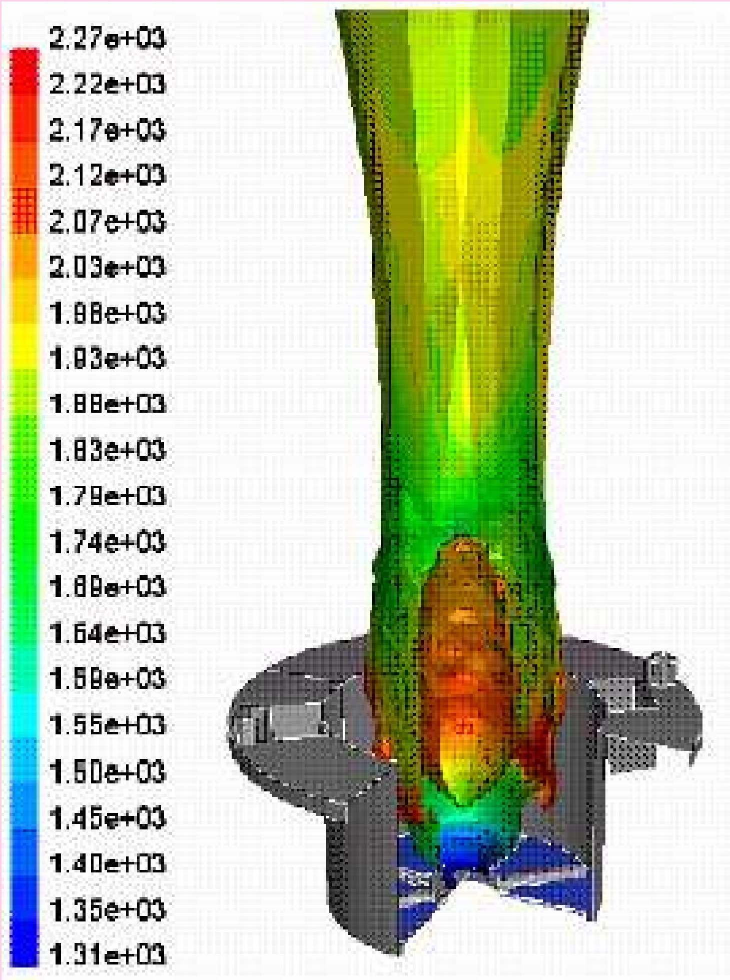
CFD5 **Computer Fluid Dynamics** 2181106 **E181107**



Transport equations, Navier Stokes equations

Remark: foils with „black background“ could be skipped, they are aimed to the more advanced courses

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Navier Stokes Equations

$\rho\vec{u}$

MOMENTUM transport

Newton's law (mass times acceleration=force)

$$\rho \frac{D\vec{u}}{Dt} = pressure\_force + viscous\_stress + gravity + centrigugal\_forces$$

