Heating of a solid cylinder immersed in an insulated bath. Thermal diffusivity and heat capacity experimental evaluation.

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1. Introduction

The problem was initiated by the following E-mail from Carter Technologies Co.:

Dear Sirs,
We are seeking a method for measurement of the thermal conductivity of a sand and cement grout used to grout heat transfer pipes into the earth. The application is for ground loop heat pumps. Presently we find that different labs give different readings. One uses a 1/8 inch diameter heated thermo couple probe inserted into a 4 inch by 8 inch cylinder of the grout. Another uses some kind of rectangular sample and a reference material with a Shotherm QTM-D2 instrument. Our application is over a temperature range of 40 degrees F to 100 degrees F. Could a accurate test be devised based on cooling a 3000 gram 2.0 specific density cylinder sample to 40 degrees F and then immersing it in an insulated water bath having 1000 cc of water at 100 degree F. and measuring the temperature at 1 minute intervals for 10 minutes? Would this work and do you have any instruments to do this. How would you calculate it?

Results of suggested experiments (time courses of water temperature) can be used for evaluation of heat capacity and thermal diffusivity of material if the following simplified description of experiment is accepted: Cylinder (radius R, height H, thermal diffusivity a) of a uniform temperature T_0 at initial time t=0 is submerged into liquid (mass M_w , specific heat capacity c_w) having different initial temperature T_{w0}. It is supposed that the faces of cylinder are insulated and the heat is exchanged only through the cylindrical surface S=2πRH. It is also assumed that the liquid is so intensively mixed, that it has a uniform temperature T_w, the same as the surface of cylinder.

2. Solution based upon linear temperature profile within a thin penetration depth

Let us assume that the temperature is changing linearly from T_w to T_0 within a thin layer of thickness δ(t) at the wall of cylinder (or any other form of body, having surface S). Taking the enthalpy of solid at initial temperature T_0 as a reference (zero), the total enthalpy of body at a time t is

$$H = S\delta(t)\rho c_p \frac{T_w(t) - T_0}{2} \tag{1}$$

This enthalpy is changing according to the heat balance

$$dH = \frac{S\rho c_p}{2}(d\delta(T_w - T_0) + \delta dT_w)\,. \tag{2}$$

The enthalpy increase of body must be the same as the enthalpy decrease of water bath

$$dH = -M_w c_w dT_w \tag{3}$$

At the same time the enthalpy change is given by Fourier law as

$$dH = S\lambda \frac{T_w - T_0}{\delta} dt\,. \tag{4}$$

Three unknowns, penetration depth δ(t), enthalpy H(t) and T_w(t) are determined uniquely by three equations (2,3,4). Enthalpy H can be eliminated by equating (2-4) and (3-4)

$$\frac{S\rho c_p}{2}(\frac{d\delta}{dt}(T_w - T_0) + \delta\frac{dT_w}{dt}) = S\lambda\frac{T_w - T_0}{\delta} \tag{5}$$

$$-\delta\frac{dT_w}{dt} = S\lambda\frac{T_w - T_0}{M_w c_w}. \tag{6}$$

Equations (5) and (6) can be integrated, giving relationship between penetration depth and time, unfortunately in the inverse form $t=f(\delta)$

$$\delta - \frac{2a}{Q}\ln(1 + \delta\frac{Q}{2a}) = Qt. \tag{7}$$

Only for a very short time the penetration depth can be expressed explicitly as

$$\delta - \frac{2a}{Q}(\delta\frac{Q}{2a} - \frac{1}{2}(\delta\frac{Q}{2a})^2 + ...) = \frac{2a}{Q}\frac{1}{2}(\delta\frac{Q}{2a})^2 + ... = \delta^2\frac{Q}{4a} = Qt\text{, giving } \delta = 2\sqrt{at}. \tag{8}$$

Substituting penetration depth Eq. (8) into Eq.(6), a short time approximation of the liquid temperature at time t can be derived in the form

$$\frac{T_w - T_0}{T_{w0} - T_0} = e^{-\frac{2\pi R^2 H\rho c_p}{M_w c_w}\sqrt{\tau}} = e^{-\frac{\sqrt{\tau}}{M}}, \tag{9}$$

where

$$M = \frac{M_w c_w}{2\pi R^2 H\rho c_p}, \qquad \tau = \frac{at}{R^2}, \qquad S = 2\pi RH \tag{10,11,12}$$

are relative heat capacity of liquid, dimensionless time and surface of cylinder, respectively. Results obtained using this "short time approximation" will be compared with exact solution in the next paragraph.

