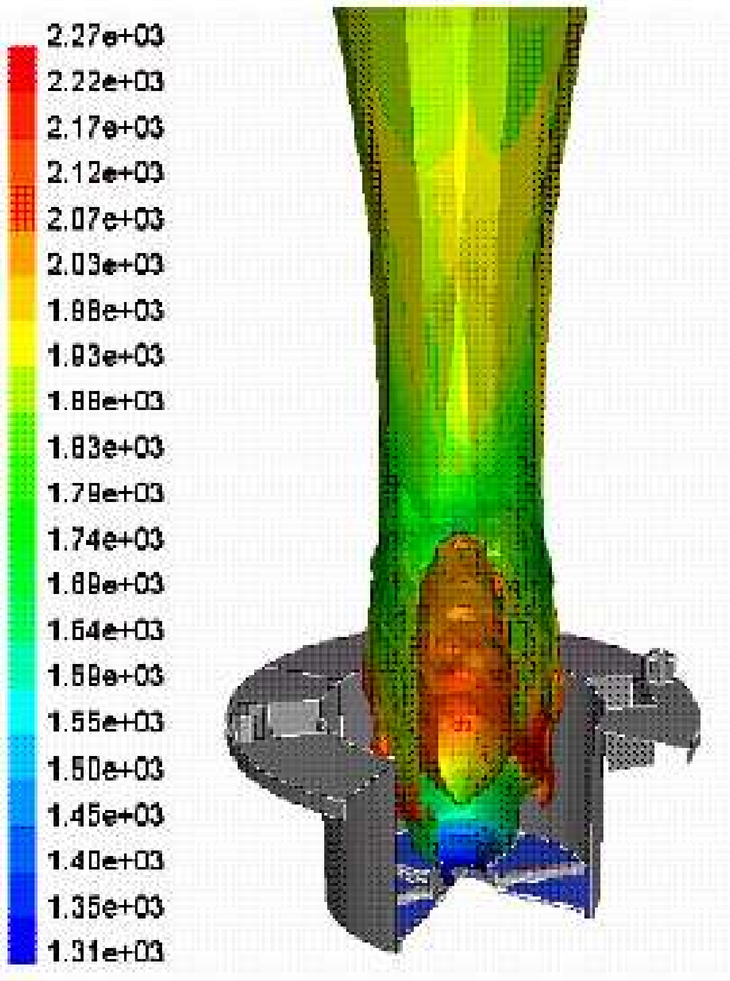
CFD3 **Computer Fluid Dynamics** 2181106 **E181107**




# Solvers, Approximation, Stability, Boundedness of Numerical schemes

Remark: foils with „black background“ could be skipped, they are aimed to the more advanced courses

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# CFD3 SOLVERS

FEM, BEM, FVM, FD transfer PDE into system of algebraic equations for $T_j$ (nodal pressures, velocities, temperature, concentrations…) solved by

- Finite methods (Gauss, SVD, LU decomposition, frontal methods)
$N^3$ operations are required – suitable for smaller systems.

- Iterative methods (GS, multigrid, GMRES, conjugated gradients).
Prevails at CFD calculations, characterized by number cells (nodes) of several millions and parallel processing (external as well as internal aerodynamics of cars requires up to $10^8$ finite volumes, solved in clusters of e.g. 512 and more processors)

Iterative methods are not so sensitive to round-off errors (that's why they can be applied for such huge systems)

# CFD3 Mathematical Requirements

Three Mathematical requirements

- **consistency** (discretized equation for $\Delta t \to 0, \Delta h \to 0$ must be identical with PDE)

$$T_{numerical} = T + O(\Delta t^m, \Delta h^n)$$

order of accuracy (m-with respect time, n-with respect to spatial approximation)

$$O(\Delta t^m, \Delta h^n) < K\Delta t^m + L\Delta h^n$$

- **stability** (attenuation of round-off errors or glitches of initial conditions)

- **convergency**. Lax theorem: consistent and stable numerical scheme converges to exact solution (but it holds only for linear systems)

# CFD3 Physical Requirements

Three Physical requirements

➢**Conservativeness**. Balance of mass should hold exactly at an element level and globally. Fulfilled by FVM (Finite Volume Method). Not exactly satisfied fy FEM.

➢**Boundedness**. Solution should not exhibit local min/max in the absence of internal sources (of mass, momentum or heat). Solution should be bounded by boundary values. Min/max principle.