mass

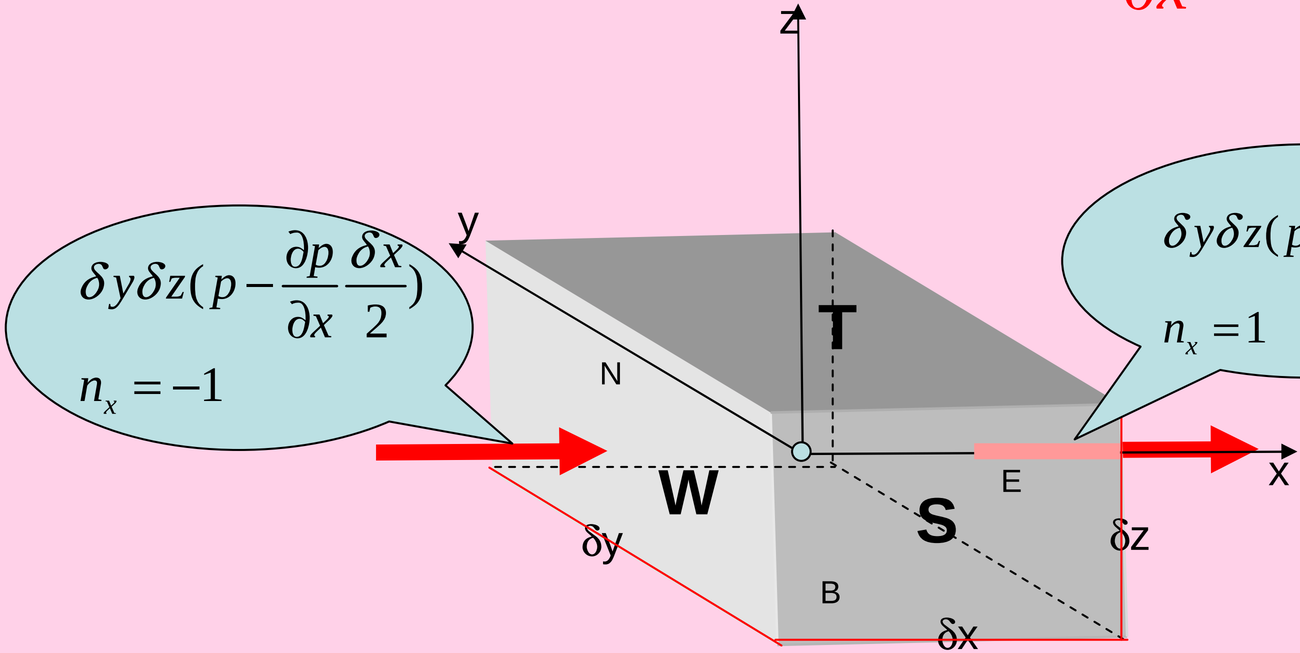
acceleration

Sum of forces on fluid particle

CFD5 **Pressure forces** on fluid element surface

Resulting pressure force acting on sides W and E in the x-direction

$$\delta x \delta y \delta z \frac{\partial p}{\partial x}$$



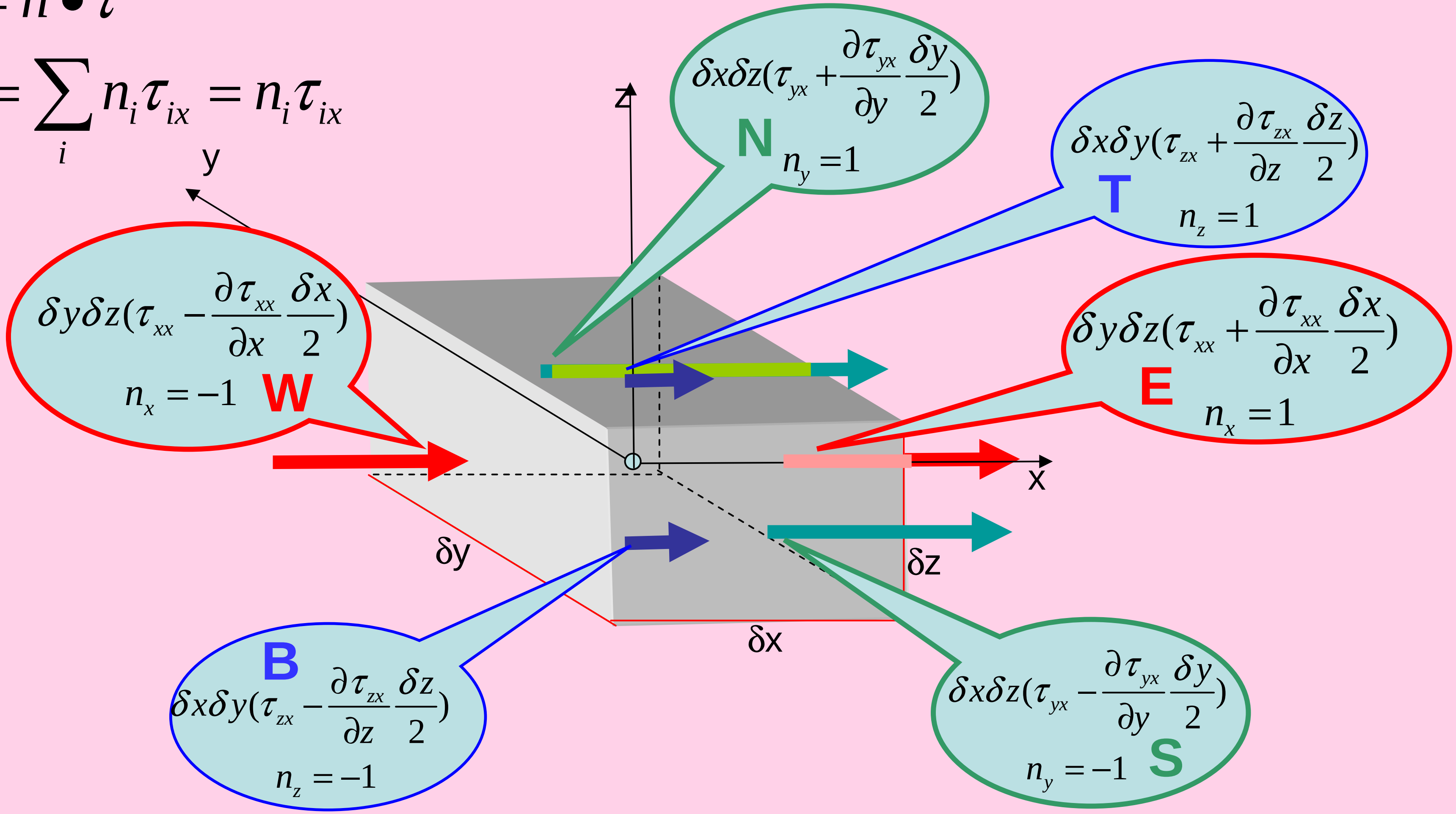
CFD5 Viscous forces on fluid element surface

Resulting viscous force acting on all sides (W,E,N,S,T,B) in the x-direction

$$\delta x \delta y \delta z \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right)$$

$$\vec{f} = \vec{n} \bullet \vec{\vec{\tau}}$$

$$f_x = \sum_i n_i \tau_{ix} = n_i \tau_{ix}$$



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Balance of forces

$$\rho \frac{D\vec{u}}{Dt}\delta x \delta y \delta z = pressure\_force + viscous\_stress + gravity$$

$$\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_x$$

$$\rho \frac{Du_y}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_y$$

$$\rho \frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + S_z$$

Cauchy's equation of momentum balances (in fact 3 equations)

