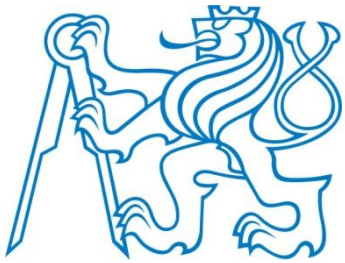
České vysoké učení technické v Praze

Fakulta strojní



DIPLOMOVÁ PRÁCE

Analysis of the Composite Beam Bending

Analýza ohybu kompozitních nosníků

2015 Tereza ZAVŘELOVÁ

Anotační list

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Název DP:	Analýza ohybu kompozitních nosníků
Anglický název:	Analysis of composite beam bending
Rok:	2015
Obor studia:	Aplikovaná mechanika
Ústav/odbor:	12 105 Ústav mechaniky, biomechaniky a mechatroniky/12 105.1 Odbor pružnosti a pevnosti
Vedoucí:	doc. Ing. Tomáš Mareš, Ph.D.
Bibliografické údaje:	počet stran: 112 počet obrázků: 63 počet tabulek: 2 počet příloh: 5
Klíčová slova:	kompozit, laminát, nosník, ohyb
Keywords:	composite, laminate, beam, deflection

Anotace:

Práce se zabývá porovnáním metod pro výpočet ohybu kompozitních nosníků. Srovnáváme výsledky výpočtů provedených pomocí Bernoulliho metody, metody výpočtu matice ABD a modelů MKP řešených pomocí klasické a objemové skořepiny i pomocí objemového modelu. Výsledkem celé práce je porovnání použitých metod a vznik programů pro výpočet ohybu v MATLABu a MKP modelů.

Abstract:

The work presents a comparison of methods for calculating the composite beams bending. We compare the results of calculations performed using the Bernoulli's method, method of calculation using ABD matrix and FEM models base on the conventional shell, the continuum shell and on the volume model. The results of the thesis is the comparison of the used methods and programs for calculating the beam deflection designed in MATLAB® and the FEM models.

Prohlašuji, že jsem svou diplomovou práci vypracovala zcela samostatně a výhradně s použitím literatury uvedené v seznamu na konci práce.

V Praze 19.6.2015 podpis:……

Poděkování:

Úvodem této práce bych chtěla poděkovat doc. Ing. Tomášovi Marešovi, Ph.D. za trpělivé a podnětné připomínky k mé práci v průběhu celého semestru.

Dále bych ráda poděkovala své rodině za veškerou podporu, díky níž mi studium umožnila.

TZ

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List of symbols

symbol	unit	name
$\boldsymbol{A}$	$N.m^{-1}$	extensional stiffness matrix
A	m^2	area
A_f	m^2	area of the fibre
A_m	m^2	area of the matrix
A_{ij}	$N.m^{-1}$	element of extensional stiffness matrix
$\boldsymbol{b}$	m	width
$\boldsymbol{B}$	$N.m^{-1}$	bending-extension coupling stiffness matrix
B_{ij}	$N.m^{-1}$	element of bending-extension coupling stiffness matrix
$\boldsymbol{C}$	Pa^{-1}	compliance matrix
$\boldsymbol{C_{xy}}$	Pa^{-1}	compliance matrix in the plane xy
d	mm	inner diameter
$\boldsymbol{D}$	N	bending stiffness matrix
D	mm	external diameter
D_{ij}	N	element of bending stiffness matrix
$\boldsymbol{E}$	Pa	modul of elasticity
E_1, E_2, E_3	Pa	module sof elasticitz in the direction 1,2,3
E_{eq}	Pa	equivalent modulus of elasticity
E_f	Pa	modulus of elasticity of the fibre
E_L	Pa	longitudinal modulus of elasticity
E_m	Pa	modulus of elasticity of the matrix
E_T	Pa	transversal modulus of elasticity
$E_{T'}$	Pa	transversal modulus of elasticity in the direction T'
E_x	Pa	modulus of elasticity in direction of the x-axis
F	N	force
$\boldsymbol{G}$	Pa	shear modulus
G_{12}, G_{13}, G_{23}	Pa	shear modulus in directions 12,13,23
G_{eq}	Pa	equivalent shear modulus
G_{xy}	Pa	shear modulus in the plane xy
G_{LT}	Pa	shear modulus in the plane LT
$G_{LT'}$	Pa	shear modulus in the plane LT'
$G_{TT'}$	Pa	shear modulus in the plane TT'
h_k	m	height
i, j, k		general indices
$\boldsymbol{J_y}$	m^4	moment of inertia in direction y
$\boldsymbol{J_z}$	m^4	moment of inertia in direction z
$\boldsymbol{k}$		vector of curvature of the midplane of the laminate
$\boldsymbol{k_x}, \boldsymbol{k_y}, \boldsymbol{k_{xy}}$		elements of vector of curvature of the midplane of the laminate
L		longitudinal direction
L	m	lenght

symbol	unit	name
m_O	N.m	moment of the dummy force
$\boldsymbol{M}$	N	vektor of resultant moments
M_x, M_y, M_{xy}	N	resultant moments in direction x, y, xy
$\boldsymbol{M_O}$	N.m	bending moment
$\boldsymbol{N}$	N.m^{-1}	vektor of resultant forces
N	-	number of layers
N_x, N_y, N_{xy}	N.m^{-1}	resultant forces in direction x, y, xy
N_1, N_2, N_3	N.m^{-1}	resultant forces
O_n		neutral axis
$\boldsymbol{Q}$	Pa	reduced stiffness matrix
Q_{ij}	Pa	element of reduced stiffness matrix
$\boldsymbol{r}$	m	diameter
$\boldsymbol{S}$	m^3	statical moment
$\boldsymbol{S}$	Pa	stifness matrix
$\boldsymbol{S_{xy}}$	Pa	stiffness matrix in the plane xy
t	m	thickness
$\boldsymbol{T}$	N	shear force
T		transverse direction
$\boldsymbol{T_\varepsilon}$		transformation matrix of the strain
$\boldsymbol{T_\sigma}$		transformation matrix of the stress
u_0, v_0, w_0	m	deflections in the directions x, y, z
$\boldsymbol{U}$	Pa	bending energy
U_{M_O}	Pa	bending energy from the moment
U_τ	Pa	bending energy from the shear
$\boldsymbol{v(x)}$		curve
$\boldsymbol{v}$	m	deflection
v_F	m	deflection under the force
$\boldsymbol{V}$	m^3	volume
V_f	-	volume of the fibre
V_m	-	volume of the matrix
W	m	width
x, y, z		directions of the axes
β		coefficient characterizing the unequal distribution of the shear stresses depending on the geometry of the cross section
$\boldsymbol{\gamma}$		shear deformation
γ_{xy}		shear deformation
γ°_{xy}		strain component of the midplane (from shear)
$\boldsymbol{\varepsilon}$		strain deformation
ε_f		strain of the fibre
ε_L		strain in direction L
ε_m		strain of the matrix
ε°_m		strain of the midplane
ε_T		strain in the transverse direction T
$\varepsilon_{T'}$		strain in the direction T'
ε_{LT}		strain in the direction LT

symbol	unit	name
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$		strain in the main directions
$\varepsilon^{\circ}{}_{xx}, \varepsilon^{\circ}{}_{yy}$		strain components of the midplane
κ		curvature
λ	Pa	density of deformation energy
v		Poisson's ratio
$\nu_{21}, \nu_{31}, \nu_{32}$		Poisson's ratio in main directions
ν_{LT}		Poisson's ratio in direction LT
ν_{TL}		Poisson's ratio in direction TL
$\nu_{LT'}$		Poisson's ratio in direction LT'
$\nu_{TT'}$		Poisson's ratio in direction TT'
θ	deg	angle of the fibres
$\boldsymbol{\sigma}$	Pa	stress
σ	Pa	stress
$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$	Pa	stresses in main directions
σ_f	Pa	stress in fiber
σ_L	Pa	stress in longitudinal direction
σ_m	Pa	stress in matrix
σ_T	Pa	stress in transverse direction
σ_{xx}, σ_{yy}	Pa	stress in direction x, y
σ_{xy}	Pa	shear stress in direction xy
$\boldsymbol{\tau}$	Pa	shear stress

Introduction

This thesis presents a comparison of methods for calculating the deflection of composite beams. The task of this thesis is to compare several methods of calculation of deflection composite beams. The objective is to compare of analytical methods with calculations made by using FEM. It compares the results of calculations performed using the Bernoulli's method, a method of calculation using ABD matrix and FEM models based on the conventional shell, the continuum shell and the volume models. The results will be used to determine the appropriate method to analyze a deformation of composite beams.

The work is created to facilitate the design of composite beams. It compares the known methods of the analysis of the deflection of composite beams for the different composition of the composite material. It is proved that the use of different calculation methods for the same composite material composition and the same geometry leads to different results. The objective of this work is to specify, which methods lead to comparable results with the experiment.

In this work, two programs designed in MATLAB® to calculate the deflection of any composite beams were created. Several models designed to calculate the deflection by FEM were created too. The comparison of all the mentioned methods yielded interesting results, which are presented in this thesis.

1 Mathematical Description of Fibre Composite Material

1.1 Description of Anisotropic Material

For anisotropic material, with general anisotropy (there is not a single plane of symmetry of elastic properties), both the stiffness matrix $\boldsymbol{S}$ and the compliance matrix $\boldsymbol{C}$ has 21 independent elements. Matrices are based on Hooke's law. [1],[6] In system $O(x_1, x_2, x_3)$ the Hooke's law is expressed as follows

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix}S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{bmatrix} . \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix}, \tag{1.1}$$

where $\boldsymbol{S}$ is a symmetric matrix. The formula can be rewritten as

$$\boldsymbol{\sigma} = \boldsymbol{S} \cdot \boldsymbol{\varepsilon} \ . \tag{1.2}$$

The equation can be expressed also in the inverse form

$$\boldsymbol{\varepsilon} = \boldsymbol{C} \cdot \boldsymbol{\sigma} \ . \tag{1.3}$$

Matrix $\boldsymbol{C}$ is also symmetric and it has a form

$$\boldsymbol{C} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix} . \tag{1.4}$$

From comparison of relations *(1.2)* and *(1.3)* follows

$$\boldsymbol{C} = \boldsymbol{S}^{-1} \ . \tag{1.5}$$

But this work will deal mainly with orthotropic or transversely isotropic materials; in those cases the numbers of independent variables are significantly reduced.

1.1.1 Orthotropic Material

Orthotropic material has three mutually perpendicular planes of symmetry of elastic properties. The stiffness matrix $\boldsymbol{S}$ (and also the compliance matrix $\boldsymbol{C}$) of orthotropic material contains only 9 independent elements.

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}. \tag{1.6}$$

When elastic modules are used and substituted to the compliance matrix $\boldsymbol{C}$, we obtain the relation

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}, \tag{1.7}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{C} \cdot \boldsymbol{\sigma} \ , \tag{1.8}$$

where E_1 , E_2, E_3, are modules of elasticity in the main directions of anisotropy;

G_{12} , G_{23}, G_{13} are shear modules in the planes parallel with the respective plane of symmetry of the elastic properties x_2x_3, x_1x_3, x_1x_2;

$\nu_{21}, \nu_{31}, \nu_{32}$ are Poisson's ratio, where the first index corresponds to the direction of the normal stress and the second direction which results in a corresponding deformation in the transverse direction.

Because the matrix S and C are symmetric matrices, these are the equalities between certain elements of the matrix

$$-\frac{\nu_{21}}{E_2} = -\frac{\nu_{12}}{E_1}; \ -\frac{\nu_{31}}{E_3} = -\frac{\nu_{13}}{E_1}; \ -\frac{\nu_{32}}{E_3} = -\frac{\nu_{23}}{E_2}. \tag{1.9}$$

From Hooke's law it is clear, that components of normal deformations are dependent only on components of normal stress and shear deformations are dependent only on shear components of stress. In this material, therefore, these shear and normal components are not tied. [6]

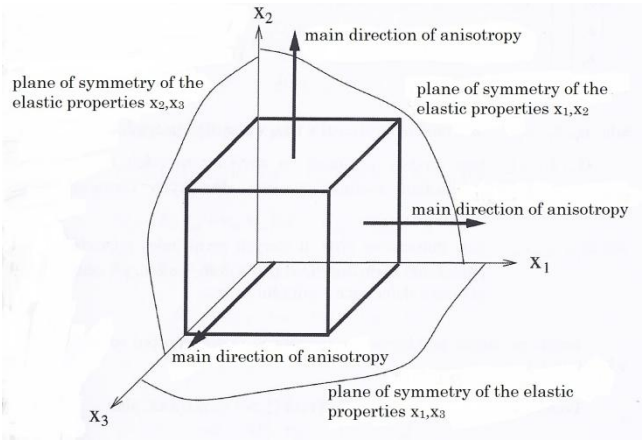


Figure 1.1: Orthotropic material [1]

1.1.2 Transversely Isotropic Material

It is a material, which has a plane of symmetry of the elastic properties. This plane is the same as a plane of isotropy, because the elastic properties in this plane in all directions are the same. [1] If we substitute material constants into compliance matrix $\boldsymbol{C}$, we get

$$\boldsymbol{C} = \begin{bmatrix} {}^{1}/_{E_1} & -{}^{\nu_{21}}/_{E_2} & -{}^{\nu_{31}}/_{E_3} & 0 & 0 & 0 \\ -{}^{\nu_{12}}/_{E_1} & {}^{1}/_{E_2} & -{}^{\nu_{32}}/_{E_3} & 0 & 0 & 0 \\ -{}^{\nu_{13}}/_{E_1} & -{}^{\nu_{23}}/_{E_2} & {}^{1}/_{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & {}^{1}/_{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & {}^{1}/_{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & {}^{1}/_{G_{12}} \end{bmatrix}. \tag{1.10}$$

Whereas the

E_1 is the modulus of elasticity in a direction perpendicular to the plane of isotropy;

$E_2 = E_3$ are modules of elasticity in the plane of isotropy;

$G_{12} = G_{13}$ are shear modules in direction perpendicular to the plane of isotropy;

$G_{23} = G_{32}$ are shear modules in the plane of isotropy;

$\nu_{12} = \nu_{21}$ are Poisson's ratios expressing the ratio shortening (elongation) in the plane of isotropy to elongation (shortening) in the main direction of anisotropy;

$\nu_{23} = \nu_{32}$ are Poisson's ratios in the plane of isotropy;

the matrix $\boldsymbol{C}$ can be written in a form

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(C_{22}-C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{1.11}$$

From the notation of matrix $\boldsymbol{C}$ it is obvious that this matrix has only five independent elements ($C_{11}, C_{12}, C_{22}, C_{23}, C_{66}$), therefore the number of independent material constants is also five ($E_1, E_2, G_{12}, \nu_{12}, \nu_{23}$). [1]

From the Hooke's law implies that the transversely isotropic material has no relation between the normal and shear components of stress and strain. [1]

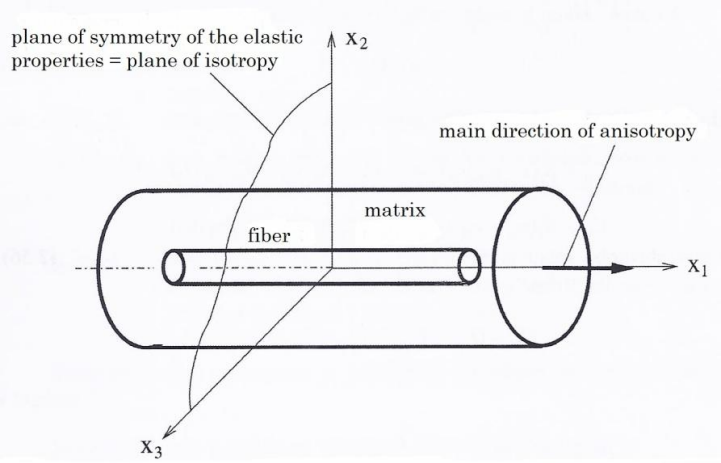


Figure 1.2: Transversely isotropic material [1]

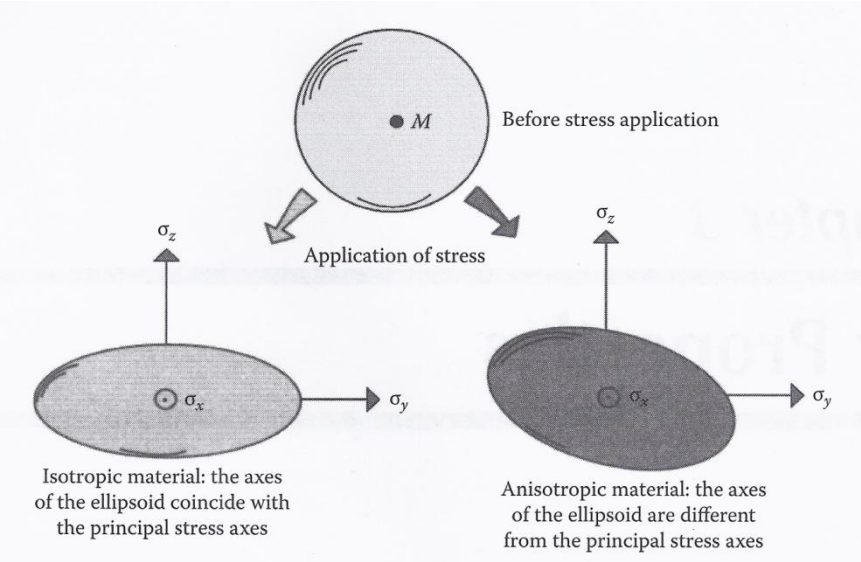


Figure 1.3: Schematic of deformation [5]

1.2 Modules of Elasticity

1.2.1 Longitudinal Modulus

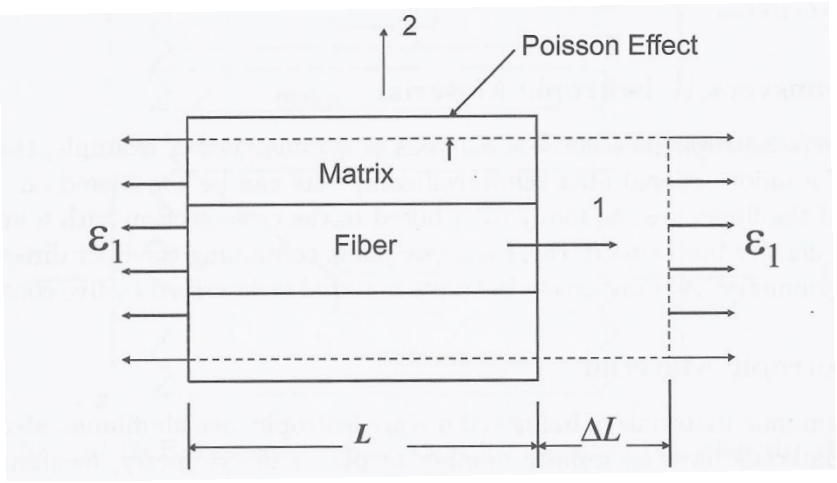


Figure 1.4: RVE subject to longitudinal uniform strain [4]

The assumption of the mathematical description of the composite material is that the two materials are bonded together. More concretely: matrix m and fiber f have the same longitudinal strain value noted ε_L. The main assumption in this formulation is that the strains in the direction of fibers are the same in the matrix and the fiber. This implies that the fiber-matrix bond is perfect. When the material is stretched along the fiber direction, the matrix m and the fibers f will elongate the same way as it is shown in the figure 1.4. This basic assumption is needed to be able to replace the heterogeneous material in the representative volume element (RVE) by a homogenous one. [4] The following derivation is based on this assumption.

