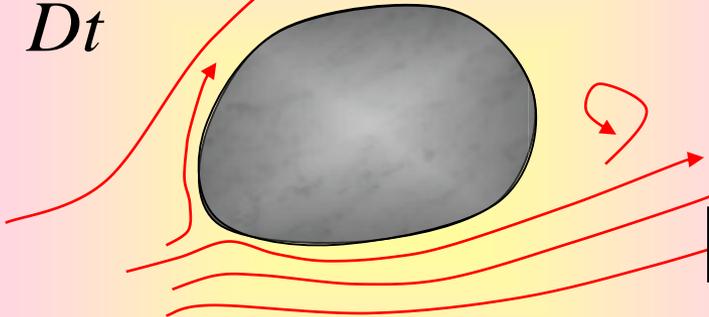


# Momentum Heat Mass Transfer

$$\frac{D\Phi}{Dt} = \nabla \cdot \Gamma \nabla \Phi + source$$



## Rules, database, Tensorial calculus

Introduction and rules (form of lectures, presentations of papers by students, requirements for exam). Preacquaintance (photography). Working with databases and primary sources. Tensors and tensorial calculus

# Momentum Heat Mass Transfer

**3P/1C, 4 credits, exam, LS2018, room 505d**

- Lectures [Prof.Ing.Rudolf Žitný, CSc.](#) (Monday 10:45-13:00)
- Tutorials [Ing.Karel Petera, PhD.](#)
  
- Rating 30(quise in test)+30(example in test)+40(oral exam) points

A	B	C	D	E	F
90+	80+	70+	60+	50+	..49
excellent	very good	good	satisfactory	sufficient	failed

# Momentum Heat Mass Transfer

2018 505

Předmět: (E181026) Momentum, Heat and Mass Transfer / FS Paralelka: Všichni studenti předmětu  Zo

Způsob zakončení: Z,ZK Rozsah: 3+1

Počet studentů : 28 Kapacita předmětu: 30

Excel Tisk E-mail Zobrazit filtr

Příjmení	Jméno	Identifikátor	Fakulta	Studijní program	Typ programu
Alsatmi	Basel	479516	FS	Bakalářský program pro výměnné studenty	B
Bemmireddy	Santhosh Reddy	473334	FS	Strojní inženýrství	N
Duque Dussan	Eduardo	472430	FS	Strojní inženýrství	N
Fenech	Daniel	479578	FS	Bakalářský program pro výměnné studenty	B
Feppon	Nicolas	472985	FS	Mobility - bakalářský	B
Gueho	Lénaig	479705	FEL	Electrical Engineering and Computer Science	B
Hanna Anz	Qais Elia	453674	FS	Strojírenství	B
Hübbers	Lena	479702	FS	Magisterský program pro výměnné studenty	N
Ipek	Murat	472154	FS	Strojní inženýrství	N
Lalonde	Timothe	479067	FS	Mobility - bakalářský	B
Lloyd	Olivia	472153	FS	Mobility - bakalářský	B
Márquez Hernández	Luis Eduardo	472087	FS	Mobility - bakalářský	B
Mcgowan	Orla	472830	FS	Magisterský program pro výměnné studenty	N
Messas	Anis	453198	FS	Strojírenství	B
Nivet	Gabriel	479697	FEL	Electrical Engineering and Computer Science	B
Perez	Ariel	479870	FEL	Electrical Engineering and Computer Science	B
Somireddy	Ashok Kumar Reddy	473153	FS	Strojní inženýrství	N
Souza	Anne	479472	FS	Mobility - bakalářský	B
Stalls	Wayne	479054	FS	Mobility - bakalářský	B
Stevenson	Craig	472700	FS	Magisterský program pro výměnné studenty	N
Stukel	Nathan	479222	FS	Mobility - bakalářský	B
Suleiman	Abubakar Shola	459336	FS	Strojírenství	B
Torres Tapia	Luis Alberto	476496	FS	Strojní inženýrství	N
Upadhyay	Sumit	473394	FS	Strojní inženýrství	N
Uzakov	Timur	453574	FS	Teoretický základ strojírenství	B
Walker	Justin Roy	479055	FS	Mobility - bakalářský	B
Yap	Jane	479062	FS	Mobility - bakalářský	B
Yücel	Utku	479428	FS	Bakalářský program pro výměnné studenty	B

MHMT1

# Momentum Heat Mass Transfer

3P/2C, 4 credits, exam, LS2018, room 505

# Literature - sources

- **Textbook:**

Šesták J., Rieger F.: Přenos hybnosti tepla a hmoty, ČVUT Praha, 2004

- **Monography:** available at **NTL**

Bird, Stewart, Lightfoot: Transport Phenomena. Wiley, 2nd edition, 2006

Welty J.R.: Fundamental of momentum, heat and mass transfer, Wiley, 2008

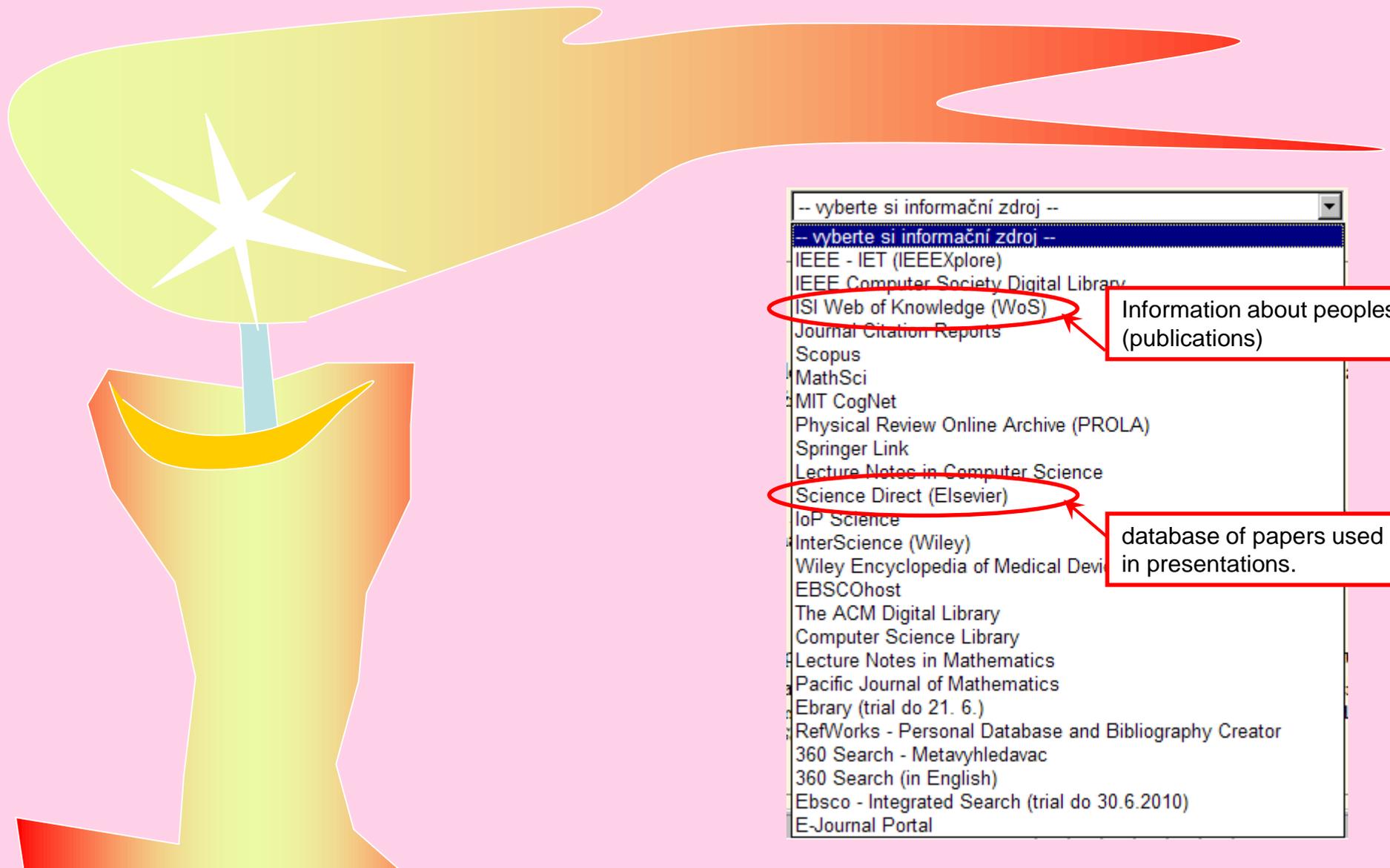
- **Databases of primary sources**

Direct access to databases (WoS, Elsevier, Springer,...)

