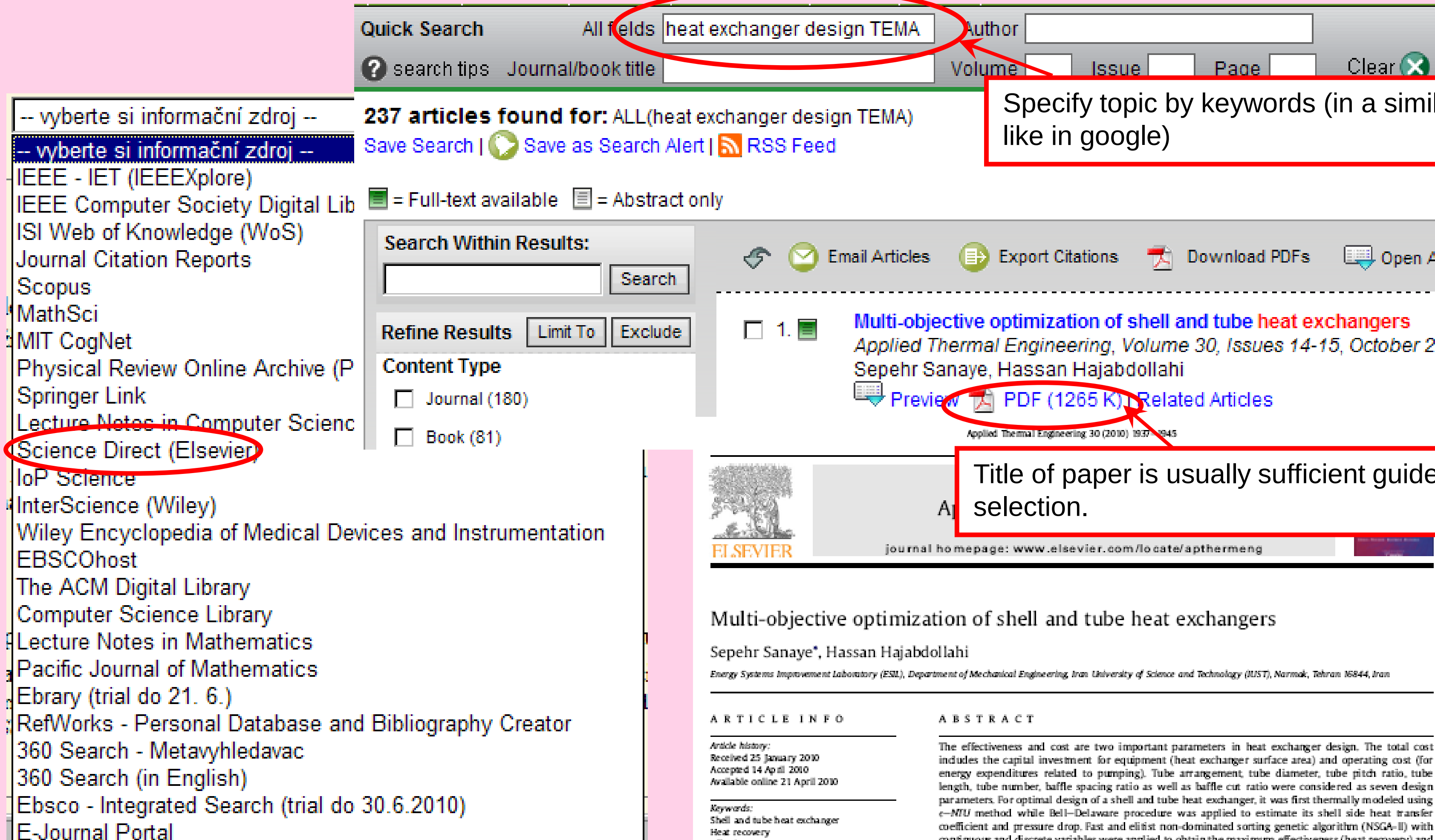
SCIENCE DIRECT



Applied Thermal Engineering 30 (2010) 1937–1945

ELSEVIER

journal homepage: www.elsevier.com/locate/apthermeng

Multi-objective optimization of shell and tube heat exchangers

Sepehr Sanaye*, Hassan Hajabdollahi

Energy Systems Improvement Laboratory (ESIL), Department of Mechanical Engineering, Iran University of Science and Technology (IUST), Narmak, Tehran 16844, Iran

ARTICLE INFO

Article history:
Received 25 January 2010
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Keywords:
Shell and tube heat exchanger
Heat recovery
Effectiveness
Total cost
Multi-objective optimization
NSGA-II

ABSTRACT

The effectiveness and cost are two important parameters in heat exchanger design. The total cost includes the capital investment for equipment (heat exchanger surface area) and operating cost (for energy expenditures related to pumping). Tube arrangement, tube diameter, tube pitch ratio, tube length, tube number, baffle spacing ratio as well as baffle cut ratio were considered as seven design parameters. For optimal design of a shell and tube heat exchanger, it was first thermally modeled using ε–NTU method while Bell–Delaware procedure was applied to estimate its shell side heat transfer coefficient and pressure drop. Fast and elitist non-dominated sorting genetic algorithm (NSGA-II) with continuous and discrete variables were applied to obtain the maximum effectiveness (heat recovery) and the minimum total cost as two objective functions. The results of optimal designs were a set of multiple optimum solutions, called 'Pareto optimal solutions'. The sensitivity analysis of change in optimum effectiveness and total cost with change in design parameters of the shell and tube heat exchanger was also performed and the results are reported.

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1. Introduction

Shell and tube heat exchanger is widely used in many industrial power generation plants as well as chemical, petrochemical, and petroleum industries. There are effective parameters in shell and tube heat exchanger design such as tube diameter, tube arrange-

as well as minimizing the total cost. Genetic algorithm optimization technique was applied to provide a set of Pareto multiple optimum solutions. The sensitivity analysis of change in optimum values of effectiveness and total cost with change in design parameters was performed and the results are reported.

As a summary, the followings are the contribution of this paper

Specify topic by keywords (in a similar way like in google)

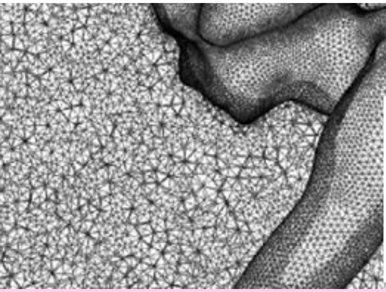
Title of paper is usually sufficient guide for selection.

