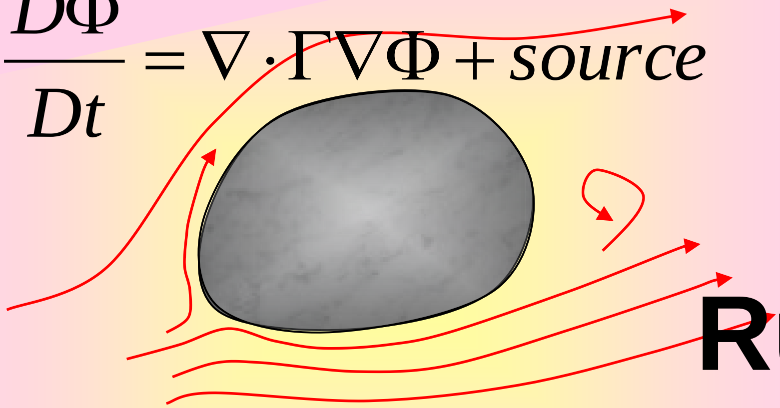
MHMT1

Momentum Heat Mass Transfer

$$\frac{D\Phi}{Dt} = \nabla \cdot \Gamma \nabla \Phi + source$$



Rules, database, Tensorial calculus

Introduction and rules (form of lectures, presentations of papers by students, requirements for exam). Preacquaintance (photography). Working with databases and primary sources. Tensors and tensorial calculus

Rudolf Žitný, Ústav procesní a zpracovatelské techniky ČVUT FS 2010

MHMT1

Momentum Heat Mass Transfer

3P/1C, 4 credits, exam, LS2018, room 505d

- Lectures Prof.Ing.Rudolf Žitný, CSc. (Monday 10:45-13:00)
- Tutorials Ing.Karel Petera, PhD.

- Rating 30(quise in test)+30(example in test)+40(oral exam) points

A	B	C	D	E	F
90+	80+	70+	60+	50+	..49
excellent	very good	good	satisfactory	sufficient	failed

MHMT1

Momentum Heat Mass Transfer

2018 **505**

Předmět: (E181026) Momentum, Heat and Mass Transfer / FS **Paralelka:** Všichni studenti předmětu Zo

Způsob zakončení: Z,ZK **Rozsah:** 3+1

Počet studentů : 28 **Kapacita předmětu:** 30

Excel | Tisk | E-mail | Zobrazit filtr

Příjmení	Jméno	Identifikátor	Fakulta	Studijní program	Typ programu
Alsatmi	Basel	479516	FS	Bakalářský program pro výměnné studenty	B
Bemmireddy	Santhosh Reddy	473334	FS	Strojní inženýrství	N
Duque Dussan	Eduardo	472430	FS	Strojní inženýrství	N
Fenech	Daniel	479578	FS	Bakalářský program pro výměnné studenty	B
Feppon	Nicolas	472985	FS	Mobility - bakalářský	B
Gueho	Lénaig	479705	FEL	Electrical Engineering and Computer Science	B
Hanna Anz	Qais Elia	453674	FS	Strojírenství	B
Hübbers	Lena	479702	FS	Magisterský program pro výměnné studenty	N
Ipek	Murat	472154	FS	Strojní inženýrství	N
Lalonde	Timothe	479067	FS	Mobility - bakalářský	B
Lloyd	Olivia	472153	FS	Mobility - bakalářský	B
Márquez Hernández	Luis Eduardo	472087	FS	Mobility - bakalářský	B
Mcgowan	Orla	472830	FS	Magisterský program pro výměnné studenty	N
Messas	Anis	453198	FS	Strojírenství	B
Nivet	Gabriel	479697	FEL	Electrical Engineering and Computer Science	B
Perez	Ariel	479870	FEL	Electrical Engineering and Computer Science	B
Somireddy	Ashok Kumar Reddy	473153	FS	Strojní inženýrství	N
Souza	Anne	479472	FS	Mobility - bakalářský	B
Stalls	Wayne	479054	FS	Mobility - bakalářský	B
Stevenson	Craig	472700	FS	Magisterský program pro výměnné studenty	N
Stukel	Nathan	479222	FS	Mobility - bakalářský	B
Suleiman	Abubakar Shola	459336	FS	Strojírenství	B
Torres Tapia	Luis Alberto	476496	FS	Strojní inženýrství	N
Upadhyay	Sumit	473394	FS	Strojní inženýrství	N
Uzakov	Timur	453574	FS	Teoretický základ strojního inženýrství	B
Walker	Justin Roy	479055	FS	Mobility - bakalářský	B
Yap	Jane	479062	FS	Mobility - bakalářský	B
Yücel	Utku	479428	FS	Bakalářský program pro výměnné studenty	B

MHMT1

Momentum Heat Mass Transfer

3P/2C, 4 credits, exam, LS2018, room 505

MHMT1

Literature - sources

- **Textbook:**

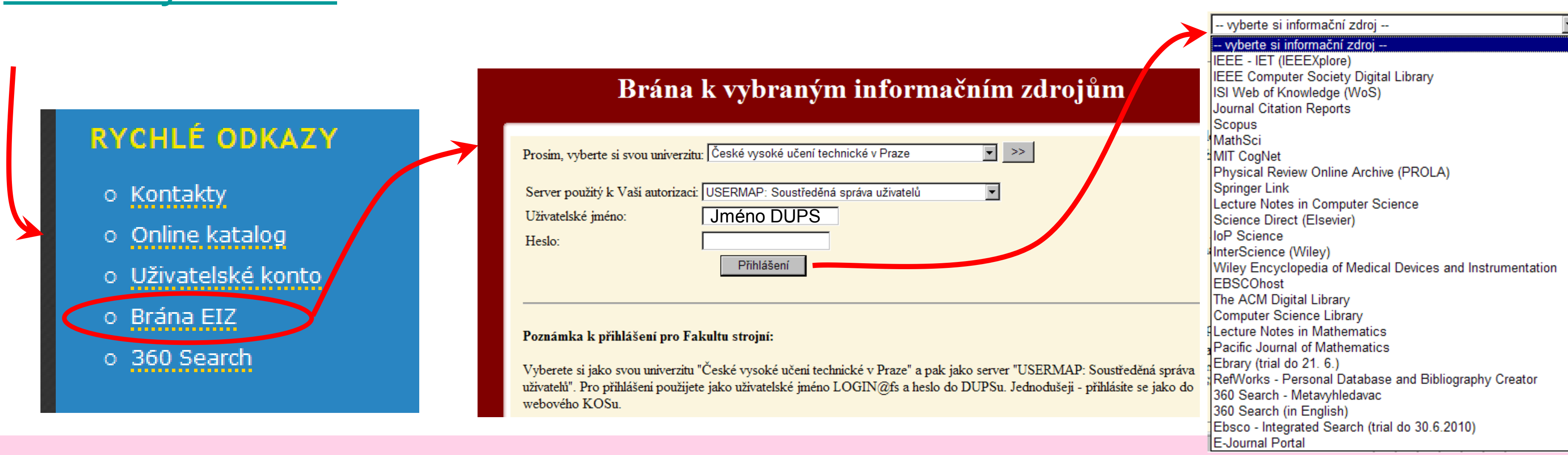
Šesták J., Rieger F.: Přenos hybnosti tepla a hmoty, ČVUT Praha, 2004

- **Monography:** available at **NTL**

Bird, Stewart, Lightfoot: Transport Phenomena. Wiley, 2nd edition, 2006
Welty J.R.: Fundamental of momentum, heat and mass transfer,Wiley,2008