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \bullet \vec{\vec{\tau}} + \vec{S}_M$$

[N/m^3]

CFD5 Balance of ENERGY

TOTAL ENERGY transport

$$\rho \frac{DE}{Dt} = heat + mechanical\_work$$

Heat added to FE by diffusion only (convective transport is included in DE/Dt)

Power of mechanical forces is velocity times force. Internal forces are pressure and viscous forces.

$$E = i + \frac{1}{2}(u^2 + v^2 + w^2)$$

Total energy [J/kg]

Internal energy all form of energies (chemical, intermolecular, thermal) independent of coordinate system

Kinetic energy [J/kg]

CFD5 Heat conduction

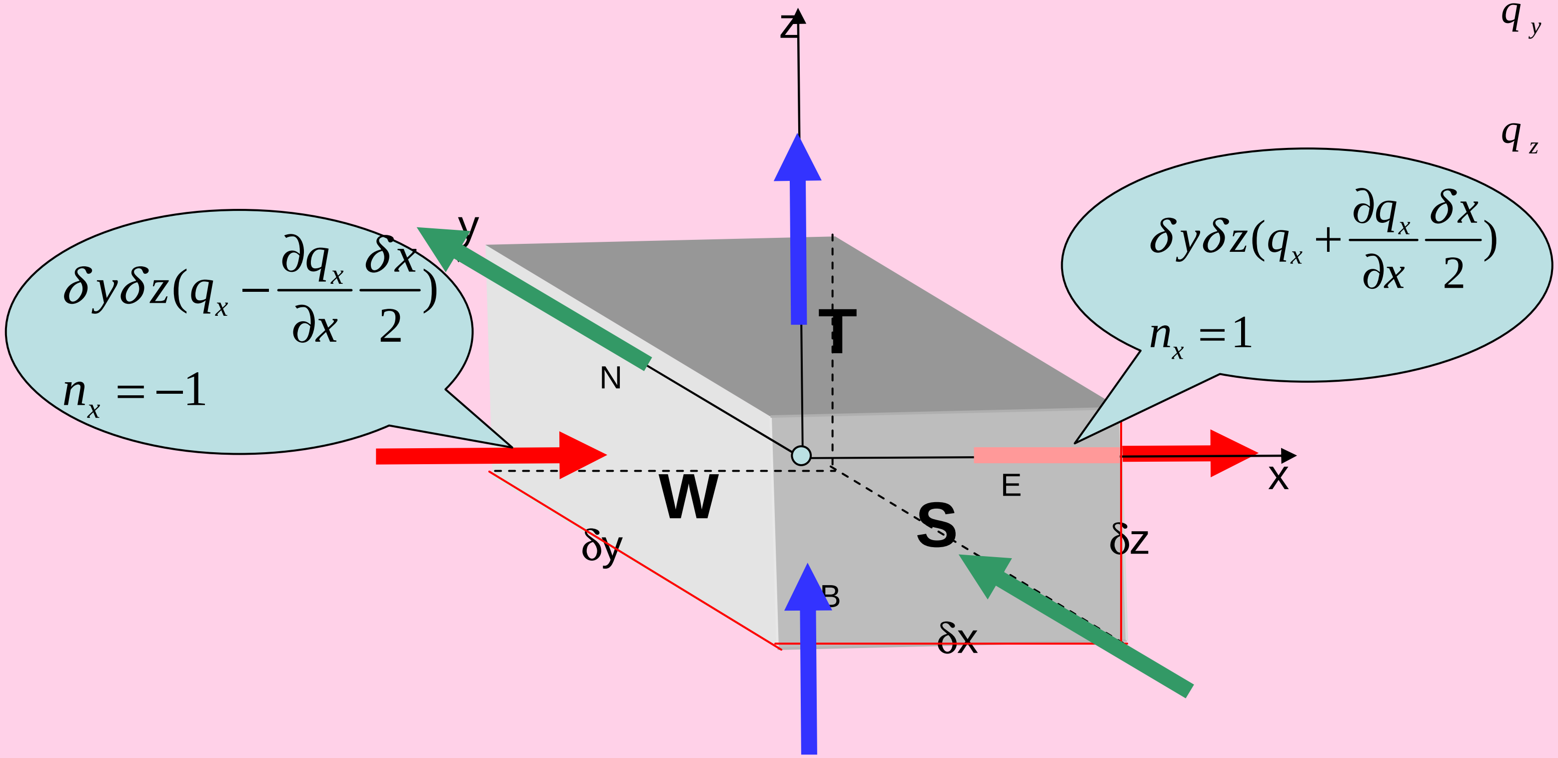
$$\vec{q} = -k\nabla T$$

$$q_x = -k\frac{\partial T}{\partial x}$$

$$q_y = -k\frac{\partial T}{\partial y}$$

