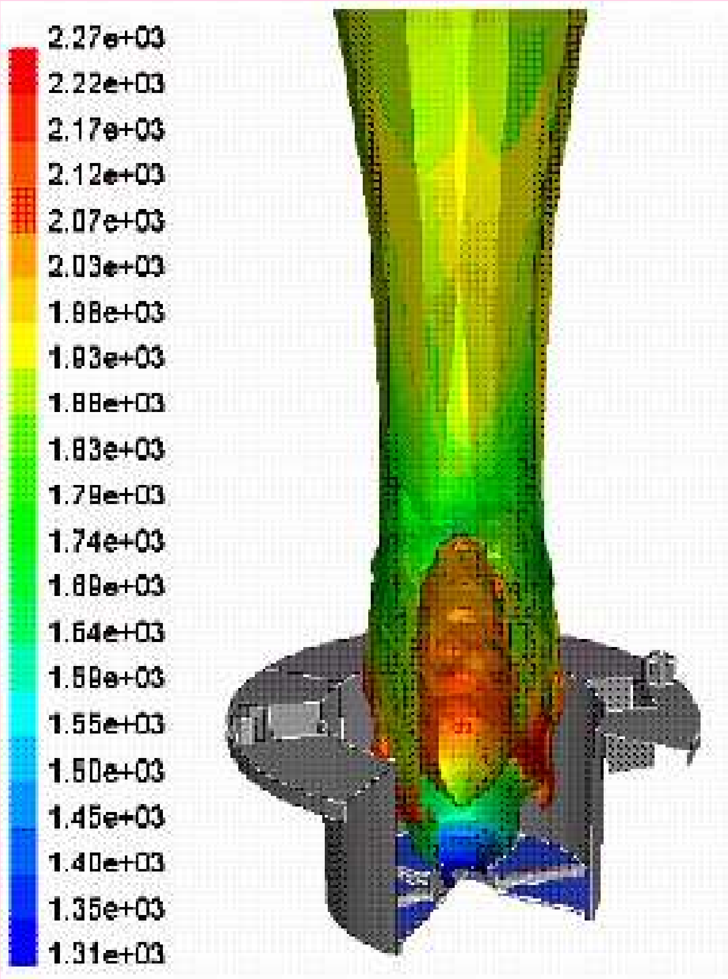
CFD4 **Computer Fluid Dynamics** 2181106 **E181107**



Balancing, transport equations

Remark: foils with „black background“ could be skipped, they are aimed to the more advanced courses

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CFD4 **Balancing**

CFD is based upong conservation laws

-conservation of mass

-conservation of momentum m.du/dt=F (second Newton.s law)

-conservation of energy dq=du+pdv (first law of thermodynamics)

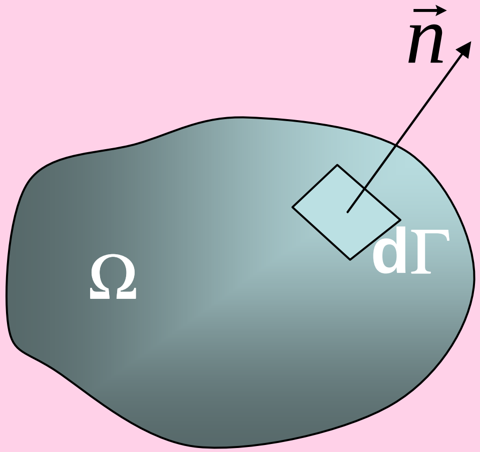
System is considered as continuum and described by macroscopic variables $\vec{u}, p, \rho, h$

CFD4 Integral balancing - Gauss

Control volume balance expressed by Gauss theorem

accumulation = flux through boundary

$$\iiint_{\Omega} \underbrace{\nabla \bullet P}_{\text{Divergence of P}} d\Omega = \iint_{\Gamma} \underbrace{\vec{n} \bullet P}_{\text{projection of P to outer normal}} d\Gamma$$



Variable $\boldsymbol{P}$ can be

- **Vector** (vector of velocity, momentum, heat flux). Surface integral represents flux of vector in the direction of outer normal.
- **Tensor** (tensor of stresses). In this case the Gauss theorem represents the balance between inner stresses and outer forces acting upon the surface, in view of the fact that $\vec{n} \bullet \vec{\vec{\sigma}} d\Gamma = d\vec{f}$ is the vector of forces acting on the oriented surface dΓ.