➢**Transportivness**. Numerical scheme should reflect directionality of information transfer (convection along streamlines)

# CFD3 Numerical method-analysis

Few examples, how to analyze  order of accuracy and stability
of suggested numerical schemes (FD methods)

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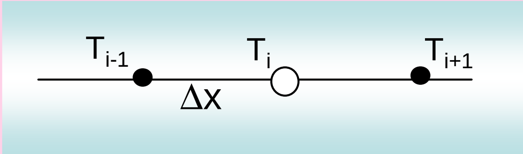
# Order of accuracy




Taylor expansion

$$T_{i+1} = T_i + \frac{dT_i}{dx}\Delta x + \frac{1}{2}\frac{d^2T_i}{dx^2}\Delta x^2 + \frac{1}{6}\frac{d^3T_i}{dx^3}\Delta x^3 + O(\Delta x^4)$$

$$T_{i-1} = T_i - \frac{dT_i}{dx}\Delta x + \frac{1}{2}\frac{d^2T_i}{dx^2}\Delta x^2 - \frac{1}{6}\frac{d^3T_i}{dx^3}\Delta x^3 + O(\Delta x^4)$$

Approximation of first derivative

Higher Order Terms

$$\frac{dT}{dx}(x_i) \cong \frac{T_{i+1} - T_{i-1}}{2\Delta x} = \frac{dT}{dx}(x_i) + \frac{d^3T}{dx^3}(x_i)\frac{\Delta x^2}{3} + HOT$$

Accurate for 2nd order polynomials $T=1,x,x^2$

$$\frac{dT}{dx}(x_i) \cong \frac{T_i - T_{i-1}}{\Delta x} = \frac{dT}{dx}(x_i) - \frac{d^2T}{dx^2}(x_i)\frac{\Delta x}{2} + HOT$$

Accurate for 1nd order polynomials $T=1,x$

Approximation of second derivative

$$\frac{d^2T}{dx^2}(x_i) \cong \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = \frac{d^2T}{dx^2}(x_i) + \frac{d^4T}{dx^4}(x_i)\frac{\Delta x^2}{12} + HOT$$

Accurate for 3rd order polynomials $T=1,x,x^2,x^3$

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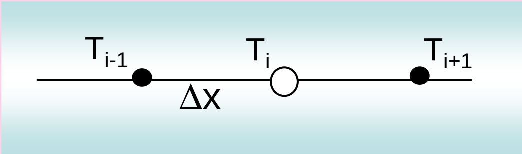
# CFD3 Order of accuracy




Therefore finite differences substituting derivatives at node $x_i$ are

- First order $$\frac{dT}{dx}(x_i) \cong \frac{T_i - T_{i-1}}{\Delta x}$$
- Second order $$\frac{dT}{dx}(x_i) \cong \frac{T_{i+1} - T_{i-1}}{2\Delta x}$$
- Third order $$\frac{d^2T}{dx^2}(x_i) \cong \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$

# CFD3 Stability example (explicit method)

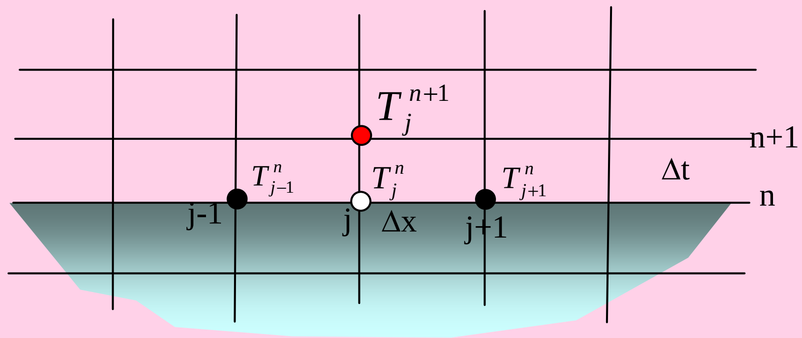
Unsteady heat transfer (Fourier equation – parabolic)

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

T-temperature, *a*-temperature diffusivity

Finite difference method **EXPLICIT** (explicit means that unknown temperatures at a new time level can be expressed explicitly, without necessity to solve a system of algebraic equations).