By the definition of strains according to the figure 1.4

$$\varepsilon_1 = \frac{\Delta L}{L} \quad . \tag{1.12}$$

Both fiber and matrix are isotropic and elastic, the Hooke's law has a form for fibre f

$$\sigma_f = E_f \varepsilon_f \tag{1.13}$$

and for matrix m

$$\sigma_m = E_m \varepsilon_m \quad . \tag{1.14}$$

The stress σ can be expressed as the loading force F divided by the area where it acts

$$\sigma = \frac{F}{A} \quad . \tag{1.15}$$

So the average stress σ_1 in the composite material acts in the entire cross section of the RVE with area

$$A = A_f + A_m \quad , \tag{1.16}$$

where A_f is the area of the cross section of the fibre and A_m is the area of the cross section of the matrix.

The applied total load is

$$F = \sigma_1 A = \sigma_f A_f + \sigma_m A_m \quad . \tag{1.17}$$

Then

$$\sigma_1 = \varepsilon_1 \left(E_f V_f + E_m V_m \right) \quad , \tag{1.18}$$

where

$$V_f = A_f / A \ \ and \ V_m = A_m / A \quad . \tag{1.19}$$

For the equivalent homogeneous material the stress is expressed as

$$\sigma_1 = E_1 \varepsilon_1 \quad . \tag{1.20}$$

Then, comparing *(1.18)* with *(1.20),* it gives the result

$$E_1 = E_f V_f + E_m V_m \quad . \tag{1.21}$$

In the most cases, the modulus of the fibers is much larger than the modulus of the matrix, so the contribution of the matrix to the composite longitudinal modulus is negligible. This indicates that the longitudinal modulus E_1 is a fiber-dominated property.

1.2.2 Transverse Modulus

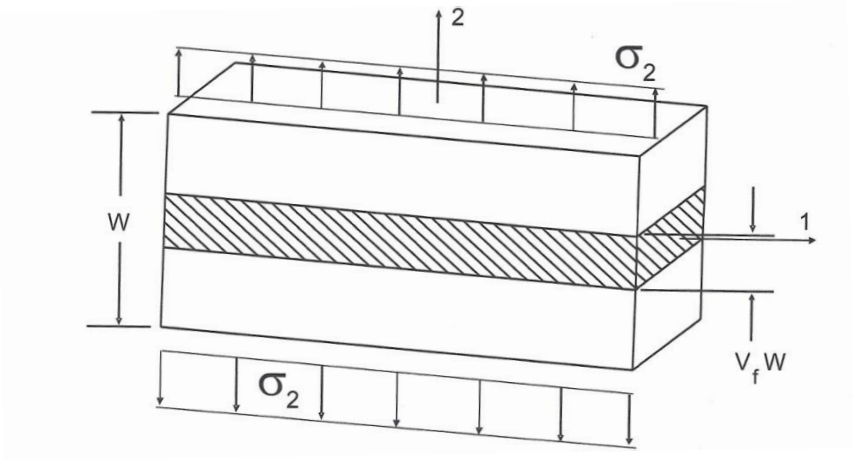


Figure 1.5: RVE subject to transverse uniform stress [4]

In the determination of the modulus in the direction transverse to the fibers, the main assumption is that the stress is the same in the fiber and the matrix. This assumption is needed to maintain equilibrium in the transverse direction. Once again, the assumption implies that the fiber-matrix bond is perfect. [4] The loaded RVE is in the figure 1.5.

The cylindrical fiber has been replaced by a rectangular one (fig. 1.5), this is for simplicity. Even micromechanics formulations do not represent the actual geometry of the fiber at all. Both the matrix and the fiber are assumed to be isotropic materials.

According to the situation in the figure 1.5, the stress in the matrix m and in the fiber f is the same

$$\sigma_2 = \sigma_f = \sigma_m \ , \tag{1.22}$$

so the strain is according to the Hooke's law for the fiber

$$\varepsilon_f = \frac{\sigma_2}{E_f} \tag{1.23}$$

and for the matrix

$$\varepsilon_m = \frac{\sigma_2}{E_m} \ . \tag{1.24}$$

These strains act over a portion of RVE; ε_f over $V_f W$, and ε_m over $V_m W$, while the average strain ε_2 acts over the entire width W. [4] The total elongation is

$$\varepsilon_2 W = \varepsilon_f \, V_f W + \varepsilon_m \, V_m W \ . \tag{1.25}$$

Cancelling W and again using Hooke's law for the constituents the relation is obtained

$$\frac{\sigma_2}{E_2} = \frac{\sigma_f}{E_f} V_f + \frac{\sigma_m}{E_m} V_m \quad . \tag{1.26}$$

Using the equation *(1.22)* it is obtained the relation for the transversal modulus E_2

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m} \quad . \tag{1.27}$$

It is evident from the figure 1.5 that the fibers do not contribute appreciably to the stiffness in the transverse direction, therefore it is said that E_2 is a matrix-dominated property. This is a simple equation and it can be used for qualitative evaluation of different candidate materials but not for design calculations. [4]

1.3 Stress and Deformation of Composite Material

Fiber reinforced composite is one of the most frequently used composite materials. Great use is mainly due to the variability of this material. The laminates usually consist of several layers of one-dimensional composite, wherein each layer is composed of fibers and matrix.

Stiffness of unidirectional composites is expressed by the same relationships, which are used for conventional materials (e.g. steel). The number of material constants is only increased. From the point of view of micromechanics it is possible to monitor tension only in the fiber or in the matrix. In this case, we compute in terms of macromechanics so we will consider tension across the whole layer of the laminate. This is called an intermediate stress in the layer.

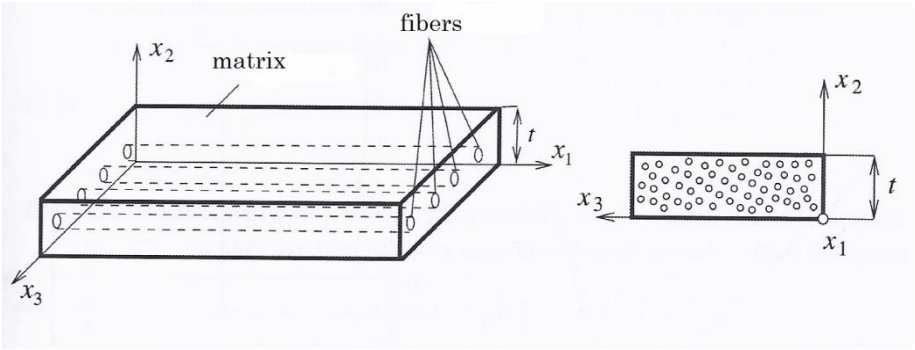


Figure 1.6: An example of the unidirectional composite material [1]

Such a composite material can be regarded as the orthotropic respectively transversely isotropic material. One-dimensional composite is represented in the coordinate system $O(x_1, x_2, x_3)$. Fibres are oriented in the direction of the axis x_1. The axis x_2 is perpendicular to the fibres. Often the coordinate system $O(L, T, T')$ is often used, where L means the longitudinal direction, T is the transverse direction and T' is the direction perpendicular to the lamina plane. Because the thickness of one lamina is much smaller than its width and length, it is possible to express the dependence between the stress and the deformation as in the case of the plane stress. This greatly simplifies the task and the results are close to reality. [1],[6]

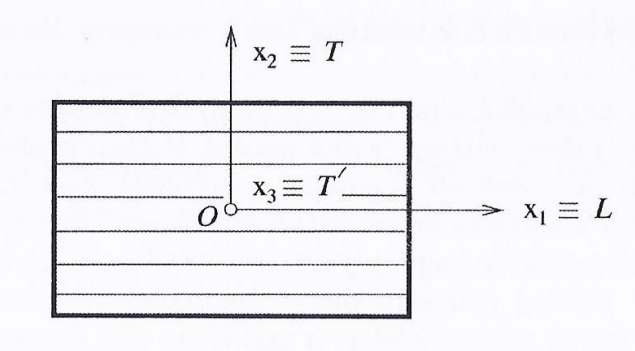


Figure 1.7: An example of the unidirectional composite in the coordinate system $O(L, T, T')$ [1]

The relation between stress and deformation is derived from assuming that the lamina is a linearly elastic material. Consider orthotropic lamina is loaded by tension $\boldsymbol{\sigma_L}$ in the fiber direction. Deformations are

$$\varepsilon_L = -\frac{1}{E_L} \cdot \sigma_L; \quad \varepsilon_T = -\frac{\nu_{LT}}{E_L} \cdot \sigma_L = -\nu_{LT} \cdot \varepsilon_L \quad , \tag{1.28}$$

where E_L is the longitudinal tensile modulus and ν_{LT} is a Poisson's ratio defined here.

In case of transversal tension the expressions are similar

$$\varepsilon_T = -\frac{1}{E_T} \cdot \sigma_T; \quad \varepsilon_L = -\frac{\nu_{TL}}{E_T} \cdot \sigma_T = -\nu_{TL} \cdot \varepsilon_T \quad , \tag{1.29}$$

where E_T is the transversal tensile modulus and ν_{TL} is the transversal Poisson's ratio.

For shear deformation we have

$$\varepsilon_{LT} = -\frac{1}{G_{LT}} \cdot \sigma_{LT} \quad , \tag{1.30}$$

where G_{LT} is shear modulus in the plane LT.

The superposition principle can be used. Then the stress components have the form

$$\varepsilon_L = -\frac{1}{E_L}\cdot\sigma_L - \frac{\nu_{TL}}{E_T}\cdot\sigma_T;\quad \varepsilon_T = -\frac{\nu_{LT}}{E_L}\cdot\sigma_L + \frac{1}{E_T}\cdot\sigma_T;\quad \varepsilon_{LT} = -\frac{1}{G_{LT}}\cdot\sigma_{LT}\ . \tag{1.31}$$

Component of deformation in the direction T' is for the case of the plane stress

$$\varepsilon_{T'} = -\frac{\nu_{LT'}}{E_L}\cdot\sigma_L - \frac{\nu_{TT'}}{E_T}\cdot\sigma_T\ , \tag{1.32}$$

where $\nu_{LT'}, \nu_{TT'}$ are transversal Poisson's ratios.

The above relations can be summarized into a matrix equation

$$\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \varepsilon_{T'}\\ 0\\ 0\\ \varepsilon_{LT}\end{bmatrix} = \begin{bmatrix} 1/E_L & -\nu_{TL}/E_T & -\nu_{T'L}/E_{T'} & 0 & 0 & 0\\ -\nu_{LT}/E_L & 1/E_T & -\nu_{T'T}/E_{T'} & 0 & 0 & 0\\ -\nu_{LT'}/E_L & -\nu_{TT'}/E_T & 1/E_{T'} & 0 & 0 & 0\\ 0 & 0 & 0 & 1/G_{TT'} & 0 & 0\\ 0 & 0 & 0 & 0 & 1/G_{LT'} & 0\\ 0 & 0 & 0 & 0 & 0 & 1/G_{LT}\end{bmatrix}\cdot\begin{bmatrix}\sigma_L\\ \sigma_T\\ 0\\ 0\\ 0\\ \sigma_{LT}\end{bmatrix}\ . \tag{1.33}$$

The compliance matrix $\boldsymbol{C}$ for orthotropic material then has a form

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\ . \tag{1.34}$$

Because the matrix $\boldsymbol{C}$ is symmetric, the following relations hold.

$$\frac{\nu_{TL}}{E_T} = \frac{\nu_{LT}}{E_L};\quad \frac{\nu_{T'L}}{E_{T'}} = \frac{\nu_{LT'}}{E_L};\quad \frac{\nu_{T'T}}{E_{T'}} = \frac{\nu_{TT'}}{E_T} \tag{1.35}$$

$$(C_{12} = C_{21};\quad C_{13} = C_{31};\quad C_{23} = C_{32})$$

As it is written in the introduction to this chapter, this is a case of the plane stress. The tension vector has only three non-zero components. Expression *(1.33)* can be rewritten as

$$\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \varepsilon_{LT}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0\\ C_{21} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\cdot\begin{bmatrix}\sigma_L\\ \sigma_T\\ \sigma_{LT}\end{bmatrix} \tag{1.36}$$

$$(\boldsymbol{\varepsilon} = \boldsymbol{C}\cdot\boldsymbol{\sigma})\ .$$

For the inverse of equation *(1.36)* one writes

$$\begin{bmatrix}\sigma_L\\ \sigma_T\\ \sigma_{LT}\end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0\\ S_{21} & S_{22} & 0\\ 0 & 0 & S_{66}\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \varepsilon_{LT}\end{bmatrix} \tag{1.37}$$

$$(\boldsymbol{\sigma} = \boldsymbol{S} \cdot \boldsymbol{\varepsilon}) \quad ,$$

where

$$\boldsymbol{S} = \boldsymbol{C}^{-1} \quad . \tag{1.38}$$

The elements specified stiffness matrix can be expressed by material constants E_L, E_T, v_{LT}, v_{TL} and G_{LT}. From these expressions it follows that for computation of stress only four independent constants E_L, E_T, v_{LT}, G_{LT} are needed.

$$S_{11} = \frac{E_L}{1 - v_{LT} \cdot v_{TL}} = \frac{E_L}{1 - \frac{E_T}{E_L} \cdot v_{LT}^2} \quad ; \tag{1.39}$$

$$S_{22} = \frac{E_T}{1 - v_{LT} \cdot v_{TL}} = \frac{E_T}{1 - \frac{E_T}{E_L} \cdot v_{LT}^2} = \frac{E_T}{E_L} \cdot S_{11} \quad ;$$

$$S_{12} = S_{21} = \frac{v_{LT} \cdot E_T}{1 - v_{LT} \cdot v_{TL}} = v_{LT} \cdot S_{22} \quad ;$$

$$S_{66} = G_{LT}$$

The specific property of unidirectional composites is their change of strength and stiffness depending on the direction in the plane $x_1 x_2$. It is necessary to transform stiffness quantities in different directions.

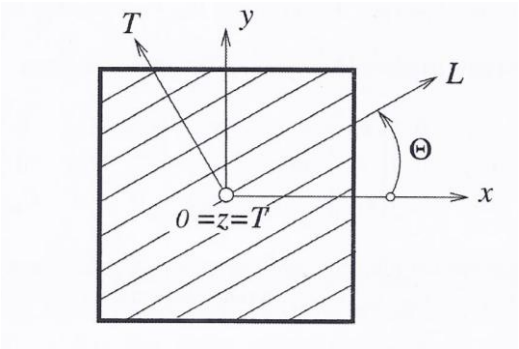


Figure 1.8: Unidirectional composite material in the two coordinate systems [1]

Figure 1.8 shows the unidirectional composite and two coordinate systems. The system $O(L, T, T')$ is rotated with respect to the system $O(x_1, x_2, x_3)$ by an angle θ around the axis $z \equiv T$. The formula for calculation of stress in the system $O(L, T, T')$ is

$$\boldsymbol{\sigma}' = \boldsymbol{T}_{\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma} \quad , \tag{1.40}$$

where $\boldsymbol{T}_{\boldsymbol{\sigma}}$ is a transformation matrix for the stress vector and $\boldsymbol{\sigma}$ is the stress vector in the coordinate system $O(x_1, x_2, x_3)$.

