Lecture notes on Mechanics of composite materials

Tomáš Mareš

January 4, 2016

1 Composite materials

What composite materials are and how they are made see
`http://en.wikipedia.org/wiki/Composite_material`.

2 Mechanics

Mechanics (in Greek $M\eta\chi\alpha\nu\iota\kappa\eta$) is a branch of physics dealing with the movement of bodies and its causes. Mechanics is based on two sets of axioms. They are either Newton's laws of motion or the principle of least action. Starting with Newton's laws we can, using variational methods, easily obtain the principle of least action and vice versa.
Just a reminder:
Newton's law of inertia (it defines inertial frame) $F = 0 \Leftrightarrow a = 0$
Newton's law of force and acceleration $F = ma$
Newton's law of action and reaction $\overrightarrow{F}$=-$\overrightarrow{R}$
and on the other hand the principle of least action says the path taken by a body (or a system) minimizes the action $S = \int_{t_1}^{t_2} L\mathrm{d}t$
where the Lagrangian $L = T - V$
In our lectures we are interested only in statics ($a = 0$) of deformable bodies of a special kind, namely bodies obeying the Hooke's law $\sigma^{ab} = E^{abcd}\varepsilon_{cd}$

3 Hooke's law

The last expression of Hooke's law is writen in tensor notation nevertheless we will use the Voigt's notation[1] $\sigma = E\varepsilon$

[1] `http://en.wikipedia.org/wiki/Voigt_notation`

i.e. $$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} \\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}$$
It seems there are 36 independent entries in E

As strain energy $$u = \frac{1}{2}\sigma'\varepsilon$$
and $$2u = \sigma'\varepsilon = \varepsilon' E'\varepsilon$$
and at the same time $2u = \varepsilon'\sigma = \varepsilon' E\sigma$

it follows $$E = E'$$
and that there are 21 independent entries

namely $$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{12} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{13} & E_{23} & E_{33} & E_{34} & E_{35} & E_{36} \\ E_{14} & E_{24} & E_{34} & E_{44} & E_{45} & E_{46} \\ E_{15} & E_{25} & E_{35} & E_{45} & E_{55} & E_{56} \\ E_{16} & E_{26} & E_{36} & E_{46} & E_{56} & E_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}$$
It holds true for every lineary elastic material. We call it Hooke's law for anisostropic material. In what follows we will study material symmetries.

4 Monoclinic material

In mechanics of composite materials we study symmetry in other way than in crystallography. What we call monoclinic material is a material that have one plane of material symmetry in point like sense. What I meen is the fact that Hooke's law in the stated form is point like and to state material symmetry it is sufficient to study this Hooke's law. We call a plane of material symmetry such a plane with respect to which both stress and strain is either symetric or anisotropic (both the same).

4.1 Monoclinic material with the plane of symmetry being plane 12

Let us say, in 123 coordinate system, the plane 12 is the plane of symmetry. Then to insure the material symmetry the entries of E that bind the entries of symmetric stress and antisymmetric strain and vice versa should be equal zero. And so the stiffness matrix must be like

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & E_{16} \\ E_{12} & E_{22} & E_{23} & 0 & 0 & E_{26} \\ E_{13} & E_{23} & E_{33} & 0 & 0 & E_{36} \\ 0 & 0 & 0 & E_{44} & E_{45} & 0 \\ 0 & 0 & 0 & E_{45} & E_{55} & 0 \\ E_{16} & E_{26} & E_{36} & 0 & 0 & E_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}$$

4.2 Monoclinic material with the plane of symmetry being plane 23

If the plane 23 is the plane of symmetry then

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 & 0 \\ E_{12} & E_{22} & E_{23} & E_{24} & 0 & 0 \\ E_{13} & E_{23} & E_{33} & E_{34} & 0 & 0 \\ E_{14} & E_{24} & E_{34} & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & E_{56} \\ 0 & 0 & 0 & 0 & E_{56} & E_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}$$

4.3 Monoclinic material with the plane of symmetry being plane 31

If the plane 31 is the plane of symmetry then

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & 0 & E_{15} & 0 \\ E_{12} & E_{22} & E_{23} & 0 & E_{25} & 0 \\ E_{13} & E_{23} & E_{33} & 0 & E_{35} & 0 \\ 0 & 0 & 0 & E_{44} & 0 & E_{46} \\ E_{15} & E_{25} & E_{35} & 0 & E_{55} & 0 \\ 0 & 0 & 0 & E_{46} & 0 & E_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}$$

A monoclinic material has 13 independent material characteristics.

5 Orthotropic materal

An orthotropic material is a material that have three mutually perpendicular planes of symmetry, let us say 12,23,31. As every one of the three above mentioned monoclinic cases holds there is just one way

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{13} & E_{23} & E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}$$

An orthotropic material thus has 9 independent material characteristics.

6 Transverse isotropic material

If there is an axis such that every plane containing this axis is a plane of material symmetry then this material is called transverse isotropic material. This material has 5 independent characteristics as may be shown using rotational transformation about the axis of symmetry.

7 Isotropic material

It is symetric with respect to every plane and there are only 2 independent material characteristics.

8 Orthotropic material in more detail

Elasticity tensor E^{abcd} and compliance tensor C_{abcd}

Ciarlet, P. G. (2005)
Mareš, T. (2006)

Isotropic material (λ, μ — Lamé coefficients)

$$E^{abcd} = \lambda g^{ab} g^{cd} + \mu g^{ac} g^{bd} + \mu g^{ad} g^{bc}$$

Orthotropic block

Young modulus in the direction ν^1

$$E_{11} = \frac{\sigma^{11}}{\varepsilon^1_1}$$

Poisson ratios

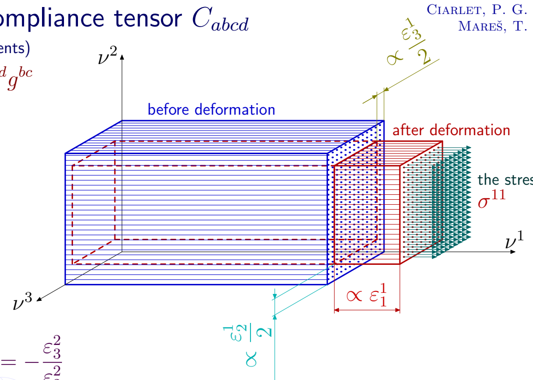
$$\nu_{12} = -\frac{\varepsilon^1_2}{\varepsilon^1_1}, \quad \nu_{13} = -\frac{\varepsilon^1_3}{\varepsilon^1_1}$$

Similarly in the direction of ν^2

$$E_{22} = \frac{\sigma^{22}}{\varepsilon^2_2}, \quad \nu_{21} = -\frac{\varepsilon^2_1}{\varepsilon^2_2}, \quad \nu_{23} = -\frac{\varepsilon^2_3}{\varepsilon^2_2}$$

and of ν^3

$$E_{33} = \frac{\sigma^{33}}{\varepsilon^3_3}, \quad \nu_{31} = -\frac{\varepsilon^3_1}{\varepsilon^3_3}, \quad \nu_{32} = -\frac{\varepsilon^3_2}{\varepsilon^3_3}$$



Strain in the ν^1 excited by all normal stresses

$$\varepsilon_{11} = \varepsilon^1_1 + \varepsilon^2_1 + \varepsilon^3_1$$

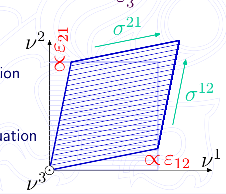
$$\varepsilon_{11} = \frac{\sigma^{11}}{E_{11}} - \nu_{21}\frac{\sigma^{22}}{E_{22}} - \nu_{31}\frac{\sigma^{33}}{E_{33}}$$

Similarly in the other directions (G_{23}, G_{31})

Pure shear

From the definition
$\varepsilon_{12} = \varepsilon_{21}$

the equilibrium equation
$\sigma^{12} = \sigma^{21}$



$$\sigma^{12} = \sigma^{12} = G_{12}(\varepsilon_{12} + \varepsilon_{21})$$

Compliance tensor C_{abcd}

Ciarlet, P. G. (2005)
Mareš, T. (2006)

in Cartesian coordinate system ν^a

alined with the principal material axes of the orthotropic material

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{21}}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{31}}{E_{33}} \\ 0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\ 0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & 0 & 0 & 0 & \frac{1}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{32}}{E_{33}} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\ 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\ -\frac{\nu_{13}}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{23}}{E_{22}} & 0 & 0 & 0 & \frac{1}{E_{33}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix}$$

$$\Updownarrow$$

$$\overset{\nu}{\varepsilon}_{ab} = \overset{\nu}{C}_{abcd}\, \overset{\nu}{\sigma}{}^{cd}$$

$$C_{abcd} = C_{cdab} = C_{bacd}$$

$\nearrow$ Energy $\qquad\qquad \nwarrow$ Equilibrium

9 Plane stress of an orthotropic material

Plane stress is a stress state where $\sigma_{3a} = 0$. Then we can, in the main coordinate system of orthotropy, write $\boldsymbol{\varepsilon} = \boldsymbol{C}\boldsymbol{\sigma}$, *i.e.*

