Dynamic Analysis of the Elastic Composite in the Shape of the Human Heart

L. Jiran, M. Stefan, T. Mares

***Abstract*—This work deals with a dynamic deformation analysis of the thick-walled closed anisotropic model whose geometry and cross-section are motivated by human heart shape. The oscillations of the model are solved using the action integral, EulerLagrange equations for the action integral and the singular value decomposition. The action integral is determined using the semi-analytical method, anisotropic elasticity in curvilinear coordinates and the Fourier series expansion.**

***Index Terms*—dynamic analysis, action integral, oscillation, heart, anisotropic.**

I. Introduction

THIS report builds on the previous articles [1], [2] which describe initial steps of our planned work to create an anisotropic model of a human heart and to determine its time dependant deformations. In [1], there are our motivation, aims, and related works of other authors briefly discussed. Article [2] describes the closed anisotropic model whose geometry and cross-section are motivated by human heart shape and presents its static deformation analysis. The same model is used in this report but its dynamic deformation analysis is included. Results of dynamic analysis are presented for free oscillations and for forced oscillations separately, no damping is considered. The linear elastic small deformation behavior of the modeled body is used for the analysis again.

II. Model

The closed model whose is used as the approximation of the real geometry of the human heart is described in detail in [2] and will be only briefly mentioned in this section.

The heart geometry is modeled via two ellipsoids and six coordinate systems are introduced for its description, Fig. 1. The local Cartesian coordinate system ν^i is aligned with the direction of the material local ortotrophy and the global circular coordinate system η^i is used as the global computational coordinate system. Whole deformation analysis is performed in this coordinate system.

Relations between the coordinate systems, ranges of coordinates, and metrics are described in [2].

L. Jiran and T. Mares are with the Department of Mechanics, Biomechanics and Mechatronics, Faculty of Mechanical Engineering, Czech Technical University in Prague, Technicka 4, 166 07 Prague 6, Czech Republic, e-mail: lukas.jiran@fs.cvut.cz, tomas.mares@fs.cvut.cz

M. Stefan is with the MBtech Bohemia s.r.o., Narozni 1400/7, CZ-158 00 Praha 13 - Stodulky, e-mail: marek.stefan@mbtech-group.com.

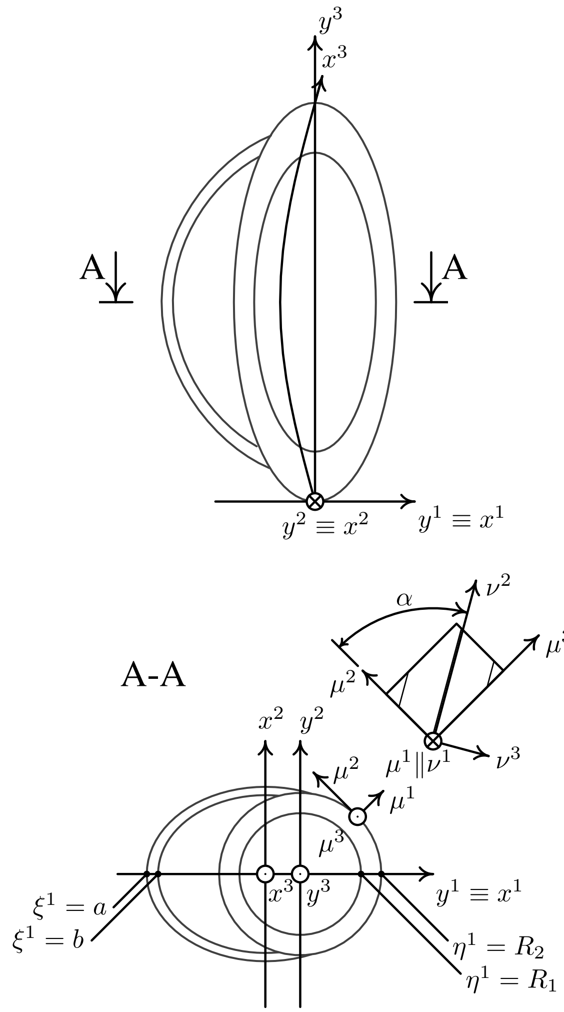

Fig. 1. The coordinate systems.

III. Oscillations

Equations of motion are derived from the action integral, which has the form

$$S = \int_0^t (T - \Pi - L)\,\mathrm{d}t, \tag{1}$$

where T is the kinetic energy of the model

$$T = \frac{1}{2} \int_\Omega \rho\ \dot{u}_k \dot{u}^k \mathrm{d}\bar{\Omega}, \tag{2}$$

where ρ is a density and u_k are the displacement functions in the global computational coordinate system η^i, which are approximated by a Fourier series expansion

$$\begin{aligned}\overset{\eta}{u}_1 &= \sum_{hkm=-K}^{K} A_1^{hkm} v_1,\\ \overset{\eta}{u}_2 &= \sum_{hkm=-K}^{K} A_2^{hkm} v_2,\\ \overset{\eta}{u}_3 &= \sum_{hkm=-K}^{K} A_3^{hkm} v_3. \qquad (3)\end{aligned}$$