CFD1 CFD Applications selected papers from Science Direct

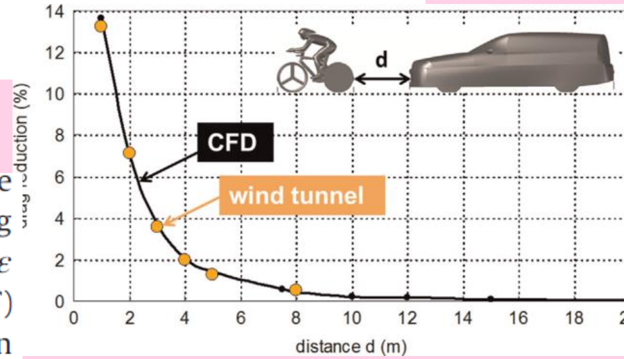
➢Aerodynamics. Keywords “Drag coefficient CFD” (126 matches 2011 , 6464 2012, **7526 2013, 10900 2016**)

☐ Development and validation of a new drag law using mechanical energy balance approach for DEM–CFD simulation of gas–solid fluidized bed Original Research Article
Chemical Engineering Journal, Volume 302, 15 October 2016, Pages 395-405
O.O. Ayeni, C.L. Wu, K. Nandakumar, J.B. Joshi
▸ Abstract | ▸ Research highlights | PDF (2264 K)

☐ A following car influences cyclist drag: CFD simulations and wind tunnel measurements Original Research Article
Journal of Wind Engineering and Industrial Aerodynamics, Volume 145, October 2015, Pages 178-186
Bert Blocken, Yasin Toparlar
▸ Abstract | ▸ Graphical abstract | ▸ Research highlights | PDF (4359 K)

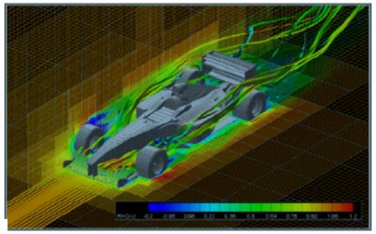


standard $k–\varepsilon$ model was made based on a previous extensive validation study for the aerodynamics of a single cyclist, including the standard, realizable and Re-normalization Group (RNG) $k–\varepsilon$ model, the standard $k–\omega$ model, the Shear–Stress Transport (SST) $k–\omega$ model and Large Eddy Simulation. This study, reported in



Keywords:“Racing car” (87 matches 2011), **October 2015 172** articles for (**Racing car CFD**)

2 ☐ CFD study of section characteristics of Formula Mazda race car wings Original Research Art
Mathematical and Computer Modelling, Volume 43, Issues 11-12, June 2006, Pages 1275-128
W. Kieffer, S. Moujaes, N. Armbya
Show preview | PDF (2411 K) | Related articles | Related reference work articles



3. Incompressible Navier–Stokes result showing u-velocity distribution and nlines around a formula-1 car model [43]. Total number of cubes is 5930 and ube has 32 × 32 × 32 cells, resulting 194,314,240 cells.

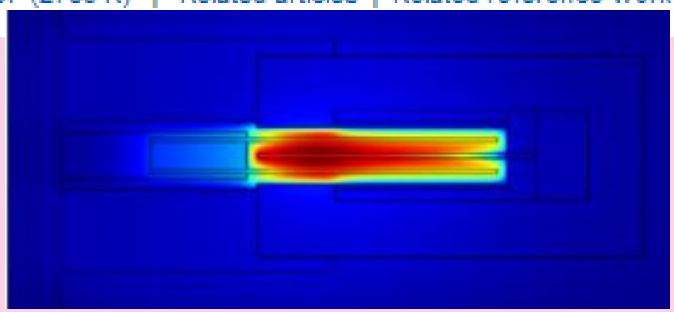
8 ☐ Aeronautical CFD in the age of Petaflops-scale computing: From unstructured to Cartesian meshes
European Journal of Mechanics - B/Fluids, Volume 40, July–August 2013, Pages 75-86
Kazuhiro Nakahashi

CFD1

CFD Applications selected papers from Science Direct

➢Hydraulic systems (fuel pumps, injectors) Keyword “Automotive magnetorheological brake design” gives 36 matches (2010), 51 matches (2011,October). 63 matches (2012,October), 74 (2013) , 88 (2015) Example

1 ☐ Design considerations for an automotive magnetorheological brake Original Research Article
Mechatronics, Volume 18, Issue 8, October 2008, Pages 434-447
Kerem Karakoc, Edward J. Park, Afzal Suleman
Show preview | PDF (2785 K) | Related articles | Related reference work articles



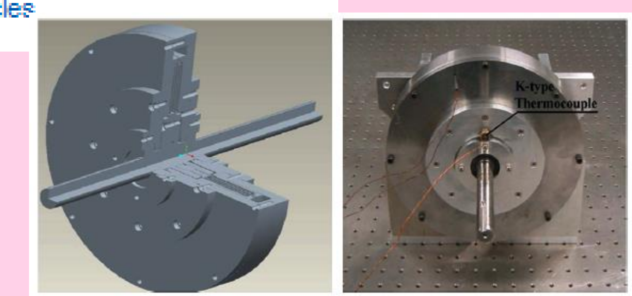


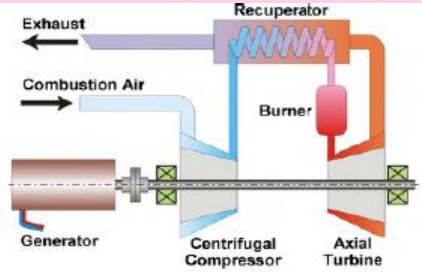
Fig. 12. CAD model (L) and prototype (R) of the proposed MRB.

☐ Two-dimensional CFD simulation of magnetorheological fluid between two fixed parallel plates applied external magnetic field Original Research Article
Computers & Fluids, Volume 63, 30 June 2012, Pages 128-134
Engin Gedik, Hüseyin Kurt, Ziyaddin Recebli, Corneliu Balan
Abstract | Graphical abstract | PDF (1052 K)

CFD1

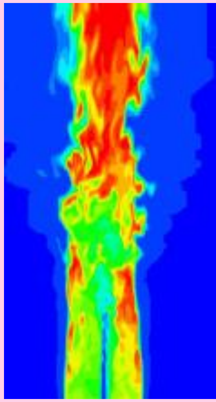
CFD Applications selected papers from Science Direct

➢Turbomachinery (gas turbines, turbocompressors)



5 Experimental and numerical investigation of a propane-fueled, catalytic mesoscale combustor Original Research Article
Catalysis Today, Volume 155, Issues 1-2, 1 October 2010, Pages 108-115
Symeon Karagiannidis, Kimon Marketos, John Mantzaras, Rolf Schaeren, Konstantinos Boulouchos
Show preview | PDF (600 K) | Related articles | Related reference work articles

➢Chemical engineering (reactors, combustion. Elsevier Direct, keywords "CFD combustion engine" 3951 papers in 2015. "CFD combustion engine spray injection droplets emission zone" 162 papers (2010), 262 articles (2011 October), 364 (October 2013). Examples of

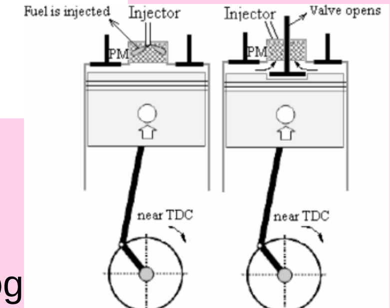


3 Modelling of instabilities in turbulent swirling flames Original Research Article
Fuel, Volume 89, Issue 1, January 2010, Pages 10-18
K.K.J. Ranga Dinesh, K.W. Jenkins, M.P. Kirkpatrick, W. Malalasekera
Show preview | PDF (1691 K) | Related articles | Related reference work articles

LES, non-premix, mixture fraction, Smagorinski subgrid scale turbulence model, laminar flamelets. These topics will be discussed in more details in this course.