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_L} & -\frac{\nu_{TL}}{E_T} & 0 \\ -\frac{\nu_{LT}}{E_L} & \frac{1}{E_T} & 0 \\ 0 & 0 & \frac{1}{G_{LT}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

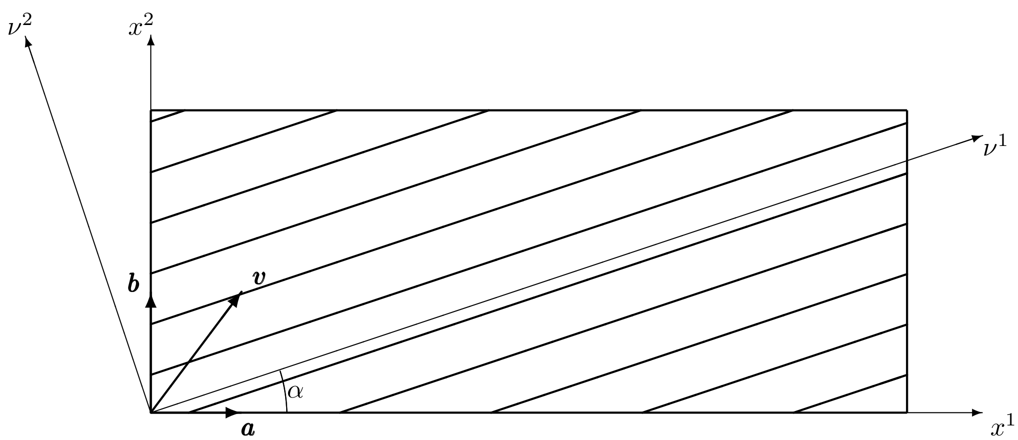
or the inverse relation

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \frac{1}{1 - \nu_{LT}\nu_{TL}} \begin{pmatrix} E_L & \nu_{LT}E_T & 0 \\ \nu_{TL}E_L & E_T & 0 \\ 0 & 0 & G_{LT}(1 - \nu_{LT}\nu_{TL}) \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

Symbolically $\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}$

The matrix $\boldsymbol{C}$ is called Compliance matrix and matrix $\boldsymbol{E}$ is called Stiffness matrix.

10 2D vector Cartesian transformation



Let us have a vector $\boldsymbol{v}$ and two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b}$

The coordinates of vectors $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ in the coordinate system x^a are respectively $\overset{x}{\boldsymbol{v}} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\overset{x}{\boldsymbol{a}} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\overset{x}{\boldsymbol{b}} = \begin{pmatrix} 0 \\ b \end{pmatrix}$

In the coordinate system ν^a the vector $\boldsymbol{a}$ has coordinates $\overset{\nu}{\boldsymbol{a}} = \begin{pmatrix} a\cos\alpha \\ -a\sin\alpha \end{pmatrix}$

vector $\boldsymbol{b}$ has coordinates $\overset{\nu}{\boldsymbol{b}} = \begin{pmatrix} b\sin\alpha \\ b\cos\alpha \end{pmatrix}$

and vector $\boldsymbol{v}$ $\overset{\nu}{\boldsymbol{v}} = \overset{\nu}{\boldsymbol{a}} + \overset{\nu}{\boldsymbol{b}} = \begin{pmatrix} a\cos\alpha + b\sin\alpha \\ -a\sin\alpha + b\cos\alpha \end{pmatrix}$

Thus we have coordinate transformation of a vector in the form $\overset{\nu}{\boldsymbol{v}} = \boldsymbol{T}_{\nu x} \overset{x}{\boldsymbol{v}}$

where the transformation matrix $\boldsymbol{T}_{\nu x} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

For the inverse transformation $\overset{x}{\boldsymbol{v}} = \boldsymbol{T}_{x\nu} \overset{\nu}{\boldsymbol{v}}$

the transformation matrix is inverse $\boldsymbol{T}_{x\nu} = \boldsymbol{T}_{\nu x}^{-1} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

11 Transformation of Voigt stress vector

As the stress is second order tensor we must at first look at second order tensor transformation. Direct multiplication of two first order tensor may be representeted as matrix multiplication of components $\overset{\nu}{\boldsymbol{v}}\overset{\nu}{\boldsymbol{v}}^T$

Using transformation rules stated above $\overset{\nu}{\boldsymbol{v}}\overset{\nu}{\boldsymbol{v}}^T = \boldsymbol{T}_{\nu x} \overset{x}{\boldsymbol{v}}\overset{x}{\boldsymbol{v}}^T \boldsymbol{T}_{\nu x}^T$

For the stress tensor then

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}_\nu = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}_x \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Executing multiplication on the right site gives (using $\sigma_{12} = \sigma_{21}$) $\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}_\nu =$

$$= \begin{pmatrix} \sigma_{11}^x \cos^2\alpha + 2\sigma_{12}^x \sin\alpha\cos\alpha + \sigma_{22}^x \sin^2\alpha & (\sigma_{22}^x - \sigma_{11}^x)\sin\alpha\cos\alpha + \sigma_{12}^x(\cos^2\alpha - \sin^2\alpha) \\ (\sigma_{22}^x - \sigma_{11}^x)\sin\alpha\cos\alpha + \sigma_{12}^x(\cos^2\alpha - \sin^2\alpha) & \sigma_{11}^x \sin^2\alpha - 2\sigma_{12}^x \sin\alpha\cos\alpha + \sigma_{22}^x \cos^2\alpha \end{pmatrix}$$

Rearranging

$$\begin{pmatrix} \sigma_{11}^\nu \\ \sigma_{22}^\nu \\ \sigma_{12}^\nu \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{pmatrix} \begin{pmatrix} \sigma_{11}^x \\ \sigma_{22}^x \\ \sigma_{12}^x \end{pmatrix}$$

Symbolically $\overset{\nu}{\boldsymbol{\sigma}} = \boldsymbol{T}_{\nu x}^\sigma \overset{x}{\boldsymbol{\sigma}}$

Inverse transformation $\overset{x}{\boldsymbol{\sigma}} = \boldsymbol{T}_{x\nu}^\sigma \overset{\nu}{\boldsymbol{\sigma}}$

can be obtained both $\boldsymbol{T}_{x\nu}^\sigma = (\boldsymbol{T}_{\nu x}^\sigma)^{-1}$

and $\boldsymbol{T}_{x\nu}^\sigma(\alpha) = \boldsymbol{T}_{\nu x}^\sigma(-\alpha)$

which leads to

$$\boldsymbol{T}_{x\nu}^\sigma = \begin{pmatrix} \cos^2\alpha & \sin^2\alpha & -2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{pmatrix}$$

12 Transformation of Voigt strain vector

Strain tensor has the same structure as stress tensor and so the transformation of Voigt strain vector would by the same as the transformation of Voigt stress vector as long as the structure of the vectors is the same. But it is not. There is $2\varepsilon_{12}$ instead of ε_{12} in the last entry. This factor of 2 must be incorporated in the transformation matrix which leads to the transformation matrices

$$\boldsymbol{T}^{\varepsilon}_{\nu x} = \begin{pmatrix} \cos^2\alpha & \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & 2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{pmatrix}$$

$$\boldsymbol{T}^{\varepsilon}_{x\nu} = \begin{pmatrix} \cos^2\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & -2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{pmatrix} = (\boldsymbol{T}^{\sigma}_{\nu x})^T$$

13 Stiffness matrix transformation

As
$$\overset{x}{\boldsymbol{\sigma}} = \overset{x}{\boldsymbol{E}}\overset{x}{\boldsymbol{\varepsilon}} = \overset{x}{\boldsymbol{E}}\,\boldsymbol{T}^{\varepsilon}_{x\nu}\,\overset{\nu}{\boldsymbol{\varepsilon}}$$
$$\overset{\nu}{\boldsymbol{\sigma}} = \overset{\nu}{\boldsymbol{E}}\overset{\nu}{\boldsymbol{\varepsilon}}$$
and
$$\overset{\nu}{\boldsymbol{\sigma}} = \boldsymbol{T}^{\sigma}_{\nu x}\,\overset{x}{\boldsymbol{\sigma}} = \boldsymbol{T}^{\sigma}_{\nu x}\,\overset{x}{\boldsymbol{E}}\,\boldsymbol{T}^{\varepsilon}_{x\nu}\,\overset{\nu}{\boldsymbol{\varepsilon}}$$
it holds
$$\overset{\nu}{\boldsymbol{E}} = \boldsymbol{T}^{\sigma}_{\nu x}\,\overset{x}{\boldsymbol{E}}\,\boldsymbol{T}^{\varepsilon}_{x\nu} = \boldsymbol{T}^{\sigma}_{\nu x}\,\overset{x}{\boldsymbol{E}}\,(\boldsymbol{T}^{\sigma}_{\nu x})^T$$
For inverse transformation...