[knihovny.cvut.cz](http://knihovny.cvut.cz)

The image shows a screenshot of a library website. On the left is a blue navigation menu titled "RYCHLÉ ODKAZY" (Quick Links) with the following items: Kontakty, Online katalog, Uživatelské konto, Brána EIZ (circled in red), and 360 Search. A red arrow points from the "Brána EIZ" link to a central login portal titled "Brána k vybraným informačním zdrojům" (Gateway to selected information sources). The login portal includes a university selection dropdown (set to "České vysoké učení technické v Praze"), a server selection dropdown (set to "USERMAP: Soustředěná správa uživatelů"), and input fields for "Uživatelské jméno" (set to "Jméno DUPS") and "Heslo". A "Přihlášení" (Login) button is located below the password field. A red arrow points from the login button to a dropdown menu on the right titled "-- vyberte si informační zdroj --" (Select your information source). This menu lists various databases including IEEE - IET, ISI Web of Knowledge, Scopus, MathSci, MIT CogNet, and others.

# EES Electronic sources



- vyberte si informační zdroj --
- vyberte si informační zdroj --
- IEEE - IET (IEEEExplore)
- IEEE Computer Society Digital Library
- ISI Web of Knowledge (WoS)
- Journal Citation Reports
- Scopus
- MathSci
- MIT CogNet
- Physical Review Online Archive (PROLA)
- Springer Link
- Lecture Notes in Computer Science
- Science Direct (Elsevier)
- IoP Science
- InterScience (Wiley)
- Wiley Encyclopedia of Medical Devices and Instruments
- EBSCOhost
- The ACM Digital Library
- Computer Science Library
- Lecture Notes in Mathematics
- Pacific Journal of Mathematics
- Ebrary (trial do 21. 6.)
- RefWorks - Personal Database and Bibliography Creator
- 360 Search - Metavyhledavac
- 360 Search (in English)
- Ebsco - Integrated Search (trial do 30.6.2010)
- E-Journal Portal

Information about peoples (publications)

database of papers used in presentations.

# Science Direct

Quick Search

Key words

- vyberte si informační zdroj --
- vyberte si informační zdroj --
- IEEE - IET (IEEEExplore)
- IEEE Computer Society Digital Lib
- ISI Web of Knowledge (WoS)
- Journal Citation Reports
- Scopus
- MathSci
- MIT CogNet
- Physical Review Online Archive (P
- Springer Link
- Lecture Notes in Computer Scienc
- Science Direct (Elsevier)**
- IoP Science
- InterScience (Wiley)
- Wiley Encyclopedia of Medical Devices and Instrumentation
- EBSCOhost
- The ACM Digital Library
- Computer Science Library
- Lecture Notes in Mathematics
- Pacific Journal of Mathematics
- Ebrary (trial do 21. 6.)
- RefWorks - Personal Database and Bibliography Creator
- 360 Search - Metavyhledavac
- 360 Search (in English)
- Ebsco - Integrated Search (trial do 30.6.2010)
- E-Journal Portal

**237 articles found for:** ALL(heat exchanger design TEMA)  
 Save Search | Save as Search Alert | RSS Feed

Search Within Results:

Refine Results

Content Type

- Journal (180)
- Book (81)

1.  **Multi-objective optimization of shell and tube heat exchangers**  
*Applied Thermal Engineering, Volume 30, Issues 14-15, October 2010*  
 Sepehr Sanaye, Hassan Hajabdollahi

Applied Thermal Engineering 30 (2010) 1937–1945

Full text available in pdf

 **Applied Thermal Engineering**  
 journal homepage: [www.elsevier.com/locate/apthermeng](http://www.elsevier.com/locate/apthermeng)

**Multi-objective optimization of shell and tube heat exchangers**  
 Sepehr Sanaye\*, Hassan Hajabdollahi  
*Energy Systems Improvement Laboratory (ESI), Department of Mechanical Engineering, Iran University of Science and Technology (IUST), Narmak, Tehran 16844, Iran*

ARTICLE INFO	ABSTRACT
<p><b>Article history:</b>            Received 25 January 2010            Accepted 14 April 2010            Available online 21 April 2010</p> <p><b>Keywords:</b>            Shell and tube heat exchanger            Heat recovery            Effectiveness            Total cost            Multi-objective optimization            NSGA-II</p>	<p>The effectiveness and cost are two important parameters in heat exchanger design. The total cost includes the capital investment for equipment (heat exchanger surface area) and operating cost (for energy expenditures related to pumping). Tube arrangement, tube diameter, tube pitch ratio, tube length, tube number, baffle spacing ratio as well as baffle cut ratio were considered as seven design parameters. For optimal design of a shell and tube heat exchanger, it was first thermally modeled using <i>c-MTU</i> method while Bell–Delaware procedure was applied to estimate its shell side heat transfer coefficient and pressure drop. Fast and elitist non-dominated sorting genetic algorithm (NSGA-II) with continuous and discrete variables were applied to obtain the maximum effectiveness (heat recovery) and the minimum total cost as two objective functions. The results of optimal designs were a set of multiple optimum solutions, called 'Pareto optimal solutions'. The sensitivity analysis of change in optimum effectiveness and total cost with change in design parameters of the shell and tube heat exchanger was also performed and the results are reported.</p> <p>© 2010 Elsevier Ltd. All rights reserved.</p>

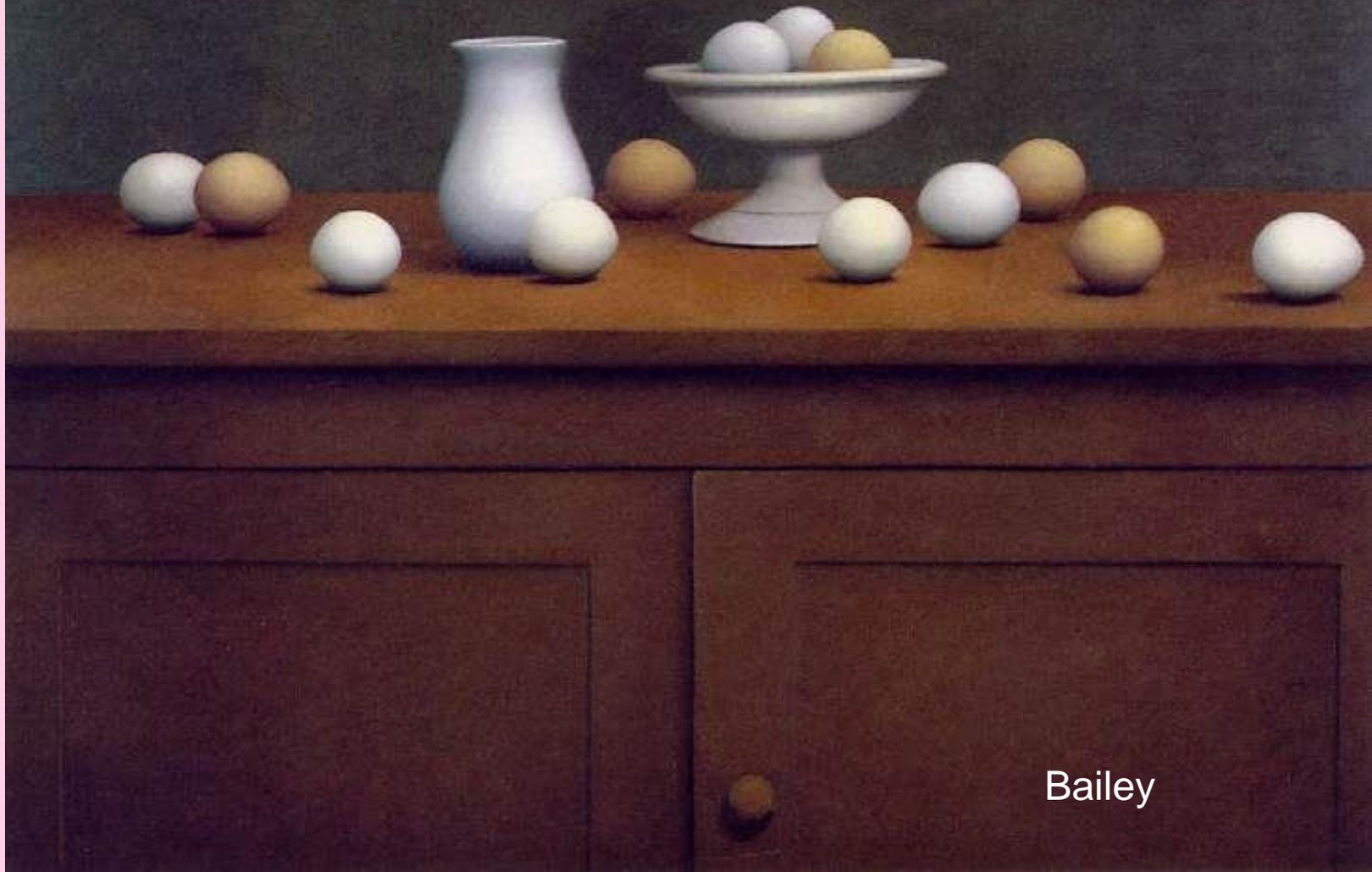
### 1. Introduction

Shell and tube heat exchanger is widely used in many industrial power generation plants as well as chemical, petrochemical, and petroleum industries. There are effective parameters in shell and tube heat exchanger design such as tube diameter, tube arrange-

as well as minimizing the total cost. Genetic algorithm optimization technique was applied to provide a set of Pareto multiple optimum solutions. The sensitivity analysis of change in optimum values of effectiveness and total cost with change in design parameters was performed and the results are reported.