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = a\frac{T_{j-1}^n - 2T_j^n + T_{j+1}^n}{\Delta x^2}$$

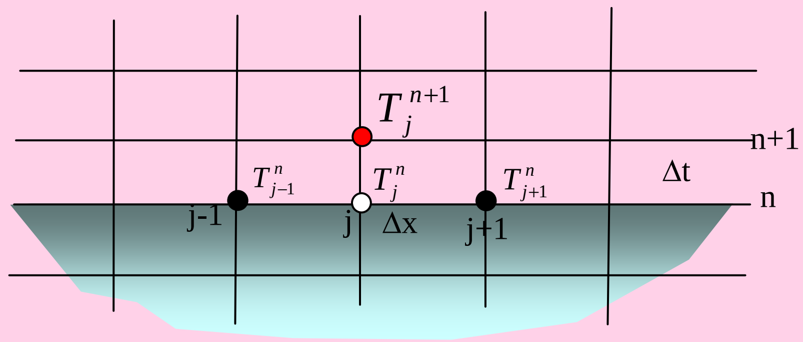



What is the order of accuracy?

# CFD3 Stability example (explicit method)

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = a\frac{T_{j-1}^n - 2T_j^n + T_{j+1}^n}{\Delta x^2}$$

$$\frac{\partial T}{\partial t} + O(\Delta t) = a\frac{\partial^2 T}{\partial x^2} + O(a\Delta x^3)$$




Residual of this PDE is therefore

$$res = O(\Delta t) + O(a\Delta x^3)$$

Scheme is consistent, linear with respect time, cubic with respect space.

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CFD3 **Stability** example (explicit method)

Rewrite the explicit formula to the following (explicit) form

$$T_j^{n+1} = \frac{a\Delta t}{\Delta x^2} T_{j-1}^n + \frac{a\Delta t}{\Delta x^2} T_{j+1}^n + (1 - \frac{2a\Delta t}{\Delta x^2}) T_j^n = A T_{j-1}^n + A T_{j+1}^n + (1 - 2A) T_j^n$$

Unknown temperature at a new time level

Known temperatures at “old” time level

$$A = \frac{a\Delta t}{\Delta x^2}$$

**Rules:**

➢Sum of coefficients must be the same on the left and on the right side (1=A+A+1-2A). Why? A constant solution must be fulfilled exactly!

➢All coefficients must be positive for bounded solution. Why?

# CFD3 **Stability** example (explicit method)

So why all coefficients must be positive for bounded solution?

Resulting value T is calculated as a weighted average of values (sum of weighting coefficients must be 1). Let us assume only two values for simplicity $T = AT_1 + (1-A)T_2$

and $T_1<T_2$. The solution is bounded if $T_1<T<T_2$. Let us assume, that the result is **not** bounded and $T<T_1$. Then

$$AT_1 + (1-A)T_2 < T_1$$

$$(1-A)T_2 < T_1(1-A)$$

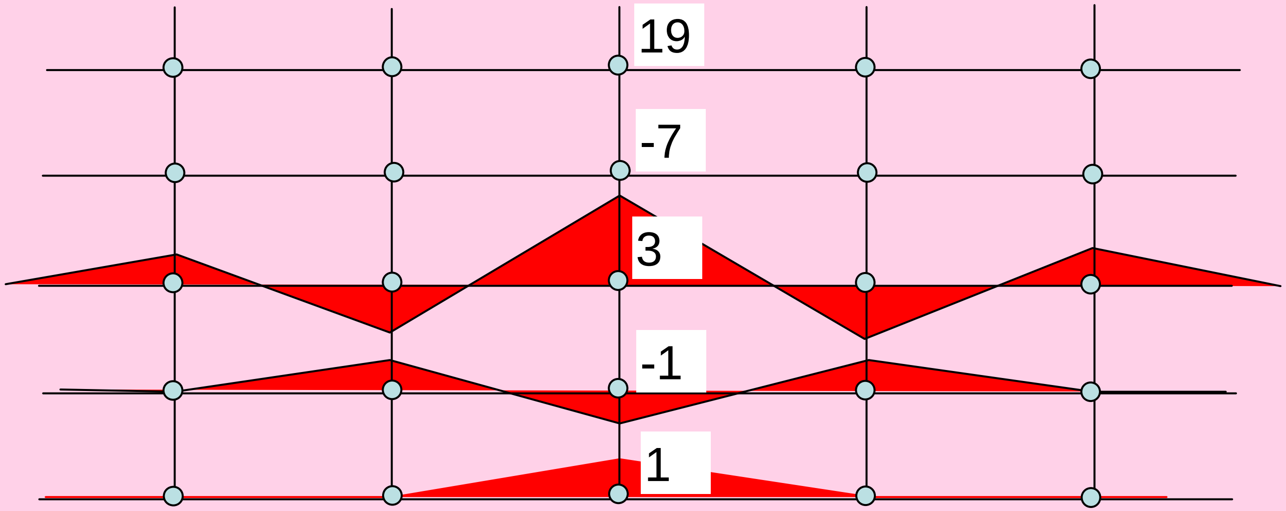
For positive value (1-A)>0 it follows that $T_1>T_2$ and this is contradiction.