14 Compliance matrix transformation

Similarly
$$\overset{x}{\boldsymbol{\varepsilon}} = \ldots$$

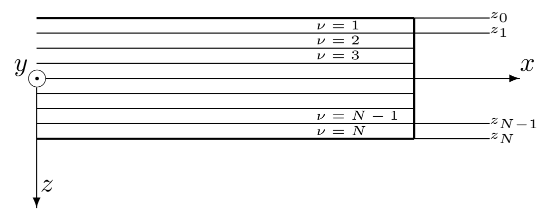
15 Composite micromechanics

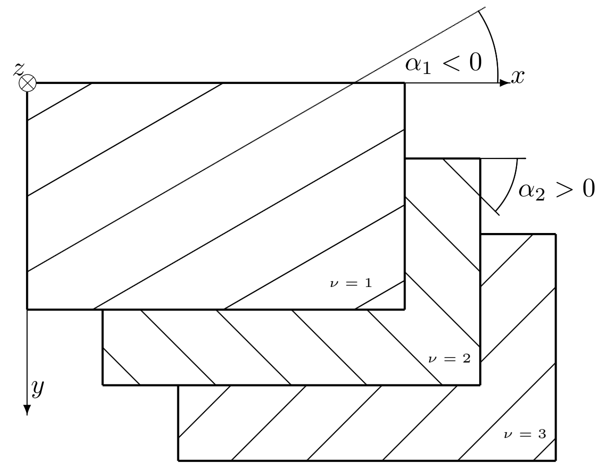
Given the micromechanical geometry and the material properties of each constituent, it is possible to estimate the effective composite material properties and the micromechanical stress/strain state of a composite material.
Thus, for fibre composite we can estimate...

16 Strength theories for filamentary composite materials

17 Composite laminate – layup nomenclature

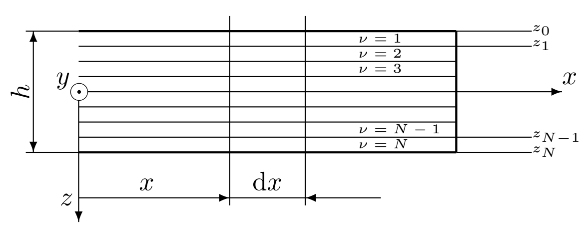
A laminate is an organized stack of uni-directional composite plies (uni-directional meaning the plies have a single fiber direction rather than a weave pattern). The stack is defined by the fiber directions of each ply like this:

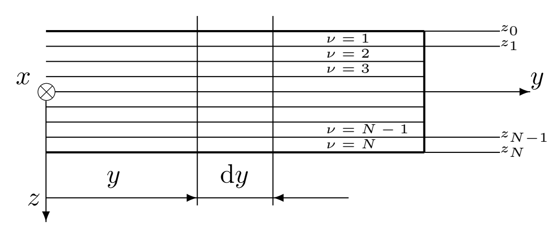


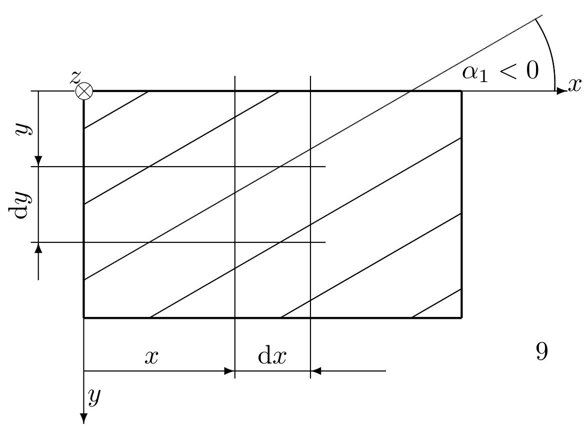


Such laminates are often described by an orientation code $[\alpha_1/\alpha_2/\alpha_3/\alpha_4]$
For example [0/-45/90/45/0/0/45/90/-45/0]
Short hand $[0/\text{-}45/90/45/0]_s$
Other examples of short hand $[0/90]_4$
or $[0/\pm 45/90]$, $[0/45_2/30]$
etc.

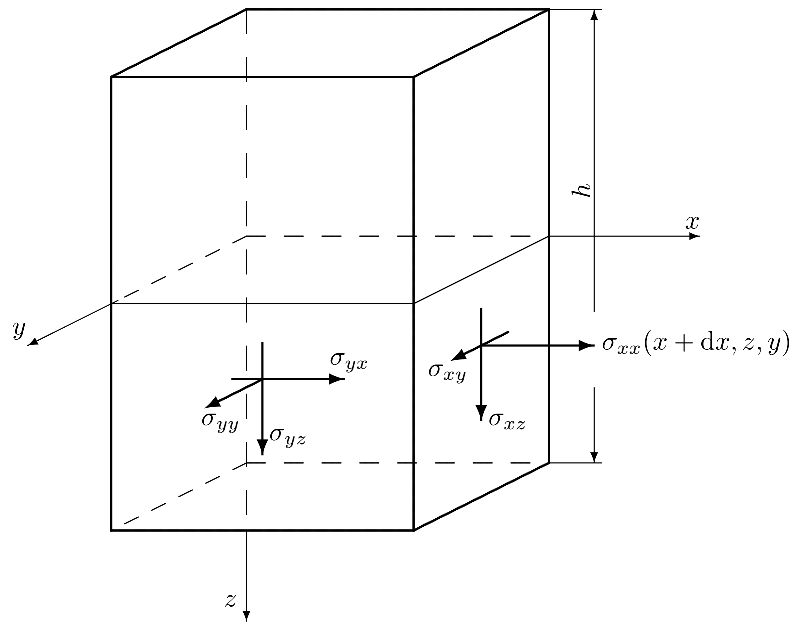
18 Equilibrium equation of a laminated plate (a laminate)







Cut out the element dx×dy
and apply the internal forces and moments
as resultants of applied stresses



As h is finite the stresses are unknown functions of z. On the other hand the dimensions $\mathrm{d}x$ and $\mathrm{d}y$ are infinitesimally small and we may approximate the functions according to Taylor series $\sigma_{ab}(x + \mathrm{d}x, y, z) = \sigma_{ab} + \sigma_{ab,x}\mathrm{d}x$ and $\sigma_{ab}(x, y + \mathrm{d}y, z) = \sigma_{ab} + \sigma_{ab,y}\mathrm{d}y$ where by σ_{ab} we understand $\sigma_{ab} = \sigma_{ab}(x, y, z)$
Now, to write equilibrium equations we need forces and moments acting upon the element. The acting generalized forces are the resultants of the stresses

$$N_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\mathrm{d}z$$

$$N_{yy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy}\mathrm{d}z$$

$$N_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy}\mathrm{d}z$$

$$N_{yx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yx}\mathrm{d}z$$

$$Q_{xz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz}\mathrm{d}z$$

$$Q_{yz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yz}\mathrm{d}z$$

$$M_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{xx}\mathrm{d}z$$

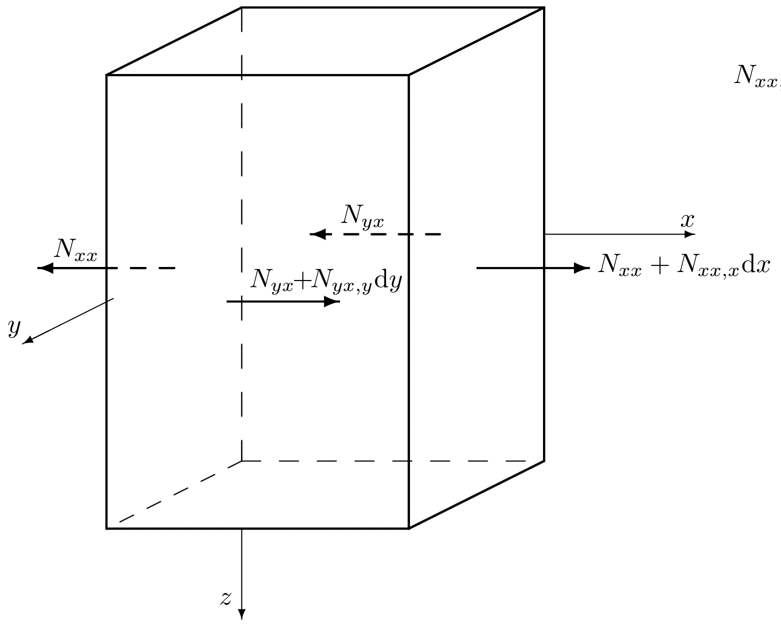
$$M_{yy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{yy}\mathrm{d}z$$

$$M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{xy}\mathrm{d}z$$

$$M_{yx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{yx}\mathrm{d}z$$

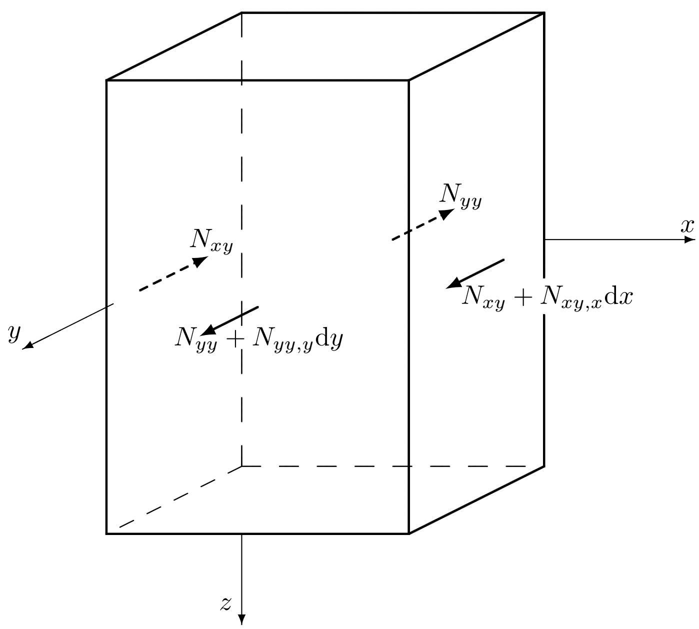
From the definition of these quantities we see that they are not forces or moments but in fact linear densities of these forces and moments. To get a real forces we need to multiply them by the width of the appropriate area of the element.