In 2D case the equation can be expressed in the form

$$\begin{bmatrix}\sigma_L\\ \sigma_T\\ \sigma_{LT}\end{bmatrix}=\begin{bmatrix}cos^2\theta & sin^2\theta & 2sin\theta cos\theta\\ sin^2\theta & cos^2\theta & -2sin\theta cos\theta\\ -sin\theta cos\theta & sin\theta cos\theta & cos^2\theta-sin^2\theta\end{bmatrix}\cdot\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}\ . \qquad (1.41)$$

A similar relation of course is applied for the transformation of strain

$$\boldsymbol{\varepsilon}' = \boldsymbol{T_\varepsilon}\cdot\boldsymbol{\varepsilon}\ , \qquad (1.42)$$

where $\boldsymbol{T_\varepsilon}$ is the transformation matrix for the strain vector. In components we get

$$\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \varepsilon_{LT}\end{bmatrix}=\begin{bmatrix}cos^2\theta & sin^2\theta & -sin\theta cos\theta\\ sin^2\theta & cos^2\theta & sin\theta cos\theta\\ 2sin\theta cos\theta & -2sin\theta cos\theta & cos^2\theta-sin^2\theta\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_L\\ \varepsilon_T\\ \varepsilon_{LT}\end{bmatrix}\ . \qquad (1.43)$$

In the previous paragraph it has been shown that the magnitude of stress and strain are dependent on the direction in which they are examined. It is seen that the stiffness matrix $\boldsymbol{S}$ and the compliance matrix $\boldsymbol{C}$ are not only dependent on materials constants, but also on the position of the selected coordinate system. We are looking for formulas of the stiffness matrix and the compliance matrix for system $O(x_1, x_2, x_3)$, which is rotated relatively to the system $O(L, T, T')$ by an angle $-\theta$. This is illustrated in the figure 1.8. The stiffness matrix and the compliance matrix in the system $O(x_1, x_2, x_3)$ are given by relations

$$\boldsymbol{S}' = \boldsymbol{T_\sigma^{-1}}\cdot\boldsymbol{C}\cdot\boldsymbol{T_\varepsilon}\ , \qquad (1.44)$$

$$\boldsymbol{C}' = \boldsymbol{T_\varepsilon^{-1}}\cdot\boldsymbol{S}\cdot\boldsymbol{T_\sigma}\ . \qquad (1.45)$$

The Hooke's law for this rotated system can be expressed in a matrix form

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ 0\\ 0\\ 0\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}S'_{11} & S'_{12} & S'_{13} & 0 & 0 & S'_{16}\\ S'_{21} & S'_{22} & S'_{23} & 0 & 0 & S'_{26}\\ S'_{31} & S'_{32} & S'_{33} & 0 & 0 & S'_{36}\\ 0 & 0 & 0 & S'_{44} & S'_{45} & 0\\ 0 & 0 & 0 & S'_{54} & S'_{55} & 0\\ S'_{61} & S'_{62} & S'_{63} & 0 & 0 & S'_{66}\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 0\\ 0\\ \gamma_{xy}\end{bmatrix}\ . \qquad (1.46)$$

Similarly, it is possible to form the relation for the deformation

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 0\\ 0\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}C'_{11} & C'_{12} & C'_{13} & 0 & 0 & C'_{16}\\ C'_{21} & C'_{22} & C'_{23} & 0 & 0 & C'_{26}\\ C'_{31} & C'_{32} & C'_{33} & 0 & 0 & C'_{36}\\ 0 & 0 & 0 & C'_{44} & C'_{45} & 0\\ 0 & 0 & 0 & C'_{54} & C'_{55} & 0\\ C'_{61} & C'_{62} & C'_{63} & 0 & 0 & C'_{66}\end{bmatrix}\cdot\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ 0\\ 0\\ 0\\ \sigma_{xy}\end{bmatrix}\ . \qquad (1.47)$$

The assumption that the width and length of the laminates are considerably greater than its thickness is still valid. In this case it is still possible to consider the plane stress. The three components of the stress can be expressed using the three components of the deformation. For example, for the first component of the tension vector σ_{xx} the following relation is valid

$$\sigma_{xx} = \left(C'_{11} - \frac{C'_{13}C'_{31}}{C'_{33}}\right) \cdot \varepsilon_{xx} + \left(C'_{12} - \frac{C'_{13}C'_{32}}{C'_{33}}\right) \cdot \varepsilon_{yy} + \left(C'_{16} - \frac{C'_{13}C'_{36}}{C'_{33}}\right) \cdot \gamma_{xy} \qquad (1.48)$$

In analogy the both the stress component σ_{yy} and σ_{xy} are obtained. These relations can be written in the matrix form

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{61} & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_L \\ \varepsilon_T \\ \varepsilon_{LT} \end{bmatrix} \quad ; \qquad (1.49)$$

$$\boldsymbol{\sigma}' = \boldsymbol{Q} \cdot \boldsymbol{\varepsilon}' \quad .$$

For reduced stiffness matrix elements Q_{ij} the following holds

$$Q_{ij} = C'_{ij} - \frac{C'_{i3}C'_{3j}}{C'_{33}} \quad ; \text{where} \quad Q_{ij} = Q_{ji} \quad , \quad i,j = 1,2,6 \quad . \qquad (1.50)$$

By comparing the equations *(1.37)* and *(1.49)* the difference between the stiffness matrix $\boldsymbol{S}$ and the reduced stiffness matrix $\boldsymbol{Q}$ is apparent. The matrix $\boldsymbol{Q}$ has generally all elements nonzero. That is, in Hooke's law *(1.49)* for off-axis components of stress and deformation, the normal components of stress (with indices xx, yy) are dependent also on the shear component (index xy), inverse is also true.

1.3.1 The Theory of the Laminate Deflection

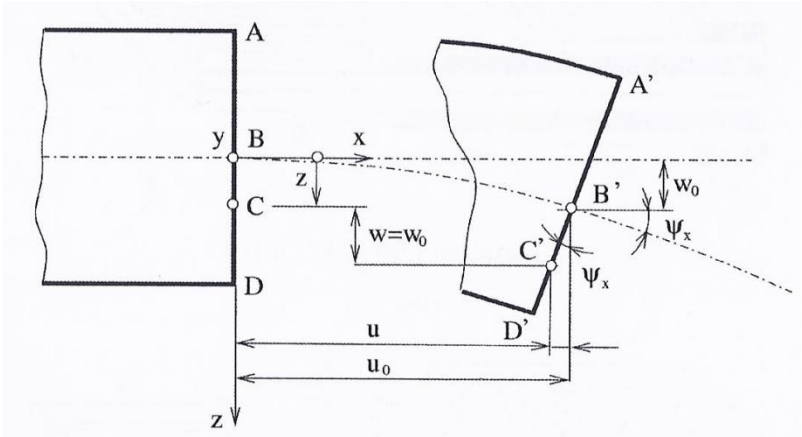


Figure 1.9: A part of laminate in the plane xz [1]

In the figure 1.9 there is a part of the laminate in the plane xz. The side AD, which is in undeformed condition straight and perpendicular to the middle surface of the laminate, remains even after deformation straight and perpendicular to the middle surface. Due to the deformation arising at mid-plane at point B displacements u_0, v_0, w_0 are corresponding to the directions of axes

x, y, z. Taking the derivatives of displacements we get the deformation field. This can be written in the matrix form

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \end{bmatrix} + z \begin{bmatrix} \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} \quad , \tag{1.51}$$

where the deformation of midplane and the curvature stands for

$$\varepsilon^{\circ}{}_m = \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{bmatrix}, \qquad \boldsymbol{k} = \begin{bmatrix} \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} \quad . \tag{1.52}$$

Tension in *k-th* layer of the laminate can be expressed by equation for off-axis strained layer of composite *(1.49)*

$$\boldsymbol{\sigma}' = \boldsymbol{Q} \cdot \boldsymbol{\varepsilon}' \quad , \tag{1.53}$$

where $\boldsymbol{Q}$ is a reduced stiffness matrix.

Using equations *(1.51)* and *(1.53)* we obtain an expression for tension in the *k-th* layer of the laminate

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{61} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \end{bmatrix} + z \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{61} & C_{66} \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} \quad . \tag{1.54}$$

Since the tension in the laminate thickness varies discontinuously, resulting forces and moments acting in cross-laminate are to be solved as a sum of the effects of all the n layers. For forces it is therefore possible to write

$$\boldsymbol{N} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} dz \tag{1.55}$$

and for the moments

$$\boldsymbol{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} z \cdot dz \quad . \tag{1.56}$$

In these relations *(1.55)* and *(1.56)* the resultants of the forces N_x, N_y, N_{xy} have a dimension $[N \cdot m^{-1}]$ i.e. the force per unit length and M_x, M_y, M_{xy} have a dimension $[N]$ i.e. the moment per unit length, because these are resultant forces and moments acting on the cross section of the *k-th* layer of the composite material. [1]

On the basis of these relations a constitutive relation of the dependence of forces and moments on deformations and curvatures can be formulated. Substituting equations *(1.55)* and *(1.56)* into the equation *(1.54)* and using the expressions for the deformation of the middle surface and the curvature of the plate *(1.52)*. The following equations are obtained

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{61} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \end{bmatrix} dz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{61} & C_{66} \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} z \cdot dz \right\} , \tag{1.57}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{61} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \end{bmatrix} z \cdot dz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{61} & C_{66} \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} z^2 \cdot dz \right\}. \tag{1.58}$$

It is obvious that multiplying the integral with elements of the reduced stiffness matrix $\boldsymbol{Q_k}$ of the individual laminas and integrating over the entire thickness of the composite we obtain following expressions

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} , \tag{1.59}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix} \begin{bmatrix} \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} , \tag{1.60}$$

where elements of the individual matrices are determined by relations

$$A_{ij} = \sum_{k=1}^{n} \left(Q_{ij}\right)_k \left(h_k - h_{k-1}\right) , \tag{1.61}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(Q_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) ,$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(Q_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) .$$

These relations can be expressed in a single equation

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ A_{61} & A_{62} & A_{66} & B_{61} & B_{62} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{21} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} \\ B_{61} & B_{62} & B_{66} & D_{61} & D_{62} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^{\circ}{}_{xx} \\ \varepsilon^{\circ}{}_{yy} \\ \gamma^{\circ}{}_{xy} \\ \boldsymbol{k}_x \\ \boldsymbol{k}_y \\ \boldsymbol{k}_{xy} \end{bmatrix} , \tag{1.62}$$

or

$$\begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \vdots & \cdots \\ \boldsymbol{B} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_{m}} \\ \cdots \\ \boldsymbol{k} \end{bmatrix} , \tag{1.63}$$

where $\boldsymbol{A}$ is the extensional stiffness matrix, $\boldsymbol{B}$ is the bending-extension coupling stiffness matrix and $\boldsymbol{D}$ is the bending stiffness matrix.

Constitutive equation of the laminate plate expresses forces and moments depending on the curvature and on the mid-plane deformations. This matrix is called the global stiffness matrix. For its notation, it is obvious that the matrix $\boldsymbol{A}$ binds force components in the median plane. The bending-extension coupling stiffness matrix $\boldsymbol{B}$ binds moment components and components of deformation in the mid-plane and also components of vector of internal forces with components of the curvature of the plate. $\boldsymbol{D}$ matrix expresses the relation between the components of moments and the curvature. This means that normal and shear forces acting in the median plane not only cause the strain in the median plane, but also the bending and the twisting of the middle area. Also components of the bending moment cause strain in the median plane. [1],[4]

The relation *(1.63)* is used to calculate forces and moments in the laminate. In practice most often stress and strain caused by external load are determined. A form, which we want to achieve, is actually the inverse equation

$$\begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_{m}} \\ \cdots \\ \boldsymbol{k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A^{*}} & \vdots & \boldsymbol{B^{*}} \\ \cdots & \vdots & \cdots \\ \boldsymbol{B^{*}} & \vdots & \boldsymbol{D^{*}} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} , \tag{1.64}$$

where

$$\begin{gathered} \boldsymbol{A^{*}} = \boldsymbol{A'} + \boldsymbol{B^{*}D^{-1}B^{*T}}; \quad \boldsymbol{A'} = \boldsymbol{A^{-1}} , \\ \boldsymbol{B^{*}} = \boldsymbol{B'D'^{-1}}; \quad \boldsymbol{B'} = \boldsymbol{A^{-1}B} , \\ \boldsymbol{D^{*}} = \boldsymbol{D'^{-1}}; \quad \boldsymbol{D'} = \boldsymbol{D - BA^{-1}B} . \end{gathered} \tag{1.65}$$

Matrices $\boldsymbol{A^{*}}, \boldsymbol{B^{*}}$ and $\boldsymbol{D^{*}}$ are called tensile, coupling and bending compliance matrices. [1]

Ties between bend and tension or torsion and tension, and also between the normal forces of the middle layer of the laminate and shear deformations are not desirable in most cases. This phenomenon should be avoided during the production of laminate's appropriate order orientation of the layers.

1.4 Common Laminate Types

The notation used to describe laminates has its roots in the description used to specify the lay-up sequence for the hand lay-up using prepreg[1]. Therefore, the laminae are numbered starting at the bottom and the angles are given from bottom up. For example, a two-lamina laminate may be [30/-30], a three-lamina one [-45/45/0], etc. [4]

If the laminate is symmetric, like [30/0/0/30], an abbreviated notation is used where only a half of the stacking sequence is given and subscript (S) is added to specify symmetry. The last example becomes $[30/0]_S$. If the thicknesses of the laminae are different, they are specified for each lamina. For example: $[\theta_{t_1}/\theta_{t_2}]$. If the different thicknesses are multiples of a single thickness t, the notation simplifies to $[\theta/-\theta_2]$, which indicates one lamina of thickness t and two laminae of the same thickness t at an angle$-\theta$. Angle-ply combinations like $[\theta/-\theta]$ can be denoted as $[\pm\theta]$. If all laminae have the same thickness, the laminate is called regular. [4]

1.4.1 Symmetric Laminate

A laminate is symmetric if laminae of the same material, thickness, and orientation are symmetrically located with respect to the middle surface of the laminate. For example: [30/0/0/30] is symmetric but not balanced, while $[30/-30/-30/30]$ is symmetric and balanced. [4]

In terms of the stress it is highly advisable to remove the coupling between the bending and the extension and between the traction and the torsion. This situation is obtained if the coupling stiffness matrix $\boldsymbol{B}$ is equal to zero. That is, with respect to equations *(1.61)* and *(1.62)*, must be true

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(Q_{ij}\right)_k\left(h_k^2 - h_{k-1}^2\right) = 0 \quad . \tag{1.66}$$

Each element of matrix $\boldsymbol{B}$ is equal to zero, if to the each contribution of the lamina above the middle surface exist the contribution from the lamina of the

[1] Prepreg is a preimpregnated fiber-reinforced material where the resin is partially cured or thickened. [4]

same properties and orientation in the same distance below the middle surface (see figure 1.10).

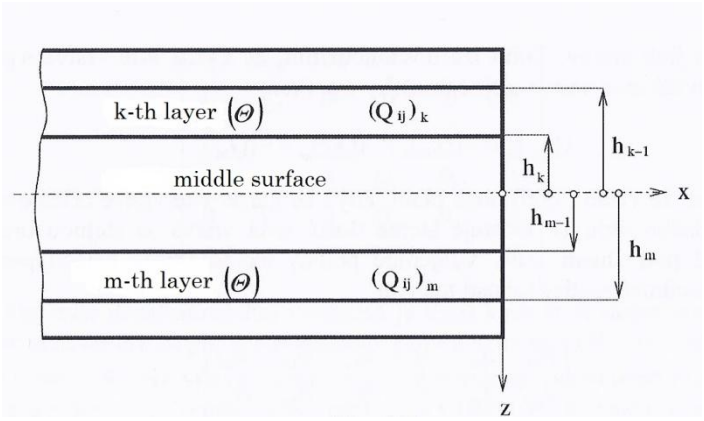


Figure 1.10: Symmetric laminate [1]

It must be true

$$\left(Q_{ij}\right)_k = \left(Q_{ij}\right)_m ; \ |-h_{k-1}| = h_m , \ |-h_k| = h_{m-1} \quad . \tag{1.67}$$

If each layer above the middle surface will correspond to the identical layer under the middle surface, it is the symmetrical laminate. The global stiffness matrix from equation *(1.63)* will be in the form

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{26} & 0 & 0 & 0 \\ A_{61} & A_{62} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{21} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{61} & D_{62} & D_{66} \end{bmatrix} . \tag{1.68}$$

A binding between tensions and the bending, which constitutes the matrix ***B***, is a result of a sequence of the layers. It does not follow from the anisotropy or the orthotropic layers. It is the result of a sequence of layers. This relation also exists in the composites made of two different metal isotropic materials (bimetal). Due to changes in temperature the bending of the composite is visible.

1.4.2 Antisymmetric Laminate

An antisymmetric laminate consists of an even number of layers (see figure 1.11). It has a pairs of laminae of opposite orientation but of the same material and thickness symmetrically located with respect to the middle surface of the laminate. For example: [30/-30/30/-30] is an antisymmetric angle-ply laminate and [0/90/0/90] is an antisymmetric cross-ply. [4]

Therefore, for each two plies of the same material properties is true

$$h_{m-1} = -h_k\,, \quad h_m = -h_{k-1}\,; \qquad \theta = -\theta \quad . \tag{1.69}$$

From this two conditions follows that both plies have the same thicknesses and they are at the same distance from the middle surface.

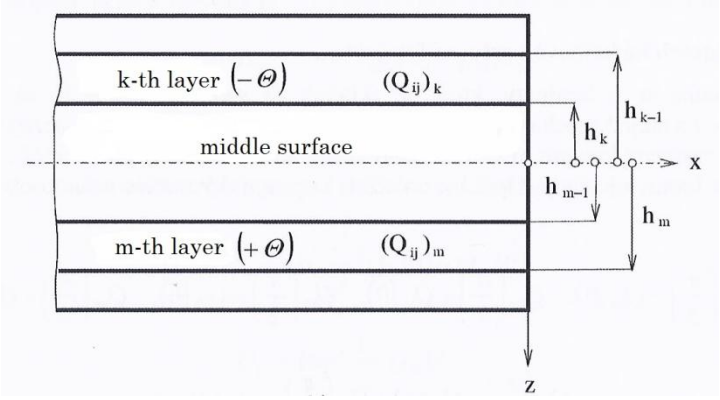


Figure 1.11: Antisymmetric laminate [1]

The global stiffness matrix from equation *(1.63)* of the antisymmetric laminates has a form

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{21} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{61} & B_{62} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{21} & D_{22} & 0 \\ B_{61} & B_{62} & 0 & 0 & 0 & D_{66} \end{bmatrix} . \tag{1.70}$$

Antisymmetric laminates have elements equal to zero

$$A_{16} = A_{61} = A_{26} = A_{62} = D_{16} = D_{61} = D_{26} = D_{62} = 0 \ , \tag{1.71}$$

but they are not particularly useful nor they are easier to analyze than general laminates because the bending extension coefficients $B_{16} = B_{61}$ and $B_{26} = B_{62}$ are not zero for these laminates. [4]

1.4.3 Quasi-isotropic Laminate

Quasi-isotropic laminates are constructed to create a composite, which behaves as an isotropic material. The in-plane behaviour of quasi-isotropic laminates is similar to that of isotropic plates but the bending behaviour of quasi-isotropic laminates is quite different than the bending behaviour of isotropic plates. [4]
In a quasi-isotropic laminate, each lamina has an orientation given by

$$\theta_k = \frac{k\pi}{N} + \theta_0 \quad , \tag{1.72}$$

where k is the lamina number, N is the number of laminae (at least three), and θ_0 is an arbitrary indicial angle. The laminate can be ordered in any order like $[60/-60/0]$ or $[60/0/-60]$ and the laminate is still quasi-isotropic.