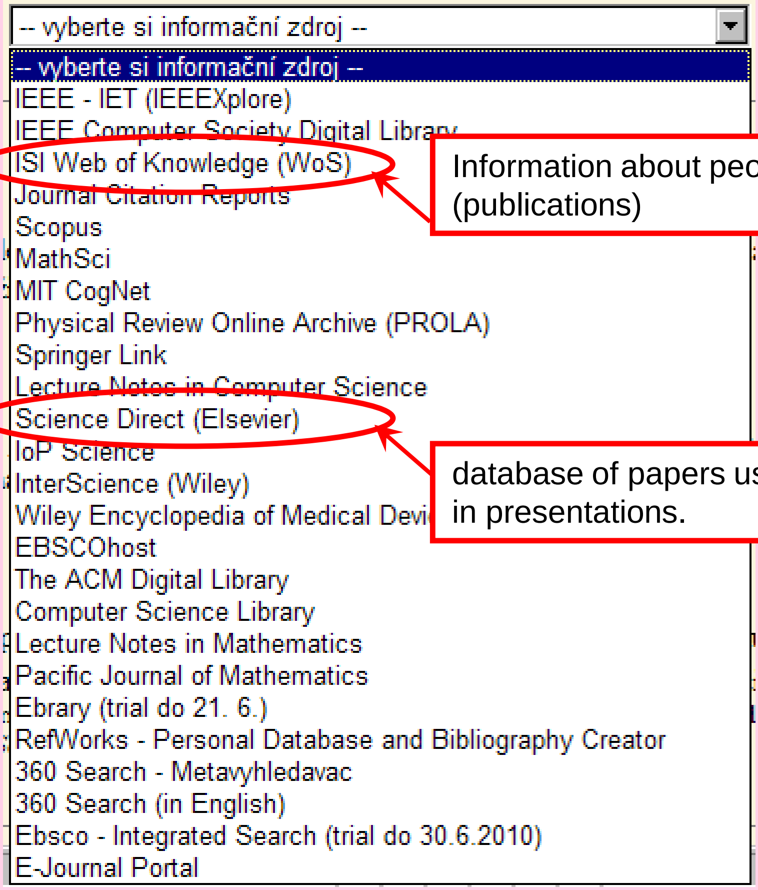
- **Databases of primary sources**

Direct access to databases (WoS, Elsevier, Springer,…)
knihovny.cvut.cz



MHMT1

EES Electronic sources

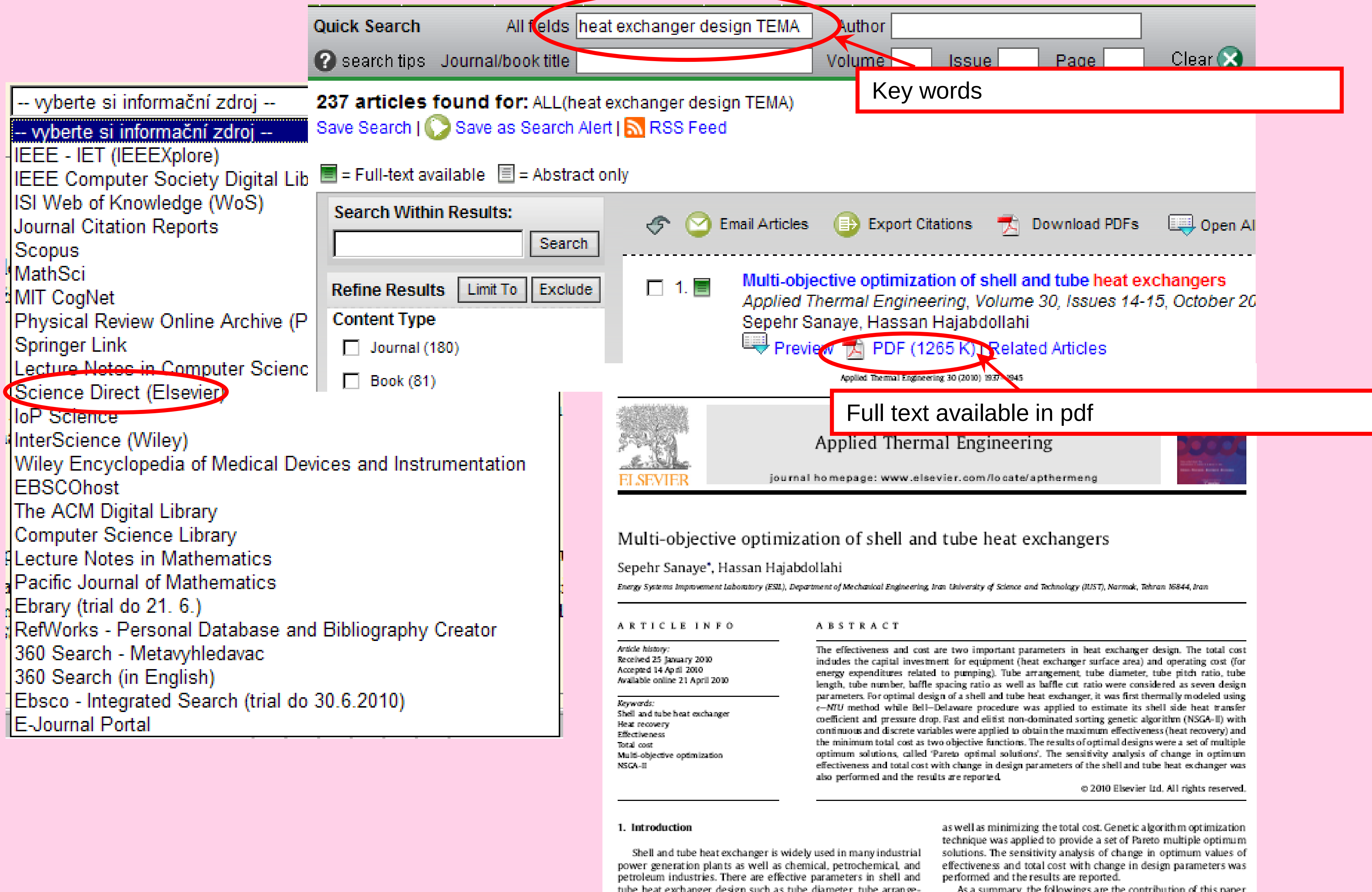


Information about peoples (publications)

database of papers used in presentations.

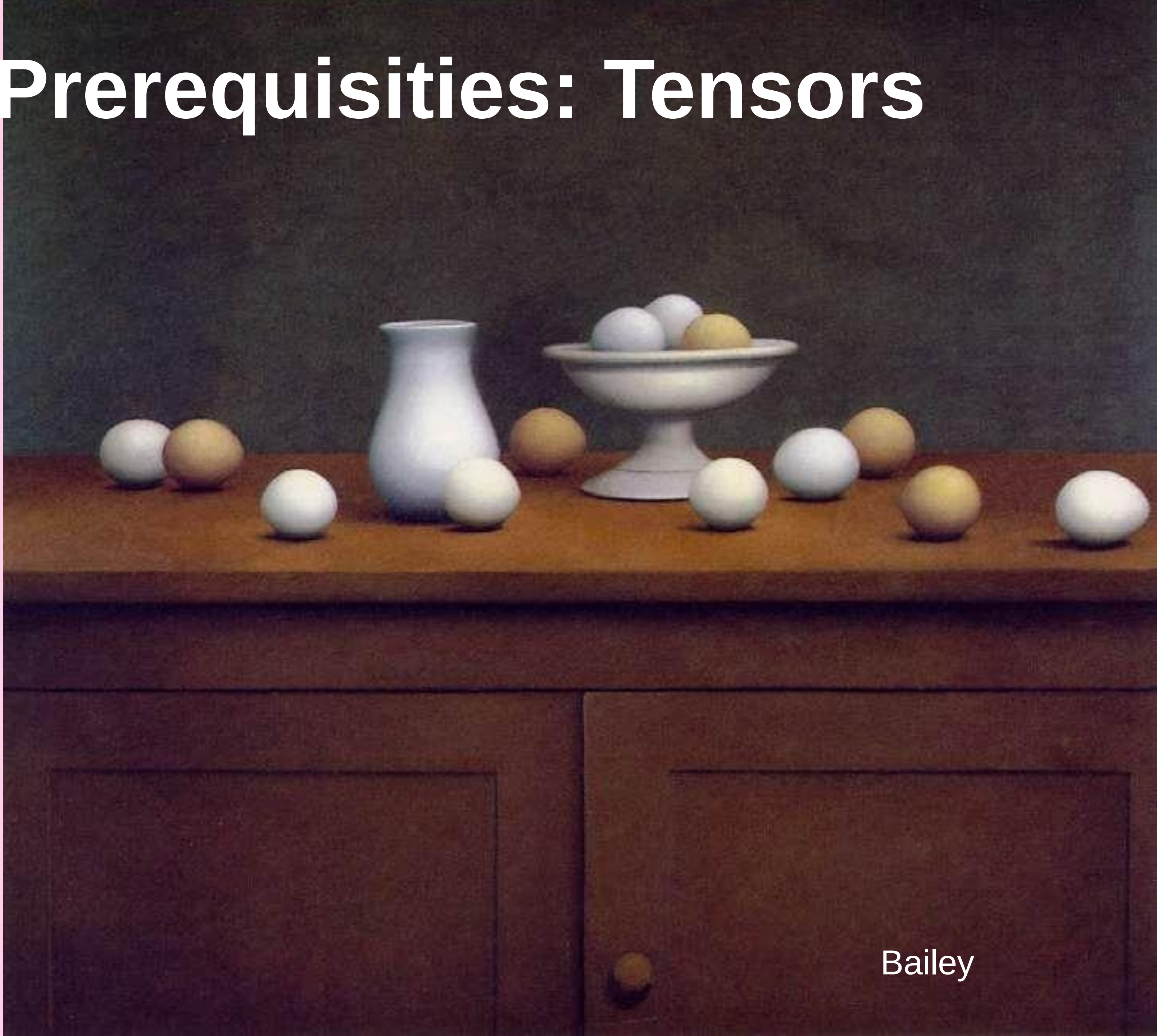
MHMT1

Science Direct



MHMT1

Prerequisities: Tensors



Bailey

MHMT1

Prerequisities: Tensors

Transfer phenomena operate with the following properties of solids and fluids (determining state at a point in space x,y,z):