Keywords "two-zone combustion model piston engine" 2100 matches (October 2012), **2,385** (October 2013)

1 Simulation of a porous medium (PM) engine using a two-zone combustion model Original Research Article
Applied Thermal Engineering, Volume 29, Issues 14-15, October 2009, Pages 3189-3197
Hongsheng Liu, Maozhao Xie, Dan Wu
Show preview | PDF (685 K) | Related articles | Related reference work articles



kinetic mechanism for iso-octane oxidation including 38 species and 69 elementary reactions was used for the chemistry simulation, which could predict satisfactorily ignition timing, burn rate and the emissions of HC, CO and NOx for HCCI engine (Homog

CFD1

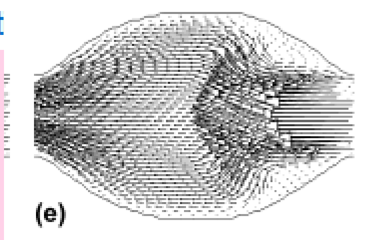
CFD Applications selected papers from Science Direct

- **Environmental** AGCM (atmospheric Global Circulation) finite differences and spectral methods, mesh 100 x 100 km, p (surface), 18 vertical layers for horizontal velocities, T, cH_2O,CH_4,CO_2, radiation modules (short wave-solar, long wave – terrestrial), model of clouds. AGCM must be combined with OGCM (oceanic, typically 20 vertical layers). FD models have problems with converging grid at poles - this is avoided by spectral methods. IPCC Intergovernmental Panel Climate Changes established by WMO World Meteorological Association.

16 Numerical weather prediction Original Research Article
Journal of Wind Engineering and Industrial Aerodynamics, Volume 90, Issues 12-15, December 2002, Pages 1403-1414
Ryuji Kimura
Show preview | PDF (557 K) | Related articles | Related reference work articles

- **Biomechanics**, blood flow in arteries (structural + fluid problem)

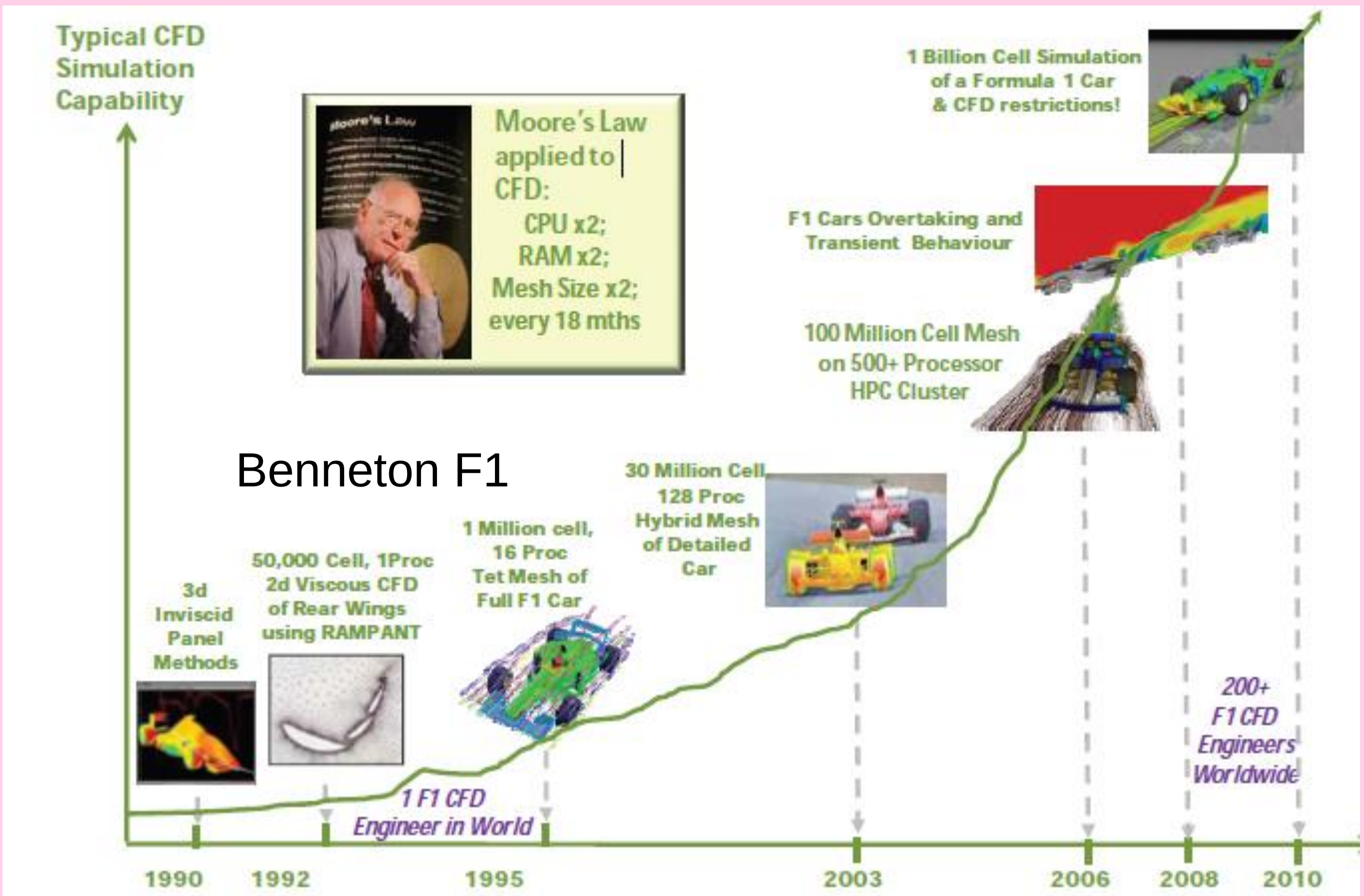
1 Steady and unsteady flow within an axisymmetric tube dilatation Original Research Article
Experimental Thermal and Fluid Science, Volume 34, Issue 7, October 2010, Pages 915-927
Ch. Stamatopoulos, Y. Papaharilaou, D.S. Mathioulakis, A. Katsamouris
Show preview | PDF (2075 K) | Related articles | Related reference work art



CFD Applications selected papers from Science Direct

- Sport

1 CFD in Sport - a Retrospective; 1992 - 2012 Original Research Article
Procedia Engineering, Volume 34, 2012, Pages 622-627
R. Keith Hanna
Show preview | PDF (571 K) | Related articles | Related reference work articles