Quasi-isotropic laminates are not symmetric, but they can be made symmetric by doubling the number of laminae in a mirror (symmetric) fashion. For e.g. the $[60/-60/0]$ can be made into a $[60/-60/0/0/-60/60]$, which is still quasi-isotropic. The advantage of the symmetric quasi-isotropic laminates is that they have the coupling stiffness matrix $\boldsymbol{B} = 0$. [4]

The tensile stiffness matrix $\boldsymbol{A}$ and the bending stiffness matrix $\boldsymbol{D}$ of isotropic plates can be written in terms of the thickness t of the plate and only two material properties, the modulus of elasticity E and the Poisson's ratio v as

$$\boldsymbol{A} = \frac{Et}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \tag{1.73}$$

and

$$\boldsymbol{D} = \frac{Et^3}{12(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \quad . \tag{1.74}$$

Quasi-isotropic laminates have, like isotropic plates, $A_{11} = A_{22}$, but they have $D_{11} \neq D_{22}, D_{16} \neq 0$ and $D_{26} \neq 0$, which makes quasi-isotropic laminates quite different from the isotropic materials as it is seen below

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \tag{1.75}$$

and

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \quad . \tag{1.76}$$

Therefore, formulas for the bending, the buckling and vibrations of isotropic plates can be used for quasi-isotropic laminates only as an approximation. The formulas for isotropic plates provide a reasonable approximation only if the laminate is designed trying to approach the characteristics of isotropic plates

with $D_{11} \approx D_{22}$ and $D_{16}, D_{26} \gg 0$. This can be achieved for symmetric quasi-isotropic laminates, which are balanced and have a large number of plies.

2 The Theory of the Deflection

The deflection is a kind of stress, in which a straight beam is curved to a plane or a three-dimensional curve. The beam is called a straight rod that it is loaded mainly to the bending. The beam bending is one of the most common types of stresses at all (e.g. all shafts are beams). The properties of the beam are substantially dependent on the type of its support. [2]

This work deals with the encastre composite beams loaded by concentrated force ***F*** at the end of its length. (Figure2.8) The bending of the beam will be solved by determination of the deformation energy due to the bending moment and the shear force. The deformation of the beam is determined using Bernoulli's method. Every cross-section of the bended beam transfers the bending moment M_O and the shear force T. The shear forces and the bending moments are caused by one common cause, namely the external loading. A relation between them is shown in the figure 2.1.

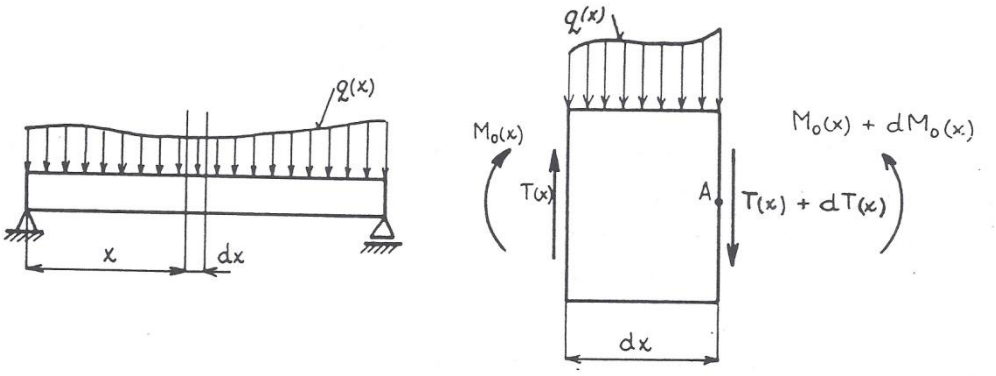


Figure 2.1: Loaded beam and out of joint element with force effects [2]

In the right side of the figure 2.1 there is an element of the beam with the force effects acting on it. The element of the beam has to be in equilibrium. The equilibrium equations of this element are known as the Schwedler theorem. [2] For the shear force we have

$$\frac{dT(x)}{dx} = -q(x) \qquad (2.1)$$

and for the bending moment

$$\frac{dM_O(x)}{dx} = T(x) \ . \qquad (2.2)$$

As a result of those effects of the shear force T and the bending moment M_O there is some tension. For simplicity, one considers only the case of load by bending momentum. This case is called a pure bending. For a pure bending is proved the validity of the Bernoulli hypothesis. This hypothesis says that the planar cuts, which were perpendicular to the longitudinal axis of the beam before the deformation, remain plane after deformation and are perpendicular to the deformed longitudinal axis of the beam. [2] This is shown in the figure 2.2 and 1.9.

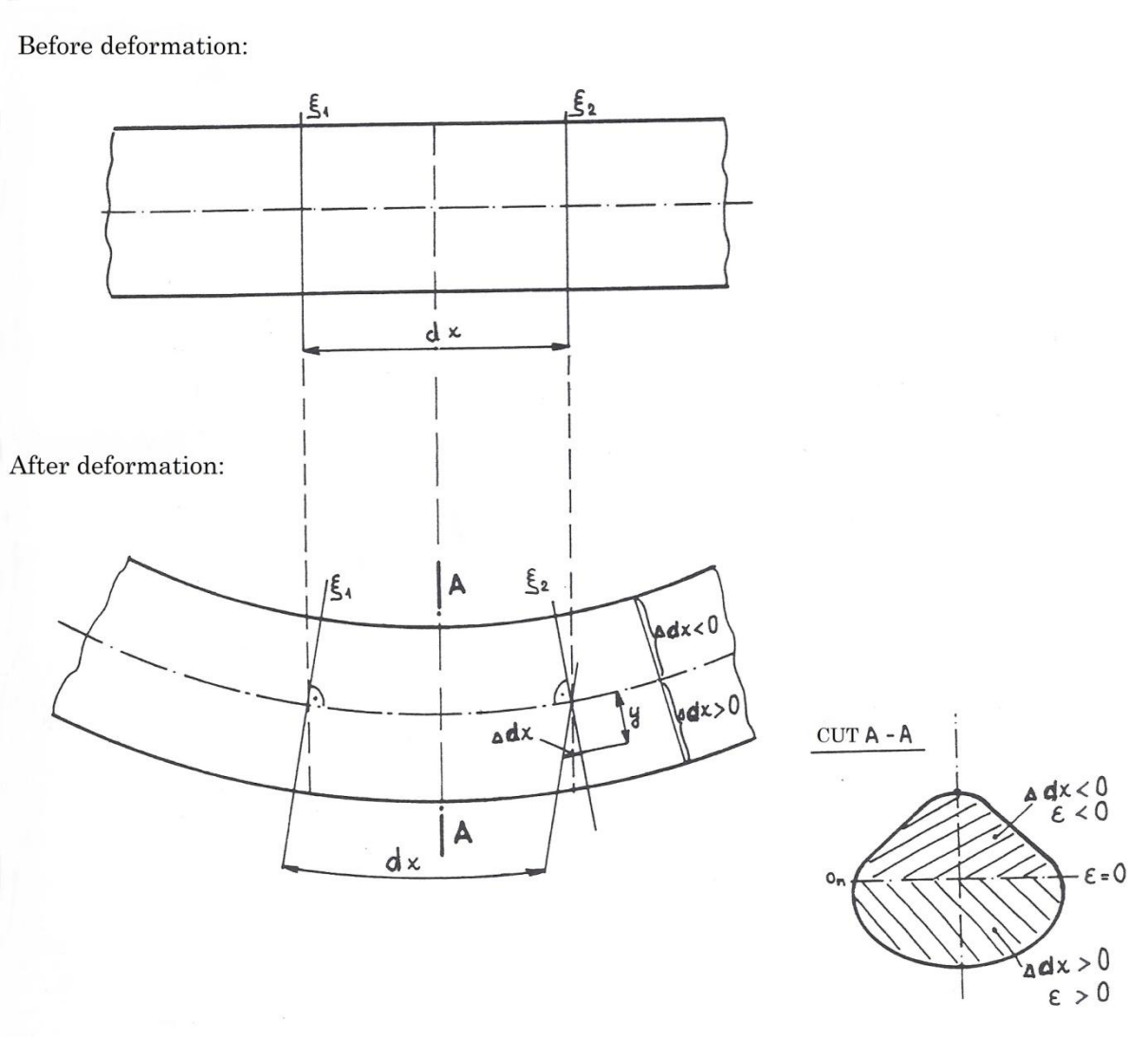


Figure 2.2: Deformation of the beam according the Bernoulli hypothesis [2]

From the figure 2.2 it is evident that an elongation $\Delta \boldsymbol{dx}$ and a relative elongation ε of the beam

$$\varepsilon = \Delta \boldsymbol{dx}/\boldsymbol{dx} \qquad (2.3)$$

are proportional to the distance y from the neutral axis O_n. As the Hooke's law is valid

$$\sigma = \boldsymbol{E} \cdot \boldsymbol{\varepsilon} \ ; \qquad (2.4)$$

one can express as

$$\sigma = -\frac{\boldsymbol{M_O}}{\boldsymbol{J_z}} \cdot y \ ,^{2} \qquad (2.5)$$

where σ is stress, M_O is the bending moment, J_z is the moment of inertia to the axis z and y is the distance from the neutral axis O_n to the top or the bottom of the section, as it is shown in the figure 2.2.

If one substitutes the relation (2.5) to the Hooke's law (2.4), one gets

$$\varepsilon_x = \frac{\boldsymbol{M_O}}{\boldsymbol{E} \cdot \boldsymbol{J_z}} \cdot y \ , \qquad (2.6)$$

where ε_x is the relative elongation in a direction of the x-axis.

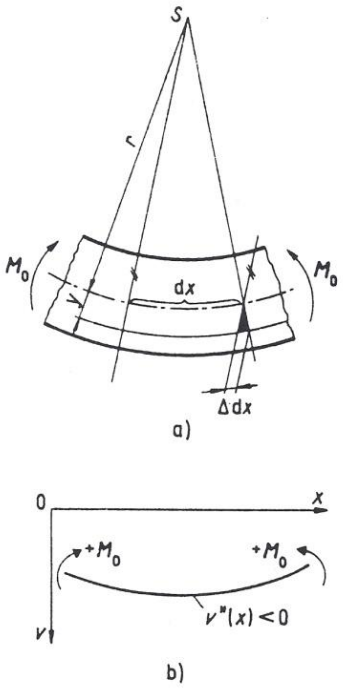


Figure 2.3: The part of the beam with marked extension [2]

The important characteristic of the deformation curve of the beam is its curvature κ. From the figure 2.3 one can express the elongation as

$$\varepsilon = \Delta \boldsymbol{dx}/\boldsymbol{dx} = \boldsymbol{y}/\boldsymbol{r} \ . \qquad (2.7)$$

The curvature of the beam is possible to express as

[2] The whole derivation is in the literature [2].

$$\kappa = \frac{1}{r} = \frac{\varepsilon_x}{y} = \frac{M_O}{E \cdot J_z} \quad . \tag{2.8}$$

From the equation *(2.8)* one derivates the differential equation of the deflection line, if one substitutes the known relation of the analytical geometry, which express the curvature κ of the planar curve $v(x)$, to the equation of the curvature of the beam.

$$\kappa = \frac{1}{r} = \pm \frac{v''(x)}{[1 + (v'(x))^2]^{3/2}} \tag{2.9}$$

To the deflection referred in the figure 2.3 corresponds the sign minus ($v''(x) < 0$). For the small values $v(x)$ one can neglect the term $(v'(x))^2$. So the simplified relation is obtained

$$\frac{1}{r} = -v''(x) = \frac{M_O(x)}{E \cdot J_z(x)} \quad . \tag{2.10}$$

The differential equation of the elastic deflection line

$$v''(x) = -\frac{M_O(x)}{E \cdot J_z(x)} \tag{2.11}$$

presented the Swiss mathematician J. Bernoulli in 1694. [2]

2.1 The Moment of Inertia

If the coordinate system is defined as in the figure 2.4, the cross section lies in the plane $y - z$.

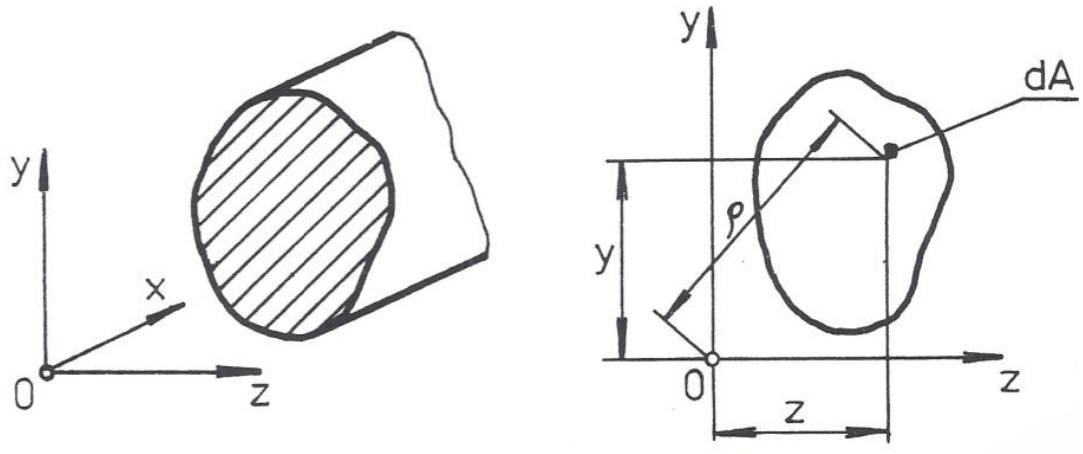


Figure 2.4: The beam placed in coordinate system and the plane of the cross section [2]

According to the figure 2.4 one removes the element $dA = dydz$ from the cross section A, which has the coordinates y and z relative to the axes y and z. [2]

The moment of inertia to the axis y is expressed by the relation

$$\boldsymbol{J_y = \int_{(A)} z^2 dA} \quad . \tag{2.12}$$

Analogically the moment of inertia to the axis z can be defined

$$\boldsymbol{J_z = \int_{(A)} y^2 dA} \quad . \tag{2.13}$$

For the beams with a circular cross section it is appropriate to establish the polar coordinates as it is shown in the figure 2.5.

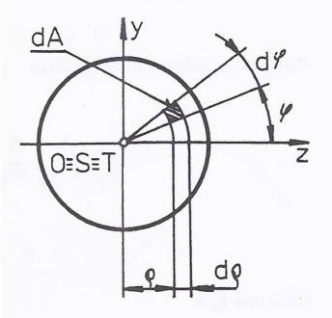


Figure 2.5: The cross section of the circular beam with the cylindrical coordinates [2]

Generally the relation

$$\boldsymbol{J_z = \int_0^{\frac{d}{2}} \int_0^{2\pi} (\rho \cdot sin\varphi)^2 \cdot \rho \, d\varphi \, d\rho = \frac{\pi d^4}{64}} \tag{2.14}$$

is valid.

To solve the moment of inertia on the wound fibreglass pipe the additivity of the moment of inertia is utilized. The moment of the whole pipe is computed as a sum of the moments of individual layers

$$\boldsymbol{J_z = \sum_i \frac{\pi {D_i}^4}{64} (D_i^4 - d_i^4)} \quad , i = 1,2, \dots, n \tag{2.15}$$

where D_i is the external diameter of the each layer, d_i is the internal diameter of the each layer and n is the number of layers.

We will use this property to computation of the bending stiffness in chapter 3.

2.2 The Determination of the Deformation Energy

2.2.1 The Deformation Energy from the Pure Bending

The pure bending is the uniaxial stress; therefore, the derivation of the deformation energy is based on the equation for density of the deformation energy

$$\lambda = \frac{\boldsymbol{\sigma}^2}{2 \cdot \boldsymbol{E}} \quad . \tag{2.16}$$

The magnitude of the stress σ is a function of a position. The relation is based on the form of the element (figure 2.6), where the stress is regarded as a constant. The energy of this element is determined, and then the value of the deformation energy in the entire beam by integration is determined. [2], [6]

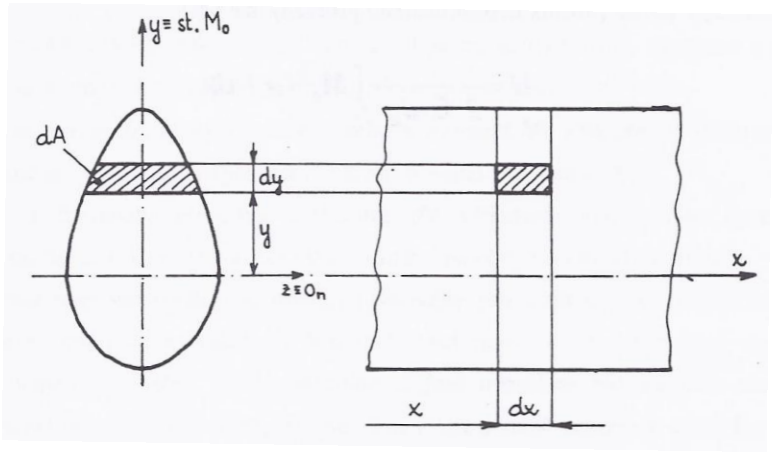


Figure 2.6: Cross section of the general beam [2]

The selected element has a volume

$$\boldsymbol{dV = dA \cdot dx} \quad , \tag{2.17}$$

where $\boldsymbol{dA}$ is an element of the area and $\boldsymbol{dx}$ is an element of the distance in the direction of x- axis. In this element the energy $\boldsymbol{dU}$ is accumulated

$$\boldsymbol{dU} = \frac{\sigma^2}{2\boldsymbol{E}}\boldsymbol{dV} = \frac{\sigma^2}{2\boldsymbol{E}}\boldsymbol{dA \cdot dx} \quad . \tag{2.18}$$

For the bending stress the relation is valid

$$\sigma(\boldsymbol{x}, \boldsymbol{y}) = -\frac{\boldsymbol{M}_o(\boldsymbol{x})}{\boldsymbol{J}_z}\boldsymbol{y} \quad . \tag{2.19}$$

After the substitution of the equation *(2.19)* to the *(2.18)* one gets

$$dU = \frac{1}{2E} \cdot \left[-\frac{M_o(x)}{J_z} \right]^2 \cdot dA \cdot dx \quad . \tag{2.20}$$

The total energy accumulated in the beam one can express by the integration with respect to x and y

$$U = \int_{(V)} dU = \int_{(l)} \int_{(A)} \frac{1}{2E} \cdot \left[-\frac{M_o(x)}{J_z} \right]^2 \cdot dA \cdot dx \quad , \tag{2.21}$$

after editing

$$U = \int_{(l)} \left[\frac{M_o^2(x)}{2E \cdot J_z^2} \cdot \int_{(A)} y^2 \cdot dA \right] \cdot dx \quad . \tag{2.22}$$

Because the relation for the moment of inertia is valid

$$\int_{(A)} y^2 \cdot dA = J_z \quad , \tag{2.23}$$

the relation for the deformation energy of the beam is

$$U = \int_{(l)} \frac{M_o^2(x)}{2EJ_z} \, dx \quad . \tag{2.24}$$

In case, that the relation $E \cdot J_z = const.$ - the beam has a constant cross-section, one can write

$$U = \frac{1}{2EJ_z} \cdot \int_{(l)} M_o^2(x) \cdot dx \quad . \tag{2.25}$$

2.2.2 The Deformation Energy by the Shear Force

In case of the deformation energy caused by the shear stress τ the relation is generally valid

$$dU = \frac{\tau^2}{2G} \cdot dV \quad , \tag{2.26}$$

where dU is an elementary deformation energy, G is a shear modulus and dV is an elementary volume. [2]

The shear stress from the shear force depends on the shape of the cross-section. The determination of the expression of the energy from the shear force we perform on the beam with the rectangular cross-section. In this case it is valid the expression

$$\tau = \frac{T \cdot S}{J_z \cdot b} = \frac{3}{2} \cdot \tau_{normal} \cdot \left[1 - \left(\frac{2 \cdot y}{h}\right)^2\right] \qquad (2.27)$$

and

$$dV = b \cdot dx \cdot dy \quad ; \qquad (2.28)$$

where T is the shear force according to the Schwedler's theorem *(2.1)*, J_z is the moment of inertia with respect to the neutral axis O_n, S is a static moment of the area ($S = \int_{(A)} y_i dA$), τ_{normal} is magnitude of shear stress which should be created if it was spread evenly over the entire cross-section and b is the width of the rectangular cross-section; if one used the designation from the figure 2.7.