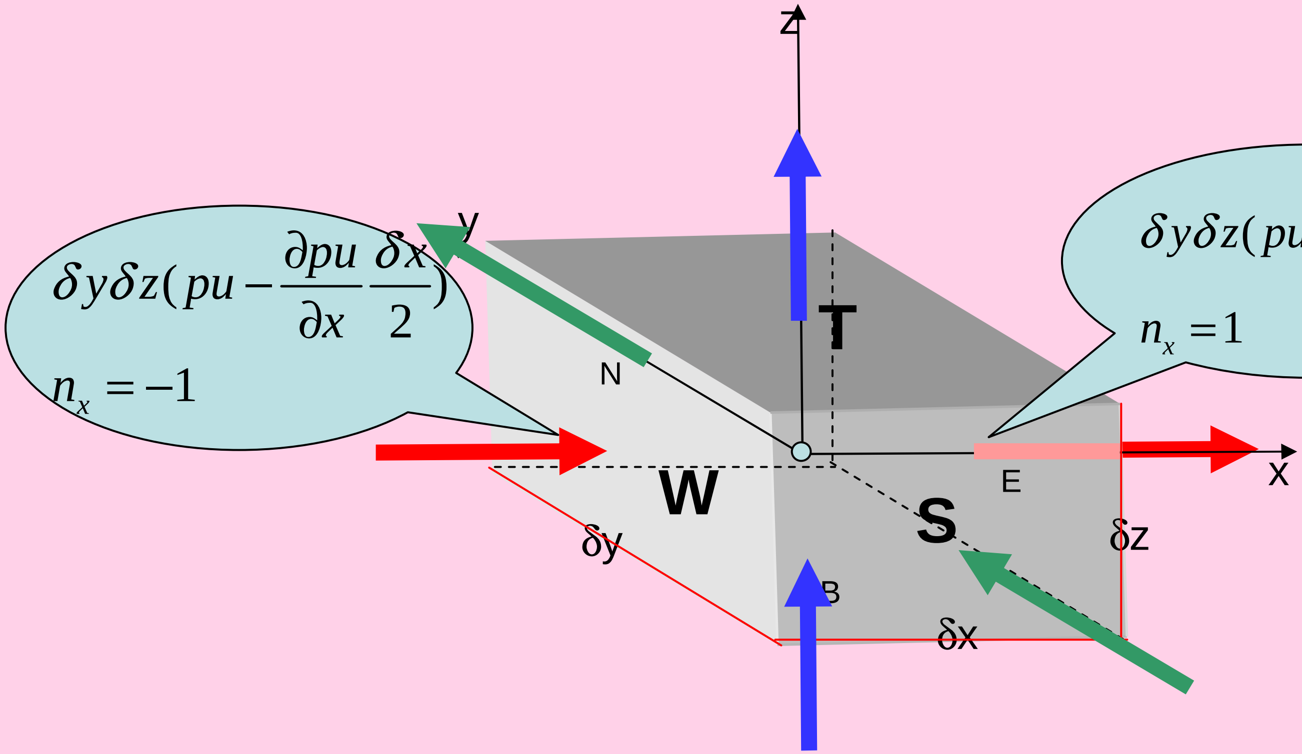
$$q_z = -k\frac{\partial T}{\partial z}$$

Heat transfer by conduction is described by Fourier's law



$$\rho\frac{DE}{Dt} = -\nabla\bullet\vec{q} = \nabla\bullet(k\nabla T) = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})$$

CFD5 Mechanical work - pressure



$$\rho \frac{DE}{Dt} = \nabla \bullet (k \nabla T) - \nabla \bullet (p\vec{u}) =$$

$$= \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x} - pu) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y} - pv) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z} - pw)$$

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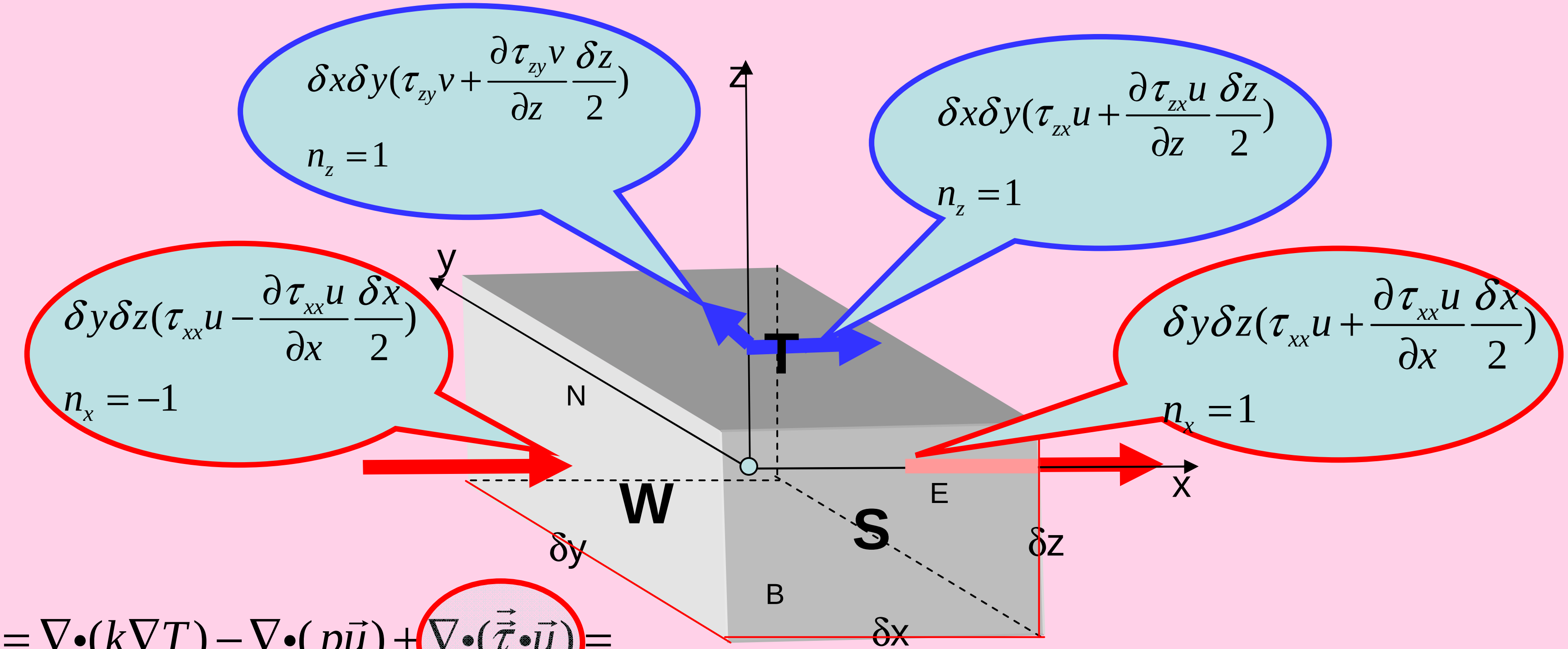
Mechanical work - stresses