# CFD3 Stability example of unbounded solution

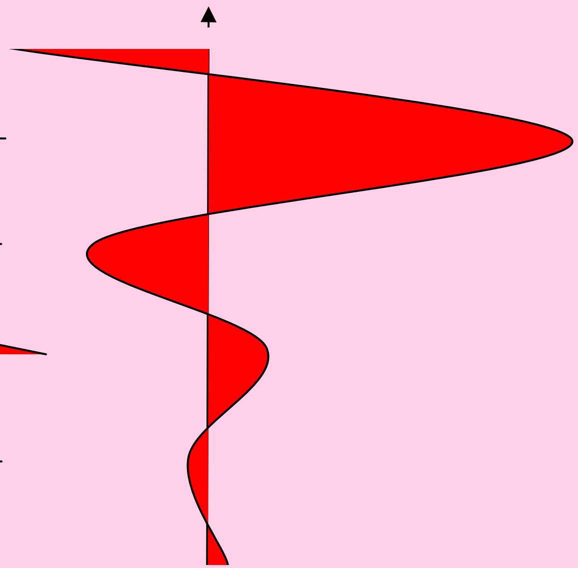
Let us consider what would happen if A=1 (negative value 1-2A)

$$T_j^{n+1} = AT_{j-1}^n + AT_{j+1}^n + (1-2A)T_j^n$$

$$T_j^{n+1} = T_{j-1}^n + T_{j+1}^n - T_j^n$$




Initial condition is 0 in all nodes and only in one node is 1.



Evolution of initial condition in node

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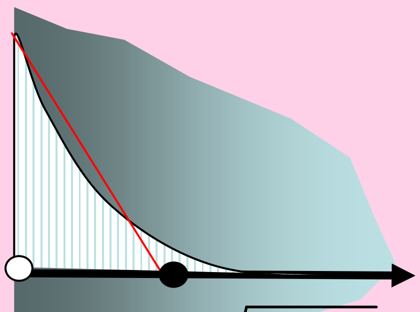
# CFD3 Stability example (explicit method)

Stability condition can be expressed as a restriction of time step

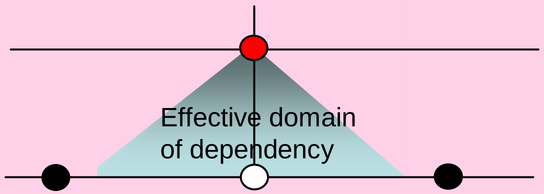
$$1-2A>0 \qquad A=\frac{a\Delta t}{\Delta x^2}$$

$$\Delta t<\frac{\Delta x^2}{2a}$$

Interpretation in terms of penetration theory. Effective velocity of a thermal pulse



$$\Delta x=\sqrt{\pi a\Delta t}$$



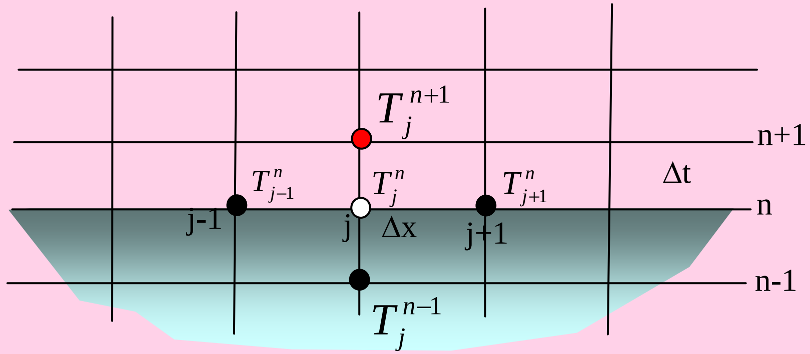

Δx-distance of penetrated thermal pulse at a time Δt

$$\Delta t<\frac{\Delta x^2}{\pi a}$$

# CFD3 Stability example of wrong scheme

Richardson's scheme for the solution of previous equation

$$\frac{T_j^{n+1}-T_j^{n-1}}{2\Delta t}=a\frac{T_{j-1}^n-2T_j^n+T_{j+1}^n}{\Delta x^2}$$