The parameter K is chosen 3 so that each displacement function is approximated by $7^3 = 343$ members of the Fourier series expansion and (3) can be written in the form

$$\begin{pmatrix} \overset{\eta}{u}_1 \\ \overset{\eta}{u}_2 \\ \overset{\eta}{u}_3 \end{pmatrix} = NA, \qquad (4)$$

where[1]

$$N = \begin{pmatrix} v_1 & \mathtt{zeros}(1,343) & \mathtt{zeros}(1,343) \\ \mathtt{zeros}(1,343) & v_2 & \mathtt{zeros}(1,343) \\ \mathtt{zeros}(1,343) & \mathtt{zeros}(1,343) & v_3 \end{pmatrix}, \qquad (5)$$

with v_1, v_2, v_3 defined by

$$v_1 = v_2 = v_3 = e^{\mathrm{i}h2\pi\frac{\eta^1 - R_1}{c - R_1}} e^{\mathrm{i}k\eta^2} e^{\mathrm{i}m\eta^3 \frac{2\pi}{l}} \eta_1 + e^{\mathrm{i}m\eta^3 \frac{2\pi}{l}}. \qquad (6)$$

Column vector A in (4) the vector of the coefficients to be determined

$$A = \begin{pmatrix} A_1^{hkm} \\ A_2^{hkm} \\ A_3^{hkm} \end{pmatrix}. \qquad (7)$$

For more information about the displacement functions $\overset{\eta}{u}_k$ and their approximation by the Fourier series expansion see [2].

For the time derivative $\dot{u}^k$ and for the contravariant components u^k we can write

$$\begin{aligned}\dot{u}^k &= N\dot{A},\\ u^k &= g^{kl}\ u_l, \qquad (8)\end{aligned}$$

with

$$g^{kl} = G = (g_{kl})^{-1}. \qquad (9)$$

The element of the volume in (2)

$$\mathrm{d}\bar{\Omega} = |\ \overset{\eta}{g}_{ij}\ |^{\frac{1}{2}} \mathrm{d}\eta^1 \mathrm{d}\eta^2 \mathrm{d}\eta^3. \qquad (10)$$

Hence, form (2) is modified

$$T = \frac{1}{2} \dot{A}^T M \dot{A}, \qquad (11)$$

where the new mass matrix

$$M = \rho \int_{\overset{\eta}{\Omega}} N^T G N |\ \overset{\eta}{g}_{ij}\ |^{\frac{1}{2}} \mathrm{d}\eta^1 \mathrm{d}\eta^2 \mathrm{d}\eta^3. \qquad (12)$$

The second term in (1) is the total potential energy of the model

$$\Pi(\overset{\eta}{u}_i) = U(\overset{\eta}{u}_i) - W(\overset{\eta}{u}_i), \qquad (13)$$

[1]Typewrite font is used for the MATLAB syntax.

with the elastic strain energy

$$U(\overset{\eta}{u}_i) = \frac{1}{2} A^{\mathrm{T}} K A, \qquad (14)$$

and with the work of the applied forces

$$W(\overset{\eta}{u}_i) = PA. \qquad (15)$$

Matrix K in (14) is the stiffness matrix of the whole model and vector P in (15) represents works done by the internal pressures. Terms K and P are described in more detail in [2].

The third term $L(\overset{\eta}{u}_i)$ in (1) is the linear combination of the left-hand sides of the following constraints: the model is fixed

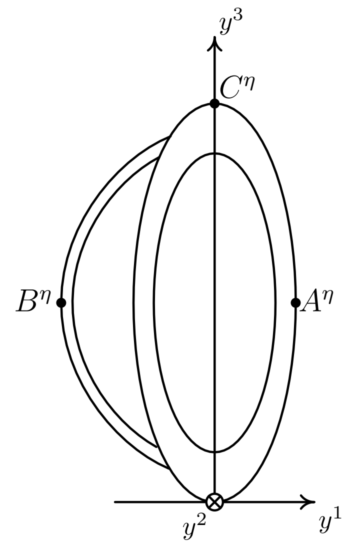

Fig. 2. The boundary conditions.

at three points, see Fig. 2, and the boundary conditions can be written in the form

$$\begin{pmatrix} \overset{\eta}{u}_{1(A)} \\ \overset{\eta}{u}_{2(A)} \\ \overset{\eta}{u}_{3(A)} \\ \overset{\eta}{u}_{2(B)} \\ \overset{\eta}{u}_{3(B)} \\ \overset{\eta}{u}_{2(C)} \end{pmatrix} = \Lambda A = 0, \qquad (16)$$

where

$$\Lambda = \begin{pmatrix} v_{1(\eta_A)} & \mathtt{zeros}(1,343) & \mathtt{zeros}(1,343) \\ \mathtt{zeros}(1,343) & v_{2(\eta_A)} & \mathtt{zeros}(1,343) \\ \mathtt{zeros}(1,343) & \mathtt{zeros}(1,343) & v_{3(\eta_A)} \\ \mathtt{zeros}(1,343) & v_{2(\eta_B)} & \mathtt{zeros}(1,343) \\ \mathtt{zeros}(1,343) & \mathtt{zeros}(1,343) & v_{3(\eta_B)} \\ \mathtt{zeros}(1,343) & v_{2(\eta_C)} & \mathtt{zeros}(1,343) \end{pmatrix}, \qquad (17)$$

with v_1, v_2, v_3 defined by (6). Hence, the linear combination of the left-hand sides of the constraints

$$L(\overset{\eta}{u}_i) = \lambda^T \Lambda A, \qquad (18)$$

where λ is the vector of Lagrange multipliers.