$$\rho\frac{DE}{Dt}=\nabla\bullet(k\nabla T)-\nabla\bullet(p\vec{u})+\nabla\bullet(\vec{\vec{\tau}}\bullet\vec{u})=$$

$$=\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}-pu+\tau_{xx}u+\tau_{xy}v+\tau_{xz}w)+$$

$$+\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}-pv+\tau_{yx}u+\tau_{yy}v+\tau_{yz}w)$$

$$+\frac{\partial}{\partial z}(k\frac{\partial T}{\partial z}-pw+\tau_{zx}u+\tau_{zy}v+\tau_{zz}w)$$

The situation is more complicated because not only the work of normal but also shear stresses must be included.

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Total energy transport

This is scalar equation for total energy, comprising internal energy (temperature) and also kinetic energy.

$$\rho \frac{DE}{Dt} = \nabla \bullet (k \nabla T) - \nabla \bullet (p \vec{u}) + \nabla \bullet (\vec{\vec{\tau}} \bullet \vec{u}) + S_E$$ [W/m^3]

$$\rho \frac{DE}{Dt} = \nabla \bullet (k \nabla T - p \vec{u} + \vec{\vec{\tau}} \bullet \vec{u}) + S_E$$

CFD5 Fourier Kirchhoff equation

Kinetic energy can be eliminated from total energy equation

$$\rho\frac{D(i+\frac{1}{2}\vec{u}\bullet\vec{u})}{Dt}=\nabla\bullet(k\nabla T)-\nabla\bullet(p\vec{u})+\nabla\bullet(\vec{\vec{\tau}}\bullet\vec{u})+S_E \qquad (1)$$

using Cauchy's equation multiplied by velocity vector (scalar product, this is the way how to obtain scalar equation from the vector equation)

$$\rho\vec{u}\bullet\frac{D\vec{u}}{Dt}=\rho\frac{D\frac{1}{2}\vec{u}\bullet\vec{u}}{Dt}=-\vec{u}\bullet\nabla p+\vec{u}\bullet\nabla\bullet\vec{\vec{\tau}}+\vec{u}\bullet\vec{S}_M \qquad (2)$$

Subtracting Eq.(2) from Eq.(1) we obtain transport equation for internal energy

$$\rho\frac{Di}{Dt}=\nabla\bullet(k\nabla T)-p\nabla\bullet\vec{u}+\vec{\vec{\tau}}:\nabla\vec{u}+S_E-\vec{u}\bullet\vec{S}_M \qquad [\mathrm{W/m^3}]$$

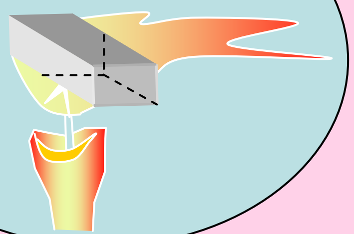
CFD5 Fourier Kirchhoff equation

Interpretation using **First law of thermodynamics**

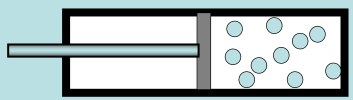
d*i* = d*q* - *p* d*v*

$$\rho \frac{Di}{Dt} = \nabla \bullet (k \nabla T) - p \nabla \bullet \vec{u} + \vec{\vec{\tau}} : \nabla \vec{u} + S_E - \vec{u} \bullet \vec{S}_M$$