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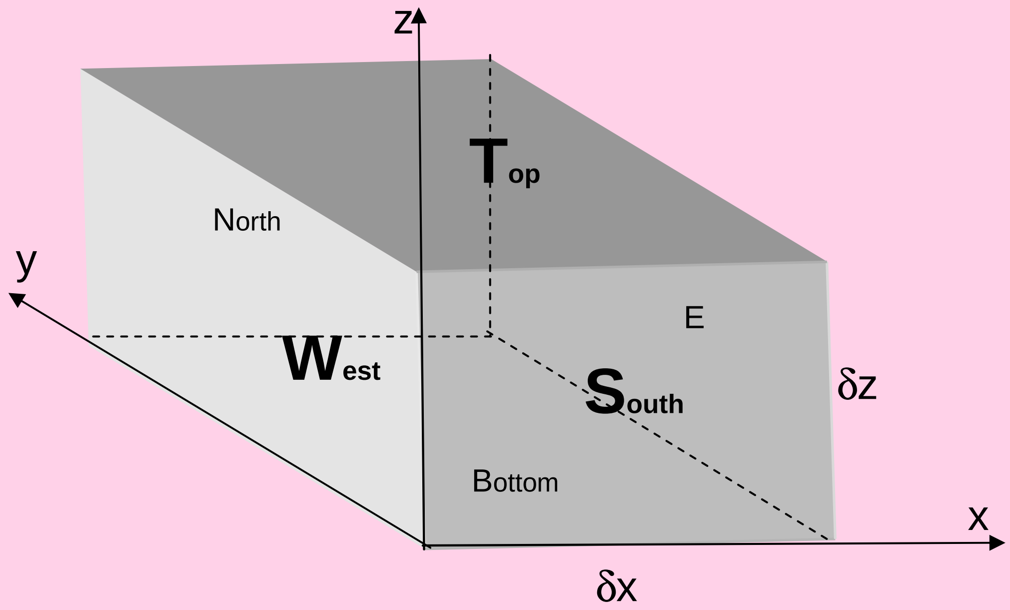
Fluid ELEMENT

Motion of fluid is described either by

- Lagrangian coordinate system (tracking individual particles along streamlines)
- Eulerian coordinate system (fixed in space, flow is characterized by velocity field)

Balances in Eulerian description are based upon identification of fluxes through sides of a box (FLUID ELEMENT) fixed in space. Sides if the box in the 3D case are usually marked by letters W/E, S/N, and B/T.



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Mass balancing (fluid element)