3. Solution using infinite series

After an infinitely long time the both temperatures of liquid and cylinder achieve equilibrium temperature

$$T_e = \frac{M_w c_w T_{w0} + \pi R^2 H\rho c_p T_0}{M_w c_w + \pi R^2 H\rho c_p} = \frac{2MT_{w0} + T_0}{2M + 1} \tag{13}$$

and this equilibrium temperature will be used in the definition of dimensionless temperature

$$T^* = \frac{T - T_e}{T_0 - T_e}. \tag{14}$$

The temperature T^* of cylinder is initially one in the whole cylinder except at its surface r=1 (where T^*_{w}=-1/(2M)), and tends to zero at equilibrium.

Temperature profile in a cylinder is described by dimensionless form of Fourier equation

$$\frac{\partial T^*}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T^*}{\partial r}), \tag{15}$$

where dimensionless radius r is related to the radius of cylinder, and dimensionless time is given by Eq.(11).

Temperature of liquid is governed by an ordinary differential equation, expressing enthalpy balance of liquid

$$(M\frac{dT^*}{d\tau} + \frac{\partial T^*}{\partial r})|_{r=1} = 0. \tag{16}$$

Dimensionless coefficient M is the ratio of heat capacities of liquid and cylinder, see Eq.(10).

The solution T^* can be expressed as an infinite series

$$T^*(\tau,r) = \sum_{i=1}^{\infty} A_i J_0(\chi_i r) e^{-\chi_i^2 \tau} \qquad (17)$$

where the boundary condition at axis (symmetry) is automatically satisfied for any χ_i (because $J_0'(0)=0$). Boundary condition at surface of cylinder (r=1), Eq.(16), is fulfilled only for eigenvalues χ_i which are roots of equation

$$J_1(\chi_i) + M\chi_i J_0(\chi_i) = 0 \qquad (18)$$

Examples of results, eigenvalues χ_i calculated numerically for M=0.5, 1, and 2 are presented in Tab.1.

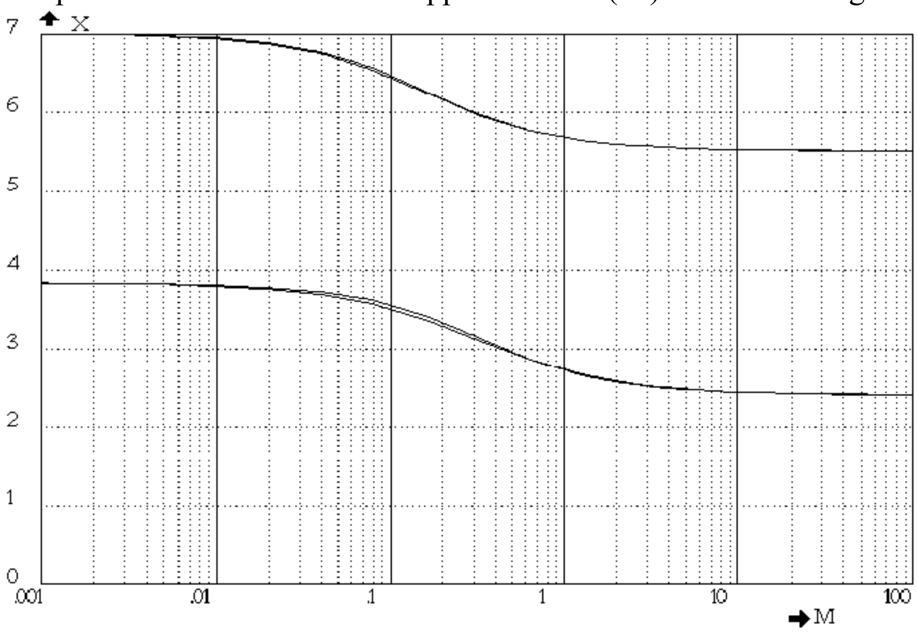
Tab.1 Roots of Eq.(30) for M=0.5,1, and 2

χ	M=0.5	M=1	M=2
0	0	0	0
1	2.9496	2.7346	2.5888
2	5.8411	5.6914	5.6083
3	8.8727	8.7666	8.7109
4	11.9561	11.8753	11.8337
5	15.0624	14.9974	14.9643
6	18.1803	18.1261	18.0987

The eigenvalues need not be solved numerically, because a simple approximation can be derived from asymptotic properties of Bessel functions for large arguments and using a mild empirical correction for low values of index i,

$$\chi_i = arctg\left[\frac{1-iM\pi}{1+iM\pi}\cdot\frac{1}{tg\left(\frac{\pi}{4}+\frac{0.06895}{iM^{0.0678}}\right)}\right] + i\pi\,, \qquad i=1,2,.... \qquad (19)$$

Comparison of exact values and approximation (19) is shown in Fig.1



Relative error of (19) within the range of M from 0.001 to 100.

i	%
1	0.37
2	0.10
3	0.045
4	0.023
5	0.014
6	0.012
7	0.011

Fig.1 Eigenvalues χ_1 and χ_2 as a function of M, according to exact solution of Eq.(18) and approximation Eq.(19), which predicts slightly higher values within the range $M\in(0.01 - 1)$.