Heat transferred by conduction into FE

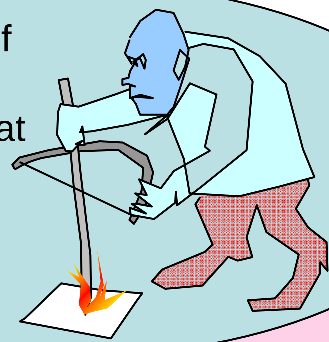


Expansion cools down working fluid



This term is zero for incompressible fluid

Dissipation of mechanical energy to heat by viscous friction



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Dissipation term

Heat dissipated in unit volume [W/m^3] by viscous forces

$$\vec{\vec{\tau}} : \nabla \vec{u} = \sum_i \sum_j \tau_{ij} \frac{\partial u_i}{\partial x_j} =$$

$$\tau_{xx} \frac{\partial u_x}{\partial x} + \tau_{xy} \frac{\partial u_x}{\partial y} + \tau_{xz} \frac{\partial u_x}{\partial z} +$$

$$\tau_{yx} \frac{\partial u_y}{\partial x} + \tau_{yy} \frac{\partial u_y}{\partial y} + \tau_{yz} \frac{\partial u_y}{\partial z} +$$

$$\tau_{zx} \frac{\partial u_z}{\partial x} + \tau_{zy} \frac{\partial u_z}{\partial y} + \tau_{zz} \frac{\partial u_z}{\partial z}$$

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Dissipation term

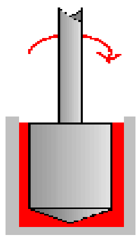
$$\vec{\vec{\tau}} : \nabla \vec{u} = \frac{1}{2}\vec{\vec{\tau}} : (\nabla \vec{u} + (\nabla \vec{u})^T) = \vec{\vec{\tau}} : \vec{\vec{e}}$$

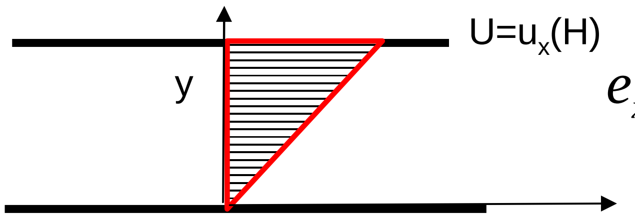
This identity follows from the stress tensor symmetry

$$\vec{\vec{e}} = \frac{1}{2}(\nabla \vec{u} + (\nabla \vec{u})^T)$$

Rate of deformation tensor

Example: Simple shear flow (flow in a gap between two plates, lubrication)





$$e_{xy} = e_{yx} = \frac{1}{2}(\nabla_x u_y + \nabla_y u_x) = \frac{1}{2}\frac{\partial u_x}{\partial y} = \frac{1}{2}\dot{\gamma}$$

$$\vec{\vec{\tau}} : \nabla \vec{u} = \vec{\vec{\tau}} : \vec{\vec{e}} = \tau_{xy} e_{yx} + \tau_{yx} e_{xy} = \tau_{xy}\dot{\gamma}$$

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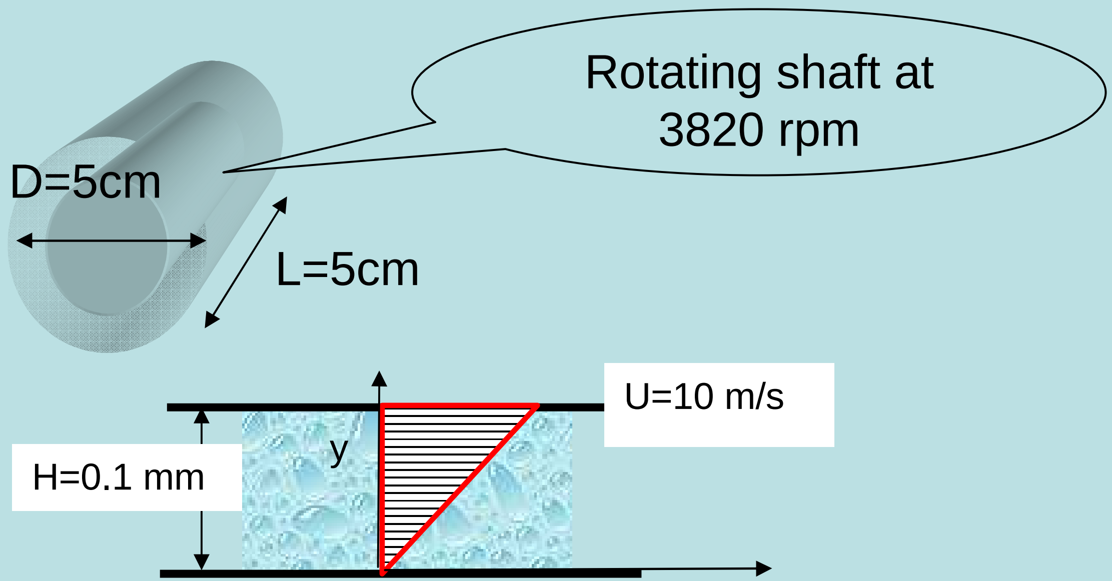
Example tutorial



Gap width H=0.1mm, U=10 m/s, oil M9ADS-II at 0^0C

μ=3.4 Pa.s, γ=10^5 1/s, τ=3.4.10^5 Pa, $\tau\gamma$= 3.4.10^{10} W/m^3

At contact surface S=0.0079 m^2 the dissipated heat is 26.7 kW !!!!