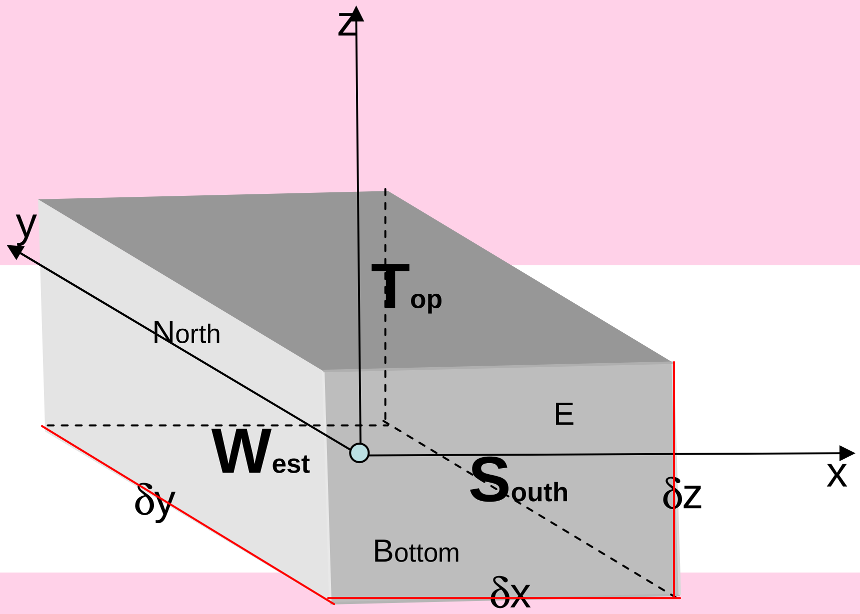
Accumulation of mass

Mass flowrate through sides W and E

$$\frac{\partial}{\partial t}(\rho\delta x\delta y\delta z)=(WE)\delta y\delta z+(SN)\delta x\delta z+(BT)\delta x\delta y=$$

$$=[(\rho u-\frac{1}{2}\frac{\partial \rho u}{\partial x}\delta x)-(\rho u+\frac{1}{2}\frac{\partial \rho u}{\partial x}\delta x)]\delta y\delta z+\ldots=-\frac{\partial \rho u}{\partial x}\delta x\delta y\delta z-\ldots$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$



CFD4 **Mass balancing**

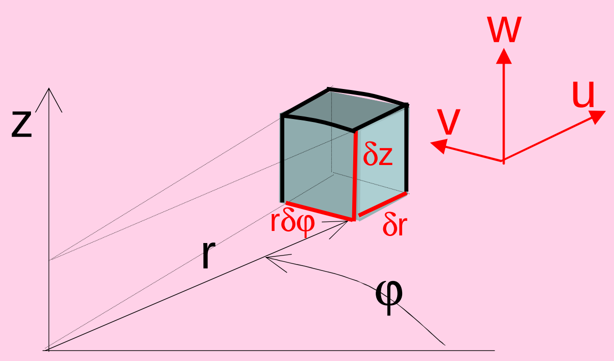
Continuity equation written in index notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

Continuity equation written in symbolic form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{u}) = 0 \qquad \frac{\partial \rho}{\partial t} + div(\rho \vec{u}) = 0$$

Symbolic notation is independent of coordinate system. For example in the cylindrical coordinate system (r,φ,z) this equation looks different



$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial r} + \frac{\rho u}{r} + \frac{1}{r}\frac{\partial \rho v}{\partial \varphi} + \frac{\partial \rho w}{\partial z} = 0$$

CFD4 **Fluid PARTICLE / ELEMENT**

Fluid element – a control volume fixed in space (filled by fluid)

Fluid particle – group of molecules at a point, characterized by property Φ (related to unit mass)

Rate of change of property $\Phi(t,x,y,z)$ during the fluid particle motion

Material derivative

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial t} =$$

$$= \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x}u + \frac{\partial \Phi}{\partial y}v + \frac{\partial \Phi}{\partial z}w =$$

$$= \frac{\partial \Phi}{\partial t} + \vec{u} \bullet \nabla \Phi$$

Projection of gradient to the flow direction

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Transported property Φ

	Φ related to unit mass	$\rho\Phi$ related to unit volume ($\rho\Phi$ is balanced in the fluid element)	Diffusive flux of property Φ through unit surface
Mass	1	ρ	
Momentum	$\vec{u}$	$\rho\,\vec{u}$	$\vec{\vec{\sigma}}$ Tensor of viscous stresses [Pa]
Total energy	E	ρE	$\vec{q}$ Heat flux [W/m^2]
Mass fraction of a component in mixture	ω_A	$\rho\omega_A$	$\vec{j}$ diffusion flux of component A [kg/m^2.s]