As a summary the followings are the contribution of this paper

# Prerequisites: Tensors



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# Prerequisites: Tensors

Transfer phenomena operate with the following properties of solids and fluids (determining state at a point in space x,y,z):

**Scalars**  $T$  (temperature),  $p$  (pressure),  $\rho$  (density),  $h$  (enthalpy),  $C_A$  (concentration),  $k$  (kinetic energy)

**Vectors**  $\vec{u}$  (velocity),  $\vec{f}$  (forces),  $\nabla T$  (gradient of scalar), and others like vorticity, displacement...

**Tensors**  $\vec{\tau}$   $\vec{\sigma}$  (stress),  $\vec{\Delta}$  (rate of deformation),  $\nabla \vec{u}$  (gradient of vector), deformation tensor...

Scalars are determined by 1 number.

Vectors are determined by 3 numbers

$$\vec{u} = (u_x, u_y, u_z) = (u_1, u_2, u_3)$$

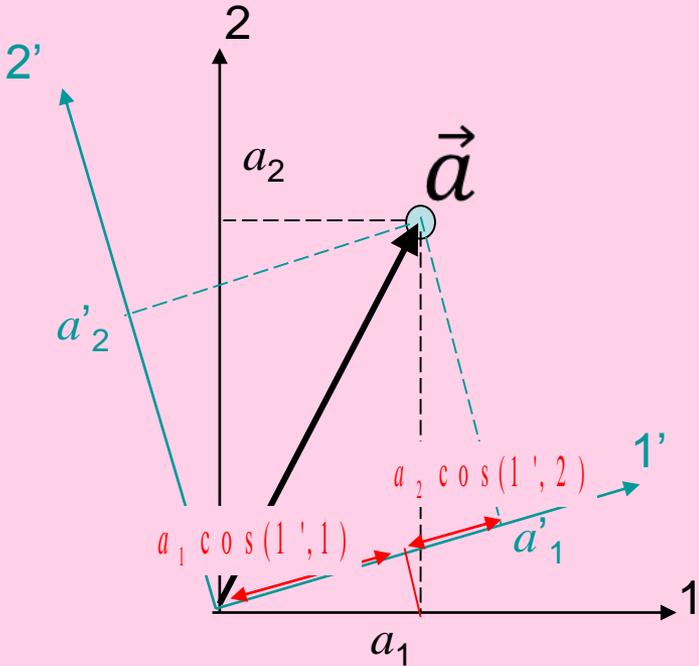
Tensors are determined by 9 numbers

$$\vec{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Scalars, vectors and tensors are independent of coordinate systems (they are objective properties). However, **components** of vectors and tensors depend upon the coordinate system. Rotation of axis has no effect upon a vector (its magnitude and the arrow direction), but coordinates of the vector are changed (coordinates  $u_i$  are projections to coordinate axis). See next slide...

# Rotation of cartesian coordinate system

Three components of a vector represent complete description (length of an arrow and its directions), but these components depend upon the choice of coordinate system. Rotation of axis of a cartesian coordinate system is represented by transformation of the vector coordinates by the matrix product



$$a'_1 = a_1 \cos(1',1) + a_2 \cos(1',2) + a_3 \cos(1',3)$$

$$a'_2 = a_1 \cos(2',1) + a_2 \cos(2',2) + a_3 \cos(2',3)$$

$$a'_3 = a_1 \cos(3',1) + a_2 \cos(3',2) + a_3 \cos(3',3)$$

$$\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = \begin{pmatrix} \cos(1',1) & \cos(1',2) & \cos(1',3) \\ \cos(2',1) & \cos(2',2) & \cos(2',3) \\ \cos(3',1) & \cos(3',2) & \cos(3',3) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

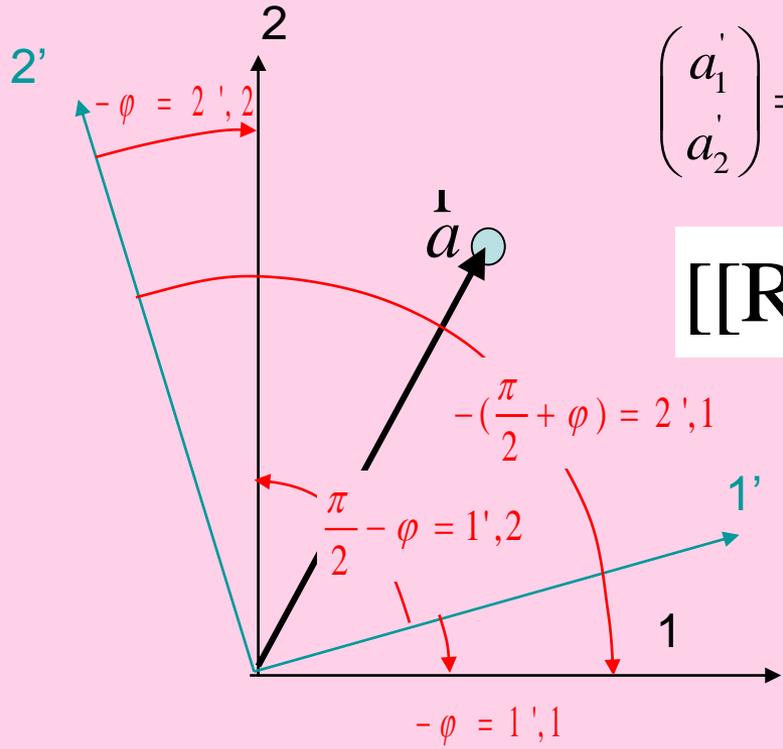
$$[a'] = [[R]][a]$$

Rotation matrix  
( $R_{ij}$  is cosine of angle between axis  $i'$  and  $j'$ )

# Rotation of cartesian coordinate system

**Example:** Rotation only along the axis 3 by the angle  $\varphi$  (positive for counter-clockwise direction)

Properties of goniometric functions  $\cos(-\varphi) = \cos \varphi$   $\cos(\frac{\pi}{2} - \varphi) = \sin \varphi$   $\cos(-(\frac{\pi}{2} + \varphi)) = -\sin \varphi$



$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \cos(1',1) = \cos \varphi & \cos(1',2) = \sin \varphi \\ \cos(2',1) = -\sin \varphi & \cos(2',2) = \cos \varphi \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$[[R]]^T [[R]] = [[I]] \rightarrow [[R]]^{-1} = [[R]]^T$$

therefore the rotation matrix is orthogonal and can be inverted just only by simple transposition (overturning along the main diagonal). Proof:

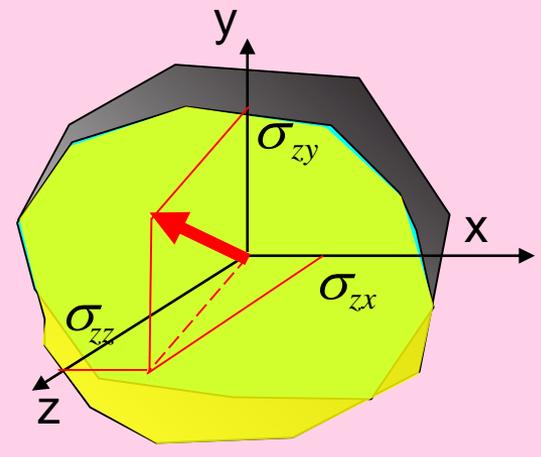
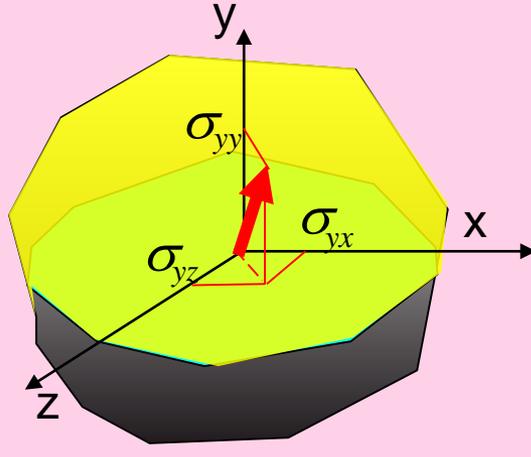
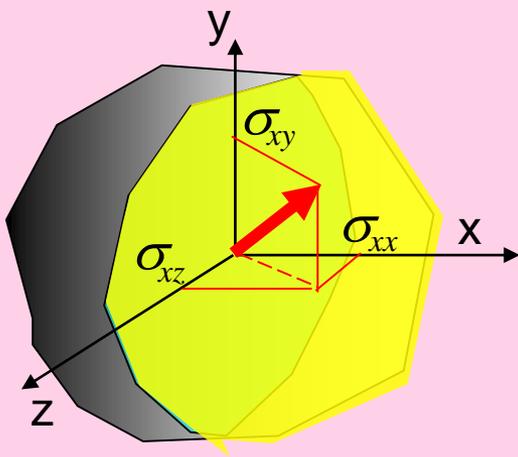
$$\begin{aligned} & \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} = \\ & = \begin{pmatrix} \cos^2 \varphi + \sin^2 \varphi & \cos \varphi \sin \varphi - \sin \varphi \cos \varphi \\ \sin \varphi \cos \varphi - \cos \varphi \sin \varphi & \sin^2 \varphi + \cos^2 \varphi \end{pmatrix} = \\ & = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