Substitution (11), (14), (15) and (18) in equation (1) leads to the form

$$S = \int_0^t \left(\frac{1}{2}\dot{A}^T M \dot{A} - \frac{1}{2} A^T K A + P A - \lambda^T \Lambda A \right) \mathrm{d}t. \quad (19)$$

This equation represents the functional

$$S = \int_0^t \mathcal{L}\left(t,\ A,\ \dot{A},\ \lambda\right) \mathrm{d}t, \quad (20)$$

and the famous necessary conditions of the action minimum

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{A}} - \frac{\partial \mathcal{L}}{\partial A} = 0, \quad (21)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\lambda}} - \frac{\partial \mathcal{L}}{\partial \lambda} = 0. \quad (22)$$

The equations of motion are, *in extenso* (21),

$$M\ddot{A} + KA + \Lambda^T \lambda = P, \quad (23)$$

and, *in extenso* (22), leads to algebraic equations which specify constraints

$$\Lambda A = 0. \quad (24)$$

Equations (23) and (24) create together the system of differential-algebraic equations

$$\begin{aligned} M\ddot{A} + KA + \Lambda^T \lambda &= P, \\ \Lambda A &= 0. \end{aligned} \quad (25)$$

Employing the singular value decomposition of matrix Λ we transform the problem into quasi-coordinates and thus we get rid of algebraic equations.

We can write in the MATLAB syntax

$$[\mathtt{a}, \mathtt{b}, \mathtt{c}] = \mathrm{svd}(\mathtt{\Lambda}), \quad (26)$$

which corresponds to

$$\mathtt{\Lambda} = \mathtt{a} * \mathtt{b} * \mathtt{c}'. \quad (27)$$

$\mathtt{\Lambda}$ is the 6-by-1029 matrix, matrix $\mathtt{b}$ has the same dimensions as $\mathtt{\Lambda}$, matrix $\mathtt{a}$ is the square matrix 6-by-6 and matrix $\mathtt{c}$ is the square matrix 1029-by-1029. When we use only the columns of matrix $\mathtt{c}$ which correspond to zero singular values (it means columns 7–1029 in our case), we get 1029-by-1023 matrix $\tilde{V}$ and the constraints (24) can be transformed to the form

$$A = \tilde{V}\alpha. \quad (28)$$

Equation (28) brings a new vector of unknown coefficients α and application of this equation in (23) changes the system of differential-algebraic equations (25) to the system of differential equations. The constraints are included in matrix $\tilde{V}$ and hence we can modify the action integral (19) to the form

$$\tilde{S} = \int_0^t \left(\frac{1}{2}\dot{\alpha}^T \tilde{M} \dot{\alpha} - \frac{1}{2}\alpha^T \tilde{K} \alpha + \tilde{P}\alpha \right) \mathrm{d}t, \quad (29)$$

where

$$\begin{aligned} \tilde{M} &= \tilde{V}^T M \tilde{V}, \\ \tilde{K} &= \tilde{V}^T K \tilde{V}, \\ \tilde{P} &= P\tilde{V}. \end{aligned}$$

Equation (29) represents the functional

$$\tilde{S} = \int_0^t \tilde{\mathcal{L}}\left(t,\ \alpha,\ \dot{\alpha}\right) \mathrm{d}t, \quad (30)$$

and the necessary conditions

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\alpha}} - \frac{\partial \tilde{\mathcal{L}}}{\partial \alpha} = 0 \quad (31)$$

lead to the system of differential equations

$$\tilde{M}\ddot{\alpha} + \tilde{K}\alpha = \tilde{P} \quad (32)$$

for the new unknown vector α.

IV. Free oscillations

This section describes how to find a solution to the problem of free oscillations. The free oscillations means that no external force acts on the model hence in (32) is $\tilde{P} = 0$.

A. *Equations of motion*

Equations of motion are a set of linear homogeneous differential equations with constant coefficients

$$\tilde{M}\ddot{\alpha} + \tilde{K}\alpha = 0. \quad (33)$$

Solution is expected in the form

$$\alpha = a e^{\mathrm{i}\Omega t}, \quad (34)$$

where a is a constant vector to be determined whose is called the amplitude of the oscillations, Ω is the frequency of the oscillations and t denotes time. Substitution (34) in the equation (33) and cancelation of $e^{i\Omega t}$ leads to a set of linear homogeneous algebraic equations

$$\begin{aligned} -\tilde{M}\Omega^2 a e^{\mathrm{i}\Omega t} + \tilde{K} a e^{\mathrm{i}\Omega t} &= 0, \\ \left(\tilde{K} - \tilde{M}\Omega^2\right) a &= 0. \end{aligned} \quad (35)$$

We obtain a non-trivial solution of this system when the determinant vanishes

$$\det |\tilde{K} - \Omega^2 \tilde{M}| = 0, \quad (36)$$

where Ω is the characteristic frequency or eigenfrequency of the system.