Scalars T (temperature), p (pressure), ρ (density), h (enthalpy), c_A (concentration), k (kinetic energy)

Vectors $\vec{u}$ (velocity), $\vec{f}$ (forces), ∇T (gradient of scalar), and others like vorticity, displacement…

Tensors $\vec{\vec{\tau}}$ $\vec{\vec{\sigma}}$ (stress), $\vec{\vec{\Delta}}$ (rate of deformation), $\nabla\vec{u}$ (gradient of vector) , deformation tensor…

Scalars are determined by 1 number.

Vectors are determined by 3 numbers

$$\vec{u} = (u_x, u_y, u_z) = (u_1, u_2, u_3)$$

Tensors are determined by 9 numbers

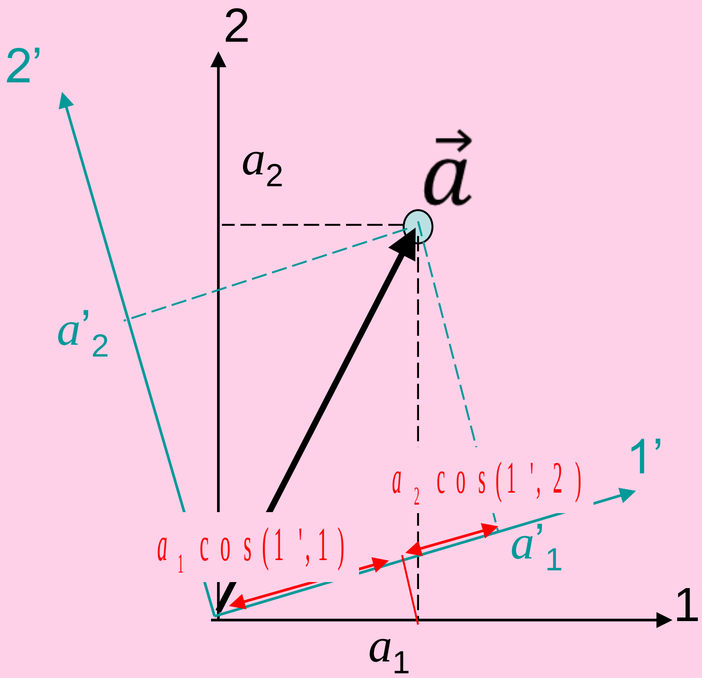
$$\vec{\vec{\sigma}} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Scalars, vectors and tensors are independent of coordinate systems (they are objective properties). However, components of vectors and tensors depend upon the coordinate system. Rotation of axis has no effect upon a vector (its magnitude and the arrow direction), but coordinates of the vector are changed (coordinates u_i are projections to coordinate axis). See next slide…

MHMT1

Rotation of cartesian coordinate system

Three components of a vector represent complete description (length of an arrow and its directions), but these components depend upon the choice of coordinate system. Rotation of axis of a cartesian coordinate system is represented by transformation of the vector coordinates by the matrix product



$$a_1' = a_1\cos(1',1) + a_2\cos(1',2) + a_3\cos(1',3)$$
$$a_2' = a_1\cos(2',1) + a_2\cos(2',2) + a_3\cos(2',3)$$
$$a_3' = a_1\cos(3',1) + a_2\cos(3',2) + a_3\cos(3',3)$$

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} \cos(1',1) & \cos(1',2) & \cos(1',3) \\ \cos(2',1) & \cos(2',2) & \cos(2',3) \\ \cos(3',1) & \cos(3',2) & \cos(3',3) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$[a'] = [[R]][a]$$

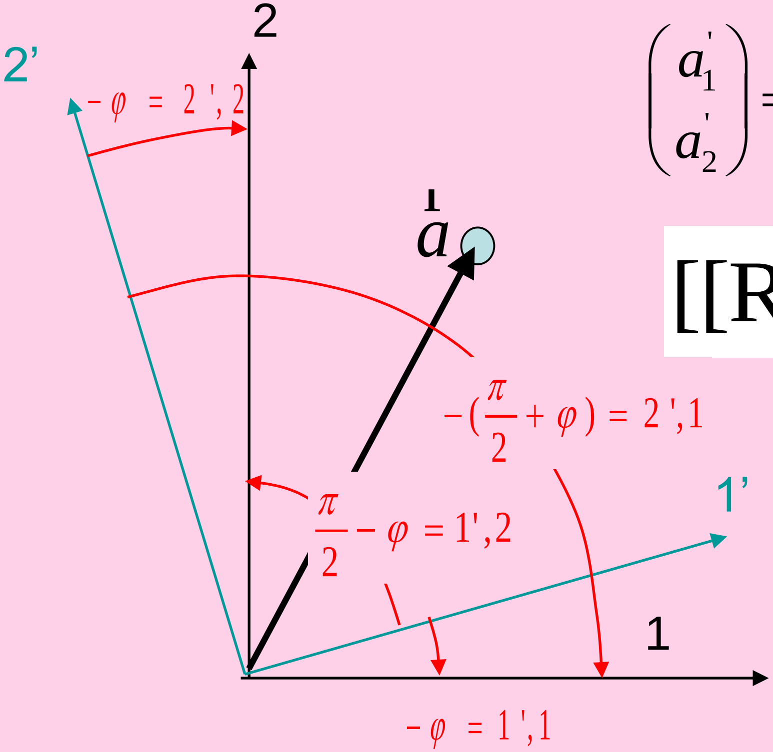
Rotation matrix
(R_{ij} is cosine of angle between axis i' and j')

MHMT1

Rotation of cartesian coordinate system

Example: Rotation only along the axis 3 by the angle φ (positive for counter-clockwise direction)

Properties of goniometric functions $\cos(-\varphi)=\cos\varphi$ $\cos(\frac{\pi}{2}-\varphi)=\sin\varphi$ $\cos(-(\frac{\pi}{2}+\varphi))=-\sin\varphi$



$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \cos(1',1)=\cos\varphi & \cos(1',2)=\sin\varphi \\ \cos(2',1)=-\sin\varphi & \cos(2',2)=\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$[[\mathrm{R}]]^T[[\mathrm{R}]]=[[\mathrm{I}]] \rightarrow [[\mathrm{R}]]^{-1}=[[R]]^T$$

therefore the rotation matrix is orthogonal and can be inverted just only by simple transposition (overturning along the main diagonal). Proof:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}=$$

$$=\begin{pmatrix} \cos^2\varphi+\sin^2\varphi & \cos\varphi\sin\varphi-\sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi-\cos\varphi\sin\varphi & \sin^2\varphi+\cos^2\varphi \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

MHMT1

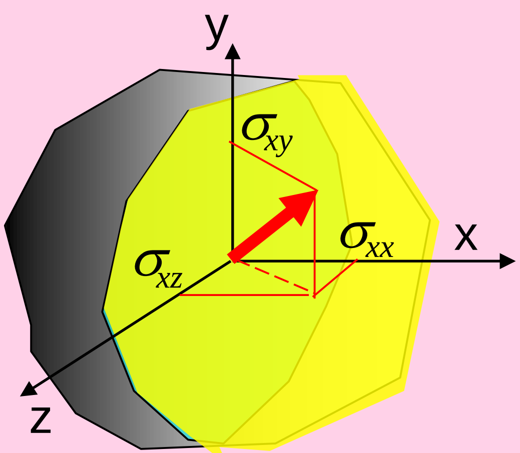
Stresses describe complete stress state at a point x,y,z