The forces acting in the x-direction and the equilibrium equation



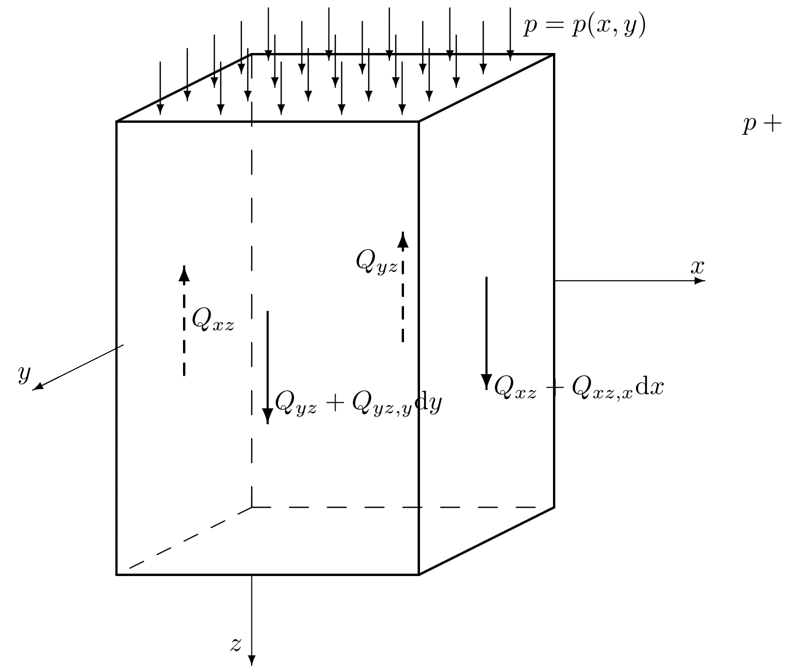
$$N_{xx,x} + N_{yx,y} = 0 \tag{x}$$

The forces acting in the y-direction and the equilibrium equation



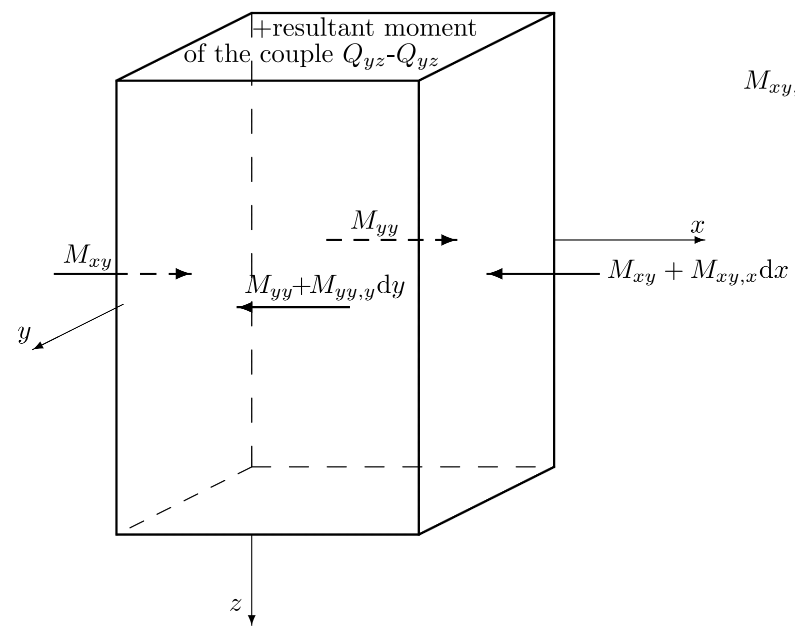
$$N_{xy,x} + N_{yy,y} = 0 \tag{y}$$

The forces acting in the z-direction and the equilibrium equation



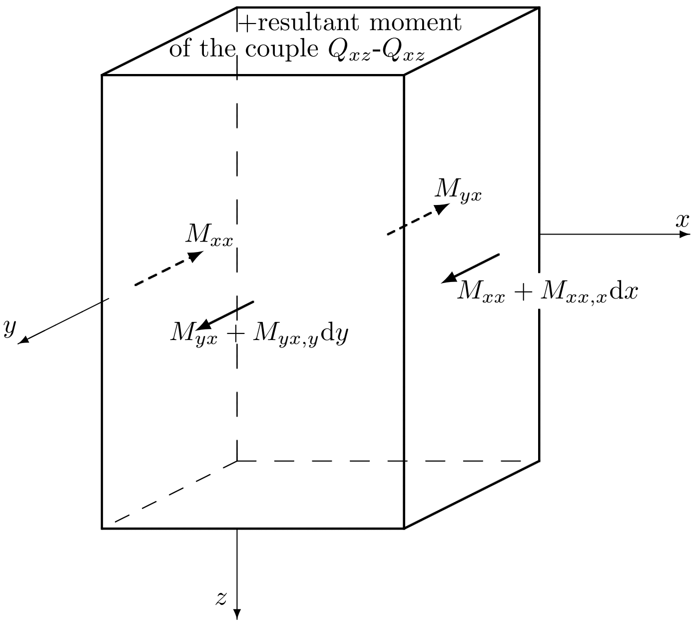
$$p + Q_{yz,y} + Q_{xz,x} = 0 \qquad \text{(z)}$$

The moments acting in the x-direction and the equilibrium equation



$$M_{xy,x} + M_{yy,y} - Q_{yz} = 0 \qquad \text{(mx)}$$

The moments acting in the y-direction and the equilibrium equation



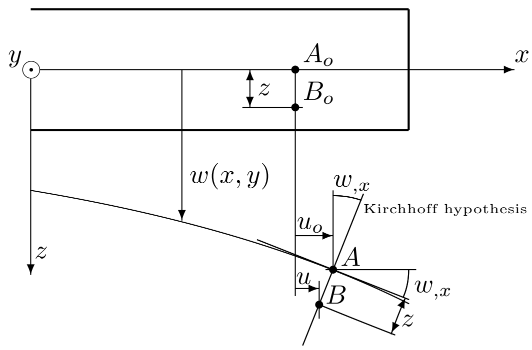
$$M_{xx,x} + M_{yx,y} - Q_{xz} = 0 \qquad \text{(y)}$$

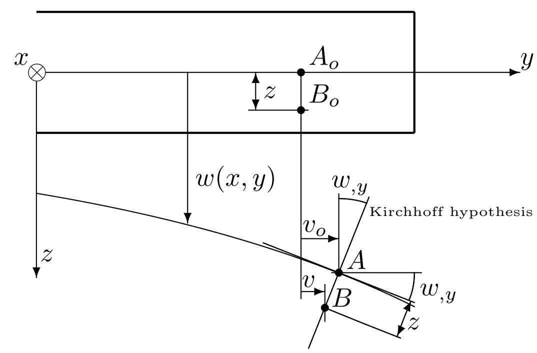
Puting these equilibrium equations together we get $M_{ab,ab} = -p$
and $N_{ab,a} = 0$ where $a, b = x, y$
There are three equations for six unknown. We need a compatibility equation. The most common one is Kirchhoff hypothesis resulting in Classical lamination theory.