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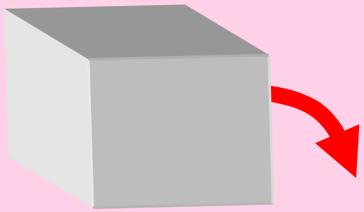
Balancing Φ in Fluid Element



[Accumulation Φ in FE] + [Outflow of Φ from FE by convection] =

intensity of inner sources or diffusional fluxes across the fluid element boundary

$$\frac{\partial \rho \Phi}{\partial t} + div(\rho \vec{u} \Phi) =$$

$$= \rho \frac{\partial \Phi}{\partial t} + \Phi \frac{\partial \rho}{\partial t} + \Phi div(\rho \vec{u}) + \rho \vec{u} \bullet grad \Phi =$$

This follows from the mass balance

$$= \rho \frac{\partial \Phi}{\partial t} + \Phi(-div(\rho \vec{u})) + \Phi div(\rho \vec{u}) + \rho \vec{u} \bullet grad \Phi =$$

These terms are cancelled

$$= \rho (\frac{\partial \Phi}{\partial t} + \vec{u} \bullet grad \Phi) = \rho \frac{D\Phi}{Dt}$$

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Balancing ρΦ in FLUID Element

$$\frac{\partial \rho \Phi}{\partial t} + div(\rho \vec{u} \Phi) = \rho \frac{D\Phi}{Dt}$$

Accumulation of Φ inside the fluid element

Flowrate of Φ out of Fluid element

Rate of Φ increase of fluid particle

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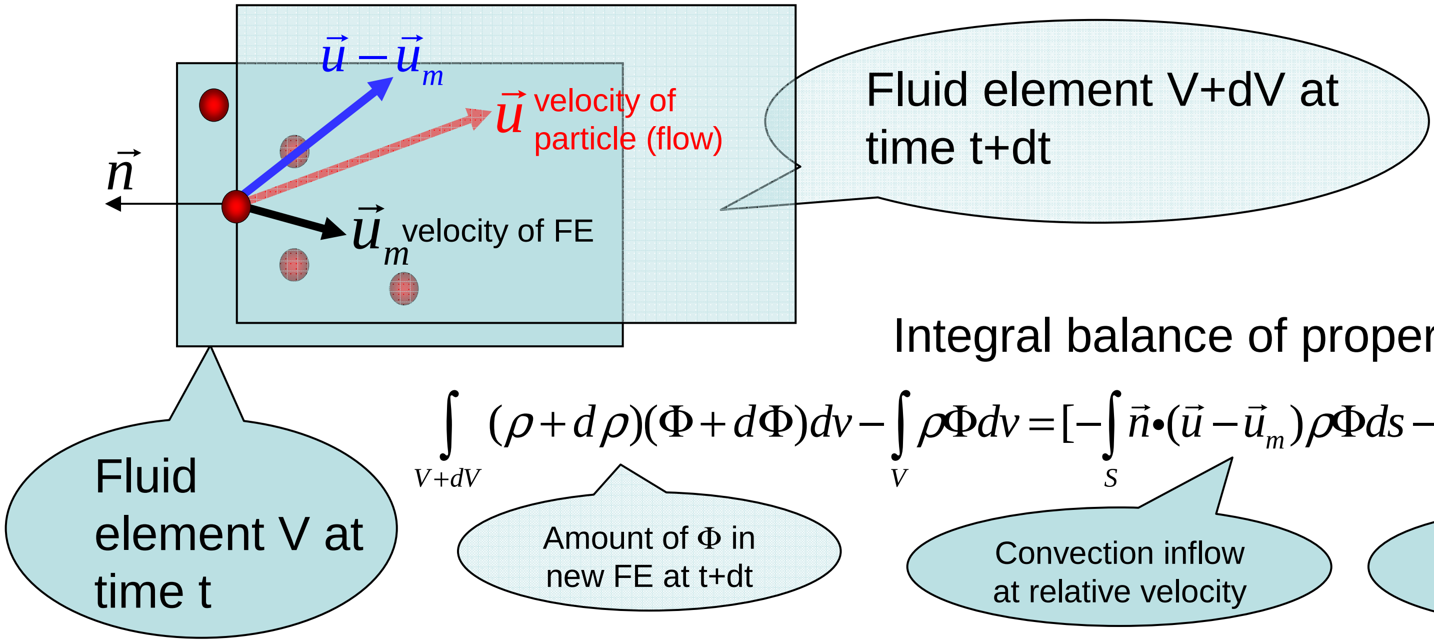