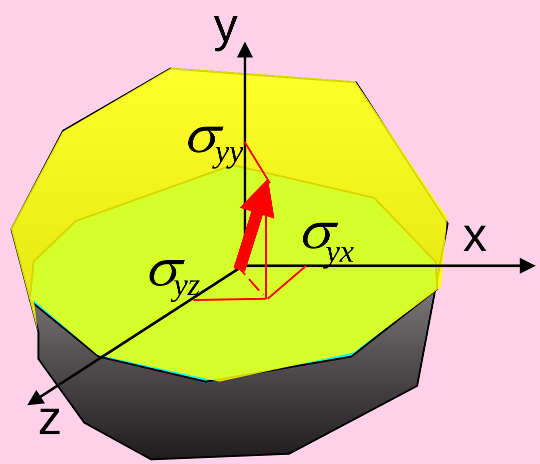
Stress tensor is a typical example of the second order tensor with a pair of indices having the following meaning

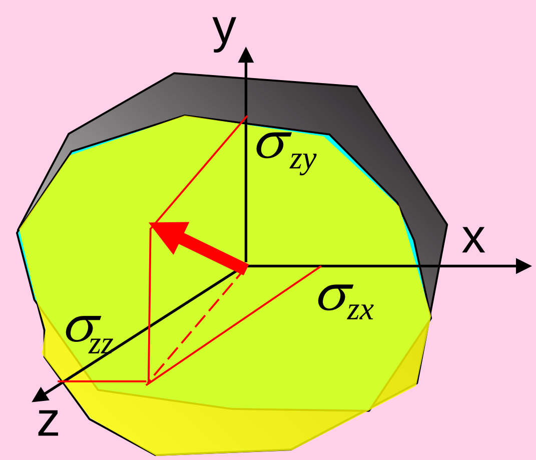
$$\sigma_{ij}$$

Index of plane (cross section) → i

index of force component (force acting upon the cross section i) → j







MHMT1

Tensor rotation of cartesian coordinate system

Later on we shall use another tensors of the second order describing kinematics of deformation (deformation tensors, rate of deformation,…)

Nine components of a tensor represent complete description of state (e.g. distribution of stresses at a point), but these components depend upon the choice of coordinate system, the same situation like with vectors. The transformation of components corresponding to the rotation of the cartesian coordinate system is given by the matrix product

$$[[\sigma']] = [[\mathrm{R}]][[\sigma]][[\mathrm{R}]]^T$$

where the rotation matrix [[R]] is the same as previously

$$[[R]] = \begin{pmatrix} \cos(1',1) & \cos(1',2) & \cos(1',3) \\ \cos(2',1) & \cos(2',2) & \cos(2',3) \\ \cos(3',1) & \cos(3',2) & \cos(3',3) \end{pmatrix}$$

MHMT1

Tensor rotation of cartesian coordinate system

Orthogonal matrix of the rotation of coordinate system [[R]] is fully determined by 3 parameters, by subsequent rotations around the x,y,z axis (therefore by 3 angles of rotations). The rotations can be selected in such a way that 3 components of the stress tensor in the new coordinate system disappear (are zero). Because the stress tensor is symmetric (usually) it is possible to anihilate all off-diagonal components

$$\begin{pmatrix} \sigma_1^{'} & 0 & 0 \\ 0 & \sigma_2^{'} & 0 \\ 0 & 0 & \sigma_3^{'} \end{pmatrix} = [[R]][[\sigma]][[R]]^T$$

Diagonal terms are normal (principal) stresses and the axis of the rotated coordinate systems are principal directions (there are no shear stresses in the cross-sections oriented in the principal directions).

MHMT1

Special tensors

Kronecker delta (unit tensor, components independent of rotation)

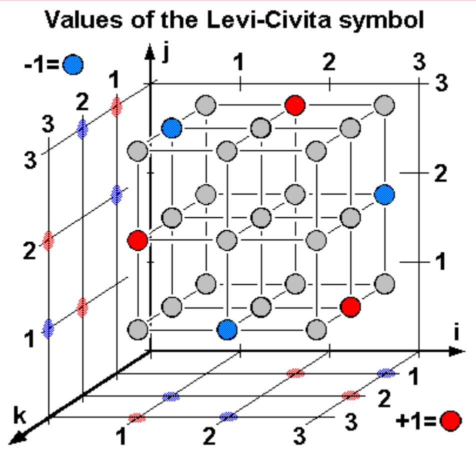
$$\delta_{ij} = 0 \text{ for } i \neq j$$

$$\delta_{ij} = 1 \text{ for } i = j$$

$$\vec{\vec{\delta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Levi Civita tensor is antisymmetric unit tensor of the third order (with 3 indices)

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2), (3,2,1) \text{ or } (2,1,3), \\ 0 & \text{otherwise: } i = j \text{ or } j = k \text{ or } k = i, \end{cases}$$

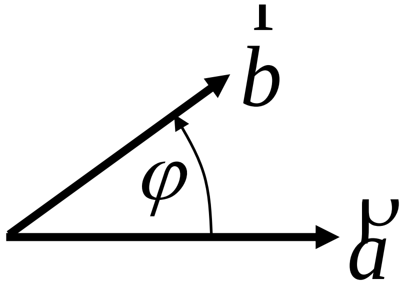


In terms of the epsilon tensor the vector product will be defined

MHMT1

Scalar product

Scalar product (operator •) of two vectors is a scalar



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|cos\varphi$$

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i = a_i b_i = a'_i b'_i$$

$a_i b_i$ is abbreviated Einstein notation. Repeated indices are summing (dummy) indices.

Proof that $a_i b_i = a'_i b'_i$

$$a'_i = R_{im} a_m \qquad b'_j = R_{jk} b_k$$

Remark: there were used dummy indices m and k in these relations. Letters of the dummy indices can be selected arbitrary, but in this case they must be different, so that to avoid appearance of four equal indices in a tensorial term in the following product a.b (there can be always max. two indices with the same name indicating a summation)

$$a'_1 b'_1 + a'_2 b'_2 + a'_3 b'_3 = a'_i b'_i = R_{im} R_{ik} a_m b_k = R^T_{mi} R_{ik} a_m b_k = \delta_{mk} a_m b_k = a_m b_m$$

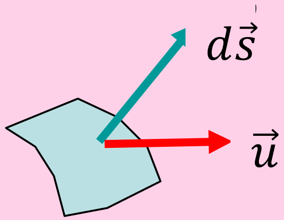
Scalar product

Example: scalar product of velocity $\vec{u}$ [m/s] and force $\vec{F}$ [N] acting at a point is power P [W] (scalar)

$$P = \vec{u} \cdot \vec{F} = u_i F_i$$

Example: Scalar product of velocity $\vec{u}$ [m/s] and the normal vector of an oriented surface $d\vec{s}$ [m^2] is the volumetric flowrate Q [m^3/s] through the surface

$$Q = d\vec{s} \bullet \vec{u}$$



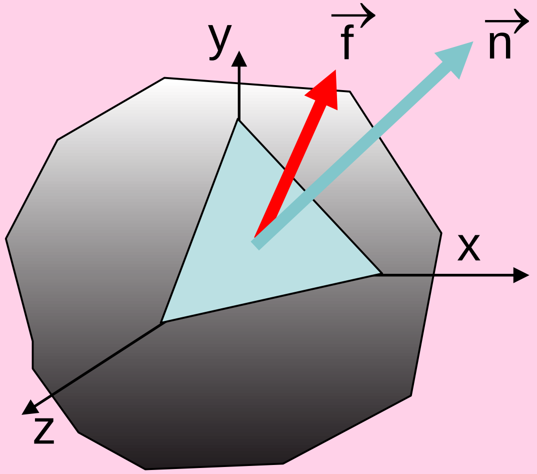
MHMT1

Scalar product

Scalar product can be applied also between tensors or between vector and tensor

$$\vec{u} \bullet \vec{\vec{\sigma}} = \vec{f} \qquad \sum_{i=1}^{3} n_i \sigma_{ij} = n_i \sigma_{ij} = f_j$$

i-is summation (dummy) index, while j-is free index



This case explains how it is possible to calculate internal stresses acting at an arbitrary cross section (determined by outer normal vector n) knowing the stress tensor.