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CFD Commertial software

Tutorials: ANSYS FLUENT

Bailey

CFD1

CFD ANSYS Fluent (CVM), CFX, Polyflow (FEM)

Single and Multiphase flows

Heat transfer & radiation

CFX(FEM) ~ Fluent

Remark: CFX is in fact CVM but using FE technology (isoparametric shape functions in finite elements)

FLUENT = Control Volume Method

Incompressible/compressible

Laminar/Turbulent flows

POLYFLOW = Finite Element Method

Incompressible flows

Laminar flows

Newtonian fluids **(air, water, oils…)**

$$\vec{\vec{\tau}} = 2\mu \vec{\vec{\Delta}}$$

tensor of viscous stress; tensor of rate of deformation

Turbulence models

RANS (Reynolds averaging) $\mu = \mu(turbulence)$

RSM (Reynolds stress) $\frac{D\vec{\vec{\tau}}}{Dt} = \vec{\vec{f}}(turbulence)$

Viscoelastic fluids **(polymers, rubbers…)**

Differential models Oldroyd type (Maxwell, Oldroyd B, Metzner White), PTT, Leonov (structural tensors)

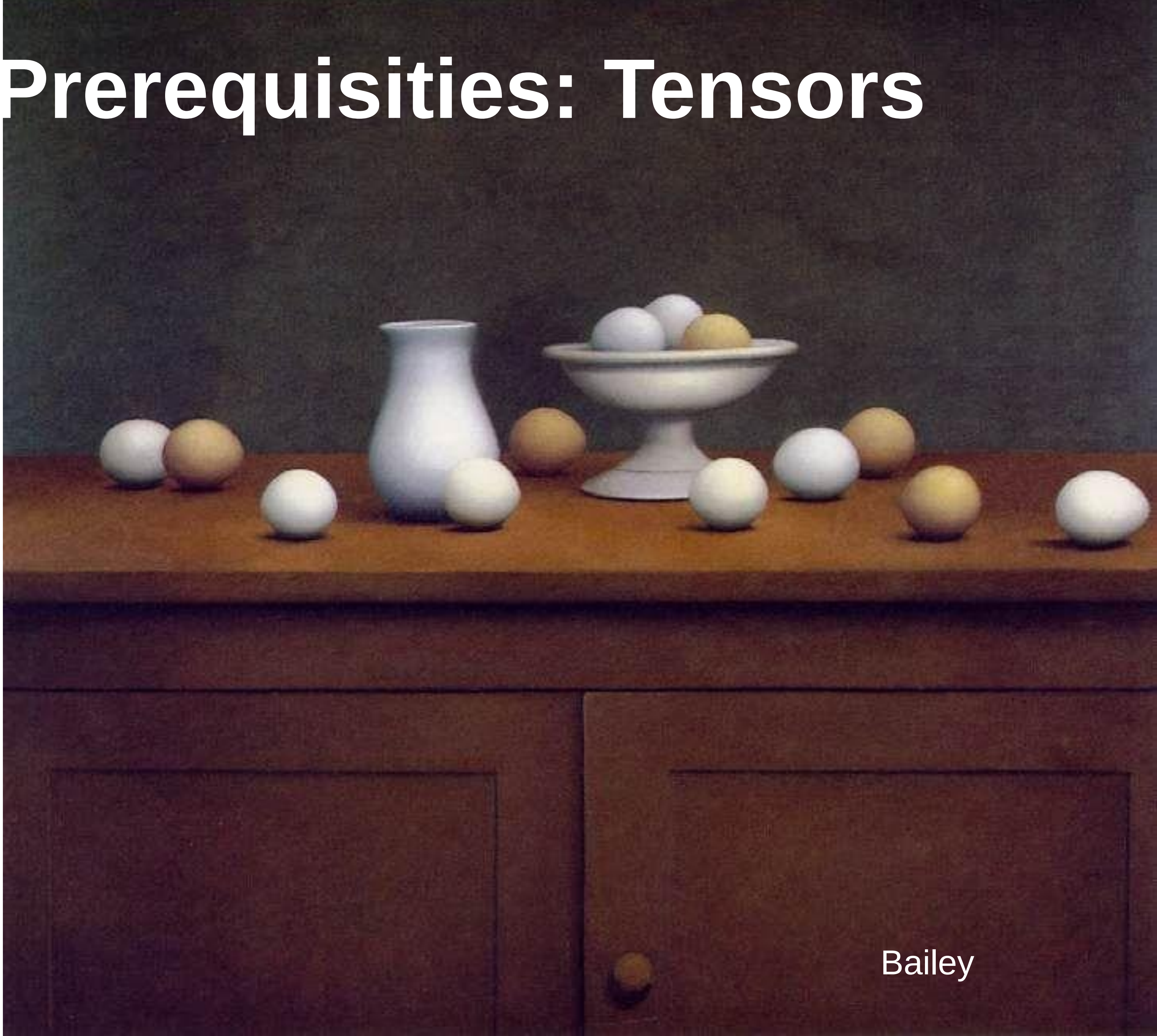
$$\lambda \frac{\delta \vec{\vec{\tau}}}{\delta t} + g\vec{\vec{\tau}} = 2\mu\vec{\vec{\Delta}}$$

Jaumann time derivative

Integral models $\vec{\vec{\tau}} = \int_0^\infty M(s) \underbrace{\vec{\vec{C}}(t-s)}_{\text{Cauchy Green deformation tensor}} ds$

CFD1

Prerequisities: Tensors



CFD1

Prerequisities: Tensors

CFD operates with the following properties of fluids (determining state at point x,y,z):

Scalars T (temperature), p (pressure), ρ (density), h (enthalpy), c_A (concentration), k (kinetic energy)

Vectors $\vec{u}$ (velocity), $\vec{f}$ (forces), ∇T (gradient of scalar)

Tensors $\vec{\vec{\tau}}$ $\vec{\vec{\sigma}}$ (stress), $\vec{\vec{\Delta}}$ (rate of deformation), $\nabla\vec{u}$ (gradient of vector)

Scalars are determined by 1 number.