19 Classical lamination theory

In Classical lamination theory we assume Kirchhoff hypothesis that says that points on a normal to an undeformed middle plane stay on a normal to the deformed middle plane.
Following the Kirchhoff hypothesis shown on the figure below

$$u_o = u_o(x, y)$$
$$v_o = v_o(x, y)$$
$$w = w_o = w(x, y)$$
$$u = u_o - zw_{,x}$$
$$v = v_o - zw_{,y}$$





From Cauchy's strain tensor formula
$$\varepsilon_{ab} = \frac{1}{2}(u_{a,b} + u_{b,a})$$
we have
$$\begin{aligned}
\varepsilon_{xx} &= u_{,x} = u_{o,x} - z w_{,xx} \\
\varepsilon_{yy} &= v_{,y} = v_{o,y} - z w_{,yy} \\
\varepsilon_{xy} &= \frac{1}{2}(u_{o,y} + v_{o,x}) - z w_{,xy} \\
\varepsilon_{zx} &= 0 \\
\varepsilon_{yz} &= 0 \\
\varepsilon_{zz} &= 0
\end{aligned}$$
The last expression is in contrariety with the assumption of plane stress...
Now, we are to express the stresses using the Hooke's law for plane stress state. Why plane stress when the Kirchhoff hypothesis leads to plane strain we will discuss later. According to (..) we have
$$\overset{x}{\boldsymbol{\sigma}} = \overset{x}{\boldsymbol{E}} \overset{x}{\boldsymbol{\varepsilon}}$$
where
$$\overset{x}{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$
and
$$\overset{x}{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} u_{o,x} \\ v_{o,y} \\ \frac{1}{2}(u_{o,y} + v_{o,x}) \end{pmatrix} - z \begin{pmatrix} w_{,xx} \\ w_{,yy} \\ w_{,xy} \end{pmatrix} = \boldsymbol{\varepsilon}_o + z\boldsymbol{\kappa}$$
Note the change in the $\pm$ sign due to the definition of the curvature vector $\boldsymbol{\kappa}$.
For the generalized forces
$$\boldsymbol{N} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \overset{x}{\boldsymbol{\sigma}} \,\mathrm{d}z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \overset{x}{\boldsymbol{E}} (\boldsymbol{\varepsilon}_o + z\boldsymbol{\kappa}) \mathrm{d}z$$
or, as $\boldsymbol{\varepsilon}_o$ and $\boldsymbol{\kappa}$ do not depend on z,
$$\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\varepsilon}_o + \boldsymbol{B}\boldsymbol{\kappa}$$
where
$$\boldsymbol{A} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \overset{x}{\boldsymbol{E}} \,\mathrm{d}z = \sum_{\nu=1}^{N} \int_{z_{\nu-1}}^{z_\nu} \boldsymbol{T}^{\sigma}_{x\nu}(\alpha_\nu)\, \overset{\nu}{\boldsymbol{E}}\, \boldsymbol{T}^{\varepsilon}_{\nu x}(\alpha_\nu) \mathrm{d}z$$

and $$\boldsymbol{B} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \overset{x}{\boldsymbol{E}} \, \mathrm{d}z = \sum_{\nu=1}^{N} \int_{z_{\nu-1}}^{z_\nu} z \boldsymbol{T}^{\sigma}_{x\nu}(\alpha_\nu) \overset{\nu}{\boldsymbol{E}} \boldsymbol{T}^{\varepsilon}_{\nu x}(\alpha_\nu) \mathrm{d}z$$

i.e. $$\boldsymbol{B} = \sum_{\nu=1}^{N} \frac{z_\nu^2 - z_{\nu-1}^2}{2} \boldsymbol{T}^{\sigma}_{x\nu}(\alpha_\nu) \overset{\nu}{\boldsymbol{E}} \boldsymbol{T}^{\varepsilon}_{\nu x}(\alpha_\nu)$$

Similarly for the moments $$\boldsymbol{M} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \overset{x}{\boldsymbol{\sigma}} \, \mathrm{d}z = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \overset{x}{\boldsymbol{E}} \, (\boldsymbol{\varepsilon}_o + z\boldsymbol{\kappa}) \mathrm{d}z$$

or, as $\boldsymbol{\varepsilon}_o$ and $\boldsymbol{\kappa}$ do not depend on z, $$\boldsymbol{M} = \boldsymbol{B}\boldsymbol{\varepsilon}_o + \boldsymbol{D}\boldsymbol{\kappa}$$

where $$\boldsymbol{D} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \overset{x}{\boldsymbol{E}} \, \mathrm{d}z = \sum_{\nu=1}^{N} \int_{z_{\nu-1}}^{z_\nu} z^2 \boldsymbol{T}^{\sigma}_{x\nu}(\alpha_\nu) \overset{\nu}{\boldsymbol{E}} \boldsymbol{T}^{\varepsilon}_{\nu x}(\alpha_\nu) \mathrm{d}z$$

i.e. $$\boldsymbol{D} = \sum_{\nu=1}^{N} \frac{z_\nu^3 - z_{\nu-1}^3}{3} \boldsymbol{T}^{\sigma}_{x\nu}(\alpha_\nu) \overset{\nu}{\boldsymbol{E}} \boldsymbol{T}^{\varepsilon}_{\nu x}(\alpha_\nu)$$

20 Symmetric laminate

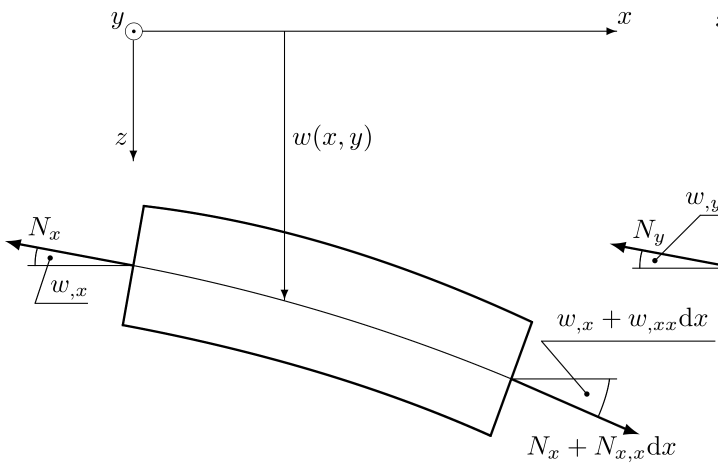
Symmetric laminate is a laminate for which for every ν there is a μ such that $$\alpha_\nu = \alpha_\mu \text{ and } z_\nu = -z_{\mu-1}$$
Then $$\boldsymbol{B} = 0$$
and $$\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\varepsilon}_o$$
$$\boldsymbol{M} = \boldsymbol{D}\boldsymbol{\kappa}$$
Using (here) and (there) we get $$D_{abcd} w_{abcd} = p$$
and $$A_{abcd} u^o_{c,ad} = 0$$
Add comment on coupling...

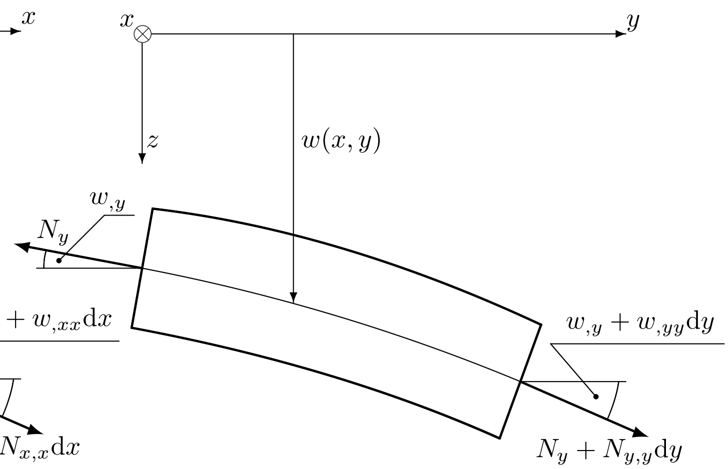
21 Balanced laminate

22 Solved problems not only on $\boldsymbol{B} = 0$ case

23 Buckling analysis of laminated plates

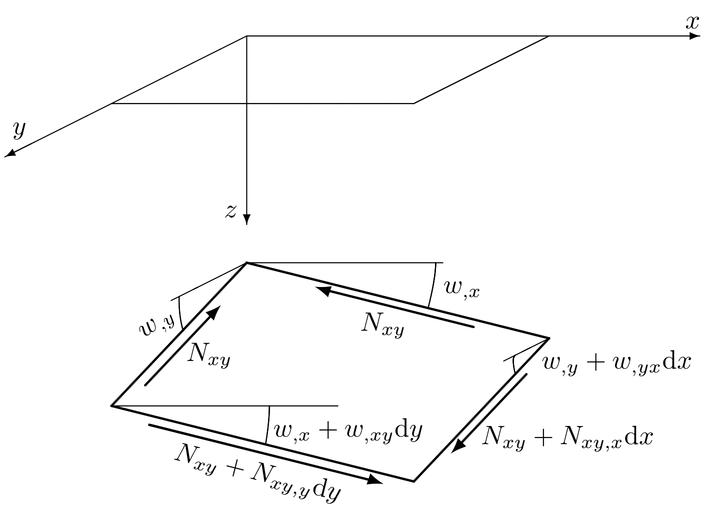
Let us consider symmetric laminate $\boldsymbol{B} = 0$. For this case we have from above
$$D_{abcd} w_{,abcd} = 0$$
$$N_{ab,a} = 0$$
These equations of equilibrium have been derived under the undeformed geometry configuration. As in the case of column buckling we need to look at the case of deformed shape.