operates at 3 time levels, n-1,n,n+1
and has higher orders of accuracy

$$res=O(\Delta t^2)+O(a\Delta x^3)$$

However, the scheme is practically useless. **WHY?**

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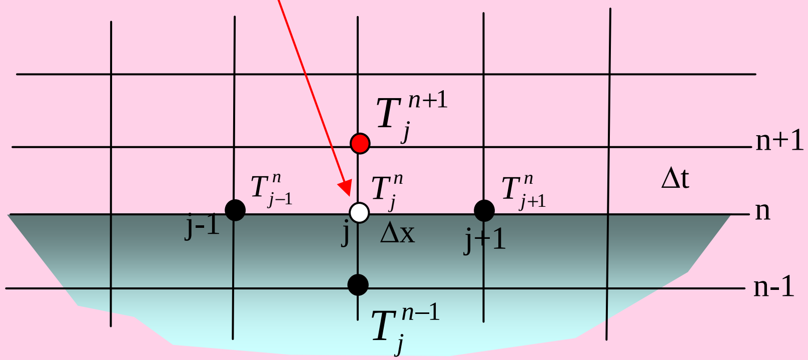
# CFD3 Stability example of wrong scheme

Because this coefficient is always negative

$$T_j^{n+1} = T_j^{n-1} + 2A(T_{j-1}^n - 2T_j^n + T_{j+1}^n)$$



# CFD3 Stability how to improve Richardson?

Richardson's scheme

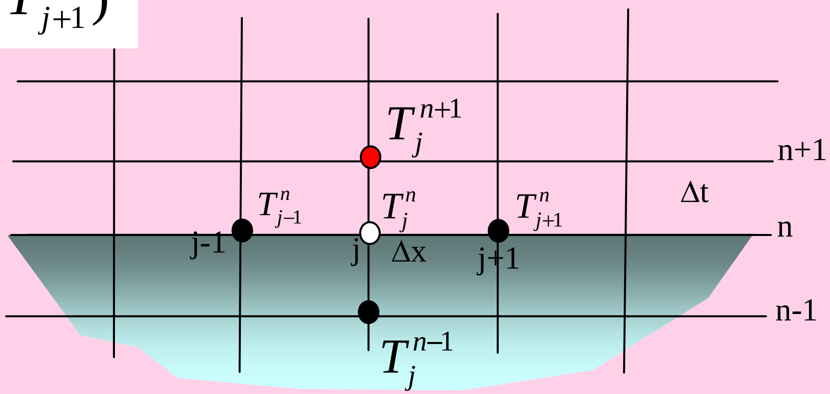
$$\frac{T_j^{n+1}-T_j^{n-1}}{2\Delta t}=a\frac{T_{j-1}^n-2T_j^n+T_{j+1}^n}{\Delta x^2}$$

duFort Frankel scheme

$$\frac{T_j^{n+1}-T_j^{n-1}}{2\Delta t}=a\frac{T_{j-1}^n-T_j^{n+1}-T_j^{n-1}+T_{j+1}^n}{\Delta x^2}$$

$$T_j^{n+1}(1+2A)=T_j^{n-1}(1-2A)+2A(T_{j-1}^n+T_{j+1}^n)$$

and this solution will be bounded for A<1/2. Order of accuracy remains high.




However:

Consistency with Fourier equation is assured only if $\lim_{\Delta x\to 0}\Delta t/\Delta x=0$

otherwise the hyperbolic equation of heat transfer would be solved

$$\frac{\partial T}{\partial t}=a\frac{\partial^2 T}{\partial x^2}-\frac{a}{c^2}\frac{\partial^2 T}{\partial t^2}$$

where $c=\frac{\Delta x}{\Delta t}$

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CFD3 **Stability Neumann**

More precise (and more complicated) is the stability analysis suggested by von Neumann. It is based upon spectral decomposition of solution, i.e. at a time level n the spatial profile is substituted by Fourier expansion

$$T^n(x) = G_k^n e^{ik\pi x/\Delta x}$$

This Fourier component is substituted into differential equation and amplification factor G is evaluated. Numerical scheme is stable, as soon as the magnitude of identified amplification factor decreases.