Vectors are determined by 3 numbers $\vec{u} = (u_x, u_y, u_z) = (u_1, u_2, u_3)$

Tensors are determined by 9 numbers

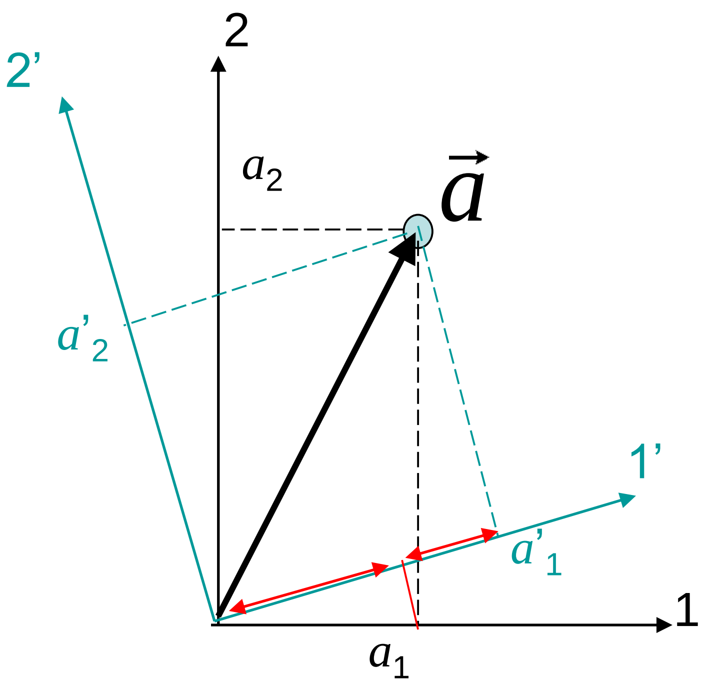
$$\vec{\vec{\sigma}} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Scalars, vectors and tensors are independent of coordinate systems (they are objective properties). However, components of vectors and tensors depend upon the coordinate system. Rotation of axis has no effect upon vector (its magnitude and arrow direction), but coordinates of the vector are changed (coordinates u_i are projections to coordinate axis).

CFD1

Rotation of cartesian coordinate system

Three components of a vector represent complete description (length of an arrow and its directions), but these components depend upon the choice of coordinate system. Rotation of axis of a cartesian coordinate system is represented by transformation of the vector coordinates by the matrix product



$$a_1' = a_1 \cos(1',1) + a_2 \cos(1',2) + a_3 \cos(1',3)$$

$$a_2' = a_1 \cos(2',1) + a_2 \cos(2',2) + a_3 \cos(2',3)$$

$$a_3' = a_1 \cos(3',1) + a_2 \cos(3',2) + a_3 \cos(3',3)$$

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} \cos(1',1) & \cos(1',2) & \cos(1',3) \\ \cos(2',1) & \cos(2',2) & \cos(2',3) \\ \cos(3',1) & \cos(3',2) & \cos(3',3) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

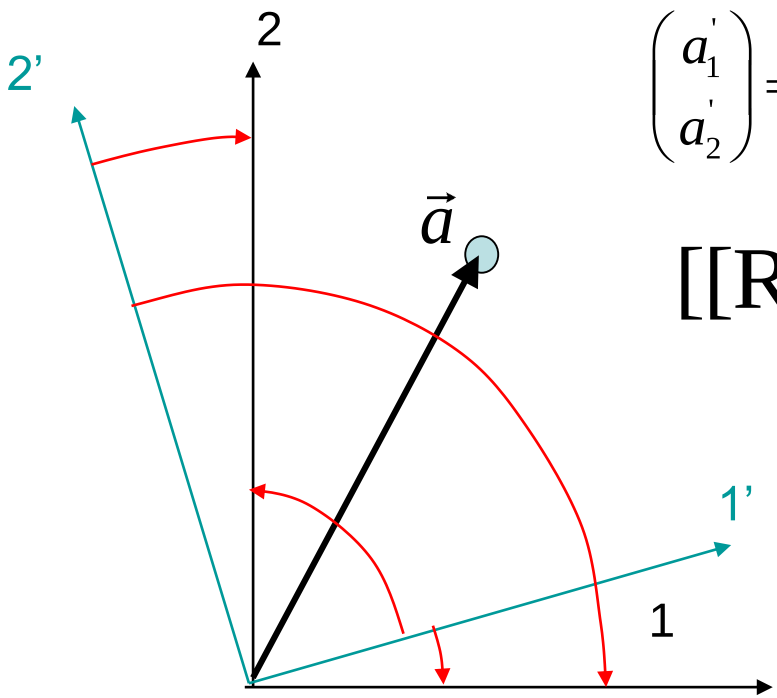
$$[a'] = [[\mathrm{R}]][a]$$

Rotation matrix (R_{ij} is cosine of angle between axis i' and j')

Rotation of cartesian coordinate system

Example: Rotation only along the axis 3 by the angle φ (positive for counter-clockwise direction)

Properties of goniometric functions $\cos(-\varphi)=\cos\varphi \quad \cos(\frac{\pi}{2}-\varphi)=\sin\varphi \quad \cos(-(\frac{\pi}{2}+\varphi))=-\sin\varphi$



$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \cos(1',1)=\cos\varphi & \cos(1',2)=\sin\varphi \\ \cos(2',1)=-\sin\varphi & \cos(2',2)=\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$[[R]]^T[[R]]=[[I]] \rightarrow [[R]]^{-1}=[[R]]^T$$

therefore the rotation matrix is orthogonal and can be inverted just only by simple transposition (overturning along the main diagonal). Proof:

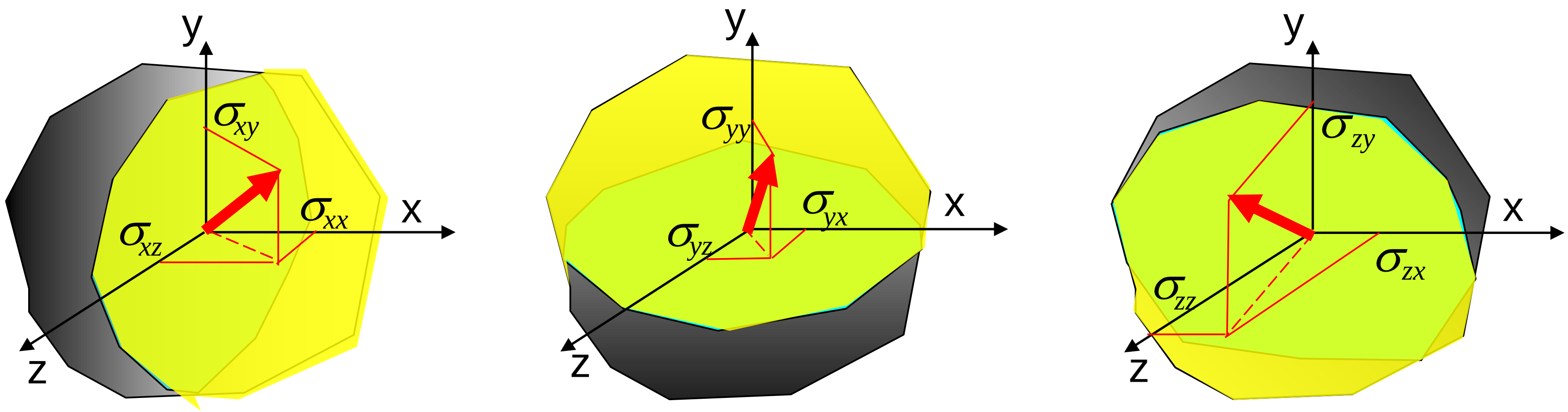
$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2\varphi+\sin^2\varphi & \cos\varphi\sin\varphi-\sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi-\cos\varphi\sin\varphi & \sin^2\varphi+\cos^2\varphi \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