Equation (36) is called characteristic equation and can be solved by using MATLAB function

$$[\mathtt{u}, \mathtt{v}] = \mathrm{eig}(\tilde{\mathtt{K}}, \tilde{\mathtt{M}}), \quad (37)$$

where $[u]$ is the matrix of eigenvectors and $[v]$ is the matrix which has on its diagonal eigenvalues $\lambda = \Omega^2$. The eigenvectors are determined for up to a multiple constant and therefore are normalized. The non-normed eigenvector is denoted a_ν, the normed eigenvector is denoted u_ν and has to fulfil the condition

$$u_\nu \tilde{M} u_\nu = 1. \tag{38}$$

The vectors are normed according the rule

$$u_\nu = \frac{a_\nu}{\sqrt{a_\nu^T \tilde{M} a_\nu}} \tag{39}$$

and than are stored in matrix U.

The solution (the ν-th solution) of the linear homogeneous differential equations (33)

$$\alpha_\nu = u_\nu \left(B_\nu \cos(\Omega_\nu t) + C_\nu \sin(\Omega_\nu t) \right) \tag{40}$$

and all solutions are briefly written in the form

$$\alpha = U \left(\cos_t B + \sin_t C \right), \tag{41}$$

where B, C are the column vectors of arbitrary constants and

$$\begin{aligned} \cos_t &= \begin{pmatrix} \cos(\Omega_1 t) & 0 & \dots \\ 0 & \cos(\Omega_2 t) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \\ \sin_t &= \begin{pmatrix} \sin(\Omega_1 t) & 0 & \dots \\ 0 & \sin(\Omega_2 t) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \tag{42}$$

The constants B and C are found from the initial conditions. The initial conditions α_0 and $\dot{\alpha}_0$ are known

$$\begin{aligned} \alpha_0 &= \texttt{pinv}(\tilde{\texttt{V}})\texttt{A}_0, \\ \dot{\alpha}_0 &= \texttt{zeros}(1023, 1), \end{aligned} \tag{43}$$

where MATLAB function `pinv()` performs a pseudoinversion of matrix $\tilde{\texttt{V}}$ and A_0 is the column vector of the coefficients, which was determined in [2] as the solution of the static analysis. For $\dot{\alpha}_0$ we have

$$\dot{\alpha}_0 = U \left(-\Omega \sin_t B + \Omega \cos_t C \right), \tag{44}$$

where

$$\Omega = \begin{pmatrix} \Omega_1 & 0 & \dots \\ 0 & \Omega_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{45}$$

Equations (41) and (44) give us for $t = 0$

$$\begin{aligned} \alpha_0 &= UB, \\ \dot{\alpha}_0 &= U\Omega C, \end{aligned} \tag{46}$$

and using

$$U^T \tilde{M} U = I, \tag{47}$$

leads to a determinations of the arbitrary constant B, C in (41)

$$\begin{aligned} U^T \tilde{M} \alpha_0 &= U^T \tilde{M} U B \\ \Rightarrow\ B &= U^T \tilde{M} \alpha_0, \\ U^T \tilde{M} \dot{\alpha}_0 &= U^T \tilde{M} U \Omega C \\ \Rightarrow\ C &= \Omega^{-1} U^T \tilde{M} \dot{\alpha}_0. \end{aligned} \tag{48}$$

The solution

$$\alpha = U \left(\cos_t U^T \tilde{M} \alpha_0 + \sin_t \Omega^{-1} U^T \tilde{M} \dot{\alpha}_0 \right), \tag{49}$$

and the coefficients to be determined

$$A = \tilde{V} \alpha. \tag{50}$$

If the coefficients A are known, the displacement functions u_k are determined by (4) and the deformed shape of the model can be plotted.

B. Results of the free oscillations

The results (Fig. 3 - Fig. 8) of the free oscillations are demonstrated for time interval $t \in \langle 0, 0.06 \rangle$ whose represents a half period of the oscillations.

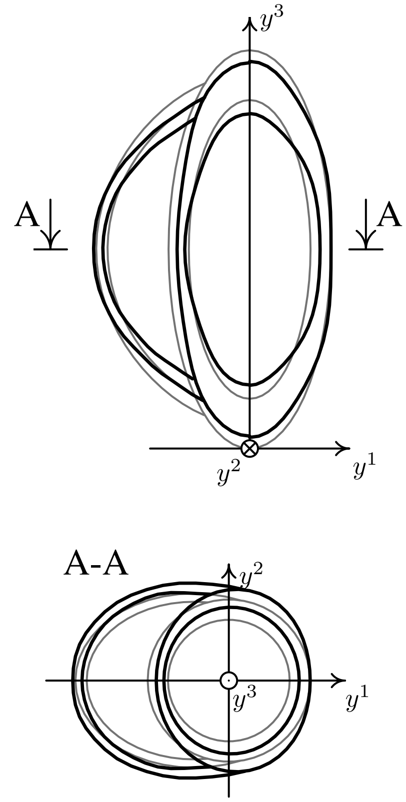

Fig. 3. Free oscillations, $t = 0$.