The contribution to the z-direction equilibrium equation

$$-N_x\mathrm{d}y\, w_{,x} + (N_x + N_{x,x}\mathrm{d}x)(w_{,x} + w_{,xx}\mathrm{d}x)\mathrm{d}y - N_y\mathrm{d}x\, w_{,x} + N_y + N_{y,y}\mathrm{d}y)(w_{,y} + w_{,yy}\mathrm{d}y)\mathrm{d}x$$



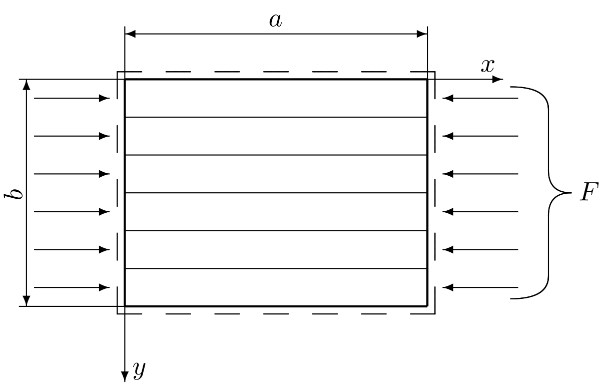
$$-N_{xy}w_{,y}\mathrm{d}y + (N_{xy} + N_{xy,x}\mathrm{d}x)(w_{,y} + w_{,yx}\mathrm{d}x)\mathrm{d}y$$

$$-N_{xy}w_{,x}\mathrm{d}x + (N_{xy} + N_{xy,y}\mathrm{d}y)(w_{,x} + w_{,xy}\mathrm{d}y)\mathrm{d}x$$

The Figures above show forces whose components in the z-direction are zero if the element is in undeformed position. Nevertheless, if deformed, as on the Figures, there are nonzero components in the direction. That means the equilibrium equation in the z-direction $D_{abcd}w_{,abcd}\mathrm{d}x\mathrm{d}y = p\mathrm{d}x\mathrm{d}y$ has the following additional terms (after using $N_{ab,a} = 0$ and $\mathcal{O}(3) = 0$) on its right side: $D_{abcd}w_{,abcd} = p + N_{ab}w_{ab}$

24 Buckling of plates–solved example

For one layered and orthotropic plate with $\nu \parallel x$ loaded as shown in the figure and simply supported we have the following.



In the Lamé equation of equilibrium $D_{abcd}w_{,abcd} = p + N_{ab}w_{ab}$

we have
$$D_{abcd} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \overset{x}{E}_{abcd}\, z^2 \mathrm{d}z$$

where $\overset{x}{E}_{abcd}=\overset{\nu}{E}_{abcd}= \begin{pmatrix} E_L^* & 0 & 0 & \nu_{TL}E_L^* \\ 0 & G_{LT} & G_{LT} & 0 \\ 0 & G_{LT} & G_{LT} & 0 \\ \nu_{LT}E_T^* & 0 & 0 & E_T^* \end{pmatrix}, E_{T,L}^* = \dfrac{E_{T,L}}{1-\nu_{LT}\nu TL}$

further $N_x = -F, N_y = 0, N_{xy} = 0, p = 0$

Thus we get $D_1 w_{,xxxx} + D_{12}w_{,xxyy} + D_2 w_{,yyyy} = -Fw_{,xx}$

where $D_1 = \dfrac{E_L^* h^3}{12}, D_{12} = \dfrac{\nu_{TL}E_L^* h^3}{12} + \dfrac{\nu_{LT}E_T^* h^3}{12} + \dfrac{4G_{LT}h^3}{12}, D_2 = \dfrac{E_T^* h^3}{12}$

Let as look for the solution using Fourier series expansion
$$w = \sum_{n,k=1}^{\infty} w_{nk} \sin\frac{n\pi x}{a} \sin\frac{k\pi y}{b}$$

Using in our Lamé equation

$$\sum_{n,k=1}^{\infty} w_{nk} \left(D_1 \left(\frac{n\pi}{a}\right)^4 + D_{12} \left(\frac{n\pi}{a}\right)^2 \left(\frac{k\pi}{b}\right)^2 + D_2 \left(\frac{k\pi}{b}\right)^4 - F \left(\frac{n\pi}{a}\right)^2 \right) \sin\frac{n\pi x}{a} \sin\frac{k\pi y}{b} = 0$$

As functions $\sin\frac{n\pi x}{a} \sin\frac{k\pi y}{b}$ are linearly independent, there are possible solutions

$$F_{nk} = \frac{D_1 \left(\frac{n\pi}{a}\right)^4 + D_{12} \left(\frac{n\pi}{a}\right)^2 \left(\frac{k\pi}{b}\right)^2 + D_2 \left(\frac{k\pi}{b}\right)^4}{\left(\frac{n\pi}{a}\right)^2}$$

The corresponding eigenmodes are

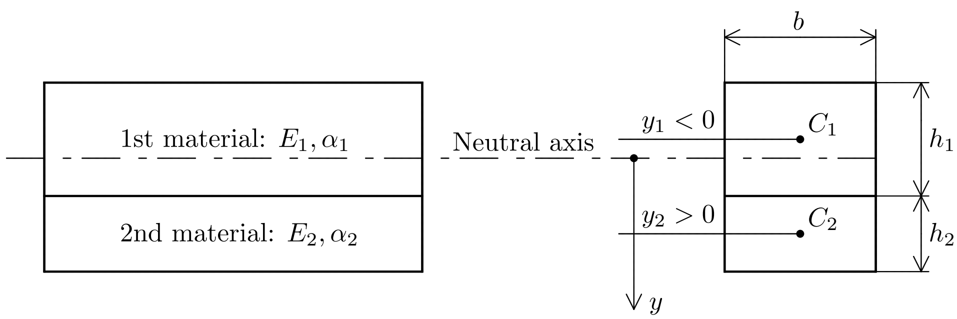
$$\sin\frac{n\pi x}{a} \sin\frac{k\pi y}{b}$$

25 Sandwich beam theory

26 Thermal deformation of simple composite beams

26.1 Bimetal–A beam made of two materials

Consider a beam made of two different materials unloaded by any force or external moment but undergoing a change in temperature (see the Fig.)



The material properties are described by the Young's modulus (E_1, E_2) and coefficient of thermal expansion (α_1, α_2).
As the beam is unloaded by external forces the overall internal normal force, N, and bending moment, M_b, are zero:

$$N = 0, M_b = 0$$

Let us suppose that the Bernoulli's hypothesis holds:

$$\varepsilon = ky$$

where k is the curvature and y the coordinate.
The strain can be decomposed into its elastic and thermal parts:

$$\varepsilon = \varepsilon_{\text{elastic}} + \varepsilon_{\text{thermal}} = \frac{\sigma}{E} + \alpha\Delta T$$

That gives

$$\sigma = Eky - E\alpha\Delta T$$

For the normal force we then have

$$N = \int_A \sigma \, \mathrm{d}A = \int_{A_1} \sigma_1 \, \mathrm{d}A + \int_{A_2} \sigma_2 \, \mathrm{d}A$$

$$N = \int_{A_1} E_1 ky \, \mathrm{d}A - \int_{A_1} E_1\alpha_1\Delta T \mathrm{d}A + \int_{A_2} E_2 ky \, \mathrm{d}A - \int_{A_2} E_2\alpha_2\Delta T \mathrm{d}A$$

i.e.

$$N = E_1 k Q_1 - E_1\alpha_1\Delta T A_1 + E_2 k Q_2 - E_2\alpha_2\Delta T A_2$$

where Q_1 and Q_2 are the first moment of area of the cross-section of the 1st and 2nd material with respect to the Neutral axis, respectively.
For the bending moment we can write

$$M_b = \int_A y\sigma \, \mathrm{d}A = \int_{A_1} y\sigma_1 \, \mathrm{d}A + \int_{A_2} y\sigma_2 \, \mathrm{d}A$$

$$M_b = \int_{A_1} E_1 ky^2 \, \mathrm{d}A - \int_{A_1} yE_1\alpha_1\Delta T \mathrm{d}A + \int_{A_2} E_2 ky^2 \, \mathrm{d}A - \int_{A_2} yE_2\alpha_2\Delta T \mathrm{d}A$$

i.e.

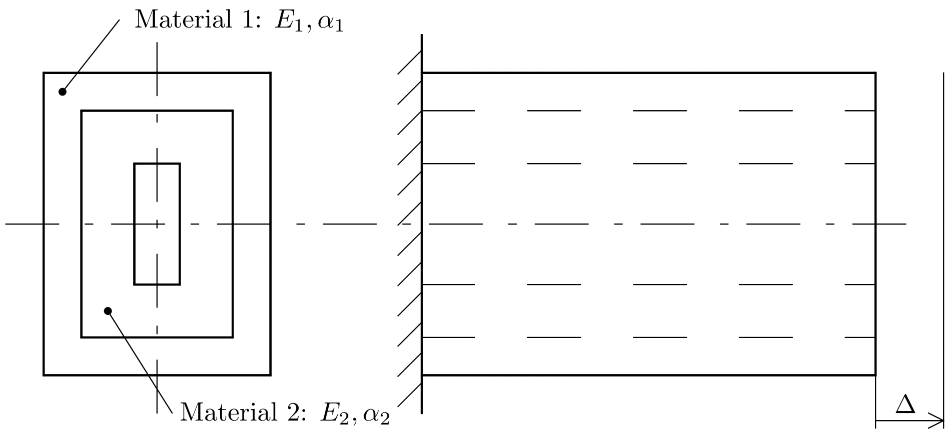
$$M_b = E_1 k I_1 - E_1\alpha_1\Delta T Q_1 + E_2 k I_2 - E_2\alpha_2\Delta T Q_2$$

where I_1 and I_2 are second moment of area with respect to the Neutral axis of the respective areas.
As $N = 0$ and $M_b = 0$ we have the conditions fixing the position of the Neutral axis and the curvature, k, which in the case of the rectangular cross-section gives

$$k = \frac{6E_1E_2(h_1+h_2)h_1h_2(\alpha_1-\alpha_2)\Delta T}{E_1^2h_1^4 + 4E_1E_2h_1^3h_2 + 6E_1E_2h_1^2h_2^2 + 4E_1E_2h_1h_2^3 + E_2^2h_2^4}$$

26.2 A two material beam with doubly symmetric cross-section

Let us study the thermal deformation of a two material beam with doubly symmetric cross section with the aim to design a beam without thermal change in its length.