CFD1

Stresses **describe complete stress state at a point x,y,z**



$$\sigma_{ij}$$

Index of plane (cross section) — i

index of force component (force acting upon the cross section i) — j

CFD1

Tensor rotation of cartesian coordinate system

Later on we shall use another tensors of the second order describing kinematics of deformation (deformation tensors, rate of deformation,…)

Nine components of a tensor represent complete description of state (e.g. distribution of stresses at a point), but these components depend upon the choice of coordinate system, the same situation like with vectors. The transformation of components corresponding to the rotation of the cartesian coordinate system is given by the matrix product

$$[[\sigma']] = [[\mathrm{R}]][[\sigma]][[\mathrm{R}]]^T$$

where the rotation matrix [[R]] is the same as previously

$$[[R]] = \begin{pmatrix} \cos(1',1) & \cos(1',2) & \cos(1',3) \\ \cos(2',1) & \cos(2',2) & \cos(2',3) \\ \cos(3',1) & \cos(3',2) & \cos(3',3) \end{pmatrix}$$

CFD1

Special tensors

Kronecker delta (unit tensor)

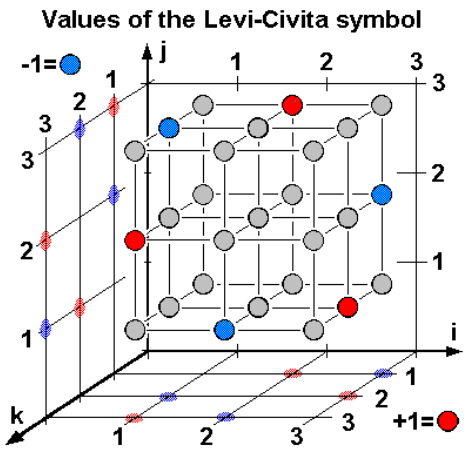
$$\delta_{ij} = 0 \ \text{ for } i \neq j$$

$$\delta_{ij} = 1 \ \text{ for } i = j$$

$$\vec{\vec{\delta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Levi Civita tensor is antisymmetric unit tensor of the third order (with 3 indices)

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2),(3,2,1) \text{ or } (2,1,3), \\ 0 & \text{otherwise: } i=j \text{ or } j=k \text{ or } k=i, \end{cases}$$



In term of epsilon tensor vector product will be defined

CFD1

Scalar product

Scalar product (operator •) of two vectors is a scalar

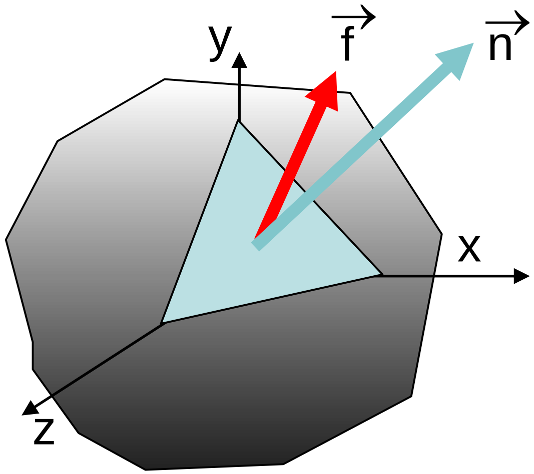
$$\vec{a}\bullet\vec{b}=a_1b_1+a_2b_2+a_3b_3=\sum_{i=1}^{3}a_ib_i=a_ib_i$$

a_ib_i is abbreviated Einstein notation (repeated indices are summing indices)

Scalar product can be applied also between tensors or between vector and tensor

$$\vec{n}\bullet\vec{\vec{\sigma}}=\vec{f} \qquad \sum_{i=1}^{3}n_i\sigma_{ij}=n_i\sigma_{ij}=f_j$$

i-is summation (dummy) index, while j-is free index



This case explains how it is possible to calculate internal stresses acting at an arbitrary cross section (determined by outer normal vector n) knowing the stress tensor.

CFD1

Scalar product

Derive dot product of delta tensor!

Define double dot product!

CFD1

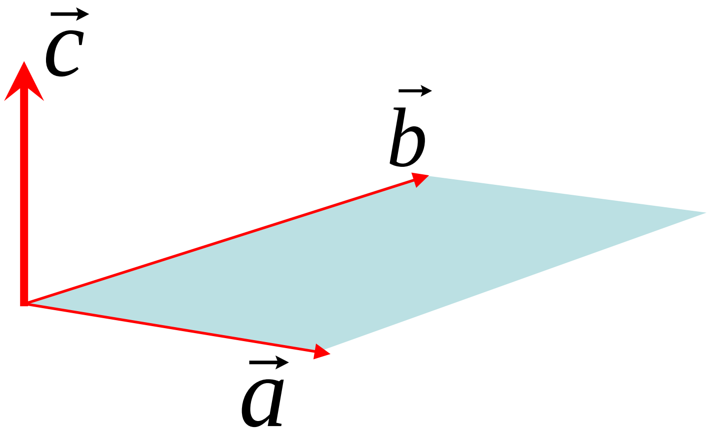
Vector product

Scalar product (operator •) of two vectors is a scalar. Vector product (operator x) of two vectors is a vector.

$$\vec{c} = \vec{a} \times \vec{b} = (\vec{\vec{\vec{\varepsilon}}} \bullet \vec{b}) \bullet \vec{a}$$

$$c_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k = \varepsilon_{ijk} a_j b_k$$

For example $$c_1 = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$$

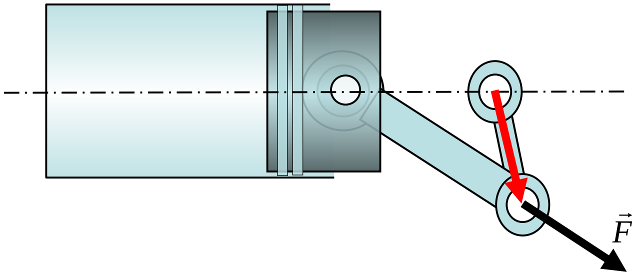


CFD 1

Vector product

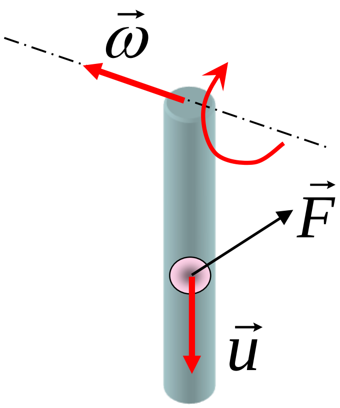
Examples of applications

Moment of force (torque) $\vec{M} = \vec{r} \times \vec{F}$

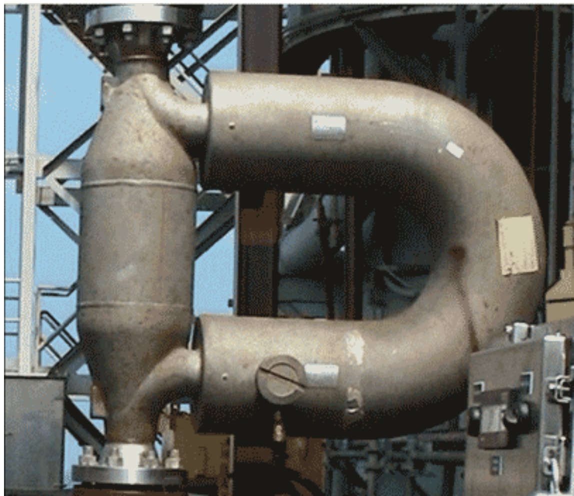


Coriolis force

$$\vec{F} = 2m\vec{u} \times \vec{\omega}$$



application: Coriolis flowmeter



CFD1

Differential operator ∇

GRADIENT

Symbolic operator ∇ represents a vector of first derivatives with respect x,y,z.