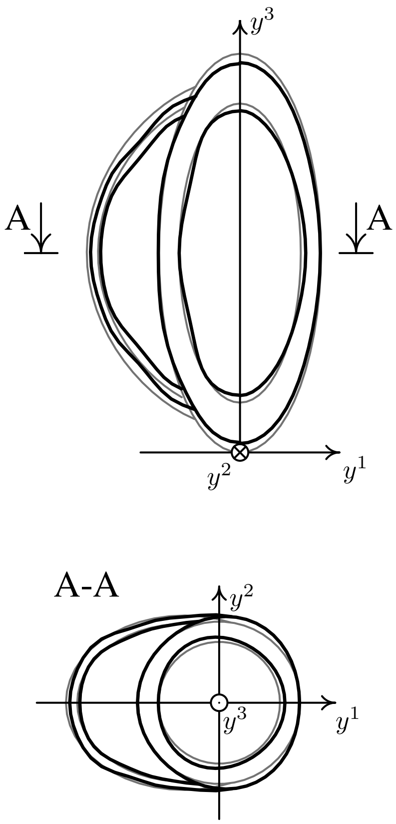

Fig. 4. Free oscillations, $t = 0.01$.

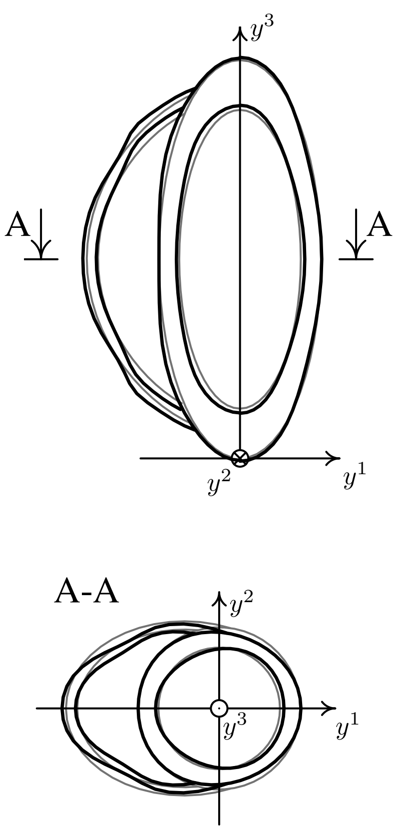

Fig. 5. Free oscillations, $t = 0.02$.

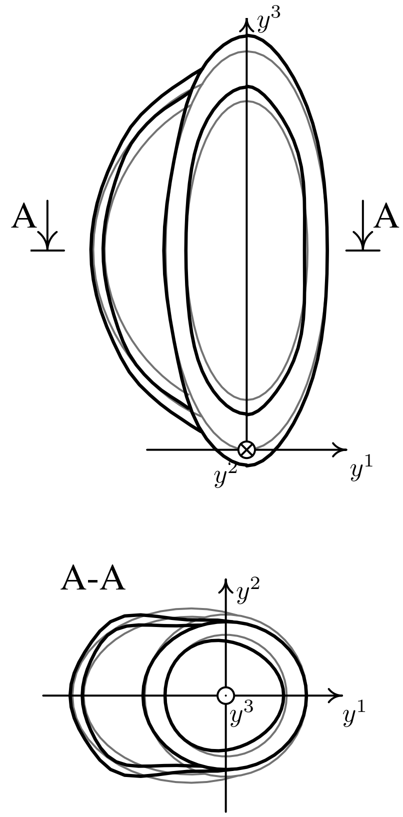

Fig. 6. Free oscillations, $t = 0.03$.

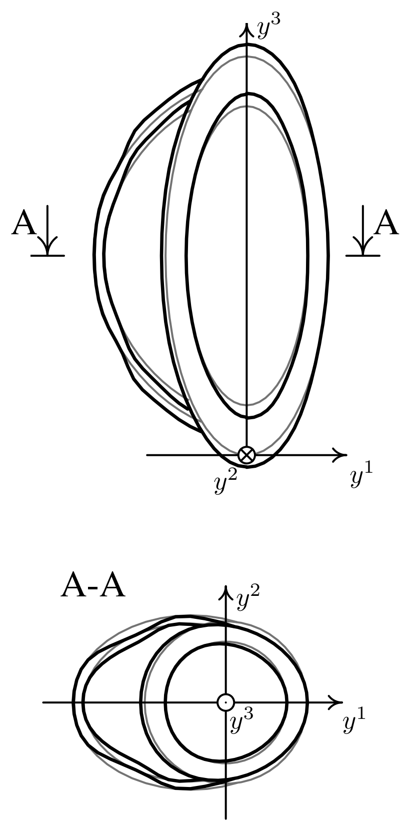

Fig. 7. Free oscillations, $t = 0.04$.