The coefficients A_i of series (17) should satisfy initial condition

$$\mathbf{1} = \sum_{i=1}^{\infty} A_i J_0(\chi_i r), \tag{20}$$

for r<0 and

$$T^*(0,1) = \frac{T_{w0} - T_e}{T_0 - T_e} = -\frac{1}{2M} = \sum_{i=1}^{\infty} A_i J_0(\chi_i) \tag{21}$$

for r=1 (at surface).

Problem is in the fact, that the system of functions (20) is not orthogonal. However, we can proceed in a standard way, i.e. multiplying the series (20) by rJ_0 and integrating

$$\int_0^1 rJ_0(\chi_j r)dr = \sum_{i=1}^{\infty} A_i \int_0^1 rJ_0(\chi_i r) J_0(\chi_j r)dr \tag{21}$$

For $i \neq j$ the integrals of product of Bessel functions are, see Jenson (1973)

$$\int_0^1 rJ_0(\chi_i r)J_0(\chi_j r)dr = \frac{-\chi_i J_0(\chi_j)J_1(\chi_i) + \chi_j J_0(\chi_i)J_1(\chi_j)}{\chi_j^2 - \chi_i^2} = \{\textbf{use } J_1(\chi) = -M\chi J_0(\chi)\} = \tag{22}$$
$$= -MJ_0(\chi_j)J_0(\chi_i)$$

For $i=j$ it is necessary to use a different formula (be aware of the fact, that the expression (22) is of the form 0/0 and for example the l'Hopital rule must be used)

$$\int_0^1 rJ_0^2(\chi_j r)dr = \frac{1}{2}(J_0^2(\chi_j) + J_1^2(\chi_j)) = \frac{1}{2}J_0^2(\chi_j)(1 + M^2\chi_j^2). \tag{23}$$

Integral on the left side of Eq.(21) follows immediately from Eq.(22) for χ_i=0,

$$\int_0^1 rJ_0(\chi_j r)dr = \frac{\chi_j J_0'(\chi_j)}{-\chi_j^2} = \frac{J_1(\chi_j)}{\chi_j} = -MJ_0(\chi_j). \tag{24}$$

Substituting Eqs.(22-24) in Eq.(21) and rearranging terms we obtain

$$MJ_0(\chi_j)[\sum_{i=1}^{\infty} A_i J_0(\chi_i) - 1] = \frac{A_j}{2} J_0^2(\chi_j)(1 + M^2\chi_j^2 + 2M) \tag{25}$$

Making use of initial condition for temperature at surface, Eq.(21), Eq.(25) can be simplified

$$MJ_0(\chi_j)[-\frac{1}{2M} - 1] = \frac{A_j}{2} J_0^2(\chi_j)(1 + M^2\chi_j^2 + 2M) \tag{26}$$

and thus the coefficient A_j can be evaluated without necessity to solve a system of equations, giving the final solution for the dimensionless temperature field in a cylinder

$$T^*(\tau, r) = -(2M+1)\sum_{i=1}^{\infty} \frac{J_0(\chi_j r)}{J_0(\chi_j)} \frac{e^{-\chi_i^2 \tau}}{(1 + 2M + M^2\chi_j^2)}. \tag{27}$$

This exact solution can be compared with the short time approximation (9), rearranged to the form

$$T_w^*(\tau) = \frac{T_w - T_e}{T_0 - T_e} = 1 - \frac{T_{w0} - T_0}{T_e - T_0} e^{-\frac{\sqrt{\tau}}{M}} = 1 - \frac{2M+1}{2M} e^{-\frac{\sqrt{\tau}}{M}} \tag{28}$$