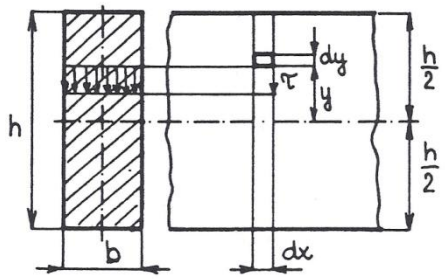


Figure 2.7: The cross section of the rectangular beam loaded with shear [2]

If the beam has a length L, then the following relation is valid

$$U = \int_0^L \left[\int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{1}{2G} \cdot \tau^2 \cdot b \cdot dy \right] \cdot dx \qquad (2.29)$$

and after substitution

$$U = \int_0^L \left\{ \frac{b}{2G} \cdot \frac{9}{4} \cdot \tau_{normal}^2 \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left[1 - \left(\frac{2 \cdot y}{h}\right)^2\right]^2 \cdot dy \right\} \cdot dx \qquad (2.30)$$

Because the centre shear stress is $\tau_{normal} = T/A$ the deformation energy is

$$U = \int_0^L \left\{ \frac{9b}{8G} \cdot \frac{T^2}{A^2} \cdot \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left[1 - \left(\frac{2 \cdot y}{h}\right)^2\right]^2 \cdot dy \right\} \cdot dx \qquad (2.31)$$

After integration and editing one obtains

$$\boldsymbol{U = \frac{6}{5} \cdot \int_0^L \frac{T^2}{2GA} \cdot dx} \tag{2.32}$$

As it is apparent from the expression of the deformation energy from the influence of the shear force, at the beam with a rectangular cross-section due to the nonlinear distribution of shear stresses the coefficient has been added. The same result one obtains for beams with other shapes of the cross-section, but with another coefficient β. [2] The general relation for the deformation energy from shear force is

$$\boldsymbol{U = \beta \cdot \int_0^L \frac{T^2}{2GA} \cdot dx} \quad . \tag{2.33}$$

The coefficient β depends on the shape of the cross-section

$$\boldsymbol{\beta = \frac{A}{J_z^2} \cdot \int_{(A)} \frac{S^2}{b^2(y)} \cdot dA} \tag{2.34}$$

For the rectangular cross-section we obtain $\beta = \frac{6}{5}$, for circular cross-section is $\beta = \frac{32}{27}$ and for eg. I-beams is $\beta = 2{,}4 \div 3{,}8$. [2]

2.3 The Deflection of the Beam

The deflection of a composite beam has two components, bending and shear

$$v = v_o + v_\tau \quad , \tag{2.35}$$

where v is the total deflection, v_o is the bending deflection and v_τ is the shear deflection. The bending deflection v_o is controlled by the bending stiffness $(E \cdot J)$ and the shear deflection v_τ by the shear stiffness $(G \cdot A)$. [4]

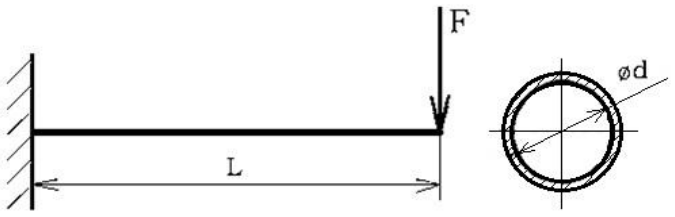


Figure 2.8: The model of the beam used for analysis

Shear deformations are neglected for metallic beams because the shear modulus is high ($G \approx E/2{,}5$), but shear deformations are important for composites because

the shear modulus is low (about $E/10$ or less). The significance of the shear deflection v_s with respect to the bending deflection varies with the span, the larger the span the lesser the influence of the shear (compared to bending). [4]
Calculation of the beam bending is made to a cantilever beam (figure 2.8), which is used for the analysis of the following methods for calculating the deflection.
First, we determine the diagram of the shear force T and the diagram of the bending moment M_O. It is illustrated in the figure 2.9.

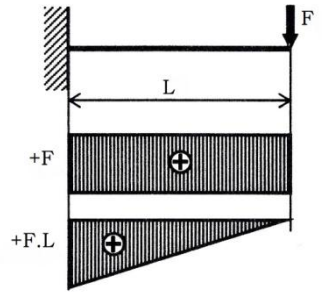


Figure 2.9: The loaded beam with the course of shear force and of the bending moment [3]

$$M_O(x) = F \cdot x \;\; ; \;\; T(x) = F = const. \tag{2.36}$$

The deformation energy U is determined as a sum of the bending deformation energy and the shear deformation energy. The equation *(2.25)* and *(2.33)* is used.

$$U = U_{M_O} + U_\tau = \int_0^L \frac{M_O^2(x)}{2 \cdot E \cdot J_z(x)} dx + \int_0^L \frac{\beta \cdot T^2(x)}{2 \cdot G \cdot A(x)} dx \tag{2.37}$$

Because the cross section A of the beam is constant, the moment of inertia J_z is constant too, so we can factor them out. For the respective deformation energies we have

$$U_{M_O} = \frac{1}{2EJ_z}\left[\int_0^L (Fx)^2 dx\right] = \frac{1}{2EJ_z}\left[\frac{F^2x^3}{3}\right]_0^L = \frac{1}{2EJ_z} \cdot \frac{F^2L^3}{3} \tag{2.38}$$

and

$$U_\tau = \frac{\beta}{2GA}\left[\int_0^L F^2 dx\right] = \frac{\beta}{2GA}\left[\frac{F^2x}{1}\right]_0^L = \frac{\beta}{2GA} \cdot F^2L \;\; . \tag{2.39}$$

The expression for the total energy is therefore

$$U = U_{M_O} + U_\tau = \frac{F^2L^3}{6EJ_z} + \frac{\beta F^2 L}{2GA} \;\; . \tag{2.40}$$

For the calculation of the deflection at the end of the beam under the force F we use the Castigliano's theorem[3]

$$v_F = \frac{\partial U}{\partial F} = \frac{\partial}{\partial F}\left(\frac{F^2L^3}{6EJ_z} + \frac{\beta F^2 L}{2GA}\right) = \frac{FL^3}{3EJ_z} + \frac{\beta FL}{GA} \tag{2.41}$$

The same result we get if we use the Mohr's Integral, which follows from the Castigliano's theorem.

$$v(x) = \int_0^L \frac{M_o(x)}{EJ_z(x)} \cdot \frac{\partial M(x)}{\partial F} dx + \int_0^L \frac{\beta T(x)}{GA(x)} \cdot \frac{\partial T(x)}{\partial F} dx \tag{2.42}$$

$$v_F(x) = \frac{1}{EJ_z} \int_0^L Fx \cdot "1" x \, dx + \frac{\beta}{GA} \int_0^L F \cdot "1" \, dx \tag{2.43}$$

The relation for the deflection at the end of the beam will have a form

$$v_F(x) = \frac{1}{EJ_z}\left[\frac{Fx^3}{3}\right]_0^L + \frac{\beta}{GA}\left[\frac{Fx}{1}\right]_0^L = \frac{FL^3}{3EJ_z} + \frac{\beta FL}{GA} \quad . \tag{2.44}$$

[3] The detailed derivation of the Castigliano's theorem can be found in the literature [2].

3 The Methods Used for the Analysis

3.1 The Used Model of the Beam

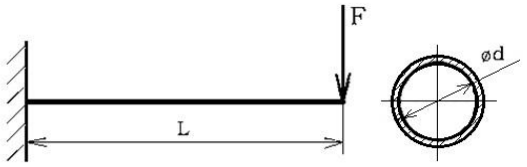


Figure 3.1: The model of the beam used for the analysis

All computations for the analysis of the composite beam bending are designed for the cantilever beam that is loaded with concentrated force at the end of its length. The beam is a wound composite pipe with a circular cross section. All data used for computations are presented in the table 3.1 below.

Input data	
Geometry	
length	$L = 1$ m
inner diameter	$d = [2, 4, 6, 8, 10]$ mm
thickness of each composite layer	$t = 1$ mm
thickness of the wound pipe	$t_p = 3$ mm
Load	
concentrated force	$F = 100$ N
Material	
density	$\rho = 1474$ kg.m^3
longitudinal modulus of elasticity	$E_L = 156.05$ GPa
transversal modulus of elasticity	$E_T = 6.045$ GPa
shear modules	$G_{LT} = 4.431$ GPa $G_{LT'} = 4.431$ GPa $G_{TT'} = 4.431$ GPa
Poisson's ratio	$\nu_{LT} = 0.328$
layup of composite material	$[90°, \alpha, -\alpha]$
angle of fiber	$\alpha = [0°, 5°, 15°, 25°, 35°, 45°, 55°, 65°, 75°, 85°, 90°]$

Table 3.1: Input data used for analysis

3.2 The Calculation of the Beam Bending by Bernoulli's Method

For the calculation of the bending by this method the Castigliano's method is used, which is the same method as the calculation of the bending of the isotropic material, with an extension for shear. The validity of the Bernoulli's hypothesis is still assumed as it is written in the previous chapter 2.

$$v = \frac{\partial U}{\partial F} \tag{3.1}$$

The Castigliano's method (3.1) is adjusted to the form (2.42) that is called Mohr's Integral.

$$v(x) = \int_0^L \frac{M_o(x)}{E_x \cdot J_y(x)} \frac{\partial M(x)}{\partial F} dx + \int_0^L \frac{T(x)\beta}{G_{xy} \cdot A(x)} \frac{\partial T(x)}{\partial F} dx \tag{3.2}$$

Shear effects cannot be ignored, because in the bending of the composite beam it has a larger share of the total bending than in the isotropic material.

The composite theory enters this computation in the calculation of the modulus of elasticity E. It is known, that the effect in each layer may be different so this problem must be included in the computation. A modulus of elasticity of each layer is calculated using the stiffness matrix $\boldsymbol{S}$ in the main coordinate system $O(L,T,T')$ of the composite material.

$$(\boldsymbol{S})_k = \begin{bmatrix} \frac{E_L}{1 - v_{LT}v_{TL}} & \frac{v_{LT}E_T}{1 - v_{LT}v_{TL}} & 0 \\ \frac{v_{LT}E_T}{1 - v_{LT}v_{TL}} & \frac{E_T}{1 - v_{LT}v_{TL}} & 0 \\ 0 & 0 & G_{LT} \end{bmatrix}_k ; \; k = 1,2,\dots,n \tag{3.3}$$

E_L, E_T are tensile modules in the direction L and T; v_{LT} and v_{TL} are Poisson's ratios; G_{LT} is a shear modulus; k is index of each layer; n is a number of layers.

The stiffness matrix must be transformed to the coordinate system $O(x,y,z)$ of the whole beam by the transformation matrix $\boldsymbol{T_{xy}}$.

$$\left(\boldsymbol{T_{xy}}\right)_k = \begin{bmatrix} cos^2\theta & sin^2\theta & -sin\theta cos\theta \\ sin^2\theta & cos^2\theta & sin\theta cos\theta \\ 2sin\theta cos\theta & -2sin\theta cos\theta & cos^2\theta - sin^2\theta \end{bmatrix}_k , \tag{3.4}$$

where θ is a rotation angle around the z- axis. (Angle θ may be different for each layer.)

Then, we can do the transformation.

$$\boldsymbol{S_{xy}} = \boldsymbol{T_{xy}} \cdot \boldsymbol{S} \cdot \boldsymbol{T'_{xy}} \tag{3.5}$$

To express the modulus of elasticity in main coordinate system $O(x,y,z)$ the compliance matrix $\boldsymbol{C_{xy}}$ is needed. The compliance matrix is the inverse matrix to the stiffness matrix $\boldsymbol{S_{xy}}$.

$$\boldsymbol{C_{xy}} = \boldsymbol{S_{xy}}^{-1} \tag{3.6}$$

From this matrix we choose the following elements to determine the modulus of elasticity E_x and the shear modulus G_{xy} for each layer.

$$C_{11} = {E_x}^{-1} \tag{3.7}$$

$$C_{66} = {G_{xy}}^{-1} \tag{3.8}$$

The resultant bending stiffness $(EJ)_{eq}$ is obtained by adding the product of the modulus of elasticity E_x and an appropriate quadratic moment of the cross section J_y for each layer

$$J_{y_k} = \frac{\pi D^4}{64}\left(1 - \left(\frac{d}{D}\right)^4\right) \;\; ; \; k = 1,2,\dots,n \tag{3.9}$$

$$(EJ)_{eq} = \sum_{k=1}^{n} E_{x_k} \cdot J_{y_k} \;\; , \tag{3.10}$$

where D is the external diameter of each layer, d is the internal diameter of each layer, k is the index of each layer and n is the number of layers. To obtain the resultant equivalent shear stiffness $(GA)_{eq}$ we proceed similarly

$$A_k = \frac{\pi}{4}(D^2 - d^2) \;\; ; \; k = 1,2,\dots,n \tag{3.11}$$

$$(GA)_{eq} = \sum_{k=1}^{n} G_{xy_k} \cdot A_k \;\; . \tag{3.12}$$

Again, D is the external diameter of each layer, d is the internal diameter of each layer, k is the index of a layer and n is the number of layers.

The deflection is calculated from the equation (3.2). The determination of the diagram of the shear force T and the bending moment M_O is the same in the section 2.3

$$M_O(x) = F \cdot (L - x) \; ; \;\; T(x) = F = const. \tag{3.13}$$

and the bending moment by a dummy force is

$$m_O(x) = 1 \cdot (L - x) \; ; \;\; T(x) = 1 \tag{3.14}$$

as it is illustrated in the figure 3.2.

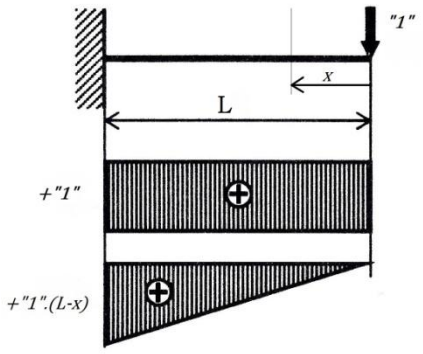


Figure 3.2: The beam loaded by the unit force with the course of the shear force and of the bending moment [3]

Then, the Mohr's integral is computed according to the equation *(2.42)*

$$v_F(x) = \frac{1}{E_x J_y}\int_0^L F(L-x)\cdot 1(L-x)\,dx + \frac{\beta}{G_{xy}A}\int_0^L F\cdot 1\,dx \quad . \tag{3.15}$$

After the integration that is

$$v_F = \frac{1}{E_x J_y}\cdot\left[FL^2x - \frac{2FLx^2}{2} + \frac{Fx^3}{3}\right]_0^L + \left[\frac{\beta F(L-x)}{G_{xy}A}\right]_0^L , \tag{3.16}$$

where F is the force load, L is the length of the beam, x is the distance on the x-axis, β is the coefficient characterizing the unequal distribution of the shear stress. The coefficient β depends on the geometry of the cross section (section 2.2.2) and A is the sectional area of the beam.

After substituting $x = 0$ we obtain the deflection in the place under the force F.

$$v_F = \frac{F\cdot L^3}{3E_x J_y} + \frac{F\cdot L\cdot \beta}{G_{xy}\cdot A} \tag{3.17}$$

This computation of the bending is based on the compliance matrix $\boldsymbol{C}$. The resultant deflection thus represents the upper limit of the safety. The deflection provides greater or equal values in the comparison with other methods of the deflection. If we use the stiffness matrix $\boldsymbol{S}$ for the calculation of the deflection, we will obtain another limit value, this time the lower limit of the beam deflection. These values correspond to the application of the longitudinal modulus of elasticity (the lower value of the deflection) and transversal modulus of elasticity (the upper value of the deflection) to the computation of bending. This fully agrees with the theory of composite materials.