V. Forced oscillations

A. Equations of motion

Let us consider a case where the external forces themselves are simple periodic functions of time of frequency ω. Motion of our system on which a variable external forces act is described

$$\tilde{M}\ddot{\alpha} + \tilde{K}\alpha = \tilde{P}e^{\mathrm{i}\omega t}. \tag{51}$$

The general solution of this inhomogeneous linear differential equations with constant coefficients is

$$\alpha = \alpha_H + \alpha_P, \tag{52}$$

where α_H is the general solution of the corresponding homogeneous equation and α_P is a particular integral of the inhomogeneous equation. The general solution α_H represents the free oscillations

$$\alpha_H = U\left(\cos_t D + \sin_t E\right). \tag{53}$$

The particular integral α_P is estimated in the form

$$\begin{aligned} \alpha_P &= re^{\mathrm{i}\omega t} \\ \ddot{\alpha_P} &= -\omega^2 re^{\mathrm{i}\omega t}. \end{aligned} \tag{54}$$

Substitution (54) in (51) gives us

$$\begin{aligned} -\tilde{M}\omega^2 re^{\mathrm{i}\omega t} + \tilde{K}re^{\mathrm{i}\omega t} &= \tilde{P}re^{\mathrm{i}\omega t}, \\ \left(\tilde{K} - \tilde{M}\omega^2\right) r &= \tilde{P}, \end{aligned} \tag{55}$$

$$r = \left(\tilde{K} - \tilde{M}\omega^2\right)^{-1} \tilde{P} \tag{56}$$

and the general solution (52) is

$$\alpha = U\left(\cos_t D + \sin_t E\right) + \left(\tilde{K} - \tilde{M}\omega^2\right)^{-1} \tilde{P}e^{\mathrm{i}\omega t}. \tag{57}$$

The arbitrary constants D and E are found from the initial conditions again,

$$\begin{aligned} \dot{\alpha} = U\left(-\Omega\sin_t D + \Omega\cos_t E\right) + \\ + \mathrm{i}\omega\left(\tilde{K} - \omega^2\tilde{M}^{-1}\right)\tilde{P}e^{\mathrm{i}\omega t} \end{aligned} \tag{58}$$

and for $t = 0$

$$\begin{aligned} \alpha_0 &= UD + \left(\tilde{K} - \tilde{M}\omega^2\right)^{-1} \tilde{P}, \\ \dot{\alpha}_0 &= U\Omega E + \mathrm{i}\omega\left(\tilde{K} - \tilde{M}\omega^2\right)^{-1} \tilde{P}. \end{aligned} \tag{59}$$

Using (47) in (59) leads to

$$D = U^T\tilde{M}\alpha_0 - U^T\tilde{M}\left(\tilde{K} - \tilde{M}\omega^2\right)^{-1} \tilde{P}, \tag{60}$$

and

$$\begin{aligned} E = \Omega^{-1}U^T\tilde{M}\dot{\alpha}_0 + \\ + \mathrm{i}\omega\Omega^{-1}U^T\tilde{M}\left(\tilde{K} - \tilde{M}\omega^2\right)^{-1} \tilde{P}. \end{aligned} \tag{61}$$

Finally, the general solution is

$$\begin{aligned} \alpha = U\cos_t\left(U^T\tilde{M}\alpha_0 - U^T\tilde{M}\left(\tilde{K} - \tilde{M}\omega^2\right)^{-1}\tilde{P}\right) + \\ + U\sin_t\Omega^{-1}U^T\tilde{M}\dot{\alpha}_0 + \\ + U\sin_t\left(\mathrm{i}\omega\Omega^{-1}U^T\tilde{M}\left(\tilde{K} - \tilde{M}\omega^2\right)^{-1}\tilde{P}\right) + \\ + \left(\tilde{K} - \tilde{M}\omega^2\right)^{-1}\tilde{P}e^{\mathrm{i}\omega t}, \end{aligned} \tag{62}$$

where

$$\begin{aligned} \alpha_0 &= \mathtt{zeros}(1023, 1), \\ \dot{\alpha}_0 &= \mathtt{zeros}(1023, 1). \end{aligned} \tag{63}$$

The coefficients to be determined A are given by (50).

B. Results

The set of figures (Fig. 9 - Fig. 15) shows the results of the forced oscillation when the variable force $f = \tilde{P}e^{\mathrm{i}\omega t}$ acts on the model. The results are shown for time interval $t \in \langle 0, 0.06\rangle$.

VI. Conclusion

This report describes the dynamic analysis of the thick-walled closed anisotropic model whose geometry and cross-section are motivated by the human heart shape and presents the results of this analysis. The equations of the motion were derived by using the principle of the action minimum.

Free oscillations of the model are analyzed for the case where an initial internal pressures load causes initial deformations. The load is than removed and the model oscillates without damping.

Forced oscillations are forced by external periodic forces which represent time variable internal pressures acting in the cardiac chambers.