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Fourier Kirchhoff equation

Internal energy can be expressed in terms of temperature as $di=c_p dT$ or $di=c_v dT$. Especially simple form of this equation holds for liquids when $c_p=c_v$ and divergence of velocity is zero (incompressibility constraint):

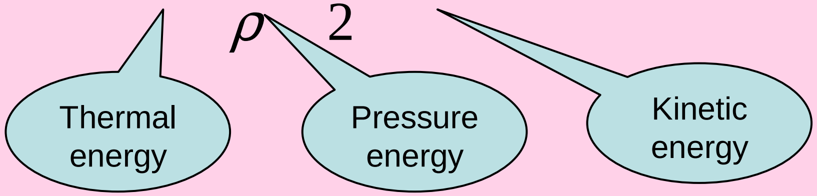
$$\rho c \frac{DT}{Dt} = \nabla \bullet (k \nabla T) + \vec{\vec{\tau}} : \nabla \vec{u} + S_i \qquad [W/m^3]$$

An alternative form of energy equation substitute internal energy by enthalpy

$$\rho \frac{DH}{Dt} = \nabla \bullet (k \nabla T) - p \nabla \bullet \vec{u} + \frac{\partial p}{\partial t} + \nabla \bullet (\vec{\vec{\tau}} \bullet \vec{u}) + S_i$$

where total enthalpy is defined as

$$H = i + \frac{p}{\rho} + \frac{1}{2} \vec{u} \bullet \vec{u}$$



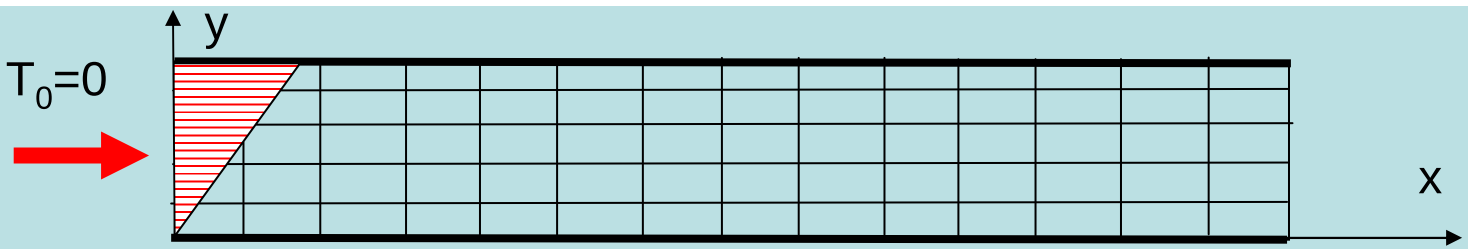
CFD5 Example tutorial

Calculate evolution of temperature in a gap assuming the same parameters as previously (H=0.1 mm, U=10 m/s, oil M9ADS-II). Assume constant value of heat production term $3.4.10^{10}$ W/m^3, uniform inlet temperature T_0=0^{o}C and thermally insulated walls, or constant wall temperature, respectively.

Parameters: density = 800 kg/m^3, c_p=1.9 kJ/(kg.K), k=0.14 W/(m.K).

Approximate FK equation in 2D by finite differences. Use upwind differences in convection terms

$$\rho c \frac{DT}{Dt} = \nabla \bullet (k \nabla T) + \vec{\vec{\tau}} : \nabla \vec{u}$$



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Summary

Mass conservation

(continuity equation)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{u}) = 0$$

Momentum balance

(3 equations)

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \bullet \vec{\vec{\tau}} + \vec{S}_M$$

Energy balance

$$\rho c \frac{DT}{Dt} = \nabla \bullet (k \nabla T) + \vec{\vec{\tau}} : \nabla \vec{u} + S_i$$

State equation F(p,T,ρ)=0, e.g.
ρ=ρ0+βT

Thermodynamic equation
di=c_p.dT

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Constitutive equations

Constitutive equations represent description of material properties

Kinematics (rate of deformation) – **stress** (dynamic response to deformation)

$$\vec{\vec{e}} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$$

$$\vec{\vec{\sigma}} = -p\vec{\vec{\delta}} + \vec{\vec{\tau}}$$

rate of deformation is symmetric part of gradient of velocity)

Gradient of velocity is tensor with components $\nabla_i u_j = \frac{\partial u_j}{\partial x_i}$

Viscous stresses affected by fluid flow. Stress is in fact momentum flux due to molecular diffusion

Second viscosity [Pa.s]

Dynamic viscosity [Pa.s]

$$\vec{\vec{\tau}} = \lambda\vec{\vec{\delta}}\nabla\bullet\vec{u} + 2\mu(II)\vec{\vec{e}}$$

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Constitutive equations

$$\vec{\vec{\tau}} = \lambda \vec{\vec{\delta}} \nabla \bullet \vec{u} + 2\mu(II)\vec{\vec{e}}$$

Rheological behaviour is quite generally expressed by viscosity function

$$\mu(II), \text{ where } II = \vec{\vec{e}} : \vec{\vec{e}} = \sum_{i=1}^{3} \sum_{j=1}^{3} e_{ij} e_{ji}$$