Examples of results are presented in Figs.2 and 3

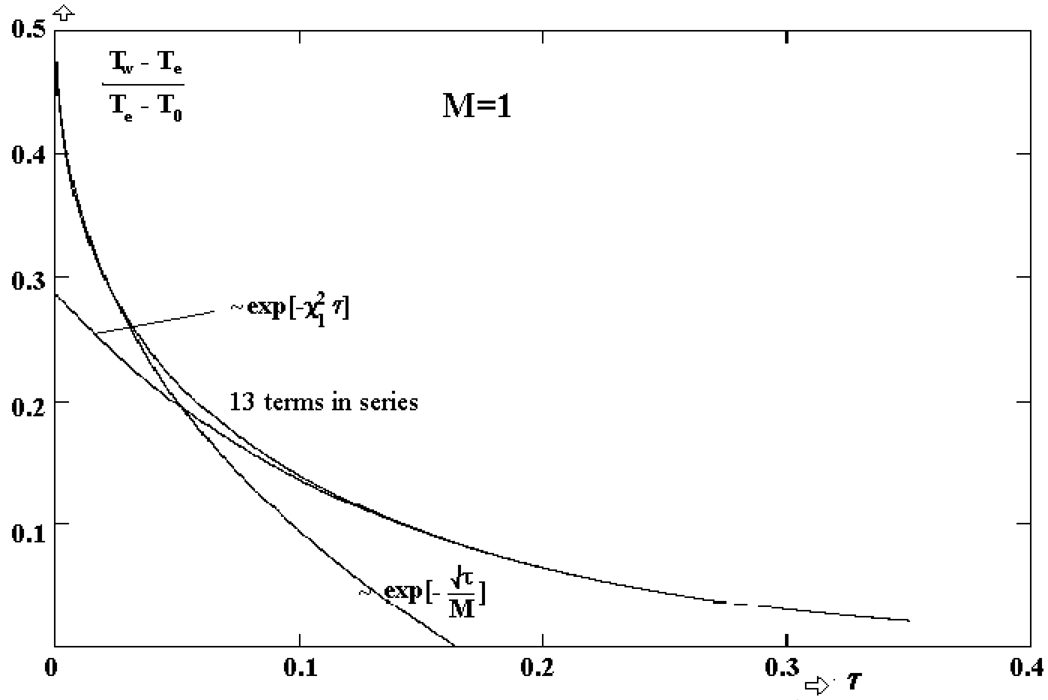


Fig.2 Dimensionless temperature of wall as a function of at/R^2 for M=1. Curves represent the short time solution Eq.(28), and the series (27) for 13 terms or just 1 term.

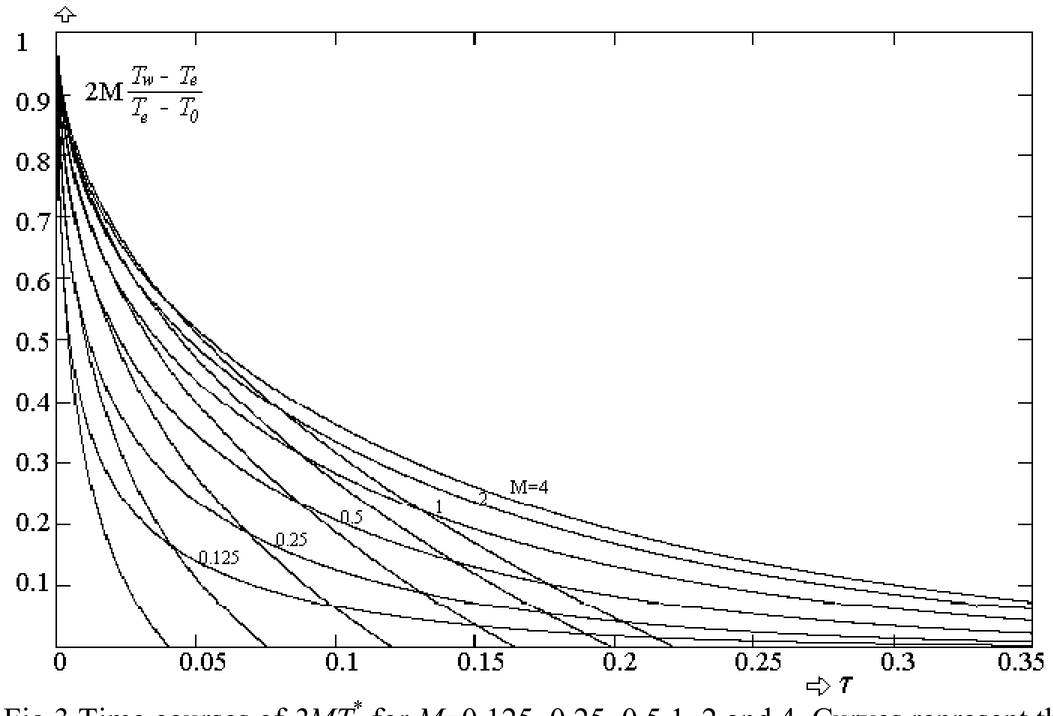


Fig.3 Time courses of $2MT^*$ for M=0.125, 0.25, 0.5,1, 2 and 4. Curves represent the short time solution Eq.(28), and series (27) were calculated with 20 terms using approximation Eq.(19).