Scalar product - examples

Dot product of delta tensors

$$\vec{\vec{\delta}} \bullet \vec{\vec{\delta}} = \vec{\vec{\delta}} \qquad \delta_{im}\delta_{mj} = \delta_{ij} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Scalar product of tensors is a tensor

$$\vec{\vec{\sigma}} \bullet \vec{\vec{\tau}} = \vec{\vec{\zeta}} \qquad \sigma_{im}\tau_{mj} = \varsigma_{ij}$$

Double dot product of tensors is a scalar

$$\vec{\vec{\sigma}} : \vec{\vec{\tau}} = \varsigma \qquad \sigma_{km}\tau_{mk} = \varsigma$$

Trace of a tensor (tensor contraction)

$$tr(\vec{\vec{\sigma}}) = \sigma_{mm} \qquad \text{example } tr(\vec{\vec{\delta}}) = 3$$

MHMT1

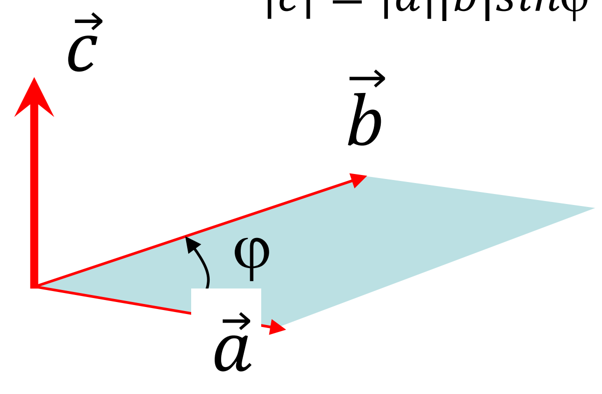
Vector product

Vector product (operator x) of two vectors is a vector

$|c| = |a||b| sin\varphi$

$$\vec{c} = \vec{a} \times \vec{b} = (\vec{\vec{\vec{\varepsilon}}} \bullet \vec{b}) \bullet \vec{a}$$

$$c_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k = \varepsilon_{ijk} a_j b_k$$



for example $c_1 = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$

Important relationship between the Levi Civita and the Kronecker delta special tensors

$$\varepsilon_{ijk}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

check for i=1,j=2,(k=3),m=1,n=2 $\varepsilon_{123}\ \varepsilon_{312} = \delta_{11}\ \delta_{22} - \delta_{12}\ \delta_{21} = 1$

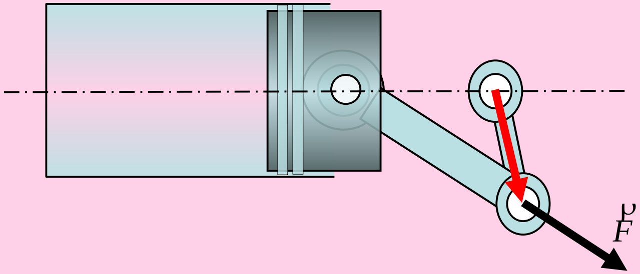
check for i=1,j=2,(k=3),m=2,n=1 $\varepsilon_{123}\ \varepsilon_{321} = \delta_{12}\ \delta_{21} - \delta_{11}\ \delta_{22} = -1$

MHMT1

Vector product

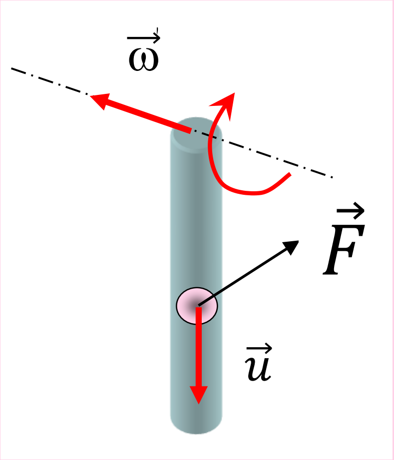
Examples of applications

Moment of force (torque) $\vec{M} = \vec{r} \times \vec{F}$



Coriolis force

$\vec{F} = 2m\,\vec{u} \times \vec{\omega}$



application: Coriolis flowmeter

MHMT1

Diadic product

Diadic product (no operator) of two vectors is a second order tensor

$$\vec{a}\vec{b} = \vec{\vec{\pi}}$$

$$a_i b_j = \pi_{ij}$$

MHMT1

Differential operator ∇ (Nabla)

Bailey

Differential operator ∇ (Nabla)

GRADIENT – measure of spatial changes

Symbolic operator ∇ represents a vector of first derivatives with respect x,y,z.

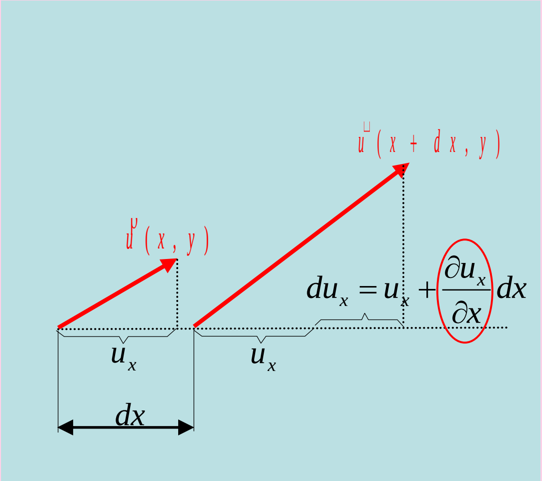
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) \qquad \nabla_{\mathrm{i}} = \frac{\partial}{\partial x_i}$$

∇ applied to scalar is a vector (gradient of scalar)

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) \qquad \nabla_i T = \frac{\partial T}{\partial x_i}$$

∇ applied to vector is a tensor (for example gradient of velocity is a tensor)

$$\nabla \vec{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix} \qquad \nabla_{\mathrm{i}} u_j = \frac{\partial u_j}{\partial x_i}$$



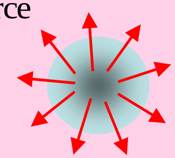
Differential operator ∇·

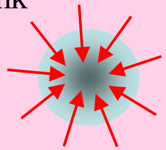
DIVERGENCY – magnitude of sources/sinks

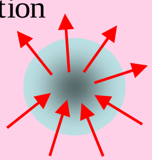
Scalar product ∇• represents intensity of source/sink of a vector quantity at a point

$$\nabla \bullet \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}$$

i-dummy index, result is a scalar







Scalar product ∇• can be applied also to a tensor giving a vector (e.g. source/sink of momentum in the direction x,y,z)

$$\vec{f} = \nabla \bullet \overline{\overline{\sigma}} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \qquad f_i = \nabla_j \sigma_{ji}$$

The vector f_i represents resulting force of stresses acting at the surface of an infinitely small volume (the force is related to this volume therefore unit is N/m^3)

MHMT1

Laplace operator ∇^2

Divergency of gradient – measure of nonuniformity

Scalar product $\nabla \bullet \nabla = \nabla^2$ is the operator of second derivatives (when applied to scalar it gives a scalar, applied to a vector gives a vector,…). Laplace operator is divergence of a gradient (gradient of temperature, gradient of velocity…)