Integral balance of property Φ

$$\int_{V+dV}(\rho+d\rho)(\Phi+d\Phi)dv-\int_V \rho\Phi dv=[-\int_S \vec{n}\bullet(\vec{u}-\vec{u}_m)\rho\Phi ds-\int_S \vec{n}\bullet\rho\vec{\Phi}ds+\int_V \rho\dot{\Phi}dv]dt$$

Amount of Φ in new FE at t+dt

Convection inflow at relative velocity

Diffusional inflow of Φ

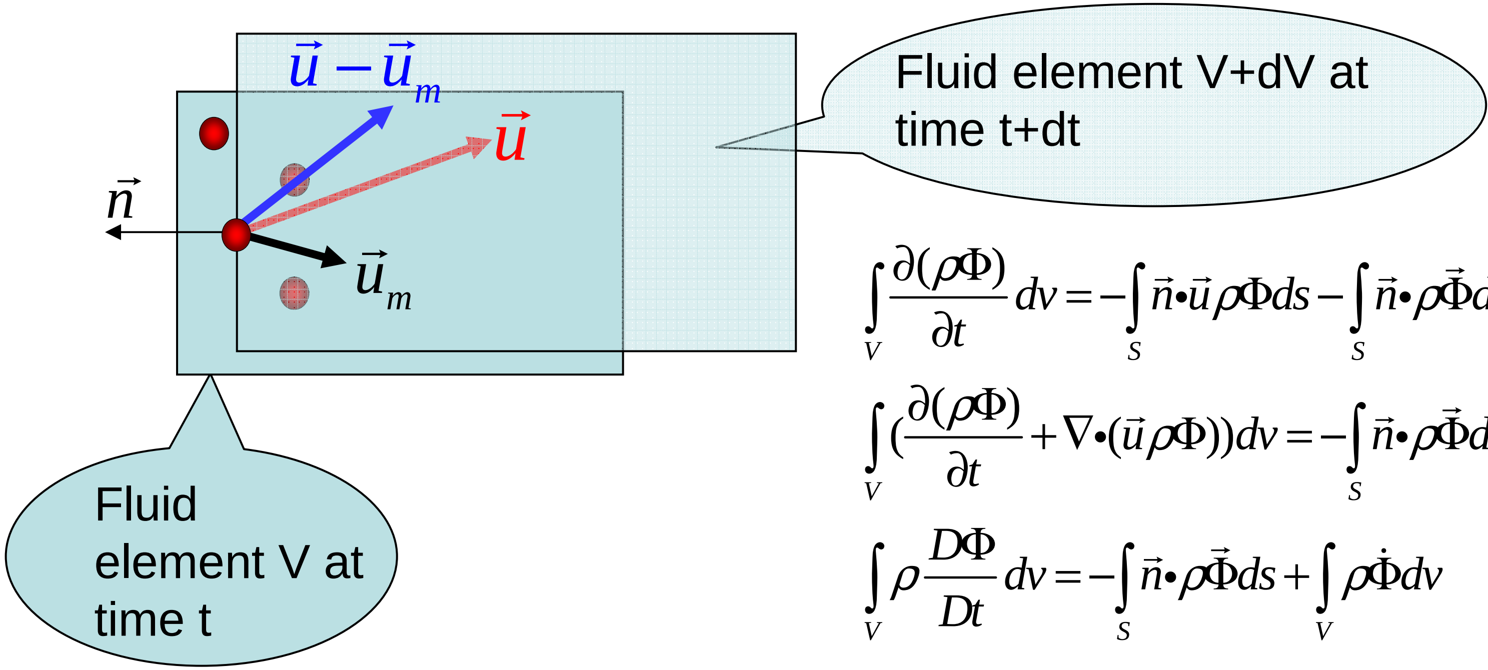
$$\int_V d(\rho\Phi)dv+\int_{dV}\rho\Phi dv=[-\int_S \vec{n}\bullet(\vec{u}-\vec{u}_m)\rho\Phi ds-\int_S \vec{n}\bullet\rho\vec{\Phi}ds+\int_V \rho\dot{\Phi}dv]dt$$