For this method of the computation bending a program DP_Trubka.m in MATLAB® was created. Program calculates the bending of the beam of the

circular cross section for any number of layers. The structure of composite material may also be arbitrary. With this program the data to the summary graphs (4.1-5) of the dependence of the bending of a beam on the angle of the direction of the fibres in the composite material have been calculated. The overall listing of the program is presented in Annex [1.1].

3.3 The Calculation of the Bending of the Composite Beam Using ABD Matrices

For the calculation of the beam bending with a circular cross section (figure 3.1) by this method the equation *(1.63)*

$$\begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \vdots & \cdots \\ \boldsymbol{B} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_{m}} \\ \cdots \\ \boldsymbol{k} \end{bmatrix} \qquad (3.18)$$

described in the chapter 1 is used.

First, we determine all input values. These are geometrical dimensions and material constants for all layers described in the table 3.1 in the section 3.1. Then, we can put together the stiffness matrix in the principal coordinate system

$$S = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} . \qquad (3.19)$$

Its elements are determined by these relations *(1.39)* in the section 1.3

$$S_{11} = \frac{E_L}{1 - \nu_{LT} \cdot \nu_{TL}} = \frac{E_L}{1 - \frac{E_T}{E_L} \cdot \nu_{LT}^2} \quad ; \qquad (3.20)$$

$$S_{22} = \frac{E_T}{1 - \nu_{LT} \cdot \nu_{TL}} = \frac{E_T}{1 - \frac{E_T}{E_L} \cdot \nu_{LT}^2} = \frac{E_T}{E_L} \cdot S_{11} \quad ;$$

$$S_{12} = S_{21} = \frac{\nu_{LT} \cdot E_T}{1 - \nu_{LT} \cdot \nu_{TL}} = \nu_{LT} \cdot S_{22} \quad ;$$

$$S_{66} = G_{LT}$$

The transformation into the global coordinate system of the beam is necessary. This is realized according to the formula *(1.44)*

$$\boldsymbol{S'} = \boldsymbol{T_{\sigma}^{-1}} \cdot \boldsymbol{C} \cdot \boldsymbol{T_{\varepsilon}} \quad . \qquad (3.21)$$

In a completely general case one needs a reduced stiffness matrix $\boldsymbol{Q}$ to compute, but this is a planar case, so the off-axis stiffness matrix is equal to the stiffness matrix $\boldsymbol{S'}$ transformed to the global coordinate system.

$$\boldsymbol{Q} = \boldsymbol{S'} \tag{3.22}$$

Now we can compute the elements of stiffness matrices A_{ij}, B_{ij} and D_{ij} according to the formulas *(1.61)*

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} \left(Q_{ij}\right)_k (h_k - h_{k-1}) \ , \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} \left(Q_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \ , \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} \left(Q_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \ . \end{aligned} \tag{3.23}$$

In our case we use the matrix $\boldsymbol{A}$ to compute the equivalent modulus of elasticity E_{eq} of our beam. The strain is planar and the matrix $\boldsymbol{B}$ is zero so the equation has a form

$$\begin{bmatrix} N_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_1} \\ \boldsymbol{\varepsilon^{\circ}_2} \\ \boldsymbol{\varepsilon^{\circ}_3} \end{bmatrix} \ . \tag{3.24}$$

Then we can divide the matrix notation to the two equations

$$N_1 = A_{11} \cdot \boldsymbol{\varepsilon^{\circ}_1} + \begin{bmatrix} A_{12} & A_{16} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_2} \\ \boldsymbol{\varepsilon^{\circ}_3} \end{bmatrix} \tag{3.25}$$

and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{61} \end{bmatrix} \cdot \boldsymbol{\varepsilon^{\circ}_1} + \begin{bmatrix} A_{22} & A_{26} \\ A_{62} & A_{66} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_2} \\ \boldsymbol{\varepsilon^{\circ}_3} \end{bmatrix} \ . \tag{3.26}$$

From the second equation (3.25) we obtain a relation of deformation to the middle area in directions y and xy

$$\begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_2} \\ \boldsymbol{\varepsilon^{\circ}_3} \end{bmatrix} = -\begin{bmatrix} A_{22} & A_{26} \\ A_{62} & A_{66} \end{bmatrix}^{-1} \begin{bmatrix} A_{21} \\ A_{61} \end{bmatrix} \cdot \boldsymbol{\varepsilon^{\circ}_1} \ . \tag{3.27}$$

This relation we substitute to the first equation *(3.24)*

$$N_1 = \left(A_{11} - \begin{bmatrix} A_{12} & A_{16} \end{bmatrix} \cdot \begin{bmatrix} A_{22} & A_{26} \\ A_{62} & A_{66} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{21} \\ A_{61} \end{bmatrix} \right) \cdot \boldsymbol{\varepsilon^{\circ}_1} \ . \tag{3.28}$$

As it is written in the section 1.3 the resultant of the force N_1 has a dimension $[N \cdot m^{-1}]$, so this is not an expression of a stress. The stress of the composite material we can express using the Hooke's law *(2.4)*. To obtain an expression of the stress from this relation *(3.28)*, it is necessary to divide the entire expression by the total thickness of the composite material. From the relation *(3.28)* it is evident, that the modulus of elasticity corresponds to the expression in brackets divided by the total thickness of the laminate t.

$$\sigma_1 = \frac{N_1}{t} = \frac{1}{t}\left(A_{11} - \begin{bmatrix} A_{12} & A_{16} \end{bmatrix} \cdot \begin{bmatrix} A_{22} & A_{26} \\ A_{62} & A_{66} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{21} \\ A_{61} \end{bmatrix}\right) \cdot \boldsymbol{\varepsilon^{\circ}_1} \quad . \tag{3.29}$$

The equivalent modulus of elasticity E_{eq} for this case one can express by the formula

$$E_{eq} = \left(A_{11} - \begin{bmatrix} A_{12} & A_{13} \end{bmatrix} \cdot \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\right) \cdot \frac{1}{t} \quad , \tag{3.30}$$

where A_{ij} are the elements of the extensional stiffness matrix and t is the thickness of the composite material.

The equivalent shear modulus G_{eq} is obtained the same way from the equation (3.18). The assumptions are similar as in the previous case. The matrix $\boldsymbol{B}$ is zero and the pure shear stress N_3 is considered. The equation has a form

$$\begin{bmatrix} 0 \\ 0 \\ N_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_1} \\ \boldsymbol{\varepsilon^{\circ}_2} \\ \boldsymbol{\varepsilon^{\circ}_3} \end{bmatrix} \quad . \tag{3.31}$$

Again we can divide the matrix notation to the two equations

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_1} \\ \boldsymbol{\varepsilon^{\circ}_2} \end{bmatrix} + \begin{bmatrix} A_{16} \\ A_{26} \end{bmatrix} \cdot \boldsymbol{\varepsilon^{\circ}_3} \tag{3.32}$$

and

$$N_3 = \begin{bmatrix} A_{61} & A_{62} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_1} \\ \boldsymbol{\varepsilon^{\circ}_2} \end{bmatrix} + A_{66} \cdot \boldsymbol{\varepsilon^{\circ}_3} \quad . \tag{3.33}$$

From the first equation (3.32) we obtain a relation of the deformation to the middle area in directions x and y

$$\begin{bmatrix} \boldsymbol{\varepsilon^{\circ}_1} \\ \boldsymbol{\varepsilon^{\circ}_2} \end{bmatrix} = -\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_{16} \\ A_{26} \end{bmatrix} \cdot \boldsymbol{\varepsilon^{\circ}_3} \quad . \tag{3.34}$$

This relation we substitute to the second equation *(3.33)*

$$N_3 = \left(A_{66} - \begin{bmatrix} A_{61} & A_{62} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{16} \\ A_{26} \end{bmatrix}\right) \cdot \boldsymbol{\varepsilon^{\circ}_3} \quad . \tag{3.35}$$

Once again the shear force N_3 has a dimension $[N \cdot m^{-1}]$, so this is not an expression of the shear stress. The shear loading we can express using the generalized Hooke's law

$$\boldsymbol{\tau} = \boldsymbol{G} \cdot \boldsymbol{\gamma} \quad . \tag{3.36}$$

To obtain an expression of the shear from the relation *(3.18)*, the entire expression *(3.35)* is necessary to divide by the total thickness of the composite material. From this relation *(3.35)* it is evident, that the shear modulus corresponds to the expression in brackets divided by the total thickness of laminate t.

$$\sigma_3 = \frac{N_3}{t} = \frac{1}{t}\left(A_{66} - [A_{61} \quad A_{62}] \cdot \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{16} \\ A_{26} \end{bmatrix}\right) \cdot \boldsymbol{\varepsilon^{\circ}_3} \quad . \tag{3.37}$$

The equivalent shear modulus G_{eq} for this case one can express by the formula

$$G_{eq} = \left(A_{66} - [A_{61} \quad A_{62}] \cdot \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{16} \\ A_{26} \end{bmatrix}\right) \cdot \frac{1}{t} \quad , \tag{3.38}$$

where A_{ij} are elements of the extensional stiffness matrix and t is the thickness of composite material.

The following is a substitution of E_{eq} and G_{eq} into the formula for the calculation of bending (2.44), which is

$$v_F = \frac{F \cdot l^3}{3E_{eq}J_y} + \frac{F \cdot l \cdot \beta}{G_{eq} \cdot A} \quad . \tag{3.39}$$

This method is only approximate, because it includes several inaccuracies. First, to calculate modulus of elasticity we assume the plane stress and the pure shear. We count with the unwound circular cross-section of the beam. Second, we assume that the bending-extension coupling stiffness matrix B is zero. This precondition is fulfilled only for specific compositions of the composite material as is the symmetric laminate (section 1.4.1). This greatly reduces the possibility of either using the method or we will admit the neglect of certain bonds in the material. [1] Although this method is approximate, it shows that it gives significant results. These assumptions do not bring a considerable mistake into the calculation.

For this method of the computation of the bending a program DP_ABD_Trubka.m in MATLAB® was created. (Annex [1.2]) The program calculates the beam bending of the circular cross section for any number of layers. The structure of the composite material may also be arbitrary. The output data are summarized into graphs of the dependence of the beam deflection on the angle of the direction of the fibres in the composite material.

3.4 The Calculation of the Beam Bending by the Finite Elements Method

In this work all FEM models are created with Abaqus CAE. There are three options that can be used for the modelling of the composite beam by the finite elements method:

- conventional shell
- continuum shell
- volume model

The specifics of each method are presented in the following sections. This is not a tutorial to get started with the modelling in Abaqus, but there are captured substantial differences of individual models and there are information to reconstruct the models of the composite beam.

For each model was created a CAE file, where was specified geometry, materials, loads and constraints. Then, it was generated a script in Python (the programming language for Abaqus) of this CAE file. In this script variables were edited (the cross-sectional size and the angle of the direction of the fibres) to obtain particular data for each combination of the diameter and the angle of fibres. The script is shown in Annexes [1.3-5]. Values of calculated deflections are shown in table 4.1. For every model the same input data have been used. This is written in the table 3.1.

3.4.1 The Calculation by Using Conventional shell

The whole geometry is represented at a reference surface. The reference surface of the shell is defined by the shell element's nodes and normal direction. Thickness is defined by section property. The input data was chosen according the table 3.1. The following is a description of the operations in Abaqus CAE to obtain a script in Python.

The following is a description of the operations in the particular modules of CAE:

Sketch

For the modelling of the geometry of the pipe as a conventional shell one used the ***Part manager →Create Part →Shell, Extrusion***. Then, a sketch is made as is shown in the figure 3.3.

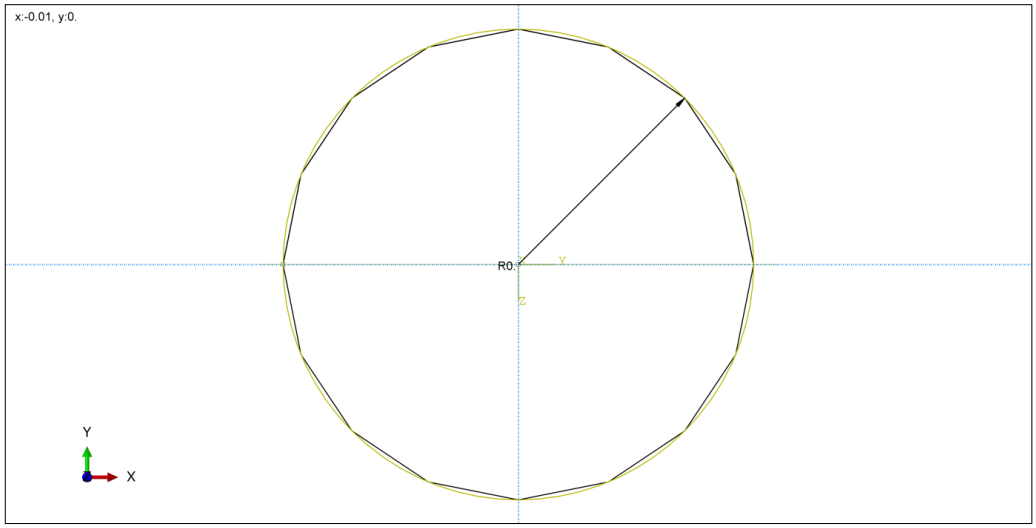


Figure 3.3: The sketch for the model of the pipe

Property

The material has to be determined. One used the ***Material manager → Create*** and set the density of the chosen material ant its constants. For the modelling of a composite material the type ***Lamina*** is used. The details are shown in a figure (3.4) below. In Abaqus we have to specify the shear modules in all three dimensions. For our task was chosen the same value in all dimensions as it is written in the table of input values (3.1).

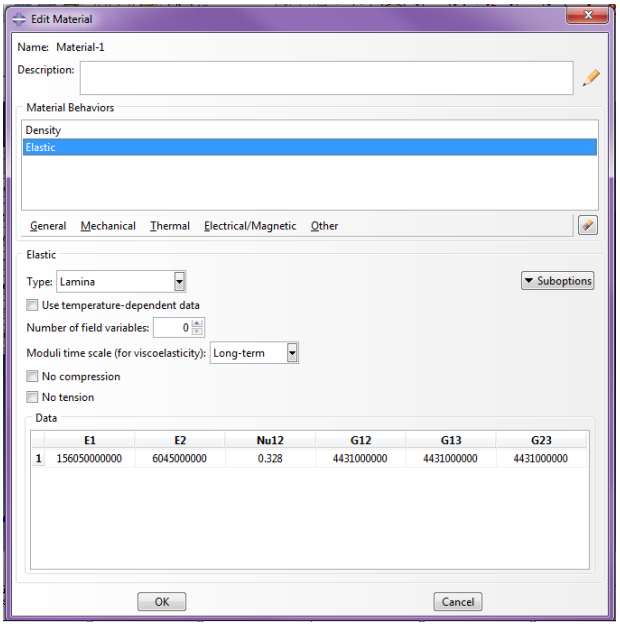


Figure 3.4: The window for editing the material

When the material is determined, the composite layup can be defined using ***Composite Layup Manager → Create →Conventional Shell***. One sets the

coordinate system for the used geometry, the normal direction, the number of plies and their properties. The thickness of plies, the material, the rotation angle and the coordinate system for all plies are specified (figure 3.5). There is also specified the geometry area for each ply.

Assembly

If the properties of material and geometry are determined one can set the assembly. In this case the assembly contains one part – the shell of the pipe. As we can see in the figure (3.6) the assembly includes two coordinate systems. It is caused by our specification of the coordinate system for the direction of fibers. The coordinate system, which is used for the computation, has the x-axis identical to the longitudinal axis of the pipe (in the figure (3.6) it is on the left side). The second one is automatically generated for the assembly (it is on the right side in the figure (3.6)).

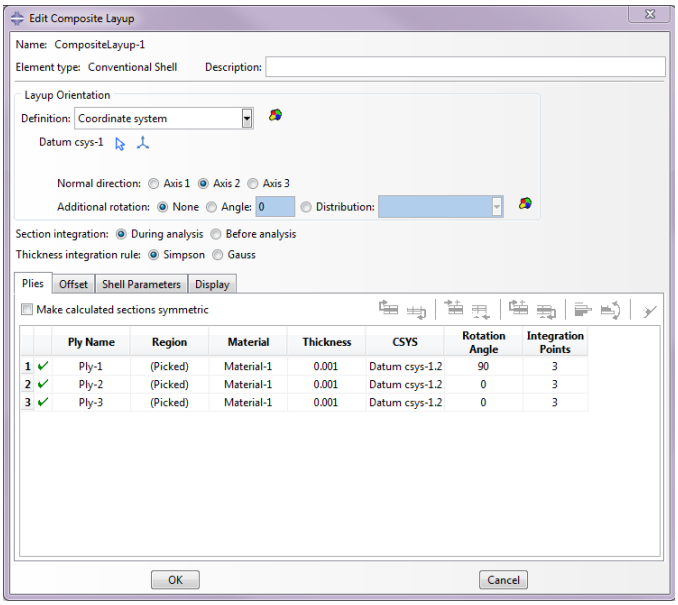


Figure 3.5: The window for editing the composite layup

Now the partition of the shell into two parts is made. It is for the better identification of the direction of the composite material and also for the better meshing of a part. The partition is made by the order ***Partition Cell: Use Datum Plane***. First one has to create the datum plane by using the order ***Create Datum Plane: 3 Points***.