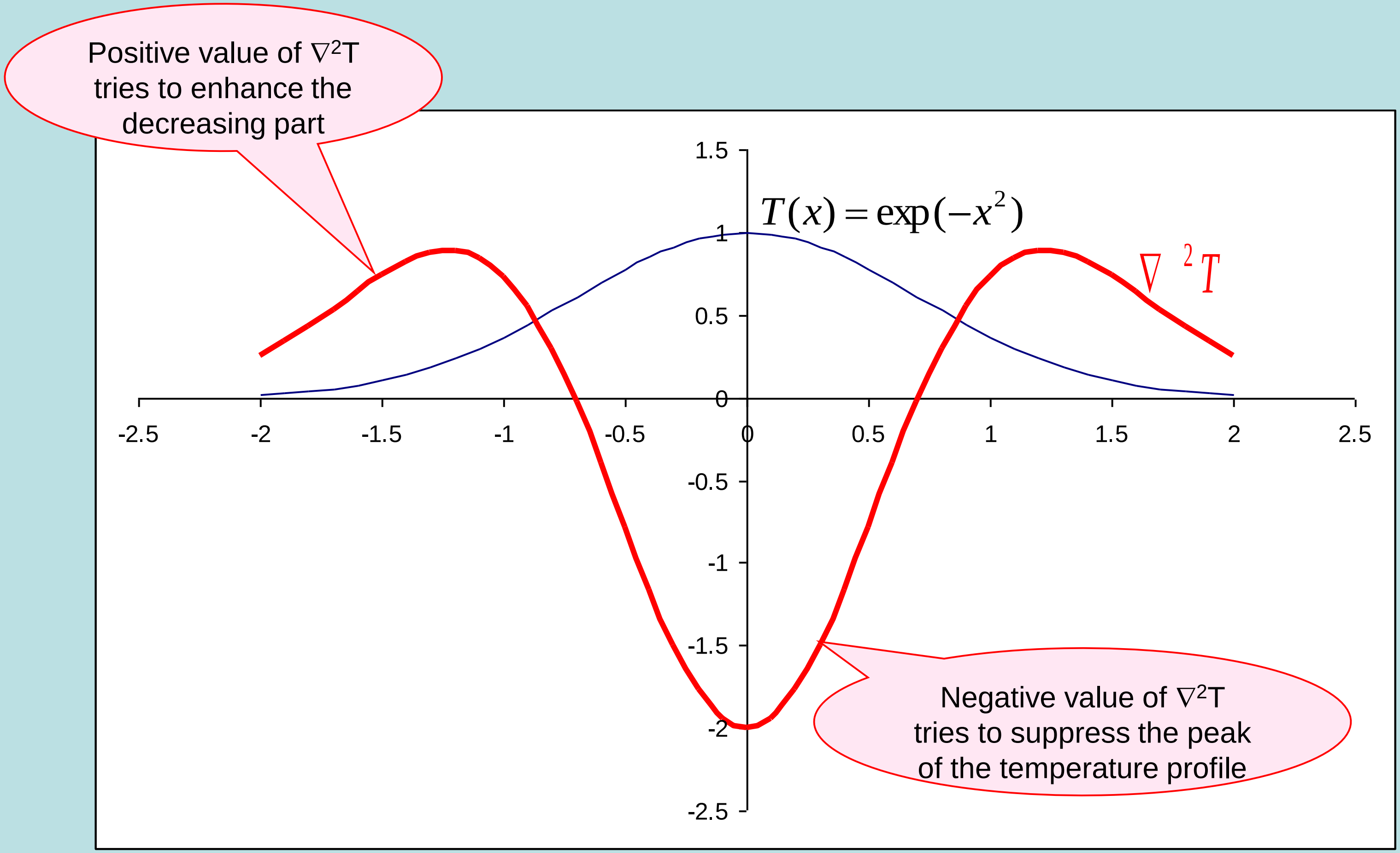
$$\nabla \bullet \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial x_i \partial x_i}$$

i-dummy index

$$\nabla \bullet \nabla\, \vec{u} = \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}, \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}, \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) = \sum_{i=1}^{3} \frac{\partial^2 u_j}{\partial x_i^2} = \frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

Physical interpretation: ∇^2 describes diffusion processes (random molecular motion). The term ∇^2 appears in the transport equations for temperatures, momentum, concentrations and its role is to smooth out all spatial nonuniformities of the transported properties.

Laplace operator ∇²

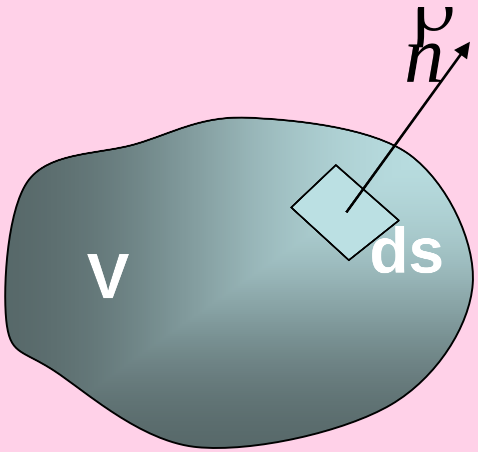


Integrals – Gauss theorem

Volume integrals with nabla operator can be converted to surface integrals (just only by replacing nabla ∇ with unit normal vector $\vec{n}$)

Physical interpretation: accumulation in volume V is overall flux through boundary

$$\iiint_V \underbrace{\nabla \cdot P}_{\text{Divergence of P}}\, dv = \iint_S \underbrace{\vec{n} \cdot P}_{\text{projection of P to outer normal}}\, ds$$



Variable $\boldsymbol{P}$ can be

- **Scalar** (typically pressure p) $\iiint_V \nabla p\, dv = \iint_S \vec{n} p\, ds$
- **Vector** (vector of velocity, momentum, heat flux). Surface integral represents flux of vector in the direction of outer normal. $\iiint_V \nabla \cdot \vec{u}\, dv = \iint_S \vec{n} \cdot \vec{u}\, ds$
- **Tensor** (tensor of stresses). In this case the Gauss theorem represents the balance between inner stresses and outer forces acting upon the surface,

$$\iiint_V \nabla \cdot \overleftrightarrow{\tau}\, dv = \iint_S \vec{n} \cdot \overleftrightarrow{\tau}\, ds$$

Transcriptions

Symbolic notation, for example $\nabla^2 T$, is compact, unique and suitable for definition of problems in terms of tensorial equations. However, if you need to solve these equation (for example $\nabla^2 T = f$) you have to rewrite symbolic form into index notation, giving equations (usually differential equations) for components of vectors or tensors, which are expressed by numbers (may be complex numbers).

Bailey

MHMT1

Symbolic → indicial notation

General procedure how to rewrite symbolic formula to index notation

- Replace each arrow → by an empty place for index _
- Replace each vector operator by (-1) • $\varepsilon_{\_\ \_\ \_}$ •
- Replace each dot • by a pair of dummy indices in the first free position left and right
- Write free indices into remaining positions

Practice examples!!

MHMT1

Coordinate systems

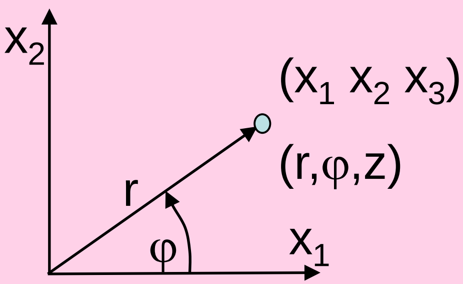
Previous conversion procedure can be applied only in a cartesian coordinate systems

Formulation in the x,y,z cartesian system is not always convenient, especially if the geometry of region is cylindrical or spherical. For example the boundary condition of a constant temperature is difficult to prescribe on the curved surface of sphere in the rectangular cartesian system. In the case that the problem formulation suppose a rotational or spherical symmetry, the number of spatial coordinates can be reduced and the problem is simplified to 1D or 2D problem, but in a new coordinate system.

In the following we shall demonstrate how to convert tensor terms from symbolic notation (which is independent to a specific coordinate system) into the cylindrical coordinate system.

Coordinate systems (cylindrical)

Cylindrical (and spherical) systems are defined by transformations



$$x_1 = rc \qquad x_2 = rs \qquad x_3 = z$$

where $s = \sin\varphi \quad c = \cos\varphi$

$$dx_i = \frac{\partial x_i}{\partial r} dr + \frac{\partial x_i}{\partial \varphi} d\varphi + \frac{\partial x_i}{\partial z} dz$$

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix}$$

$$\begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -\frac{s}{r} & \frac{c}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$\frac{\partial r}{\partial x_1} = c, \; \frac{\partial r}{\partial x_2} = s, \; \frac{\partial \varphi}{\partial x_1} = -\frac{s}{r}, \; \frac{\partial \varphi}{\partial x_2} = \frac{c}{r}$$

Using this it is possible to express partial derivatives with respect x_1,x_2,x_3 in terms of derivatives with respect the coordinates of cylindrical system

$$\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x_1} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x_1} = \frac{\partial T}{\partial r} c - \frac{\partial T}{\partial \varphi}\frac{s}{r}$$

$$\frac{\partial T}{\partial x_2} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x_2} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x_2} = \frac{\partial T}{\partial r} s + \frac{\partial T}{\partial \varphi}\frac{c}{r}$$

$$\frac{\partial T}{\partial x_3} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x_3} + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x_3} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x_3} = \frac{\partial T}{\partial z}$$

MHMT1

Coordinate systems (cylindrical)