The hypothesis is that the displacement, Δ, is constant across the cross-section and consequently the strain, ε, is constant along the whole body:

$$\varepsilon = \varepsilon_{\text{elastic}} + \varepsilon_{\text{thermal}} = \frac{\sigma}{E} + \alpha\Delta T = \text{a constant}$$

The internal normal force, N, is zero as there are not external forces applied:

$$N = \int_A \sigma \, \mathrm{d}A = \int_{A_1} \sigma_1 \, \mathrm{d}A + \int_{A_2} \sigma_2 \, \mathrm{d}A$$

$$N = \int_{A_1} E_1(\varepsilon - \alpha_1 \Delta T) \, \mathrm{d}A + \int_{A_2} E_2(\varepsilon - \alpha_2 \Delta T) \, \mathrm{d}A = 0$$

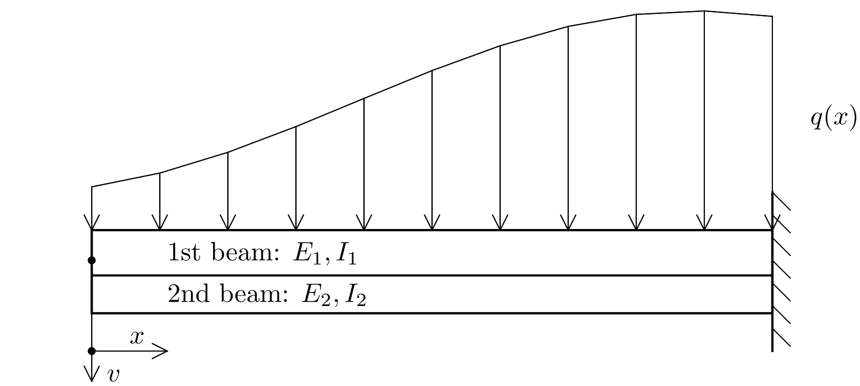
Consequently,

$$\varepsilon = \frac{\alpha_1 E_1 A_1 + \alpha_2 E_2 A_2}{E_1 A_1 + E_2 A_2} \Delta T$$

As there are carbon fibres with a negative coefficient of thermal expansion it is possible to arrange the dimensions and composition of the beam in such a way that the fraction vanishes and the beam has a zero thermal expansion.

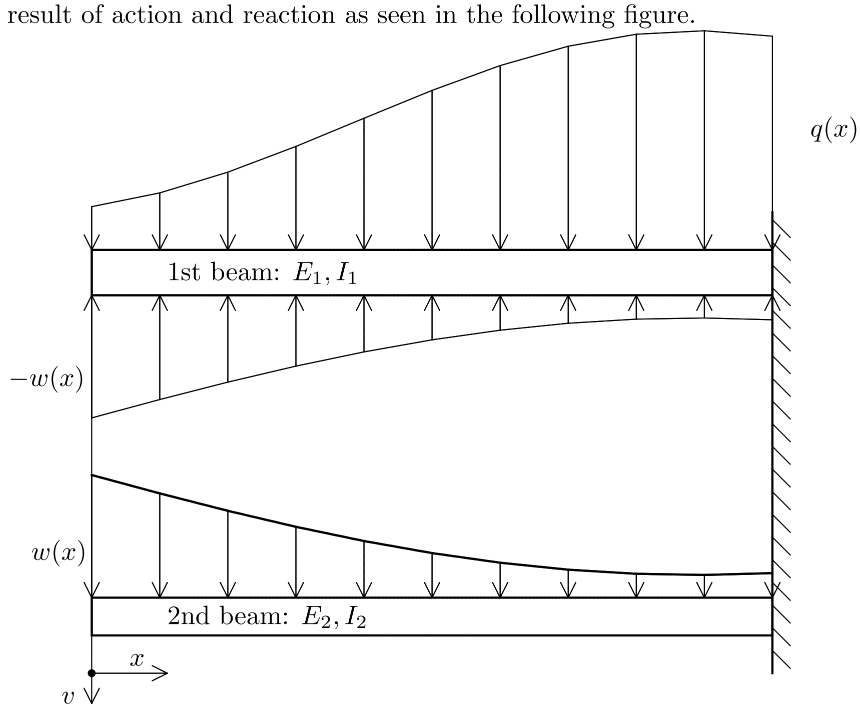
27 Deformation of loaded beams made of two parallel parts

Let us consider a beam composed of two parallel beams as shown at the Figure:



27.1 Unbound case

First, consider the case of free conection, *i.e.* the case when the two parts can freely slice on each orther surface. In this case we can regard it as two Bernoulli beams with an identical displacements and an additional distributed load as a result of action and reaction as seen in the following figure.



For the two beams we have two equilibrium equations (valid for Bernoulli's hypothesis and constant EI along the length of the beam)

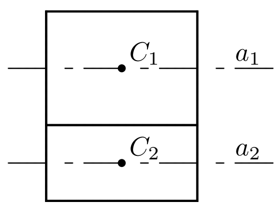
$$E_1 I_1 v_1^{IV} = q - w \tag{1}$$

$$E_2 I_2 v_2^{IV} = w \tag{2}$$

and compatibility conditions

$$v_1 = v_2 = v \text{ and, consequently, } v_1^{IV} = v_2^{IV} \tag{3}$$

where v_1 and v_2 are the displacements, E_1 and E_2 are the Young's moduli, and I_1 and I_2 are the second moments of area of the upper and lower beam, respectively.



C_1 is the centre of the cross-sectional area of the first beam

I_1 is the second moment of the first beam's area with respect to the axis a_1

C_2 is the centre of the cross-sectional area of the first beam

I_2 is the second moment of the first beam's area with respect to the axis a_2

Using the equations of equilibrium (5 and 7) in the compatibility condition (3) gives

$$\frac{q - w}{E_1 I_1} = \frac{w}{E_2 I_2}$$

i.e.

$$w = q \frac{E_2 I_2}{E_1 I_1 + E_2 I_2} \tag{4}$$

As $q > 0 \Rightarrow w > 0$ there is not a gap between the two beams.
Inserting w given by (4) into either (5) or (7) leads to

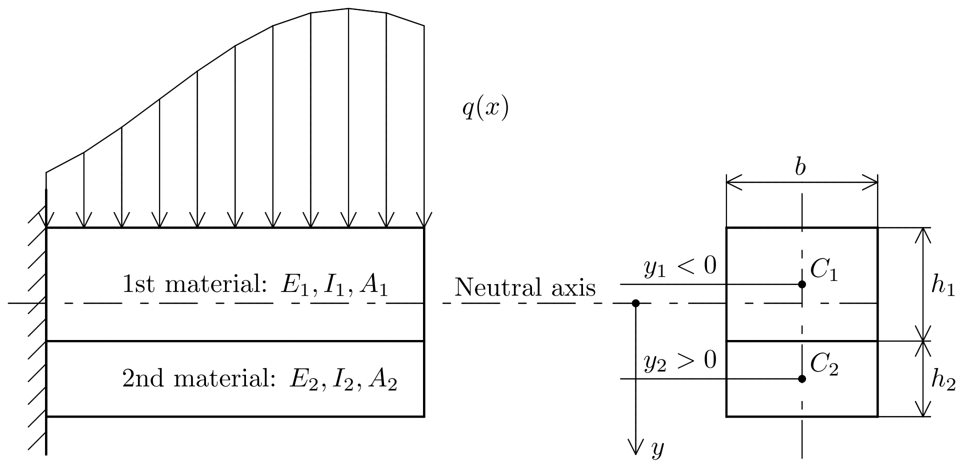
$$v^{IV} = \frac{q}{(EI)_{\text{eq}}}$$

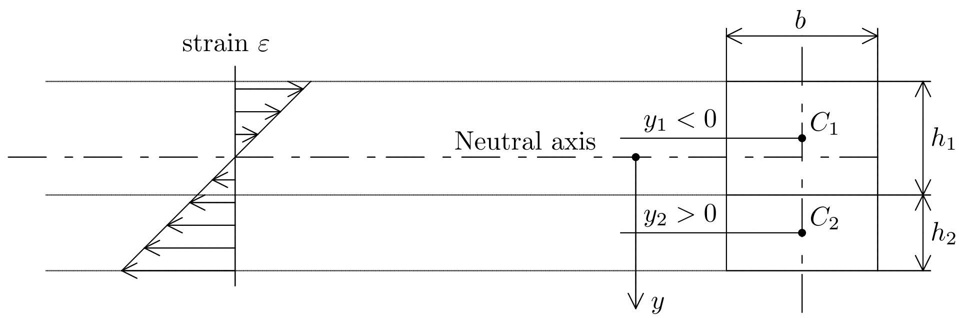
where

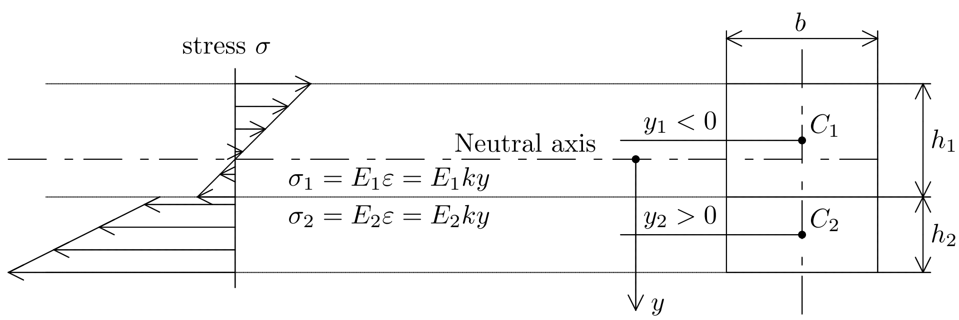
$$(EI)_{\text{eq}} = E_1 I_1 + E_2 I_2$$

