Simple Algorithm for Impulse Response Identification

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Abstract

Identification of an impulse response of a system from the time courses of tracer concentration measured at inlet and at outlet of the system leads to the integral Volterra equation. This equation can be solved for example by Fourier transformation, Laplace transformation or by method of splines, which is rather complicated. However, in the case, that the stimulus function is relatively narrow peak, the corresponding response is close to the impulse response. This article describes simple methods how to perform correction of the measured response if the stimulus function is so short, that it can be substituted by a rectangular pulse having the same variance. The first approach transforms the Volterra equation to a differential equation of the second order – correction is proportional to the second derivative of measured response. Numerical solution of this differential equation is stable only if the variance of the stimulus function is very small. The second approach solves the Volterra's equation iteratively – this method is simpler and has been used in data acquisition program PCL818 for routine evaluation of experimental data.

1. Problem description

Let $x(t)$ and $y(t)$ are experimentally determined stimulus and response functions of a system characterized by impulse response $f(t)$. Evaluation of impulse response $f(t)$ is a difficult problem of deconvolution

$$y(t) = \int_{-\infty}^{t} f(t-\tau)x(\tau)d\tau \tag{1}$$

which can be solved for example by FFT or by spline methods, see [1].

2. Procedure of deconvolution for the stimulus function in form of a pulse

In the case that the stimulus function $x(t)$ is very narrow pulse, the response $y(t)$ can be considered as a good approximation of $f(t)$ and no identification is necessary. Nevertheless, even if the experiment is realized by short injection of tracer the stimulus function has a form of pulse with finite duration, and if its mean duration time is higher than 5% of the mean residence time, the solution of Volterra equation (1) is necessary. This solution can be simplified in the case that the $x(t)$ is not wide and can be approximated by a short step function. For the sake of simplicity we shall assume, that the stimulus function has the form of a pulse of the width 2σ and height $1/(2\sigma)$, where σ^2 is variance of $x(t)$ (this *wide pulse* is acceptable approximation of arbitrary short function $x(t)$). It is always possible to shift both $x(t)$ and $y(t)$ to the left so that the stimulus function (pulse) will be centered around origin.

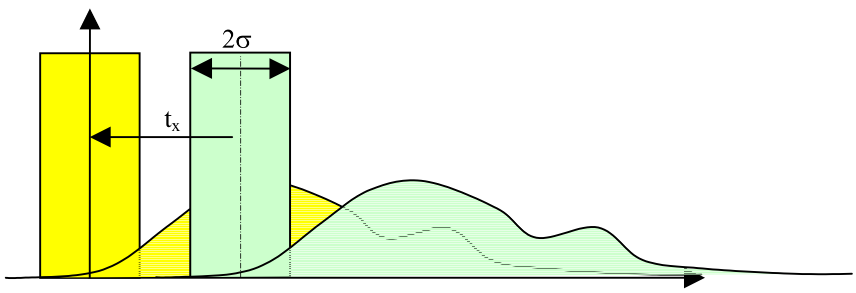

Fig.1 *Wide pulse stimulus function and response*

In this case the Volterra equation (1) reduces to

$$y(t) = \frac{1}{2\sigma} \int_{\max(0,t-\sigma)}^{t+\sigma} f(\tau) d\tau \tag{2}$$

Let us suppose that the $g(t)$ is a primitive function of $f(t)$, therefore

$$2\sigma y(t) = g(t+\sigma) - g(t-\sigma) . \tag{3}$$

This function can be approximated by the Taylor's expansion

$$g(t+\sigma) = g(t) + f(t)\sigma + \frac{df}{dt}\frac{\sigma^2}{2!} + \frac{d^2 f}{dt^2}\frac{\sigma^3}{3!} + HOT$$

$$g(t-\sigma) = g(t) - f(t)\sigma + \frac{df}{dt}\frac{\sigma^2}{2!} - \frac{d^2 f}{dt^2}\frac{\sigma^3}{3!} + HOT \tag{4}$$

Substituting into Eq.(3) and neglecting high order terms we arrive to the differential equation

$$y(t) = f(t) + \frac{d^2 f}{dt^2}\frac{\sigma^2}{6} \tag{5}$$

which reduces to the trivial case $y(t)=f(t)$ if the variance of stimulus function is negligibly small. This equation is linear and numerical solution for the case that only boundary conditions at infinity f=df/dt=0 are used is quite simple

$$f = \frac{y + \frac{\sigma^2}{6\Delta t}\left(\frac{f_0}{\Delta t} + g_0\right)}{1 + \frac{\sigma^2}{6\Delta t}} \qquad \text{and} \qquad g = \frac{f - f_0}{\Delta t} \tag{6}$$

where f_0 and g_0 are known values at a time level t_k while f,g are calculated values at a new time level t_{k-1}.