In the same way also the second derivatives can be expressed

$$\frac{\partial^2 T}{\partial x_1^2} = \frac{\partial^2 T}{\partial r^2} c^2 - \frac{2cs}{r} \frac{\partial}{\partial \varphi} (\frac{\partial T}{\partial r} - \frac{T}{r}) + \frac{\partial T}{\partial r} \frac{s^2}{r} + \frac{\partial^2 T}{\partial \varphi^2} \frac{s^2}{r^2}$$

$$\frac{\partial^2 T}{\partial x_2^2} = \frac{\partial^2 T}{\partial r^2} s^2 + \frac{2cs}{r} \frac{\partial}{\partial \varphi} (\frac{\partial T}{\partial r} - \frac{T}{r}) + \frac{\partial T}{\partial r} \frac{c^2}{r} + \frac{\partial^2 T}{\partial \varphi^2} \frac{c^2}{r^2}$$

$$\frac{\partial^2 T}{\partial x_3^2} = \frac{\partial^2 T}{\partial z^2}$$

giving expression for the **Laplace operator in the cylindrical coordinate system**

(use goniometric identity $s^2+c^2=1$)

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} = \frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{\partial r} \frac{1}{r} + \frac{\partial^2 T}{\partial \varphi^2} \frac{1}{r^2} + \frac{\partial^2 T}{\partial z^2}$$

MHMT1

Coordinate systems (cylindrical)

Previous example demonstrated how to solve the problem of transformations to cylindrical coordinate system with scalars. However, how to calculate gradients or divergence of a vector and a tensor field? Vectors and tensors are described by 3 or 3x3 values of components now expressed in terms of new unit vectors

$$\vec{u} = u_m \vec{i}_m = u_r \vec{e}^{(r)} + u_\varphi \vec{e}^{(\varphi)} + u_z \vec{e}^{(z)}$$

Einstein summation applied in the cartesian coordinate system

unit vectors in the cylindrical coordinate system

$$\vec{\vec{\sigma}} = \sigma_{mn} \vec{i}_m \vec{i}_n =$$

$$= \sigma_{rr} \vec{e}^{(r)} \vec{e}^{(r)} + \sigma_{r\varphi} \vec{e}^{(r)} \vec{e}^{(\varphi)} + \sigma_{rz} \vec{e}^{(r)} \vec{e}^{(z)} + \ldots + \sigma_{zz} \vec{e}^{(z)} \vec{e}^{(z)}$$

Unit vectors in orthogonal coordinate system are orthogonal, which means that

$$\vec{e}^{(r)} \cdot \vec{e}^{(r)} = 1 \quad \vec{e}^{(r)} \cdot \vec{e}^{(\varphi)} = 0 \quad \ldots \vec{e}^{(z)} \cdot \vec{e}^{(z)} = 1$$

and therefore the components in the new coordinate system are for example

$$u_\varphi = \vec{u} \cdot \vec{e}^{(\varphi)} = u_m e_m^{(\varphi)} \qquad \sigma_{rz} = \vec{e}^{(r)} \cdot \vec{\vec{\sigma}} \cdot \vec{e}^{(z)} = e_m^{(r)} \sigma_{mn} e_n^{(z)}$$

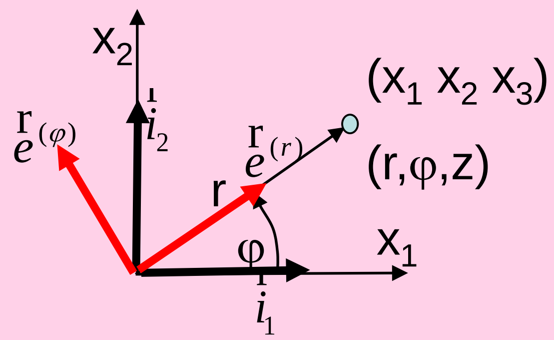
$$u_i = u_r e_i^{(r)} + u_\varphi e_i^{(\varphi)} + u_z e_i^{(z)}$$

$$e_i^{(z)} = \vec{e}^{(z)} \bullet \vec{i}_i$$

this is i-th cartesian component of the unit vector

Coordinate systems (cylindrical)

Transformation of unit vectors



$$\begin{pmatrix} \vec{e}^{(r)} \\ \vec{e}^{(\varphi)} \\ \vec{e}^{(z)} \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{pmatrix}$$

$$\begin{pmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{pmatrix} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{e}^{(r)} \\ \vec{e}^{(\varphi)} \\ \vec{e}^{(z)} \end{pmatrix}$$

Example: **gradient of temperature** can be written in the following way (alternatively in the cartesian and the cylindrical coordinate system)

substitute by using previously derived $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial r}c - \frac{\partial T}{\partial \varphi}\frac{s}{r}$

$$\nabla T = \frac{\partial T}{\partial x_1}\vec{i}_1 + \frac{\partial T}{\partial x_2}\vec{i}_2 + \frac{\partial T}{\partial x_3}\vec{i}_3 = (c\frac{\partial T}{\partial x_1} + s\frac{\partial T}{\partial x_2})\vec{e}^{(r)} + (s\frac{\partial T}{\partial x_1} - c\frac{\partial T}{\partial x_2})\vec{e}^{(\varphi)} + \frac{\partial T}{\partial x_3}\vec{e}^{(z)} =$$

$$= \frac{\partial T}{\partial r}\vec{e}^{(r)} + \frac{1}{r}\frac{\partial T}{\partial \varphi}\vec{e}^{(\varphi)} + \frac{\partial T}{\partial z}\vec{e}^{(z)}$$

Nabla operator in the cylindrical coordinate system

$$\nabla \equiv \vec{e}^{(r)}\frac{\partial}{\partial r} + \vec{e}^{(\varphi)}\frac{1}{r}\frac{\partial}{\partial \varphi} + \vec{e}^{(z)}\frac{\partial}{\partial z}$$

Coordinate systems (cylindrical)

Example: **Divergence of a vector** $\vec{u} = u_m \vec{i}_m = u_r \vec{e}^{(r)} + u_\varphi \vec{e}^{(\varphi)} + u_z \vec{e}^{(z)}$

$$\nabla \cdot \vec{u} = \frac{\partial u_m}{\partial x_m} = \frac{\partial (u_r e_m^{(r)} + u_\varphi e_m^{(\varphi)} + u_z e_m^{(z)})}{\partial x_m} =$$

$$= \frac{\partial u_r}{\partial x_m} e_m^{(r)} + u_r \frac{\partial e_m^{(r)}}{\partial x_m} + \frac{\partial u_\varphi}{\partial x_m} e_m^{(\varphi)} + u_\varphi \frac{\partial e_m^{(\varphi)}}{\partial x_m} + \frac{\partial u_z}{\partial x_m} e_m^{(z)} + u_z \frac{\partial e_m^{(z)}}{\partial x_m}$$

components of unit vectors follow from the previously derived

$$\begin{pmatrix} \vec{e}^{(r)} \\ \vec{e}^{(\varphi)} \\ \vec{e}^{(z)} \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{pmatrix}$$

$$e_1^{(r)} = c \qquad e_1^{(\varphi)} = -s \qquad e_1^{(z)} = 0$$

$$e_2^{(r)} = s \qquad e_2^{(\varphi)} = c \qquad e_2^{(z)} = 0$$

$$e_3^{(r)} = 0 \qquad e_3^{(\varphi)} = 0 \qquad e_3^{(z)} = 1$$

Note the fact, that the partial derivatives of these components with respect to x_m are not zero and can be calculated using the previously derived relationships

$$\frac{\partial \varphi}{\partial x_1} = -\frac{s}{r}, \quad \frac{\partial \varphi}{\partial x_2} = \frac{c}{r}$$

giving

$$\frac{\partial e_1^{(r)}}{\partial x_1} = \frac{\partial c}{\partial x_1} = \frac{\partial c}{\partial \varphi}\frac{\partial \varphi}{\partial x_1} = -s\frac{s}{r}$$

$$\frac{\partial e_1^{(\varphi)}}{\partial x_1} = -\frac{\partial s}{\partial x_1} = -\frac{\partial s}{\partial \varphi}\frac{\partial \varphi}{\partial x_1} = c\frac{s}{r}$$

…and so on

MHMT1

Coordinate systems (cylindrical)