We are going to improve our model using the viscoelastic material in the next step of our planned work to perform viscoelasto dynamic analysis of the considered model.

Acknowledgment

The authors would like to thank MSM 6840770012 and GAČR 101/08/H068.

References

[1] L. Jiran, K. Doubrava, M. Stefan, and T. Mareš. Analysis of the Heart Motivated Anisotropic Elastic Tube. *Bulletin of Applied Mechanics*, 6(23):57–65, 10 2010.

[2] L. Jiran and T. Mareš. Analysis of the Elastic Composite in the Shape of the Human Heart. *Accepted for publication in Bulletin of Applied Mechanics*.

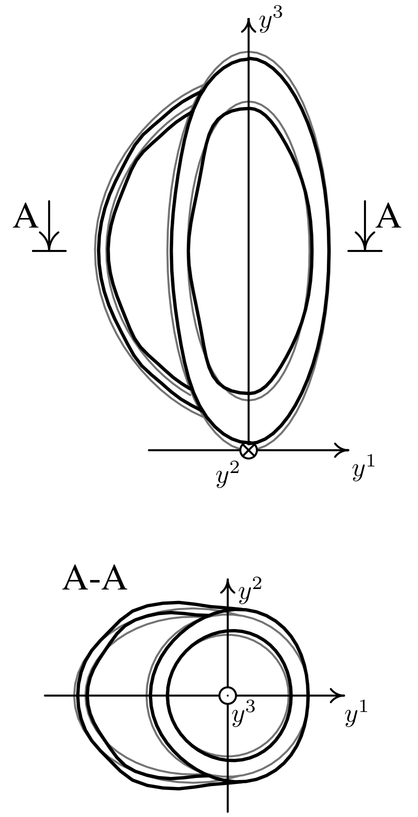

Fig. 8. Free oscillations, $t = 0.06$.

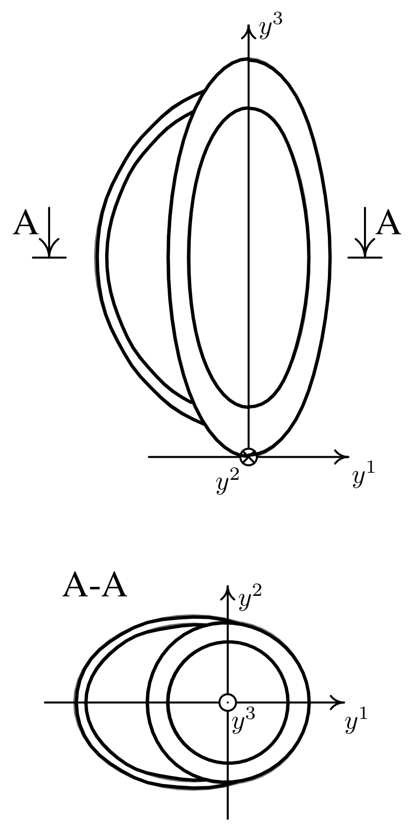

Fig. 9. Forced oscillations, $t = 0$.

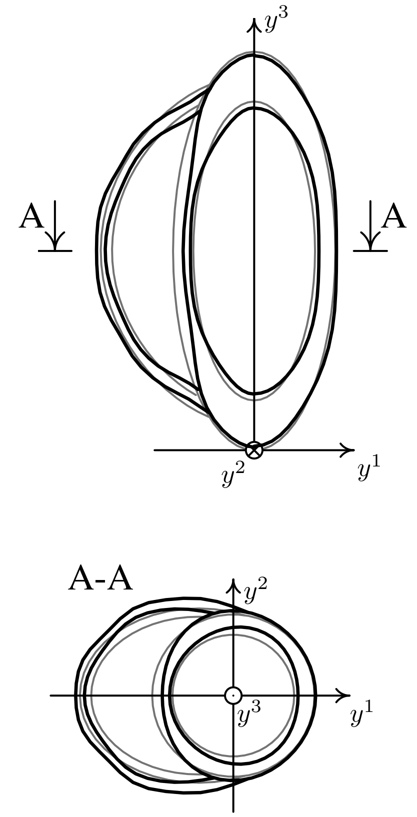

Fig. 10. Forced oscillations, $t = 0.01$.

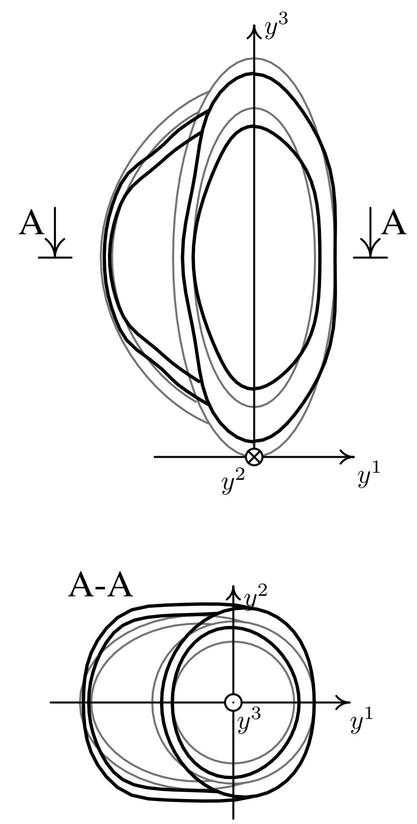

Fig. 11. Forced oscillations, $t = 0.02$.