# Stresses describe complete stress state at a point x,y,z

Stress tensor is a typical example of the second order tensor with a pair of indices having the following meaning

$$\sigma_{ij}$$

Index of plane (cross section)      index of force component  
(force acting upon the cross section i)



# Tensor rotation of cartesian coordinate system

Later on we shall use another tensors of the second order describing kinematics of deformation (deformation tensors, rate of deformation,...)

Nine components of a tensor represent complete description of state (e.g. distribution of stresses at a point), but these components depend upon the choice of coordinate system, the same situation like with vectors. The transformation of components corresponding to the rotation of the cartesian coordinate system is given by the matrix product

$$[[\sigma']] = [[R]][[\sigma]][[R]]^T$$

where the rotation matrix  $[[R]]$  is the same as previously

$$[[R]] = \begin{pmatrix} \cos(1',1) & \cos(1',2) & \cos(1',3) \\ \cos(2',1) & \cos(2',2) & \cos(2',3) \\ \cos(3',1) & \cos(3',2) & \cos(3',3) \end{pmatrix}$$

# Tensor rotation of cartesian coordinate system

Orthogonal matrix of the rotation of coordinate system  $[[R]]$  is fully determined by 3 parameters, by subsequent rotations around the x,y,z axis (therefore by 3 angles of rotations). The rotations can be selected in such a way that 3 components of the stress tensor in the new coordinate system disappear (are zero). Because the stress tensor is symmetric (usually) it is possible to annihilate all off-diagonal components

$$\begin{pmatrix} \sigma'_1 & 0 & 0 \\ 0 & \sigma'_2 & 0 \\ 0 & 0 & \sigma'_3 \end{pmatrix} = [[R]][[\sigma]][[R]]^T$$

Diagonal terms are normal (principal) stresses and the axis of the rotated coordinate systems are principal directions (there are no shear stresses in the cross-sections oriented in the principal directions).

# Special tensors

Kronecker delta (unit tensor, components independent of rotation)

$$\delta_{ij} = 0 \text{ for } i \neq j$$

$$\delta_{ij} = 1 \text{ for } i = j$$

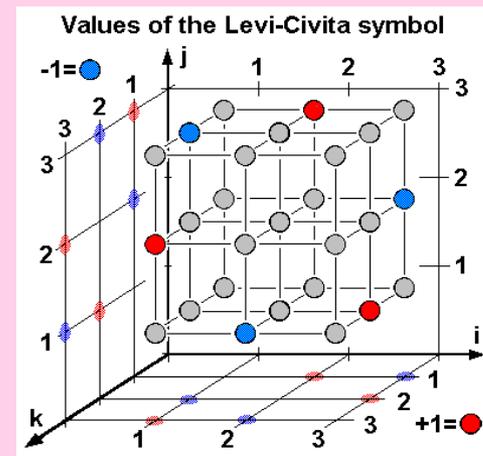


$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Levi Civita tensor is antisymmetric unit tensor of the third order (with 3 indices)

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (3, 1, 2) \text{ or } (2, 3, 1), \\ -1 & \text{if } (i, j, k) \text{ is } (1, 3, 2), (3, 2, 1) \text{ or } (2, 1, 3), \\ 0 & \text{otherwise: } i = j \text{ or } j = k \text{ or } k = i, \end{cases}$$

In terms of the epsilon tensor the vector product will be defined

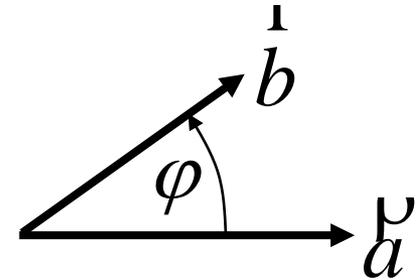


# Scalar product

Scalar product (operator  $\bullet$ ) of two vectors is a scalar

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \varphi$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^3 a_i b_i = a_i b_i = a'_i b'_i$$



$a_i b_i$  is abbreviated **Einstein notation**. Repeated indices are summing (dummy) indices.

Proof that  $a_i b_i = a'_i b'_i$

$$a'_i = R_{im} a_m \quad b'_j = R_{jk} b_k$$

Remark: there were used dummy indices **m** and **k** in these relations. Letters of the dummy indices can be selected arbitrary, but in this case they must be different, so that to avoid appearance of four equal indices in a tensorial term in the following product  $a \cdot b$  (there can be always max. two indices with the same name indicating a summation)

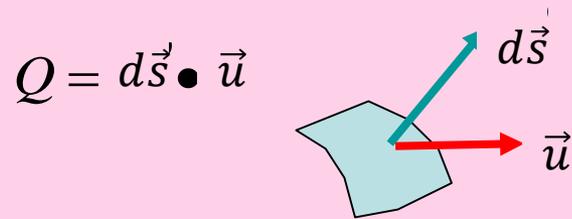
$$a'_1 b'_1 + a'_2 b'_2 + a'_3 b'_3 = a'_i b'_i = R_{im} R_{ik} a_m b_k = R_{mi}^T R_{ik} a_m b_k = \delta_{mk} a_m b_k = a_m b_m$$

# Scalar product

Example: scalar product of velocity  $\vec{u}$  [m/s] and force  $\vec{F}$  [N] acting at a point is power  $P$  [W] (scalar)

$$P = \vec{u} \cdot \vec{F} = u_i F_i$$

Example: Scalar product of velocity  $\vec{u}$  [m/s] and the normal vector of an oriented surface  $d\vec{s}$  [m<sup>2</sup>] is the volumetric flowrate  $Q$  [m<sup>3</sup>/s] through the surface

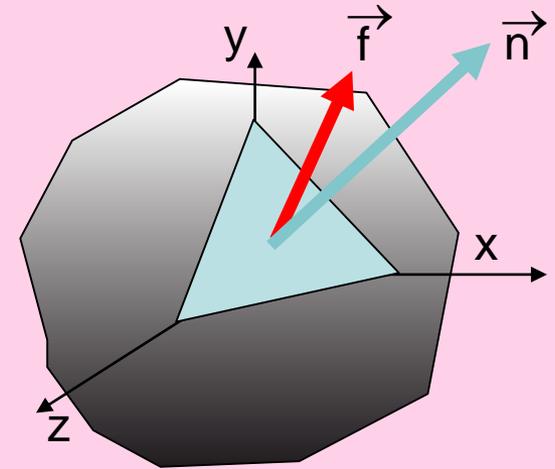


# Scalar product

Scalar product can be applied also between tensors or between vector and tensor

$$\vec{u} \bullet \vec{\sigma} = \vec{f} \quad \sum_{i=1}^3 n_i \sigma_{ij} = n_i \sigma_{ij} = f_j$$

i-is summation (dummy) index, while j-is free index



This case explains how it is possible to calculate internal stresses acting at an arbitrary cross section (determined by outer normal vector n) knowing the stress tensor.