However, this method is limited only to very narrow pulse, otherwise solution diverges. Much more reliable and also very simple is solution of integral equation (2) by iterated kernel

$$f(t) = y_0(t) + y_1(t) + y_2(t) + ... \tag{7}$$

where

$$y_{k+1}(t) = y_k(t) - \frac{1}{2\sigma} \int_{\max(0,t-\sigma)}^{t+\sigma} y_k(\tau) d\tau \qquad \text{and} \quad y_0(t) = y(t) . \tag{8}$$

It has been shown that several iterations is sufficient to achieve acceptable results.

3. Stepwise stimulus function

The algorithm of iterated kernel can be used in a similar way also for a general stepwise form of stimulus function

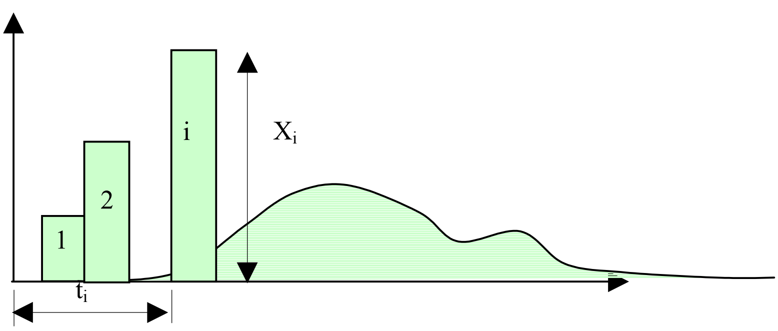

Fig.2 *Step wise stimulus function and response*

$$y(t) = \int_0^\infty f(t-\tau) x(\tau) d\tau = \sum_i X_i \int_{t-t_{i+1}}^{t-t_i} f(u) du \tag{9}$$

and $f(t)$ has the form of (7), where

$$y_{k+1}(t) = y_k(t) - \sum_i X_i \int_{t-t_{i+1}}^{t-t_i} y_k(u)du \qquad (10)$$

This is a general idea, which has not been tested yet.

4. Numerical experiments

Simulated data were created using RTD1 program [3].

Case 1.

As a stimulus function x(t) the model of N=5 mixers with mean time 0.2 s, and as a residence time distribution the model of two parallel series N_1=10, N_2=5, $\alpha=t_1/t_2$=3, $f=Q_1/Q_2$=0.3 with mean residence time 0.5 s, were used. Response function y(t) was calculated by numerical integration of convolution integral using splines. Stimulus function x(t) and response y(t) are shown in the following Fig.3

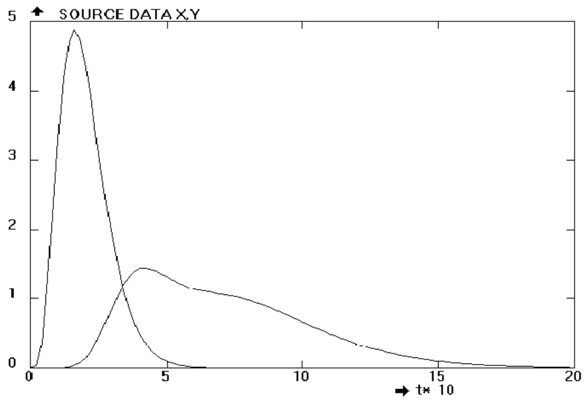

Fig.3 Stimulus and response (case 1)

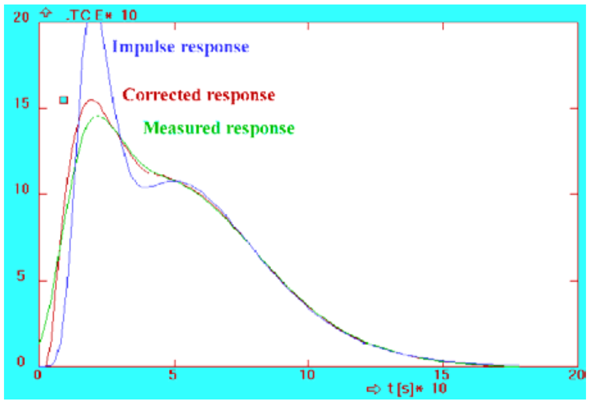

Fig.4 Comparison of impulse responses (case 1)