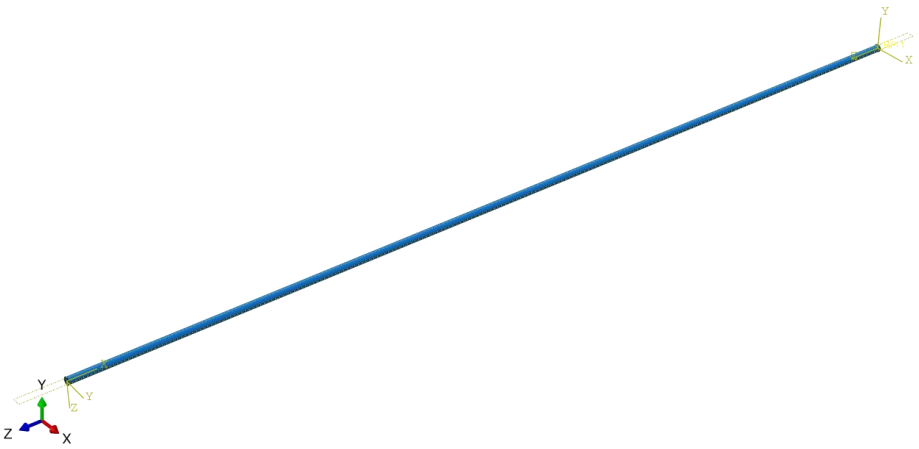


Figure 3.6: The assembly of the beam

Step

In the mode Step the procedure type ***Static, General*** is defined, because we are computing the static case of the beam bending. The point of our interest is a size of the bending at the end of the beam, so the only calculation of the translation is needed. This we specify in the ***Field Output Request Manager*** → ***Cerate*** → ***Continue*** → ***Edit Field Output Request***, where we choose the possibilities as it is shown in the figure (3.7 b).

a) b)

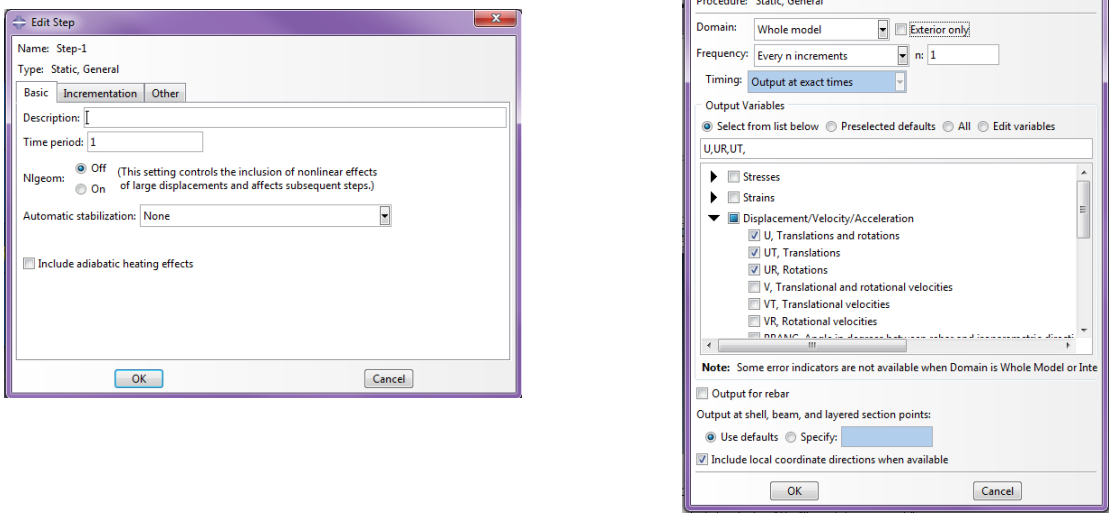


Figure 3.7: a) The window to specify the calculating step
b) The window for choosing the outputs

Interaction

The beam is loaded by a concentrated force, therefore it is appropriate to place the force into the centre of the circular cross-section at the end of the beam. The

centre of the cross-section has to be connected with a cross-section at the end of the beam. This we can define by the ***Constraint Manager*** → ***Create*** → ***Coupling*** →***Edit Constraint***. Before that we determine the reference point (***RP-1*** in the figure (3.8 a)) for a coupling by the order ***Create Reference Point*** in the centre of the cross-section at the end of the beam.

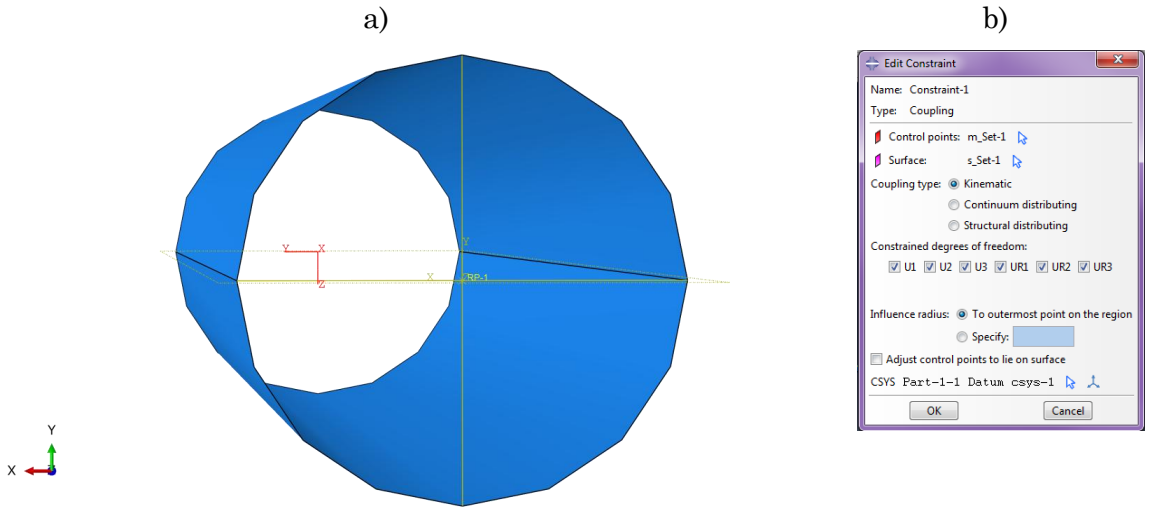


Figure 3.8: a) The pipe with the shown coupling properties
b) The window for editing the coupling properties

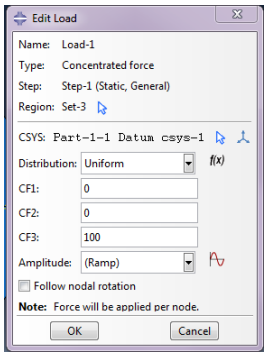
Than we set a coordinate system for the coupling, we choose the same one we have used for the determination of the composite material specification (the red one in the figure (3.8 a)).

Load

In this module the size of the concentrated force and its location is specified as well as fixation of the beam. The concentrated force is defined in ***Load Manager*** → ***Create*** → ***Concentrated Force*** → ***Edit Load***. We define a coordinate system that we used (the red one in the figure (3.9 b) and a force in the direction of the z -axis as it is shown in the figure (3.9 a).

The fixation of the beam is specified in ***Boundary Condition Manager*** → ***Create*** → ***Symmetry/Antisymmetry/Encastre*** → ***Encastre*** ***(***$U1 = U2 = U3 = UR1 = UR2 = UR3 = 0$***)***. Again the coordinate system has to be defined.

a)



b)

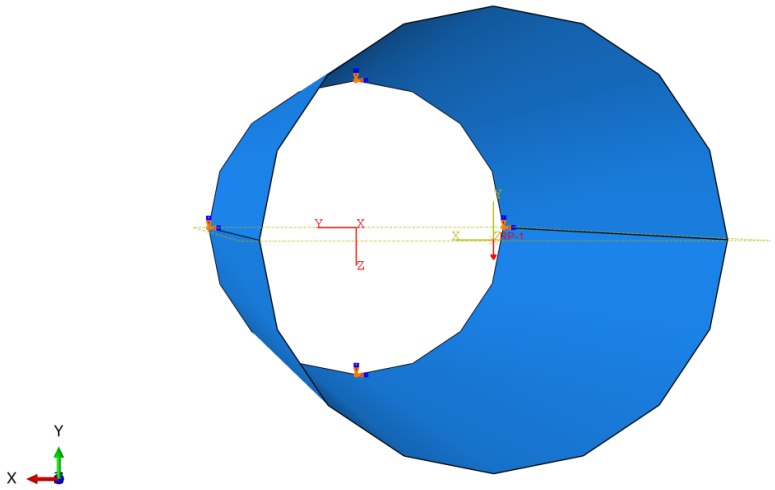


Figure 3.9: a) The window for editing the load
b) The pipe with the shown load and the fixation

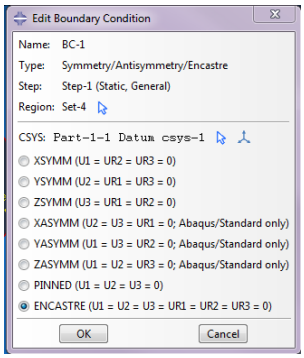


Figure 3.10: The window for editing the boundary conditions

Mesh

The element type S4R has been used for the meshing of the pipe. S4R is a robust, general-purpose element that is suitable for a wide range of applications. The size of elements has been chosen 0,005 x 0,001. It is not an optimal choice of a size for this type of elements, because the aspect ratio should be less than three. But we do not review some local effects on the geometry, for calculating the deflection of the whole beam this selection of the size of elements does not distort the calculation.

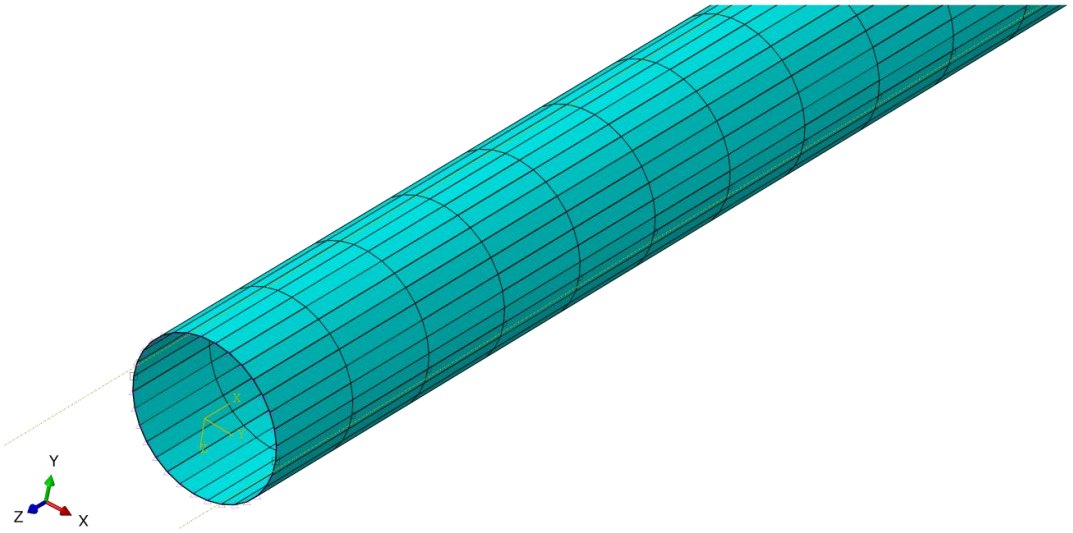


Figure 3.11: The meshed beam

Job

The calculation was carried out without errors. Details can be seen in the figure (3.12).

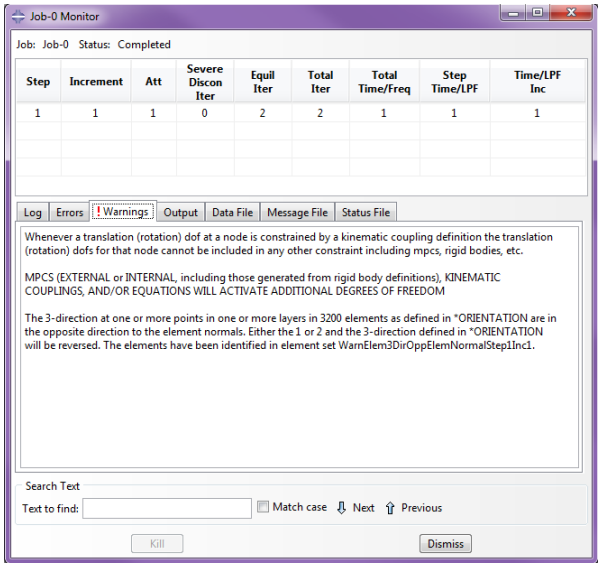


Figure 3.12: The listing of the calculation

Visualization

The deformation of the beam was most reflected as it has been expected in the direction of the concentrated force. It is evident from the figure (3.13 a).

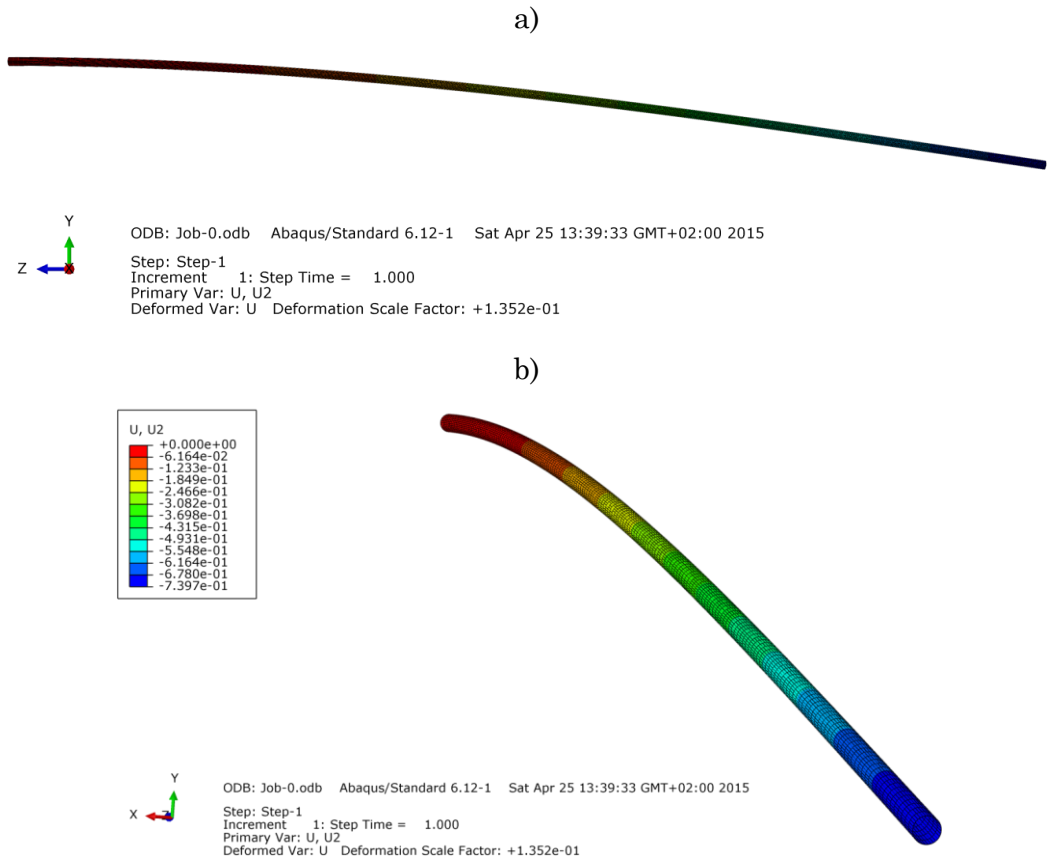


Figure 3.13: a) The deformed beam shown in the *yz* plane
b) The deformed beam in a general perspective with the scale

The value of the deflection is different in individual nodes in the cross-section. This is shown in the figure (3.14). So the numerical size of the bending was calculated as the average of values of the deformation in the direction of the y-axis of individual nodes at the end of the beam.

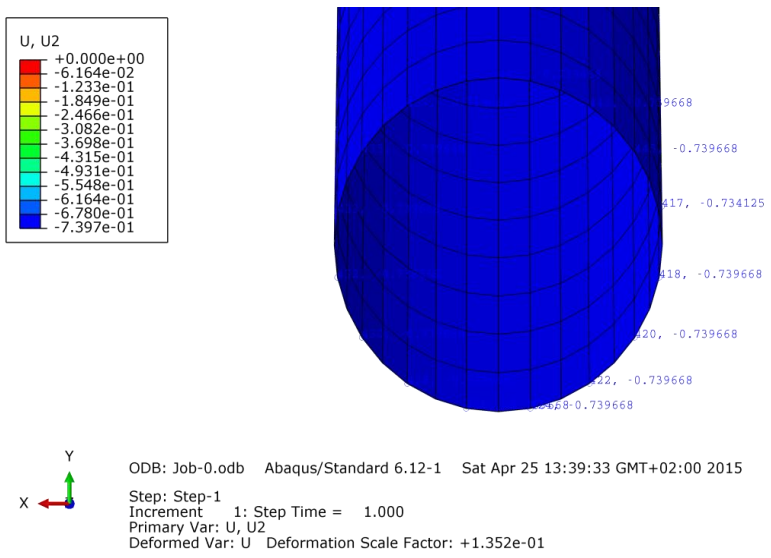


Figure 3.14: The detail of the end of the deformed beam with the values of the deflection

3.4.2 The Calculation by Using the Continuum Shell

For the calculation using the continuum shell the full 3-D geometry is specified. The element thickness is defined by the nodal geometry. Continuum shell captures more accurately the through-thickness response for composite laminate structures. It has a high aspect ratio between in-plane dimensions and the thickness. The input data were chosen according the table 3.1. The following is a description of the operations in Abaqus CAE to obtain a script in Python.

The following is a description of the operations in the particular modules of CAE:

Sketch

For modelling the geometry of the pipe as a continuum shell one used the ***Part manager → Create Part →Solid, Extrusion***. Then a sketch is made by two circles as it is shown in the figure (3.15).

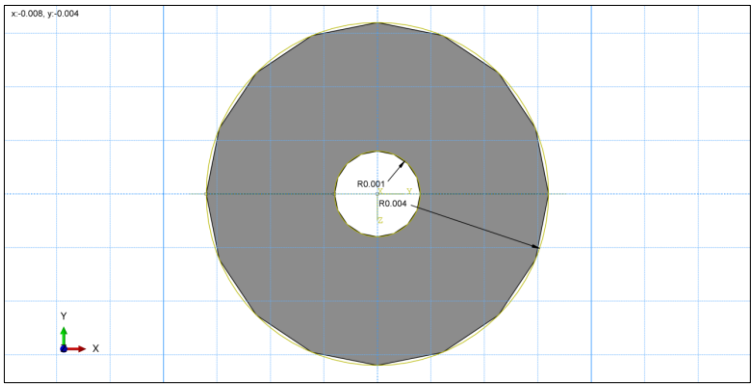


Figure 3.15: The sketch for model of the pipe

Property

In the module ***Property*** the material has to be determined. One used the ***Material manager → Create*** and set the density of the chosen material and its constants. For the modelling of a composite material the type ***Lamina*** is used. The details are shown in the figure (3.16) below. This part is the same as in the section 3.4.1 (modelling of the conventional shell).