# Scalar product - examples

## Dot product of delta tensors

$$\vec{\delta} \cdot \vec{\delta} = \vec{\delta} \quad \delta_{im} \delta_{mj} = \delta_{ij} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Scalar product of tensors is a tensor

$$\vec{\sigma} \cdot \vec{\tau} = \vec{\zeta} \quad \sigma_{im} \tau_{mj} = \zeta_{ij}$$

## Double dot product of tensors is a scalar

$$\vec{\sigma} : \vec{\tau} = \zeta \quad \sigma_{km} \tau_{mk} = \zeta$$

## Trace of a tensor (tensor contraction)

$$tr(\vec{\sigma}) = \sigma_{mm} \quad \text{example } tr(\vec{\delta}) = 3$$

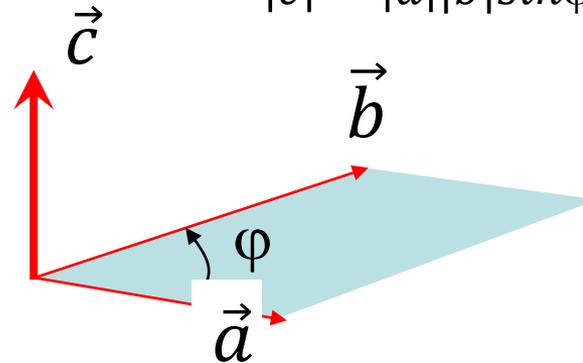
# Vector product

Vector product (operator  $\times$ ) of two vectors is a vector

$$|c| = |a||b|\sin\phi$$

$$\vec{c} = \vec{a} \times \vec{b} = (\vec{\varepsilon} \bullet \vec{b}) \bullet \vec{a}$$

$$c_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} a_j b_k = \varepsilon_{ijk} a_j b_k$$



for example  $c_1 = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$

Important relationship between the Levi Civita and the Kronecker delta special tensors

$$\varepsilon_{ijk} \varepsilon_{kmn} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

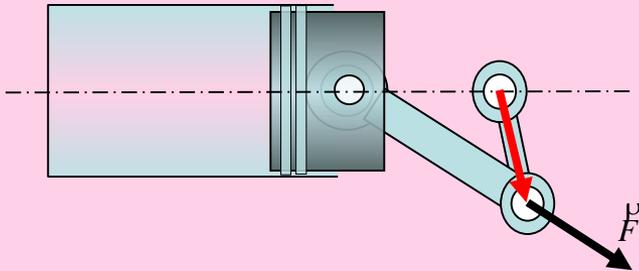
check for  $i=1, j=2, (k=3), m=1, n=2$       $\varepsilon_{123} \varepsilon_{312} = \delta_{11} \delta_{22} - \delta_{12} \delta_{21} = 1$

check for  $i=1, j=2, (k=3), m=2, n=1$       $\varepsilon_{123} \varepsilon_{321} = \delta_{12} \delta_{21} - \delta_{11} \delta_{22} = -1$

# Vector product

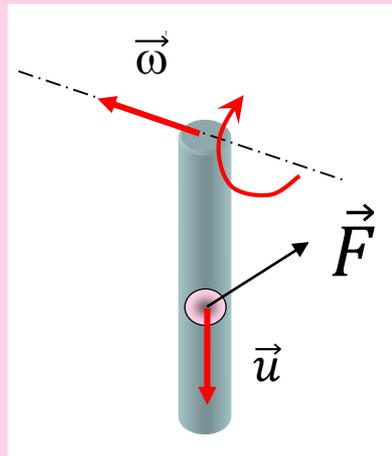
Examples of applications

Moment of force (torque)  $\vec{M} = \vec{r} \times \vec{F}$



Coriolis force

$$\vec{F} = 2m \vec{u} \times \vec{\omega}$$



application: Coriolis flowmeter



# Diadic product

Diadic product (no operator) of two vectors is a second order tensor

$$\vec{a}\vec{b} = \vec{\pi}$$

$$a_i b_j = \pi_{ij}$$

# Differential operator $\nabla$ (Nabla)



Bailey

# Differential operator $\nabla$ (Nabla)

## GRADIENT – measure of spatial changes

Symbolic operator  $\nabla$  represents a vector of first derivatives with respect x,y,z.

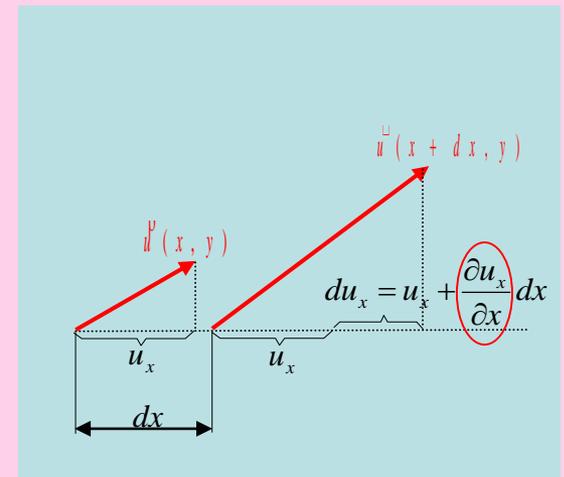
$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad \nabla_i = \frac{\partial}{\partial x_i}$$

$\nabla$  applied to scalar is a vector (gradient of scalar)

$$\nabla T = \left( \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) \quad \nabla_i T = \frac{\partial T}{\partial x_i}$$

$\nabla$  applied to vector is a tensor (for example gradient of velocity is a tensor)

$$\nabla \vec{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix} \quad \nabla_i u_j = \frac{\partial u_j}{\partial x_i}$$



# Differential operator $\nabla \cdot$ .

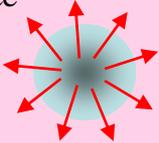
## DIVERGENCY – magnitude of sources/sinks

Scalar product  $\nabla \cdot$  represents intensity of source/sink of a vector quantity at a point

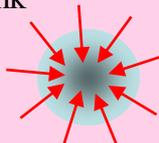
$$\nabla \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}$$

i-dummy index, result is a scalar

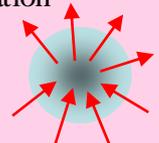
$\nabla \cdot \vec{u} > 0$  source



$\nabla \cdot \vec{u} < 0$  sink



$\nabla \cdot \vec{u} = 0$  conservation



Scalar product  $\nabla \cdot$  can be applied also to a tensor giving a vector (e.g. source/sink of momentum in the direction x,y,z)

$$\vec{f} = \nabla \cdot \boldsymbol{\sigma} = \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \quad \mathbf{f}_i = \nabla_j \sigma_{ji}$$

The vector  $\mathbf{f}_i$  represents resulting force of stresses acting at the surface of an infinitely small volume (the force is related to this volume therefore unit is N/m<sup>3</sup>)

# Laplace operator $\nabla^2$

## Divergency of gradient – measure of nonuniformity

Scalar product  $\nabla \bullet \nabla = \nabla^2$  is the operator of second derivatives (when applied to scalar it gives a scalar, applied to a vector gives a vector,...). Laplace operator is divergence of a gradient (gradient of temperature, gradient of velocity...)

$$\nabla \bullet \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial x_i \partial x_i}$$