Result of correction by iterative method (7),(8) $f=y_0+y_1+...+y_8$ is shown in Fig.4. It is obvious, that the correction (red curve) improves results (measured response - green curve), nevertheless it is still far from the correct impulse response (blue curve). It is not possible to improve results by increasing number of terms in series (7), because stability problems beginning from 15 terms in series (7).

Case 2.

Stimulus N=20, mean time 0.2s, unimodal impulse response N=5, mean time 0.5 s.

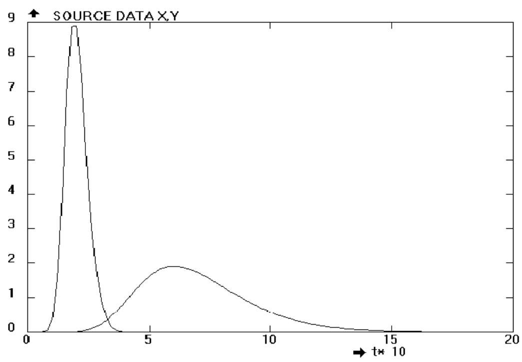

Fig.5 Stimulus and response (case 2)

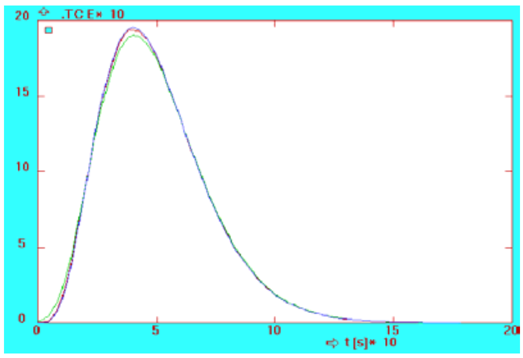

Fig.6 Comparison of impulse responses (case 2)

The ratio of mean times (stimulus:response) is the same as in the previous case (relative duration of stimulus function 40%), however, the agreement between measured response (green) and impulse response (blue) is much better. Corrected response (red) fits impulse response perfectly.

Case 3.

While no noise has been considered in the previous cases, an artificial noise (1%) is superposed on the predicted response in the case 3. Stimulus function N=20 and mean time 0.2 s is the same as in the case 2., impulse response corresponds to parallel series N_1=5, N_2=10, $\alpha=t_1/t_2=3$, $f=Q_1/Q_2=0.2$.

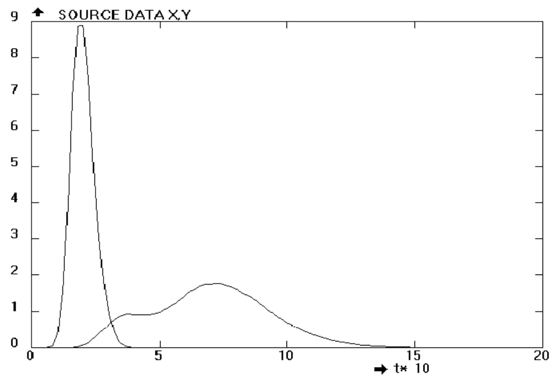

Fig.7 Stimulus and response (case 3)

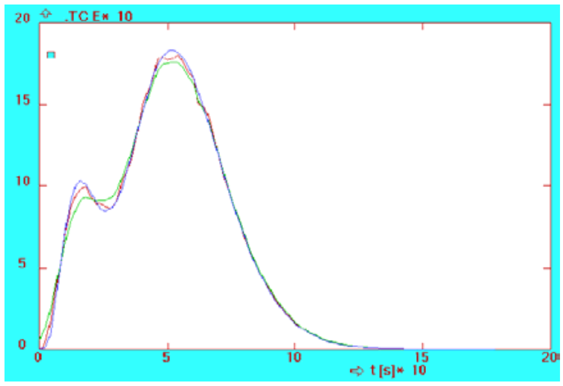

Fig.8 Comparison of impulse responses (case 3)

Even in this case the correction (7) is better approximation of the impulse response than the measured response without correction.