Comparison of methods results in the conclusion that the "short time approximation" is suitable for dimensionless time $\tau<0.03$ at M=0.5. This value is not so low, because e.g. for $a=10^{-7}$ m^2 /s and for radius 0.05 m the corresponding time is 750 s (more than 10 minutes). Reducing series (28) only to the first term of expansion is acceptable for $\tau>0.1$.

4. Application, experimental procedure

Let us assume that the whole temperature course $T_w(t)$ has been recorded, and therefore the equilibrium temperature need not be calculated, see Fig.4.

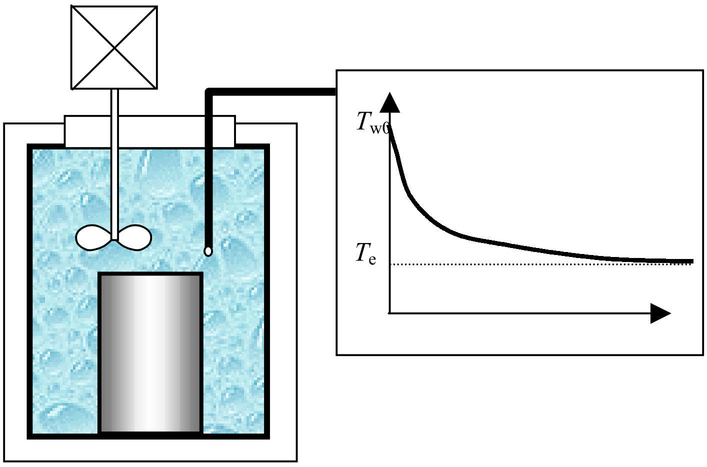


Fig.4 Experimental setup.

Knowing initial temperature of solid T_0 and initial temperature of water bath T_{w0} the dimensionless coefficient M, see Eq.(10), can be calculated as

$$M = \frac{1}{2}\frac{T_e - T_0}{T_{w0} - T_e}. \tag{29}$$

For this M the smallest positive eigenvalue χ_1 have to be solved numerically from Eq.(18) or expressed from approximation (19). The temperature of water can be described by only the first term of expansion (27) at a sufficiently long time (τ>0.1) as

$$\frac{T_w(\tau) - T_e}{T_0 - T_e} = -\frac{2M+1}{1+2M+M^2\chi_1^2} e^{-\chi_1^2 \tau} \tag{30}$$

This relationship enables to evaluate thermal diffusivity a (e.g. from the slope of $T(t)$ in a semilogarithmic plot). Thermal conductivity is related to the thermal diffusivity as

$$a = \frac{\lambda}{\rho c_p} \tag{31}$$

Because the mass and thermal capacity of water is known, the product ρc_p can be evaluated from the measured value of M

$$\rho c_p = \frac{M_w c_w}{2\pi R^2 HM} \quad . \tag{32}$$

Thus the experiment recording initial temperature of cylindrical sample T_0 and the time course of liquid temperature $T_w(t)$ yields the ρc_p value from Eq.(32), and thermal conductivity from Eq.(31).

References

Jenson,V.G., Jeffreys,G.V.: Matematické metódy v chemickom inžinierstve. 1973. Alfa Bratislava. Překlad z originálu Mathematical Methods in Chemical Engineering, Academic Press Inc. Ltd., London, 1963.

List of symbols

A_i coefficient of expansion (17)
a thermal diffusivity [$m^2.s^{-1}$]
c_w specific heat capacity of liquid [$J.kg^{-1}.K^{-1}$]
c_p specific heat capacity of cylinder [$J.kg^{-1}.K^{-1}$]
H height of cylinder [m]
$J_0(x)$ Bessel function
M relative heat capacity of liquid, Eq.(10)
M_w mass of liquid [kg]
R radius of cylinder [m]
r dimensionless radial coordinate (r/R)
S surface of sample [m^2]
T temperature of cylinder [K]
T_w temperature of liquid [K]
T_e equilibrium temperature [K]
T^* dimensionless temperature, see Eq.(14) [-]
t time [s]
χ eigenvalue, see Eq.(30)
δ penetration depth [m]
λ thermal conductivity of cylinder [$W.m^{-1}.K^{-1}$]
ρ density of cylinder [$kg.m^{-3}$]
τ dimensionless time (Fourier number)