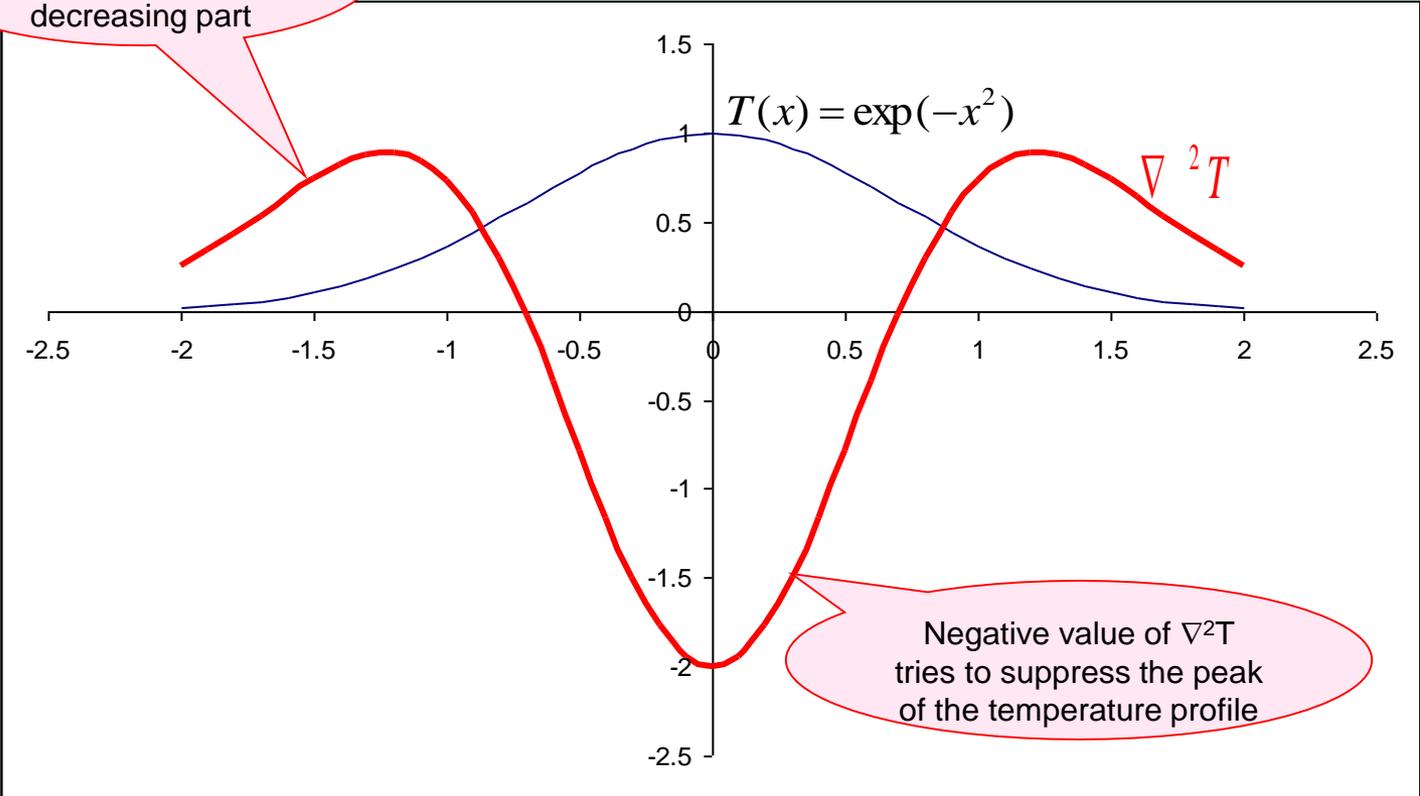
i-dummy index

$$\nabla \bullet \nabla \vec{u} = \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}, \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}, \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = \sum_{i=1}^3 \frac{\partial^2 u_j}{\partial x_i^2} = \frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

Physical interpretation:  $\nabla^2$  describes diffusion processes (random molecular motion). The term  $\nabla^2$  appears in the transport equations for temperatures, momentum, concentrations and its role is to smooth out all spatial nonuniformities of the transported properties.

# Laplace operator $\nabla^2$

Positive value of  $\nabla^2 T$  tries to enhance the decreasing part



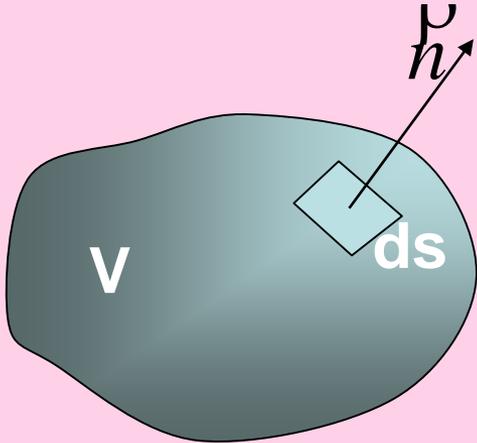
Negative value of  $\nabla^2 T$  tries to suppress the peak of the temperature profile

# Integrals – Gauss theorem

Volume integrals with nabla operator can be converted to surface integrals (just only by replacing nabla  $\nabla$  with unit normal vector  $\hat{n}$  )

Physical interpretation: accumulation in volume V is overall flux through boundary

$$\iiint_V \underbrace{\nabla \cdot \mathbf{P}}_{\text{Divergence of P}} dv = \iint_S \underbrace{\hat{n} \cdot \mathbf{P}}_{\text{projection of P to outer normal}} ds$$



Variable **P** can be

➤ **Scalar** (typically pressure p)  $\iiint_V \nabla p dv = \iint_S \hat{n} p ds$

➤ **Vector** (vector of velocity, momentum, heat flux). Surface integral represents flux of vector in the direction of outer normal.  $\iiint_V \nabla \cdot \mathbf{u} dv = \iint_S \hat{n} \cdot \mathbf{u} ds$

➤ **Tensor** (tensor of stresses). In this case the Gauss theorem represents the balance between inner stresses and outer forces acting upon the surface,

$$\iiint_V \nabla \cdot \boldsymbol{\tau} dv = \iint_S \hat{n} \cdot \boldsymbol{\tau} ds$$

# Transcriptions

Symbolic notation, for example  $\nabla^2 T$ , is compact, unique and suitable for definition of problems in terms of tensorial equations. However, if you need to solve these equation (for example  $\nabla^2 T = f$ ) you have to rewrite symbolic form into index notation, giving equations (usually differential equations) for components of vectors or tensors, which are expressed by numbers (may be complex numbers).



Bailey

# Symbolic $\rightarrow$ indicial notation

General procedure how to rewrite symbolic formula to index notation

- Replace each arrow  $\rightarrow$  by an empty place for index  $\_$
- Replace each vector operator by  $(-1) \cdot \varepsilon_{\_ \_ \_} \cdot$
- Replace each dot  $\cdot$  by a pair of dummy indices in the first free position left and right
- Write free indices into remaining positions

**Practice examples!!**

# Coordinate systems

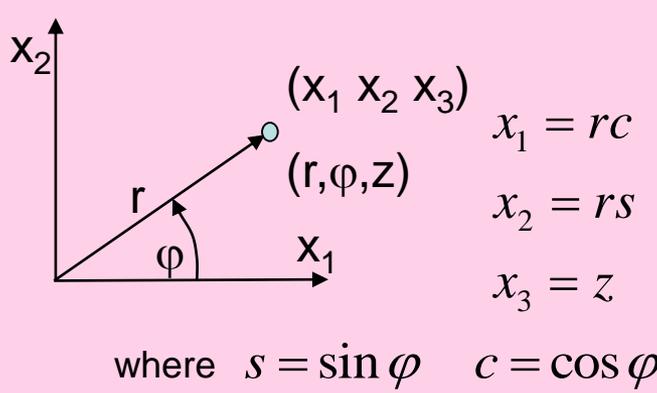
Previous conversion procedure can be applied only in a cartesian coordinate systems

Formulation in the  $x,y,z$  cartesian system is not always convenient, especially if the geometry of region is cylindrical or spherical. For example the boundary condition of a constant temperature is difficult to prescribe on the curved surface of sphere in the rectangular cartesian system. In the case that the problem formulation suppose a rotational or spherical symmetry, the number of spatial coordinates can be reduced and the problem is simplified to 1D or 2D problem, but in a new coordinate system.

In the following we shall demonstrate how to convert tensor terms from symbolic notation (which is independent to a specific coordinate system) into the cylindrical coordinate system.

# Coordinate systems (cylindrical)

Cylindrical (and spherical) systems are defined by transformations



$$dx_i = \frac{\partial x_i}{\partial r} dr + \frac{\partial x_i}{\partial \varphi} d\varphi + \frac{\partial x_i}{\partial z} dz$$

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} c & -rs & 0 \\ s & rc & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix}$$

$$\begin{pmatrix} dr \\ d\varphi \\ dz \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -\frac{s}{r} & \frac{c}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$\frac{\partial r}{\partial x_1} = c, \quad \frac{\partial r}{\partial x_2} = s, \quad \frac{\partial \varphi}{\partial x_1} = -\frac{s}{r}, \quad \frac{\partial \varphi}{\partial x_2} = \frac{c}{r}$$

Using this it is possible to express partial derivatives with respect  $x_1, x_2, x_3$  in terms of derivatives with respect the coordinates of cylindrical system

$$\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial r} \frac{\partial r}{\partial x_1} + \frac{\partial T}{\partial \varphi} \frac{\partial \varphi}{\partial x_1} + \frac{\partial T}{\partial z} \frac{\partial z}{\partial x_1} = \frac{\partial T}{\partial r} c - \frac{\partial T}{\partial \varphi} \frac{s}{r}$$

$$\frac{\partial T}{\partial x_2} = \frac{\partial T}{\partial r} \frac{\partial r}{\partial x_2} + \frac{\partial T}{\partial \varphi} \frac{\partial \varphi}{\partial x_2} + \frac{\partial T}{\partial z} \frac{\partial z}{\partial x_2} = \frac{\partial T}{\partial r} s + \frac{\partial T}{\partial \varphi} \frac{c}{r}$$

$$\frac{\partial T}{\partial x_3} = \frac{\partial T}{\partial r} \frac{\partial r}{\partial x_3} + \frac{\partial T}{\partial \varphi} \frac{\partial \varphi}{\partial x_3} + \frac{\partial T}{\partial z} \frac{\partial z}{\partial x_3} = \frac{\partial T}{\partial z}$$