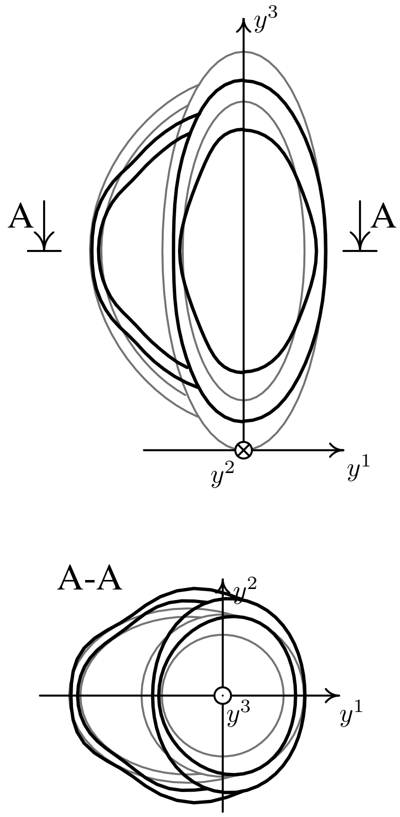

Fig. 12. Forced oscillations, $t = 0.03$.

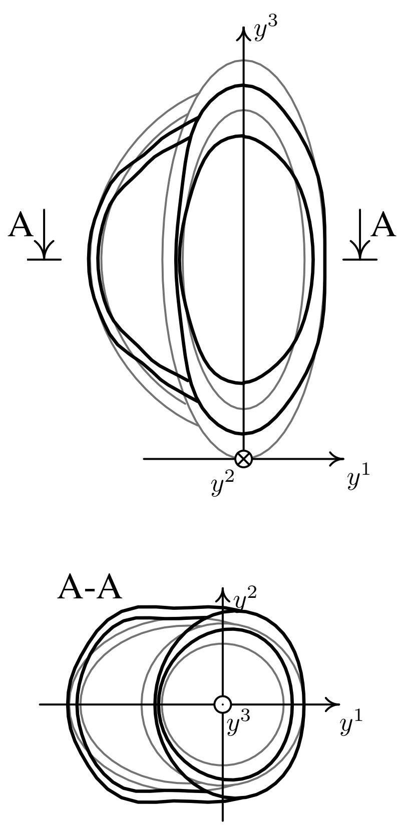

Fig. 13. Forced oscillations, $t = 0.04$.

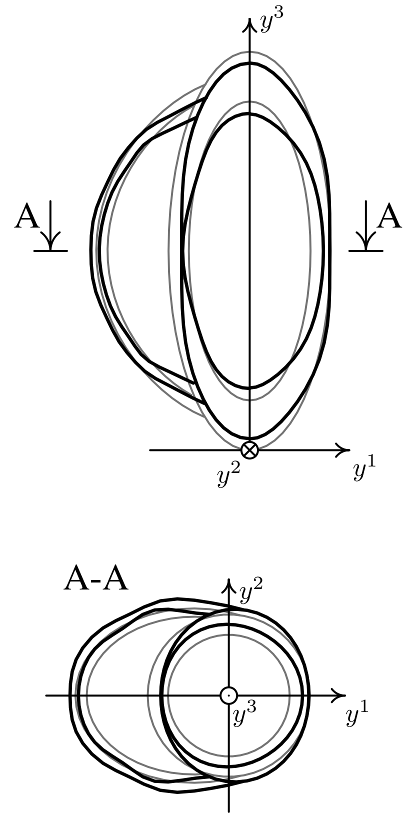

Fig. 14. Forced oscillations, $t = 0.06$.

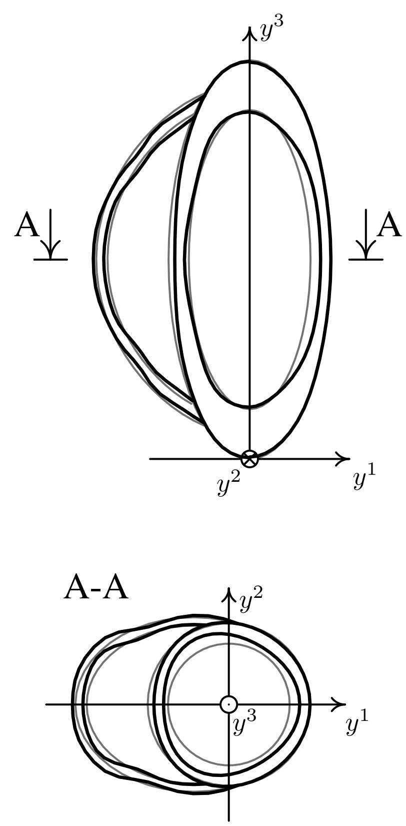

Fig. 15. Forced oscillations, $t = 0.06$.