27.2 Ideally bound case

Now, let us consider the same beam but ideally bounded together. Once more we assume the Bernoulli hypothesis, $\varepsilon = ky$, only this time for the whole beam with one common Neutral axis that is not generally passing through the centroid of the cross-section.







The position of the neutral axis is given by the fact that the resultant axial force, N, is zero due to the chosen supports:

$$N = \int_{A_1} \sigma_1 \, \mathrm{d}A + \int_{A_2} \sigma_2 \, \mathrm{d}A = 0$$

where A_1 and A_2 is the cross-sectional area of material 1 and 2, respectively. Thus

$$E_1 \int_{A_1} y \, \mathrm{d}A + E_2 \int_{A_2} y \, \mathrm{d}A = 0$$

i.e.

$$E_1 Q_1 + E_2 Q_2 = 0$$

where Q_1 and Q_2 is the first moment of the cross-sectional area 1 and 2 with respect to the Neutral axis, respectively. The last equation gives the origin of coordinate y.

Moment-curvature relationship is based on the expression of the bending moment as an integral of elementary bending moments

$$M_b = \int_{A_1} y\sigma_1 \, \mathrm{d}A + \int_{A_2} y\sigma_2 \, \mathrm{d}A = kE_1 \int_{A_1} y^2 \, \mathrm{d}A + kE_2 \int_{A_2} y^2 \, \mathrm{d}A$$

that is

$$M_b = k(E_1 J_1 + E_2 J_2)$$

where

$$J_1 = I_1 + y_1^2 A_1$$

is the second moment of the cross-sectional area A_1 with respect to the Neutral axis and

$$J_2 = I_2 + y_2^2 A_2$$

is the second moment of the cross-sectional area A_2 with respect to the same Neutral axis.

In the case of small deformations the curvature can be approximated by $k = -v''$ and the differential equation for deflection is

$$v'' = -\frac{M_b}{(EJ)_{\mathrm{eq}}}$$

where the equivalent stiffness

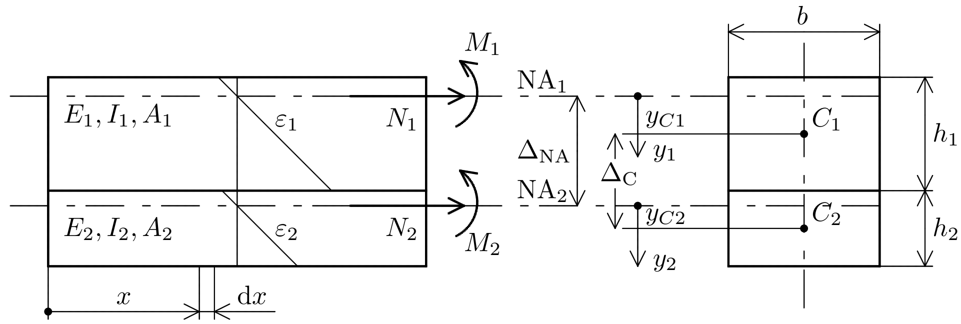
$$(EJ)_{\mathrm{eq}} = E_1 J_1 + E_2 J_2 = E_1(I_1 + y_1^2 A_1) + E_2(I_2 + y_2^2 A_2)$$

Using the curvature k in the stress formulas gives

$$\sigma_1 = M_b \frac{E_1}{(EJ)_{\mathrm{eq}}} y$$

$$\sigma_2 = M_b \frac{E_2}{(EJ)_{\mathrm{eq}}} y$$

27.3 Elastically bound case



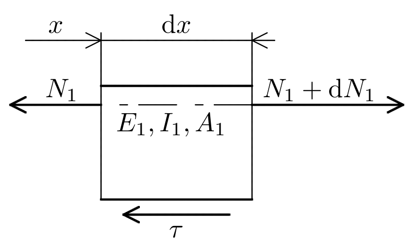
Let us assume that both parts of the beam obey the Bernoulli's hypothesis

$$\varepsilon_1 = ky_1 \text{ and } \varepsilon_2 = ky_2$$

where the curvature, k, is the same for both parts as for small deformations $k = -v''$ and the deflection, v, is supposed to be the same for both parts. The coordinates y_1 and y_2 originates at the two respective neutral axes, NA_1 and NA2, as seen at the Figure above.
The connection between the two parts is assumed to be elastic with the shear stress at the interface given by

$$\tau = g(\varepsilon_1^{\text{interface}} - \varepsilon_2^{\text{interface}})$$

where g is a spring constant and $\varepsilon_a^{\text{interface}}$ $(a = 1, 2)$ are the respective normal strains at the two parts at the point of the interface.
Let us cut two elements of the length $\mathrm{d}x$. First, from the upper part of the beam.



The equilibrium equation

$$\mathrm{d}N_1 - \tau b \mathrm{d}x = 0$$

where b is a width of the beam at the interface

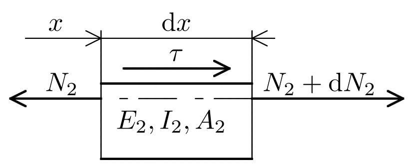
The internal normal force, N_1 is given by integrating the stress, $\sigma_1 = E_1\varepsilon_1$, over the cross-sectional area of the upper part of the beam, A_1

$$N_1 = \int_{A_1} \sigma_1 \,\mathrm{d}A = E_1 k \int_{A_1} y_1 \,\mathrm{d}A = E_1 k Q_1$$

where $Q_1 = y_{C1} A_1$ is the first moment of the area A_1 with respect to the neutral axis of the upper part of the beam, NA_1.
Using N_1 in the equilibrium equation leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}(k y_{C1}) E_1 A_1 - bgk(y_1^{\text{interface}} - y_2^{\text{interface}}) = 0 \tag{5}$$

where note that y_{C1} (as well as y_{C2} and $\Delta_{\mathrm{NA}} = y_1^{\text{interface}} - y_2^{\text{interface}}$) is a function of x, *i.e.*, the neutral axis is not, generally, at the same place at every cross-section.
Now, cut the element of the lower part:



The equilibrium equation

$$\mathrm{d}N_2 + \tau b \mathrm{d}x = 0$$

The two equilibrium equations imply $N_1 = -N_2$ that is

$$E_1 A_1 y_{C1} + E_2 A_2 y_{C2} = 0 \tag{6}$$

Also using

$$N_2 = \int_{A_2} \sigma_2 \,\mathrm{d}A = E_2 k \int_{A_2} y_2 \,\mathrm{d}A = E_2 k Q_2$$

where $Q_2 = y_{C2} A_2$, in the last equilibrium equation gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(k y_{C2}) E_2 A_2 + bgk\Delta_{\mathrm{NA}} = 0 \tag{7}$$

As $N_1 = -N_2$ there is a force couple, $N_2 \Delta_{\mathrm{NA}}$, adding to the resulting **bending moment**

$$M_b = M_{b1} + M_{b2} + N_2 \Delta_{\mathrm{NA}}$$

Using the stress expression above leads to

$$M_b = E_1 k \int_{A_1} y_1^2 \,\mathrm{d}A + E_2 k \int_{A_2} y_2^2 \,\mathrm{d}A + E_2 k y_{C2} A_2 \Delta_{\mathrm{NA}}$$

and

$$M_b = k(E_1 J_1 + E_2 J_2 + E_2 y_{C2} A_2 \Delta_{\mathrm{NA}}) \tag{8}$$

where

$$J_1 = I_1 + y_{C1}^2 A_1 \text{ and } J_2 = I_2 + y_{C2}^2 A_2$$

I_1 and I_2 being the second moments of area with respect to the two principal central axis.
There is also a geometric condition (as seen in the Figure)

$$y_{C1} + \Delta_C - y_{C2} = \Delta_{\mathrm{NA}} \tag{9}$$

There are five equations, four of them linearly independent, (5–9) for four unknown functions, k, y_{C1}, y_{C2} and Δ_{NA}.