# Coordinate systems (cylindrical)

In the same way also the second derivatives can be expressed

$$\frac{\partial^2 T}{\partial x_1^2} = \frac{\partial^2 T}{\partial r^2} c^2 - \frac{2cs}{r} \frac{\partial}{\partial \phi} \left( \frac{\partial T}{\partial r} - \frac{T}{r} \right) + \frac{\partial T}{\partial r} \frac{s^2}{r} + \frac{\partial^2 T}{\partial \phi^2} \frac{s^2}{r^2}$$

$$\frac{\partial^2 T}{\partial x_2^2} = \frac{\partial^2 T}{\partial r^2} s^2 + \frac{2cs}{r} \frac{\partial}{\partial \phi} \left( \frac{\partial T}{\partial r} - \frac{T}{r} \right) + \frac{\partial T}{\partial r} \frac{c^2}{r} + \frac{\partial^2 T}{\partial \phi^2} \frac{c^2}{r^2}$$

$$\frac{\partial^2 T}{\partial x_3^2} = \frac{\partial^2 T}{\partial z^2}$$

giving expression for the Laplace operator in the cylindrical coordinate system

(use goniometric identity  $s^2+c^2=1$ )

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} = \frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{\partial r} \frac{1}{r} + \frac{\partial^2 T}{\partial \phi^2} \frac{1}{r^2} + \frac{\partial^2 T}{\partial z^2}$$

# Coordinate systems (cylindrical)

Previous example demonstrated how to solve the problem of transformations to cylindrical coordinate system with scalars. However, how to calculate gradients or divergence of a vector and a tensor field? Vectors and tensors are described by 3 or 3x3 values of components now expressed in terms of new unit vectors

$$\mathbf{u} = u_m \mathbf{i}_m = u_r \mathbf{e}^{(r)} + u_\varphi \mathbf{e}^{(\varphi)} + u_z \mathbf{e}^{(z)}$$

Einstein summation applied in the cartesian coordinate system

unit vectors in the cylindrical coordinate system

$$\mathbf{\sigma} = \sigma_{mn} \mathbf{i}_m \mathbf{i}_n =$$

$$= \sigma_{rr} \mathbf{e}^{(r)} \mathbf{e}^{(r)} + \sigma_{r\varphi} \mathbf{e}^{(r)} \mathbf{e}^{(\varphi)} + \sigma_{rz} \mathbf{e}^{(r)} \mathbf{e}^{(z)} + \dots + \sigma_{zz} \mathbf{e}^{(z)} \mathbf{e}^{(z)}$$

Unit vectors in orthogonal coordinate system are orthogonal, which means that

$$\mathbf{e}^{(r)} \mathbf{g}^{(r)} = 1 \quad \mathbf{e}^{(r)} \mathbf{g}^{(\varphi)} = 0 \quad \dots \quad \mathbf{e}^{(z)} \mathbf{g}^{(z)} = 1$$

and therefore the components in the new coordinate system are for example

$$u_\varphi = \mathbf{u} \mathbf{g}^{(\varphi)} = u_m e_m^{(\varphi)}$$

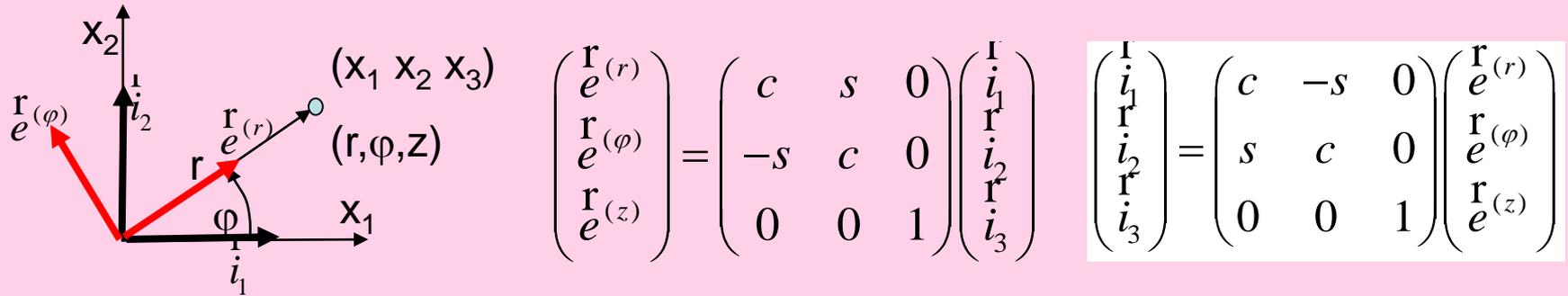
$$\sigma_{rz} = \mathbf{e}^{(r)} \mathbf{\sigma} \mathbf{g}^{(z)} = e_m^{(r)} \sigma_{mn} e_n^{(z)}$$

$$u_i = u_r e_i^{(r)} + u_\varphi e_i^{(\varphi)} + u_z e_i^{(z)}$$

$e_i^{(z)} = \mathbf{e}^{(z)} \cdot \mathbf{i}_i$  this is i-th cartesian component of the unit vector

# Coordinate systems (cylindrical)

Transformation of unit vectors



Example: **gradient of temperature** can be written in the following way (alternatively in the cartesian and the cylindrical coordinate system)

substitute by using previously derived  $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial r} c - \frac{\partial T}{\partial \varphi} s$

$$\begin{aligned} \nabla T &= \frac{\partial T}{\partial x_1} \mathbf{i}_1 + \frac{\partial T}{\partial x_2} \mathbf{i}_2 + \frac{\partial T}{\partial x_3} \mathbf{i}_3 = \left( c \frac{\partial T}{\partial x_1} + s \frac{\partial T}{\partial x_2} \right) \mathbf{r}^{(r)} + \left( s \frac{\partial T}{\partial x_1} - c \frac{\partial T}{\partial x_2} \right) \mathbf{e}^{(\varphi)} + \frac{\partial T}{\partial x_3} \mathbf{r}^{(z)} = \\ &= \frac{\partial T}{\partial r} \mathbf{r}^{(r)} + \frac{1}{r} \frac{\partial T}{\partial \varphi} \mathbf{e}^{(\varphi)} + \frac{\partial T}{\partial z} \mathbf{r}^{(z)} \end{aligned}$$

Nabla operator in the cylindrical coordinate system

$$\nabla \equiv \mathbf{r}^{(r)} \frac{\partial}{\partial r} + \mathbf{e}^{(\varphi)} \frac{1}{r} \frac{\partial}{\partial \varphi} + \mathbf{r}^{(z)} \frac{\partial}{\partial z}$$

# Coordinate systems (cylindrical)

Example: Divergence of a vector  $\mathbf{u} = u_m \mathbf{i}_m = u_r \mathbf{e}^{(r)} + u_\phi \mathbf{e}^{(\phi)} + u_z \mathbf{e}^{(z)}$

$$\begin{aligned} \nabla_{\mathbf{g}} \mathbf{u} &= \frac{\partial u_m}{\partial x_m} = \frac{\partial(u_r e_m^{(r)} + u_\phi e_m^{(\phi)} + u_z e_m^{(z)})}{\partial x_m} = \\ &= \frac{\partial u_r}{\partial x_m} e_m^{(r)} + u_r \frac{\partial e_m^{(r)}}{\partial x_m} + \frac{\partial u_\phi}{\partial x_m} e_m^{(\phi)} + u_\phi \frac{\partial e_m^{(\phi)}}{\partial x_m} + \frac{\partial u_z}{\partial x_m} e_m^{(z)} + u_z \frac{\partial e_m^{(z)}}{\partial x_m} \end{aligned}$$

components of unit vectors follow from the previously derived

$$\begin{aligned} e_1^{(r)} &= c & e_1^{(\phi)} &= -s & e_1^{(z)} &= 0 \\ e_2^{(r)} &= s & e_2^{(\phi)} &= c & e_2^{(z)} &= 0 \\ e_3^{(r)} &= 0 & e_3^{(\phi)} &= 0 & e_3^{(z)} &= 1 \end{aligned}$$

$$\begin{pmatrix} \mathbf{e}^{(r)} \\ \mathbf{e}^{(\phi)} \\ \mathbf{e}^{(z)} \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix}$$

Note the fact, that the partial derivatives of these components with respect to  $x_m$  are not zero and can be calculated using the previously derived relationships

$$\frac{\partial \phi}{\partial x_1} = -\frac{s}{r}, \quad \frac{\partial \phi}{\partial x_2} = \frac{c}{r} \quad \text{giving} \quad \frac{\partial e_1^{(r)}}{\partial x_1} = \frac{\partial c}{\partial x_1} = \frac{\partial c}{\partial \phi} \frac{\partial \phi}{\partial x_1} = -s \frac{s}{r}$$

$$\frac{\partial e_1^{(\phi)}}{\partial x_1} = -\frac{\partial s}{\partial x_1} = -\frac{\partial s}{\partial \phi} \frac{\partial \phi}{\partial x_1} = c \frac{s}{r}$$