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}) \qquad \nabla_{\mathrm{i}} = \frac{\partial}{\partial x_i}$$

∇ applied to scalar is a vector (gradient of scalar)

$$\nabla T = (\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}) \qquad \nabla_i T = \frac{\partial T}{\partial x_i}$$

∇ applied to vector is a tensor (for example gradient of velocity is a tensor)

$$\nabla \vec{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix} \qquad \nabla_{\mathrm{i}} u_j = \frac{\partial u_j}{\partial x_i}$$

Differential operator ∇

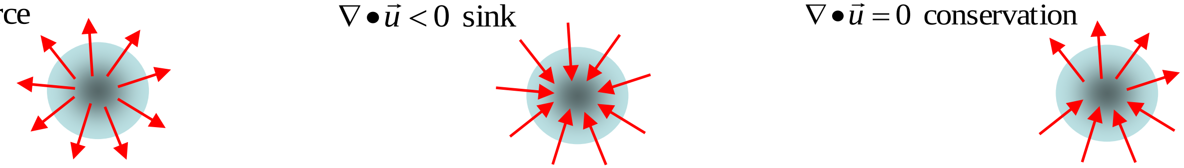
DIVERGENCY

Scalar product ∇• represents intensity of source/sink of a vector quantity at a point

$$\nabla \bullet \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}$$

i-dummy index, result is a scalar

$\nabla \bullet \vec{u} > 0$ source $\quad \nabla \bullet \vec{u} < 0$ sink $\quad \nabla \bullet \vec{u} = 0$ conservation



Scalar product ∇• can be applied also to a tensor giving a vector (e.g. source/sink of momentum in the direction x,y,z)

$$\vec{f} = \nabla \bullet \vec{\vec{\sigma}} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \qquad \mathrm{f}_\mathrm{i} = \nabla_\mathrm{j} \sigma_{ji}$$

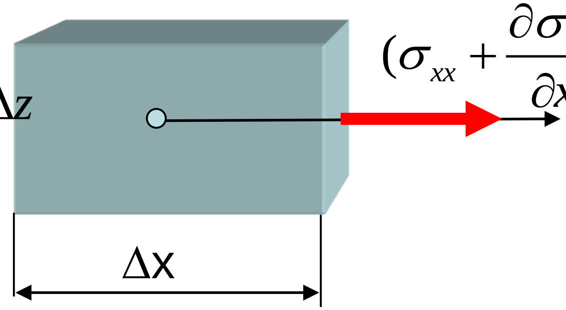
CFD1

Differential operator ∇

DIVERGENCY of stress tensor

(special case with only one non zero component σ_{xx})

$$(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2})\Delta y \Delta z$$



$$(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2})\Delta y \Delta z$$

$$\nabla \bullet \vec{\vec{\sigma}} = \underbrace{\frac{\partial \sigma_{xx}}{\partial x}\Delta x \Delta y \Delta z}_{\text{resulting surface force acting to small cube}} / \underbrace{\Delta x \Delta y \Delta z}_{\text{volume of cube}}$$

CFD1

Laplace operator ∇^2

Scalar product $\nabla \bullet \nabla = \nabla^2$ is the operator of second derivatives (when applied to scalar it gives a scalar, applied to a vector gives a vector,…). Laplace operator is divergence of a gradient (gradient temperature, gradient of velocity…)

$$\nabla \bullet \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial x_i \partial x_i}$$

i-dummy index

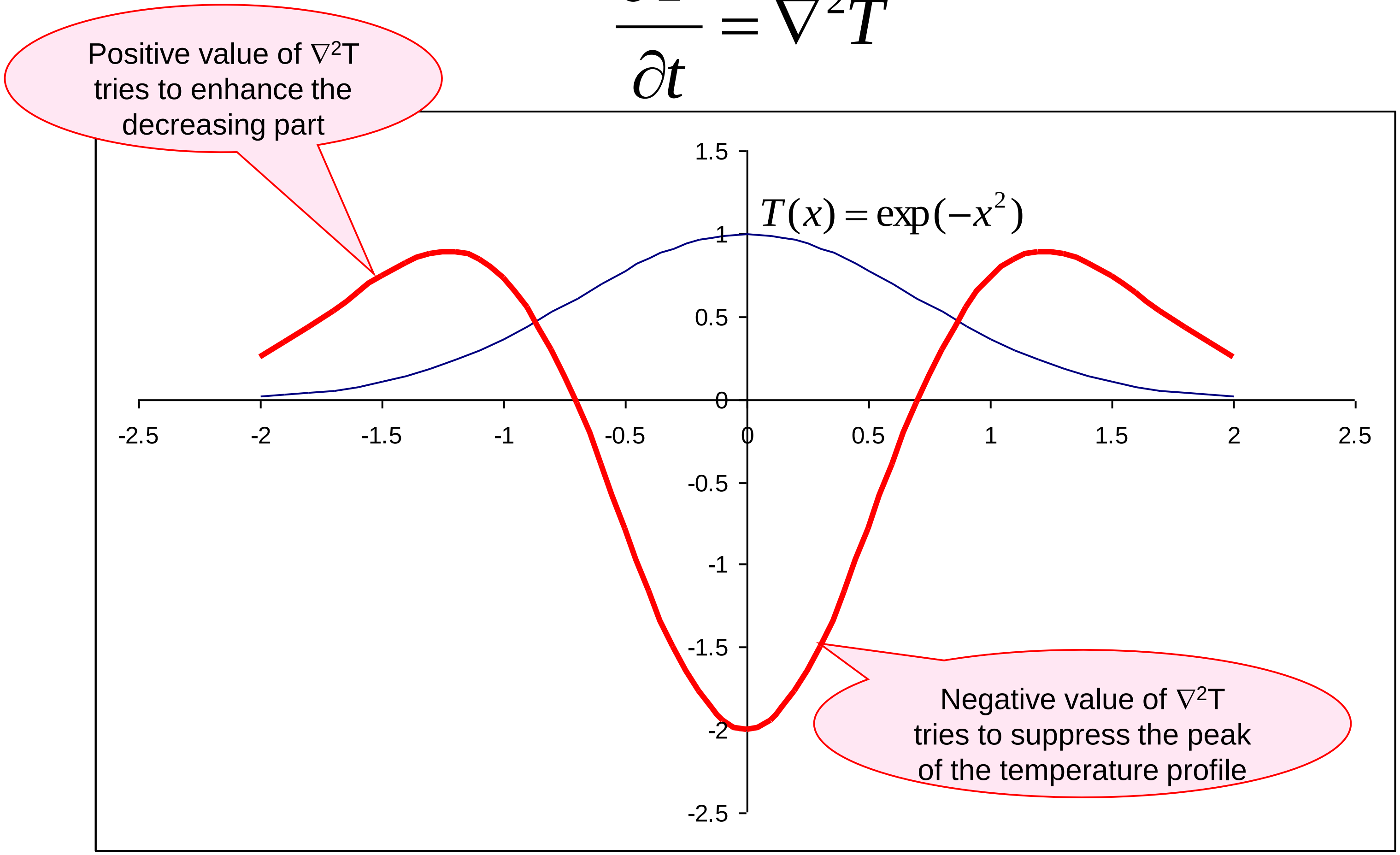
$$\nabla \bullet \nabla \vec{u} = \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}, \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}, \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) = \sum_{i=1}^{3} \frac{\partial^2 u_j}{\partial x_i^2} = \frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