Substituting these expression we obtain

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_r}{\partial r} c^2 - \frac{\partial u_r}{\partial \varphi} \frac{cs}{r} + u_r \frac{s^2}{r} - \frac{\partial u_\varphi}{\partial r} cs + \frac{\partial u_\varphi}{\partial \varphi} \frac{s^2}{r} + u_\varphi \frac{cs}{r}$$

$$\frac{\partial u_2}{\partial x_2} = \frac{\partial u_r}{\partial r} s^2 + \frac{\partial u_r}{\partial \varphi} \frac{cs}{r} + u_r \frac{c^2}{r} + \frac{\partial u_\varphi}{\partial r} cs + \frac{\partial u_\varphi}{\partial \varphi} \frac{c^2}{r} - u_\varphi \frac{cs}{r}$$

$$\frac{\partial u_3}{\partial x_3} = \frac{\partial u_z}{\partial z}$$

Summing together, the final form of divergence in the cylindrical coordinate system is obtained

$$\nabla \bullet \vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}$$

MHMT1

Coordinate systems (cylindrical)

Example: **Gradient of a vector** $\vec{u} = u_m \vec{i}_m = u_r \vec{e}^{(r)} + u_\varphi \vec{e}^{(\varphi)} + u_z \vec{e}^{(z)}$

$$\overline{\overline{\pi}} = \nabla \vec{u}$$

$$\pi_{rr} = e_m^{(r)} \frac{\partial (u_r e_n^{(r)} + u_\varphi e_n^{(\varphi)} + u_z e_n^{(z)})}{\partial x_m} e_n^{(r)} \qquad \pi_{r\varphi} = e_m^{(r)} \frac{\partial (u_r e_n^{(r)} + u_\varphi e_n^{(\varphi)} + u_z e_n^{(z)})}{\partial x_m} e_n^{(\varphi)} \ldots$$

m and *n* are dummy indices (summing is required)

Substituting previous expressions for unit vector and their derivatives results to final expression for the velocity gradient tensor in a cylindrical coordinate system

$$\nabla \vec{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\varphi}{\partial r} & \frac{\partial u_z}{\partial r} \\ \frac{1}{r}(\frac{\partial u_r}{\partial \varphi} - u_\varphi) & \frac{1}{r}(\frac{\partial u_\varphi}{\partial \varphi} + u_r) & \frac{1}{r}\frac{\partial u_z}{\partial \varphi} \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\varphi}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

MHMT1

Coordinate systems (general)

Procedure how to derive tensorial equations in a general coordinate system.

1. Rewrite equation from the symbolic notation to the index notation for cartesian coordinate system, for example $\nabla\vec{u} = \overset{\leftrightarrow}{\pi} \rightarrow \frac{\partial u_j}{\partial x_i} = \pi_{ij}$

2. Define transformation $x_1(r_1,r_2,r_3),\ldots$ $r_1(x_1,x_2,x_3),\ldots$ and $\frac{\partial r_i}{\partial x_j} = f_{ij}(r_1,r_2,r_3)$ therefore also the first and the second derivatives of scalar values, for example

$$\frac{\partial u_j}{\partial x_1} = \frac{\partial u_j}{\partial r_1} f_{11} + \frac{\partial u_j}{\partial r_2} f_{21} + \frac{\partial u_j}{\partial r_3} f_{31},\ldots$$

3. Define unit vectors of coordinate systems r_i and express their cartesian coordinates $e_m^{(ri)} = g_m^{(ri)}(r_1,r_2,r_3)$
Calculate their derivatives with respect to the cartesian coordinates

$$\frac{\partial e_m^{(r)}}{\partial x_i} = \frac{\partial g_m^r}{\partial x_1} = \frac{\partial g_m^r}{\partial r_1}\frac{\partial r_1}{\partial x_i} + \frac{\partial g_m^r}{\partial r_2}\frac{\partial r_2}{\partial x_i} + \frac{\partial g_m^r}{\partial r_3}\frac{\partial r_3}{\partial x_i}$$

4. In case that the result is a vector, for example the gradient of scalar, calculate its components from

$$u_\varphi = \vec{u}\cdot\vec{e}^{(\varphi)} = u_m e_m^{(\varphi)}$$

In the case that result is a tensor calculate its components

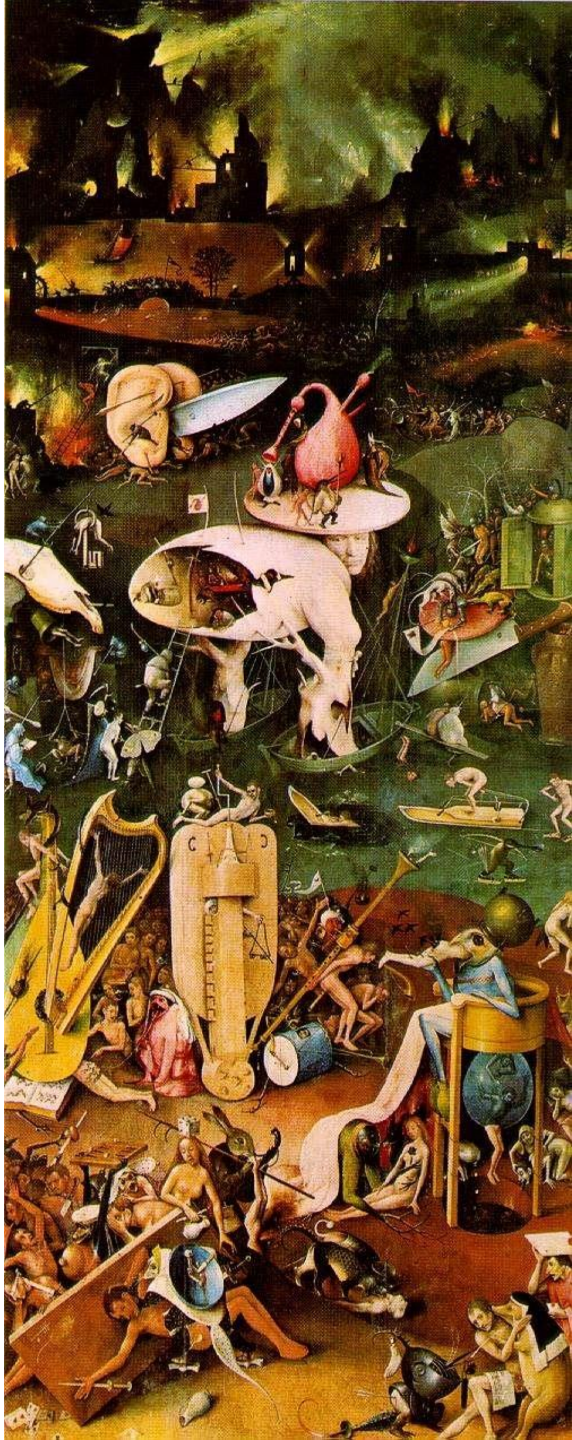
$$\pi_{rz} = \vec{e}^{(r)}\cdot\overset{\leftrightarrow}{\pi}\cdot\vec{e}^{(z)} = e_i^{(r)}\pi_{ij}e_j^{(z)}$$

Remark: This suggested procedure (transform everything, including new unit vectors, to the cartesian coordinate system) is straightforward and seemingly easy. This is not so, it is „crude“, lenghty (the derivation of velocity gradient is on several lists of paper) and without finesses. Better and more sophisticated procedures are described in standard books, e.g. Aris R: Vectors, tensors…N.J.1962, or Bird,Stewart,Lightfoot:Transport phenomena.

MHMT1

EXAM

Tensors



MHMT1

What is important (at least for exam)

You should know what is it scalar, vector, tensor and transformations at rotation of coordinate system

$$[a'] = [[R]][a] \qquad [[\sigma']] = [[R]][[\sigma]][[R]]^T$$

(and how is defined the rotation matrix R?)

Scalar and vector products

$$\vec{a} \bullet \vec{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i$$

$$\vec{c} = \vec{a} \times \vec{b} = (\vec{\vec{\varepsilon}} \bullet \vec{b}) \bullet \vec{a}$$

$$c_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k = \varepsilon_{ijk} a_j b_k$$

(and what is it Kronecker delta and Levi Civita tensors?)

MHMT1

What is important (at least for exam)

Nabla operator. Gradient

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) \qquad \nabla_{\mathrm{i}} = \frac{\partial}{\partial x_i}$$

Divergence

$$\nabla \bullet \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}$$

Laplace operator

$$\nabla \bullet \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial x_i \partial x_i}$$

MHMT1

What is important (at least for exam)

Gauss integral theorem

$$\iiint_V \nabla \bullet P \, dv = \iint_S \vec{n} \bullet P \, ds$$

(demonstrate for the case that P is scalar, vector, tensor)