This is second invariant of rate of deformation tensor – scalar value (magnitude of shear rate)

and by the coefficient of second viscosity, that represents resistance of fluid to volumetric expansion or compression. According to Lamb's hypothesis the second (volumetric) viscosity can be expressed in terms of dynamic viscosity μ

$$\lambda = -\frac{2}{3}\mu$$

This follows from the requirement that the mean normal stresses are zero (this mean value is absorbed in the pressure term)

$$trace \vec{\vec{\tau}} = \tau_{xx} + \tau_{yy} + \tau_{zz} = 3\lambda \nabla \bullet \vec{u} + 2\mu \nabla \bullet \vec{u} = 0$$

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Constitutive equations

$$\vec{\vec{\tau}} = 2\mu(II)(\vec{\vec{e}} - \frac{\nabla \bullet \vec{u}}{3}\vec{\vec{\delta}})$$

The simplest form of rheological model is NEWTONIAN fluid, characterized by viscosity independent of rate of deformation. Example is water, oils and air.

More complicated constitutive equations exist for fluids exhibiting

- **yield stress** (fluid flows only if stress exceeds a threshold, e.g. ketchup, tooth paste, many food products),
- **generalized newtonian fluids** (viscosity depends upon the actual state of deformation rate, example are power law fluids $\mu = K(\sqrt{2II})^{n-1}$)
- **thixotropic** fluids (viscosity depends upon the whole deformation history, examples thixotropic paints, plasters, yoghurt)
- **viscoelastic** fluids (exhibiting recovery of strains and relaxation of stresses). Examples are polymers.

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Unknowns / Equations

There are **13** unknowns:

u,v,w, (3 velocities), **p, T, ρ, i, τ_{xx}, τ_{xy},…**(6 components of symmetric stress tensor)

And the same number of equations

Continuity equation $\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{u}) = 0$

3 Cauchy's equations $\rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \bullet \vec{\vec{\tau}} + \vec{S}_M$

Energy equation $\rho \frac{Di}{Dt} = \nabla \bullet (k \nabla T) - p \nabla \bullet \vec{u} + \vec{\vec{\tau}} : \nabla \vec{u} + S_E - \vec{u} \bullet \vec{S}_M$

State equation p/ρ=RT

Thermodynamic equation di=c_p.dT

6 Constitutive equations $\vec{\vec{\tau}} = 2\mu(II)(\vec{\vec{e}} - \frac{\nabla \bullet \vec{u}}{3} \vec{\vec{\delta}})$

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Navier Stokes equations

Using constitutive equation the divergence of viscous stresses can be expressed

$$\nabla\bullet\vec{\vec{\tau}} = 2\nabla\bullet(\mu(II)(\vec{\vec{e}}-\frac{\nabla\bullet\vec{u}}{3}\vec{\vec{\delta}})) = \nabla\bullet(\mu(II)(\nabla\vec{u}+(\nabla\vec{u})^T)) - \frac{2}{3}\nabla\bullet(\mu(II)(\nabla\bullet\vec{u})\vec{\vec{\delta}})$$

This is the same, but written in the index notation (you cannot make mistakes when calculating derivatives)

$$\frac{\partial\tau_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}(\mu\frac{\partial u_j}{\partial x_i}) + \frac{\partial}{\partial x_i}(\mu\frac{\partial u_i}{\partial x_j}) - \frac{2}{3}\frac{\partial}{\partial x_i}(\mu(\frac{\partial u_k}{\partial x_k})\delta_{ij}) =$$

$$= \frac{\partial}{\partial x_i}(\mu\frac{\partial u_j}{\partial x_i}) + \frac{\partial}{\partial x_i}(\mu\frac{\partial u_i}{\partial x_j}) - \frac{2}{3}\frac{\partial}{\partial x_j}(\mu(\frac{\partial u_i}{\partial x_i})) =$$

$$= \frac{\partial}{\partial x_i}(\mu\frac{\partial u_j}{\partial x_i}) + \frac{\partial\mu}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_i} - \frac{2}{3}(\frac{\partial\mu}{\partial x_j}\frac{\partial u_i}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_i}) =$$

$$= \frac{\partial}{\partial x_i}(\mu\frac{\partial u_j}{\partial x_i}) + \frac{\partial\mu}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \frac{\mu}{3}\frac{\partial^2 u_i}{\partial x_j\partial x_i} - \frac{2}{3}\frac{\partial\mu}{\partial x_j}\frac{\partial u_i}{\partial x_i}$$

$$\nabla\bullet\vec{\vec{\tau}} = \nabla\bullet(\mu(II)\nabla\vec{u}) + \nabla\mu(II)\bullet(\nabla\vec{u})^T + \frac{\mu(II)}{3}\nabla(\nabla\bullet\vec{u}) - \frac{2}{3}(\nabla\bullet\vec{u})\nabla\mu(II)$$

These terms are small and will be replaced by a parameter s_m

These terms are ZERO for incompressible fluids

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Navier Stokes equations

General form of Navier Stokes equations valid for compressible/incompressible Non-Newtonian (with the exception of viscoelastic or thixotropic) fluids

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \bullet (\mu(II)\nabla \vec{u}) + \vec{s}_m + \vec{S}_M$$

Special case – Newtonian liquids with constant viscosity

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \vec{S}_M$$

Written in the cartesian coordinate system

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + \rho g_x$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) + \rho g_y$$

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) + \rho g_z$$