Case 4.

An obvious drawback of the proposed method is sensitivity to fluctuations of response, which is demonstrated in Figs.9,10. Stimulus function (N=20, mean time 0.2 s), and impulse response corresponding to parallel series (N_1=10, N_2=5, $\alpha=t_1/t_2=3$, $f=Q_1/Q_2=0.2$) are similar as in previous cases. However, the level of noise is increased substantially to 5%. Trend of correction is correct, but fluctuations should have to be filtered.

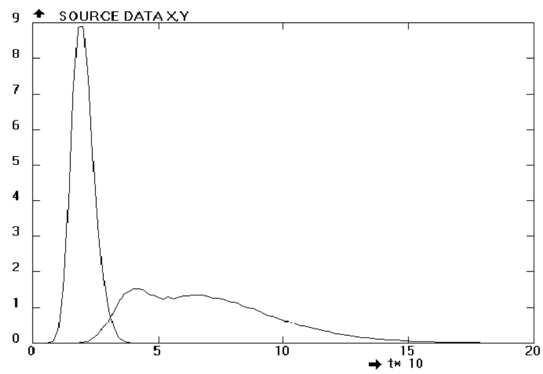

Fig.9 Stimulus and response (case 4)

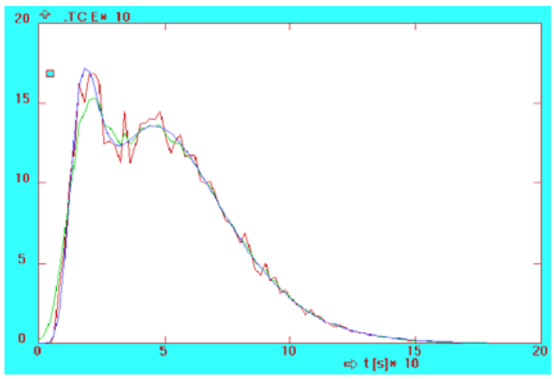

Fig.10 Comparison of impulse responses (case 4)

5. Implementation

The procedures described in the previous paragraph 2 has been implemented in PCL818 program, and can be activated during data acquisition using directives (commands)

SRTD inlet outlet

ERTD 2

The part of the PCL818 source program solving the initial value problem, Eq.(6) (ordinary differential equation /implemented in PCL818b) reads as follows

```
         s=0
         do i=1,n-1
           s=s+((t(i)-tmean(1))**2*c(i,1)+(t(i+1)-tmean(1))**2*c(i+1,1))
     /         *0.5*(t(i+1)-t(i))
         enddo
         s2=sqrt(s)
c Solution of differential equation
c f-impulse response, dg/dt=f
         f0=0
         g0=0
         do i=n-1,1,-1
             d=t(i+1)-t(i)
             f=(c(i,2)+s/(6*d)*(f0/d+g0))/(1+s/(6*d**2))
             g0=(f-f0)/d
             f0=f
             c(i,1)=f
         enddo
```

Method based upon iterated kernel (equations 7,8), implemented in PCL818

```
         do iter=1,3
            call kernel(y0,yi,n,ns)
            do i=1,n
               e(i)=e(i)+yi(i)
               y0(i)=yi(i)
            enddo
         enddo

      subroutine kernel(y,yi,n,ns)
      dimension y(n),yi(n)
      do i=1,n
        ym=0
        do j=i-ns,i+ns
            if(j.ge.1.and.j.le.n)ym=ym+y(j)
        enddo
        yi(i)=y(i)-ym/(2*ns+1)
      enddo
      end
```

Notation

f(t) impulse response
g(t) indefinite integral of impulse response
t time
x(t) stimulus function
y(t) response function
σ^2 variance (second central moment)

References:

[1] Thýn J., Žitný R., et al: Analysis and diagnostics of industrial processes by radiotracers and radioisotope sealed sources, CTU Prague, 2000, ISBN 80-01-02241-2
[2] Delves L.M., Mohamed J.L.: Computational methods for integral equations, Cambridge Univ.Press, N.Y.1985
[3] Zitny R., Thýn J.: Residence Time Distribution Software Analysis, Computer Manual Series 11, IAEA, Vienna 1996, p.218