$$\int_V \frac{\partial(\rho\Phi)}{\partial t}dv+\int_S n\bullet\vec{u}_m\rho\Phi ds=-\int_S \vec{n}\bullet(\vec{u}-\vec{u}_m)\rho\Phi ds-\int_S \vec{n}\bullet\rho\vec{\Phi}ds+\int_V \rho\dot{\Phi}dv$$

$$\int_V \frac{\partial(\rho\Phi)}{\partial t}dv=-\int_S \vec{n}\bullet\vec{u}\rho\Phi ds-\int_S \vec{n}\bullet\rho\vec{\Phi}ds+\int_V \rho\dot{\Phi}dv$$

Terms describing motion of FE are canceled

CFD4 Moving Fluid element



$$\int_V \frac{\partial(\rho\Phi)}{\partial t} dv = -\int_S \vec{n}\bullet\vec{u}\rho\Phi ds - \int_S \vec{n}\bullet\rho\vec{\Phi} ds + \int_V \rho\dot{\Phi} dv$$

$$\int_V \left(\frac{\partial(\rho\Phi)}{\partial t} + \nabla\bullet(\vec{u}\rho\Phi)\right) dv = -\int_S \vec{n}\bullet\rho\vec{\Phi} ds + \int_V \rho\dot{\Phi} dv$$

$$\int_V \rho\frac{D\Phi}{Dt} dv = -\int_S \vec{n}\bullet\rho\vec{\Phi} ds + \int_V \rho\dot{\Phi} dv$$

$$\int_V \frac{\partial(\rho\Phi)}{\partial t} dv + \int_S \vec{n}\bullet\vec{u}_m\rho\Phi ds = -\int_S \vec{n}\bullet(\vec{u} - \vec{u}_m)\rho\Phi ds - \int_S \vec{n}\bullet\rho\vec{\Phi} ds + \int_V \rho\dot{\Phi} dv$$

$$\int_V \left(\frac{\partial(\rho\Phi)}{\partial t} + \nabla\bullet(\vec{u}\rho\Phi)\right) dv = -\int_S \vec{n}\bullet\rho\vec{\Phi} ds + \int_V \rho\dot{\Phi} dv$$

Lagrangian fluid particle corresponds to $u=u_m$ but result is the same as with fixed FE

CFD4 Moving Fluid element

You can imagine that the FE moves with fluid particles, with the same velocity, that it expands or contracts according to changing density (therefore FE represents a moving cloud of fluid particle), however the same resulting integral balance is obtained as for the case of the fixed FE in space:

$$\int_V \left(\frac{\partial(\rho\Phi)}{\partial t} + \nabla \bullet (\vec{u}\rho\Phi)\right) dv = -\int_S \vec{n} \bullet \rho\vec{\dot{\Phi}} ds + \int_V \rho\dot{\Phi} dv$$

$$\int_V \rho \frac{D\Phi}{Dt} dv = -\int_S \vec{n} \bullet \rho\vec{\dot{\Phi}} ds + \int_V \rho\dot{\Phi} dv$$

Diffusive flux of Φ superposed to the fluid velocity u

Internal volumetric sources of Φ (e.g. gravity, reaction heat, microwave…)