Laplace operator describes diffusion processes, dispersion of temperature, concentration, effects of viscous forces.

MHMT1

Laplace operator ∇²

$$\frac{\partial T}{\partial t} = \nabla^2 T$$



CFD1

Symbolic $\rightarrow$ indicial notation

General procedure how to rewrite symbolic formula to index notation

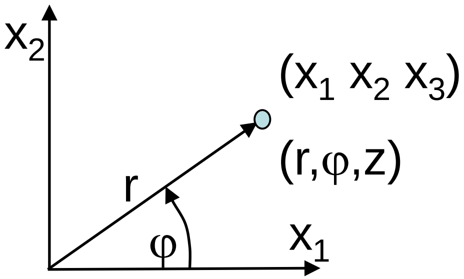
- Replace each arrow by an empty place for index
- Replace each vector operator by - • ε •
- Replace each dot • by a pair of dummy indices in the first free position left and right
- Write free indices into remaining positions

Practice examples!!

Orthogonal coordinates

Previous formula hold only in a cartesian coordinate systems

Cylindrical and spherical systems require transformations



$$x_1 = rc \qquad x_2 = rs \qquad x_3 = z$$

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix} \qquad \begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ \frac{s}{r} & -\frac{c}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

where $s = \sin\varphi \quad c = \cos\varphi$

Using this it is possible to express the same derivatives in different coordinate systems, for example

$$\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x_1} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x_1} = \frac{\partial T}{\partial r} c + \frac{\partial T}{\partial \varphi}\frac{s}{r}$$

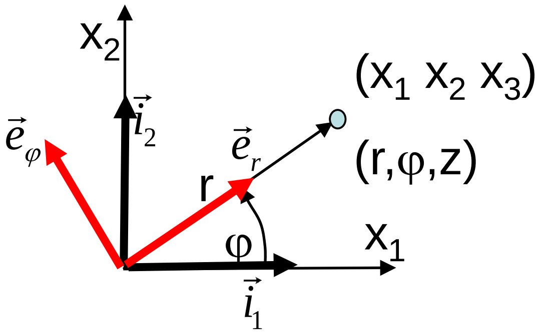
$$\frac{\partial T}{\partial x_2} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x_2} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x_2} = \frac{\partial T}{\partial r} s - \frac{\partial T}{\partial \varphi}\frac{c}{r}$$

$$\frac{\partial T}{\partial x_3} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_3} + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x_3} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x_3} = \frac{\partial T}{\partial z} s$$

CFD1

Orthogonal coordinates

Transformation of unit vectors



$$\begin{pmatrix}\vec{e}_r\\ \vec{e}_\varphi\\ \vec{e}_z\end{pmatrix}=\begin{pmatrix}c & s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\vec{i}_1\\ \vec{i}_2\\ \vec{i}_3\end{pmatrix}\qquad \begin{pmatrix}\vec{i}_1\\ \vec{i}_2\\ \vec{i}_3\end{pmatrix}=\begin{pmatrix}c & -s & 0\\ s & c & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\vec{e}_r\\ \vec{e}_\varphi\\ \vec{e}_z\end{pmatrix}$$

Example: gradient of temperature can written in any of the following ways

$$\nabla T=\frac{\partial T}{\partial x_1}\vec{i}_1+\frac{\partial T}{\partial x_2}\vec{i}_2+\frac{\partial T}{\partial x_3}\vec{i}_3=(c\frac{\partial T}{\partial x_1}+s\frac{\partial T}{\partial x_2})\vec{e}_r+(-s\frac{\partial T}{\partial x_1}+c\frac{\partial T}{\partial x_2})\vec{e}_\varphi+\frac{\partial T}{\partial x_3}\vec{e}_z=$$

$$=\frac{\partial T}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial T}{\partial \varphi}\vec{e}_\varphi+\frac{\partial T}{\partial z}\vec{e}_z$$

follows from transformation of unit vectors

follows from transformation of derivatives (previous slide)

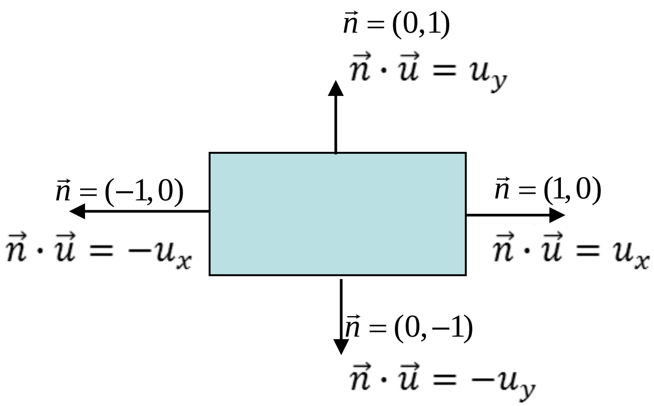
CFD1

Integral theorems

Gauss

$$\iint_{\Omega} \nabla \cdot \vec{u}\, d\Omega = \int_{\Gamma} \vec{n} \cdot \vec{u}\, d\Gamma$$

$$\iint_{\Omega} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) dxdy = \int_{\Gamma} (u_x n_x + u_y n_y)\, d\Gamma$$



Green (generalised per partes integration)

$$\iint_{\Omega} (\nabla^2 u) v\, d\Omega = -\iint_{\Omega} (\nabla u) \cdot (\nabla v)\, d\Omega + \int_{\Gamma} (\vec{n} \cdot \nabla u) v\, d\Gamma$$

$$\int_a^b \frac{d^2 u}{dx^2} v\, dx = -\int_a^b \frac{du}{dx} \frac{dv}{dx}\, dx + \left[v \frac{du}{dx}\right]_a^b$$