...and so on

# Coordinate systems (cylindrical)

Substituting these expression we obtain

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_r}{\partial r} c^2 - \frac{\partial u_r}{\partial \varphi} \frac{cs}{r} + u_r \frac{s^2}{r} - \frac{\partial u_\varphi}{\partial r} cs + \frac{\partial u_\varphi}{\partial \varphi} \frac{s^2}{r} + u_\varphi \frac{cs}{r}$$

$$\frac{\partial u_2}{\partial x_2} = \frac{\partial u_r}{\partial r} s^2 + \frac{\partial u_r}{\partial \varphi} \frac{cs}{r} + u_r \frac{c^2}{r} + \frac{\partial u_\varphi}{\partial r} cs + \frac{\partial u_\varphi}{\partial \varphi} \frac{c^2}{r} - u_\varphi \frac{cs}{r}$$

$$\frac{\partial u_3}{\partial x_3} = \frac{\partial u_z}{\partial z}$$

Summing together, the final form of divergence in the cylindrical coordinate system is obtained

$$\nabla \bullet \vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}$$

# Coordinate systems (cylindrical)

Example: Gradient of a vector  $\mathbf{u} = u_m \mathbf{i}_m = u_r \mathbf{e}^{(r)} + u_\varphi \mathbf{e}^{(\varphi)} + u_z \mathbf{e}^{(z)}$

$$\mathbf{\pi} = \nabla \mathbf{u}$$

$$\pi_{rr} = e_m^{(r)} \frac{\partial(u_r e_n^{(r)} + u_\varphi e_n^{(\varphi)} + u_z e_n^{(z)})}{\partial x_m} e_n^{(r)} \quad \pi_{r\varphi} = e_m^{(r)} \frac{\partial(u_r e_n^{(r)} + u_\varphi e_n^{(\varphi)} + u_z e_n^{(z)})}{\partial x_m} e_n^{(\varphi)} \dots$$

$m$  and  $n$  are dummy indices (summing is required)

Substituting previous expressions for unit vector and their derivatives results to final expression for the velocity gradient tensor in a cylindrical coordinate system

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\varphi}{\partial r} & \frac{\partial u_z}{\partial r} \\ \frac{1}{r} \left( \frac{\partial u_r}{\partial \varphi} - u_\varphi \right) & \frac{1}{r} \left( \frac{\partial u_\varphi}{\partial \varphi} + u_r \right) & \frac{1}{r} \frac{\partial u_z}{\partial \varphi} \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\varphi}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

# Coordinate systems (general)

Procedure how to derive tensorial equations in a general coordinate system.

1. Rewrite equation from the symbolic notation to the index notation for cartesian coordinate system, for example  $\nabla_{\mathbf{u}}^{\mathbf{r}} = \frac{\mathbf{f}}{\pi} \rightarrow \frac{\partial u_j}{\partial x_i} = \pi_{ij}$

2. Define transformation  $x_1(r_1, r_2, r_3), \dots, r_1(x_1, x_2, x_3), \dots$  and  $\frac{\partial r_i}{\partial x_j} = f_{ij}(r_1, r_2, r_3)$  therefore also the first and the second derivatives of scalar values, for example

$$\frac{\partial u_j}{\partial x_1} = \frac{\partial u_j}{\partial r_1} f_{11} + \frac{\partial u_j}{\partial r_2} f_{21} + \frac{\partial u_j}{\partial r_3} f_{31}, \dots$$

3. Define unit vectors of coordinate systems  $r_i$  and express their cartesian coordinates  $e_m^{(ri)} = g_m^{(ri)}(r_1, r_2, r_3)$   
Calculate their derivatives with respect to the cartesian coordinates

$$\frac{\partial e_m^{(r)}}{\partial x_i} = \frac{\partial g_m^r}{\partial x_1} = \frac{\partial g_m^r}{\partial r_1} \frac{\partial r_1}{\partial x_i} + \frac{\partial g_m^r}{\partial r_2} \frac{\partial r_2}{\partial x_i} + \frac{\partial g_m^r}{\partial r_3} \frac{\partial r_3}{\partial x_i}$$

4. In case that the result is a vector, for example the gradient of scalar, calculate its components from

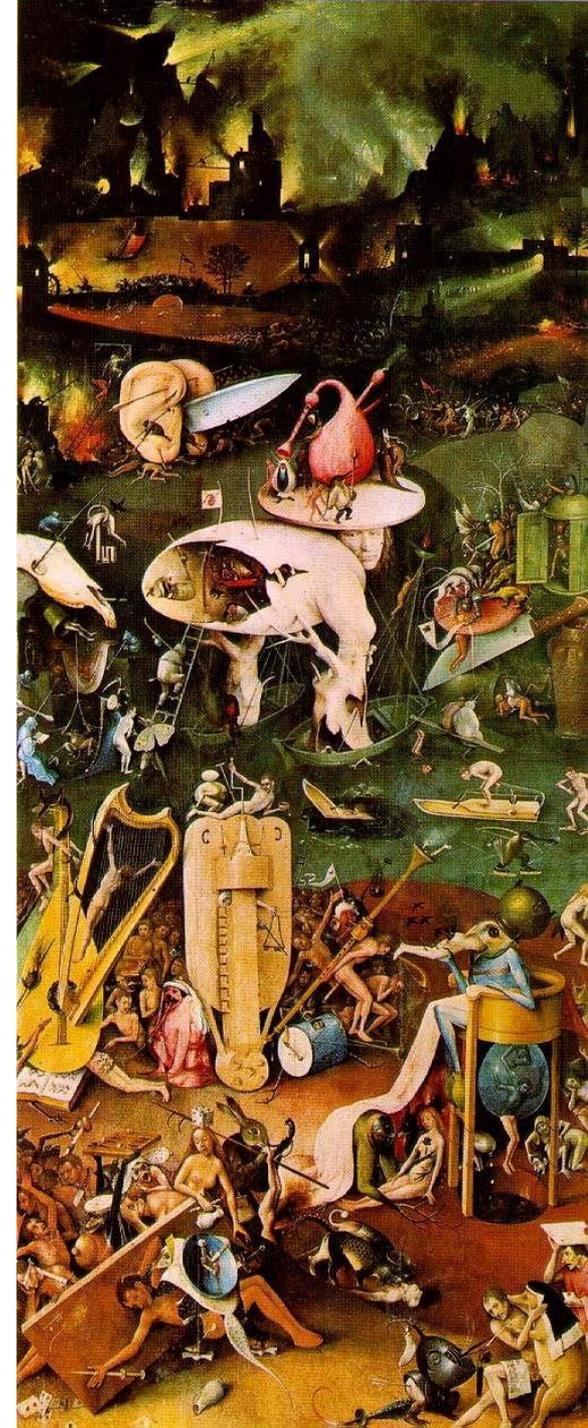
$$u_{\varphi} = \mathbf{u}^{\mathbf{r}} \mathbf{g}^{\mathbf{r}(\varphi)} = u_m e_m^{(\varphi)}$$

In the case that result is a tensor calculate its components

$$\pi_{rz} = \mathbf{e}^{\mathbf{r}(r)} \mathbf{g}^{\mathbf{r}} \mathbf{g}^{\mathbf{r}(z)} = e_i^{(r)} \pi_{ij} e_j^{(z)}$$

Remark: This suggested procedure (transform everything, including new unit vectors, to the cartesian coordinate system) is straightforward and seemingly easy. This is not so, it is „crude“, lengthy (the derivation of velocity gradient is on several lists of paper) and without finesses. Better and more sophisticated procedures are described in standard books, e.g. Aris R: Vectors, tensors...N.J.1962, or Bird,Stewart,Lightfoot:Transport phenomena.

## Tensors



# What is important (at least for exam)

You should know what is it scalar, vector, tensor and transformations at rotation of coordinate system

$$[a'] = [[R]][a] \qquad [[\sigma']] = [[R]][[\sigma]][[R]]^T$$

(and how is defined the rotation matrix R?)

Scalar and vector products

$$\overset{\mathbf{r}}{a} \cdot \overset{\mathbf{r}}{b} = \sum_{i=1}^3 a_i b_i = a_i b_i$$

$$\overset{\rho}{c} = \overset{\rho}{a} \times \overset{\rho}{b} = (\overset{\rho}{\varepsilon} \cdot \overset{\rho}{b}) \cdot \overset{\rho}{a}$$

$$c_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} a_j b_k = \varepsilon_{ijk} a_j b_k$$

(and what is it Kronecker delta and Levi Civita tensors?)

# What is important (at least for exam)

Nabla operator. Gradient

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad \nabla_i = \frac{\partial}{\partial x_i}$$

Divergence

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}$$

Laplace operator

$$\nabla \cdot \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial x_i \partial x_i}$$

# What is important (at least for exam)

Gauss integral theorem

$$\iiint_V \nabla \bullet P dv = \iint_S \overset{\mathbf{r}}{n} \bullet P ds$$

(demonstrate for the case that P is scalar, vector, tensor)