When the material is determined, the composite layup can be defined using ***Composite Layup Manager → Create → Continuum Shell*** as it is shown in the figure (3.17). One sets the coordinate system for the used geometry, the normal direction, the number of plies and their properties. The thickness of plies, the material, the rotation angle and the coordinate system for all plies are specified. There is also specified the geometry area for properties of each ply. The

properties specification of the model is the same as in the previous case in the section 3.4.1.

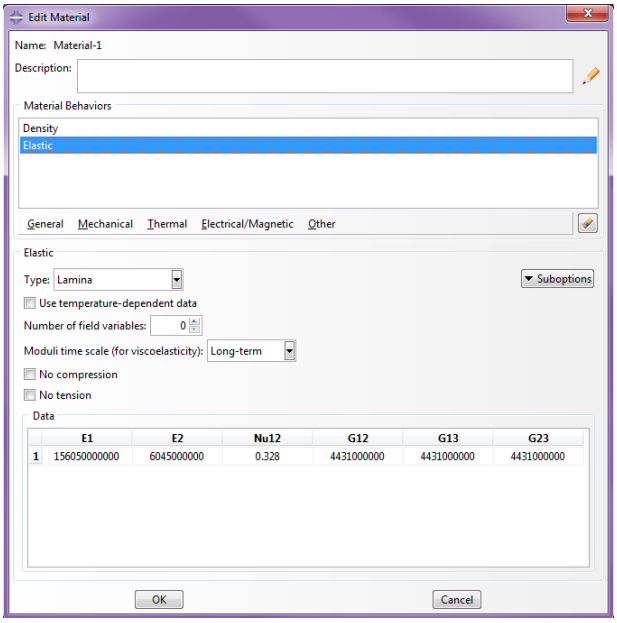


Figure 3.16: The window for editing the material

a) b)

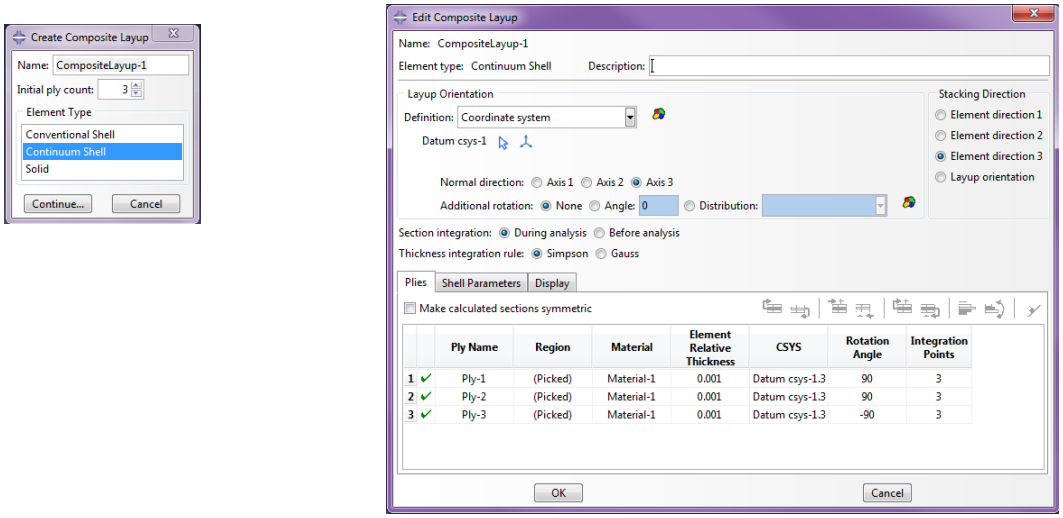


Figure 3.17: a) The window for creating the composite layup;
b) The window for editing the composite layup

Assembly

The part has to be put into the assembly. In this case the assembly contains one part –the body of the pipe. The coordinate system of the pipe is determinated as

we can see in the figure (3.18). It is the coordinate system, which is used for the determination of the direction of fibres. The coordinate system that is used for the computation has the x-axis identical to the longitudinal axis of the pipe.

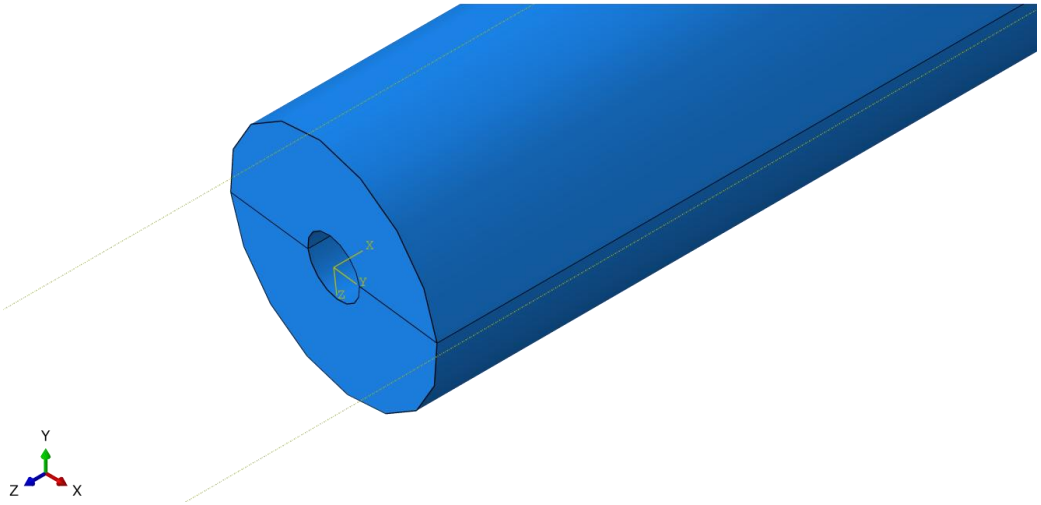


Figure 3.18: The assembly of the beam

Now the partition of the geometry into two parts is made. It is for the better identification of the direction of the composite material and also for the better meshing of the part. The partition is made by order ***Partition Cell: Use Datum Plane***. First, one has to create the datum plane by using order ***Create Datum Plane: 3 Points***.

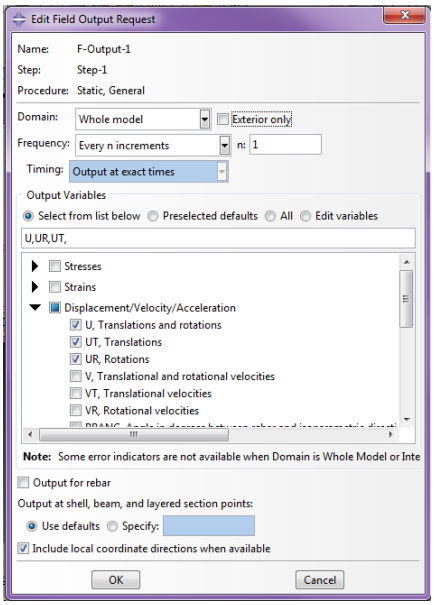


Figure 3.19: The window for choosing the outputs

Step

As it is written in the section 3.4.1, the procedure type ***Static, General*** is defined. The outputs are set as in the previous case, where we choose the possibilities of the displacement as is shown in the figure (3.19).

Interaction

Again the one end of the beam is prepared for the loading using ***Constraint Manager → Create → Coupling →Edit Constraint***. The reference point (***RP-1*** in the figure (3.20 a)) is set. According to our model the beam is loaded by a concentrated force, which is placed into the centre of the circular cross-section at the end of the beam. The centre of the cross-section has to be connected with a cross-section at the end of the beam by constraint. The coordinate system is selected as in the previous case according to the specification of the composite material.

a) b)

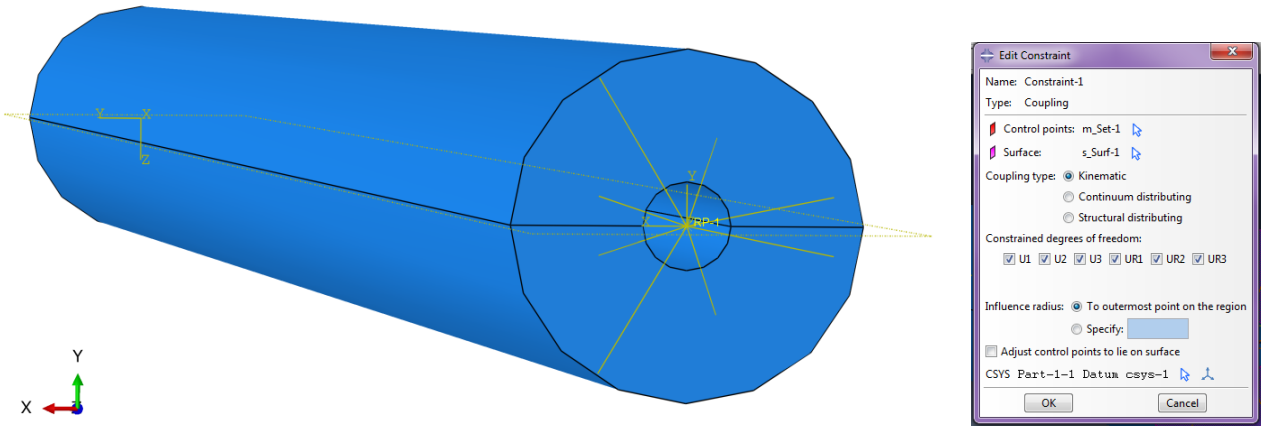


Figure 3.20: a) The pipe with the shown coupling properties
b) The window for editing the coupling properties

Load

Further, the concentrated force and the fixation of the beam are determined. The concentrated force is defined in ***Load Manager →Create →Concentrated Force →Edit Load***. We define a coordinate system that we have used before (the red one in the figure (3.21 a)) and a concentrated force in the direction of the *z*-axis as it is shown in the figure (3.21 b).

The support of the beam is specified in ***Boundary Condition Manager*** → ***Create*** → ***Symmetry/Antisymmetry/Encastre*** → ***Encastre (*****U1 = U2 = U3 = UR1 = UR2 = UR3 = 0*****)***. The coordinate system has to be defined. The fixation is accomplished in the same way as in the previous section 3.4.1.

Mesh

Continuum shell elements are 3-D stress/displacement elements for the use in the modelling of structures that are generally slender, with a shell-like response but the continuum element topology. They capture more accurately the through-thickness response for composite laminate structures. The element type SC8R

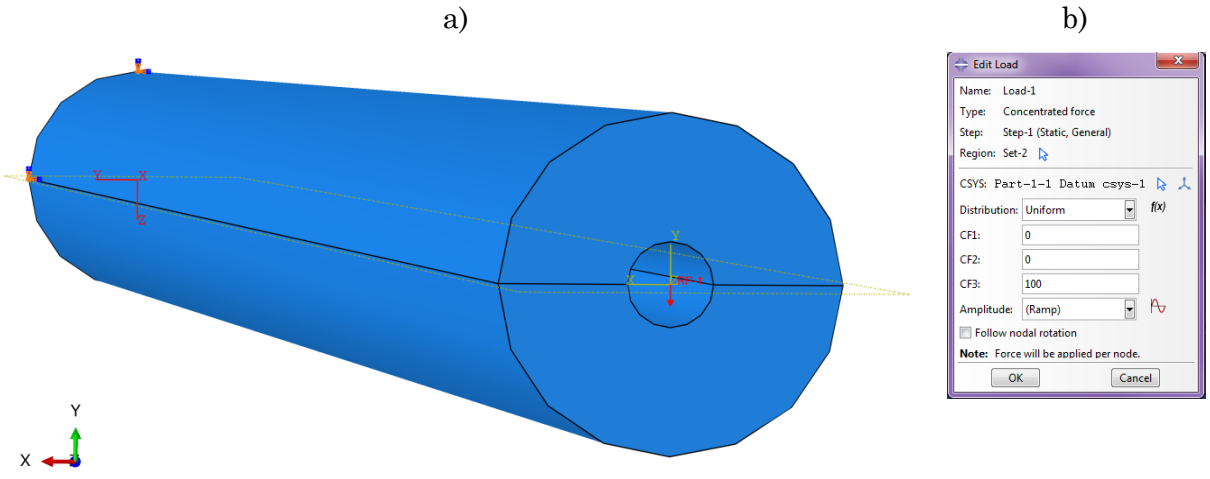


Figure 3.21: a) The pipe with the shown load and the fixation
b) The window for editing the load

(8-node hexahedron for a general purpose) has been used for the meshing the geometry. The thickness direction can be ambiguous for the SC8R element. Any of the 6-faces could be a bottom face, therefore it is important to define the bottom side of the elements and assign the stack direction. This can be determined by order ***Assign Stack Direction*** as it is shown in the figure (3.22). The size of elements has been chosen *0,005m* along the x-axis and the quantity 40 elements along the perimeter.

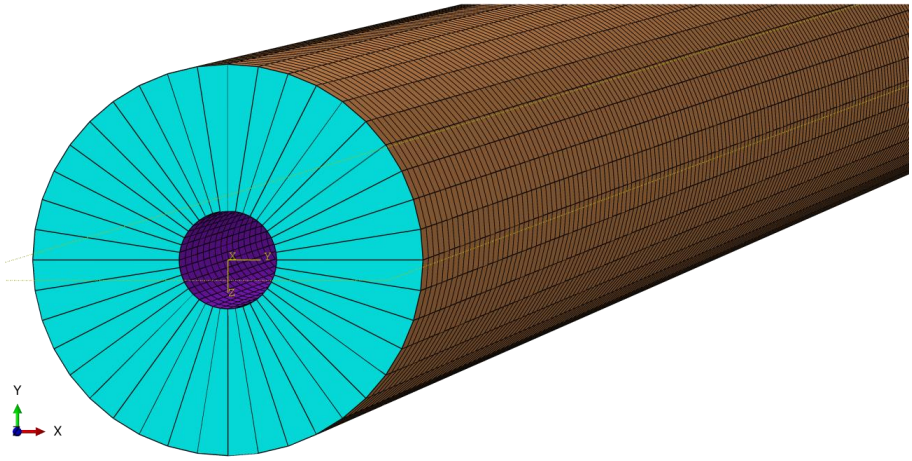


Figure 3.22: The meshed beam with the layup orientation

Job

The calculation was carried out without errors. Details can be seen in the figure (3.23).

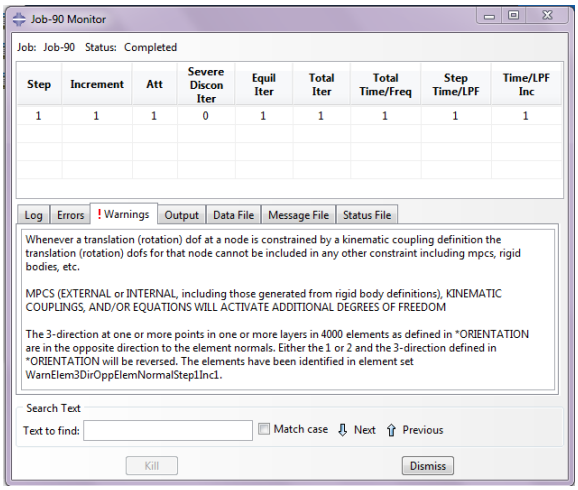


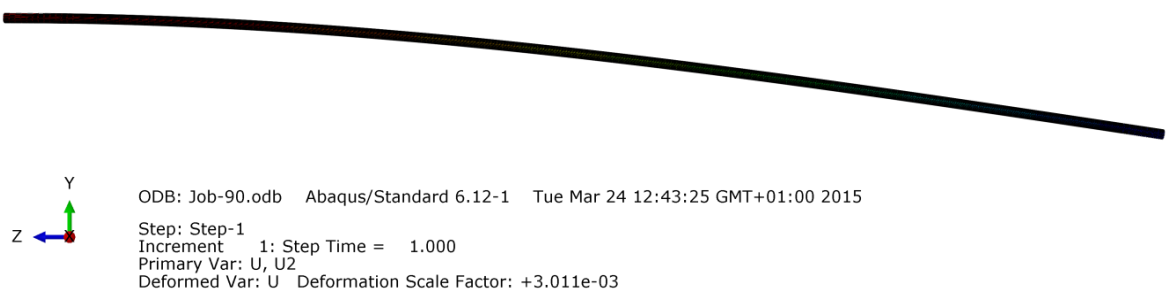
Figure 3.23: The listing of the calculation

Visualization

Deformation of the beam was most reflected as it has been expected in the direction of the concentrated force. It is evident from the figure (3.24 a).

The value of the deflection is the same in individual nodes in the whole cross-section. This is shown in the figure (3.25). So the numerical size of the deflection was taken as the value of the deformation in one node at the end of the beam.

a)



b)

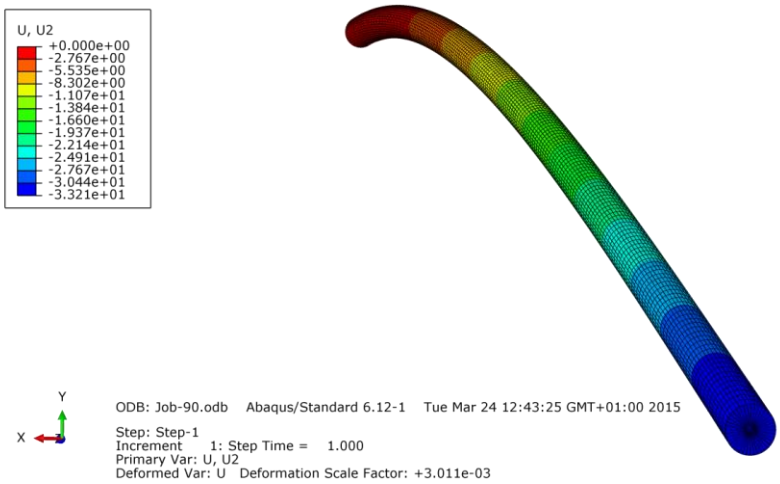


Figure 3.24: a) The deformed beam shown in the *yz* plane
b) The deformed beam in a general perspective with the scale

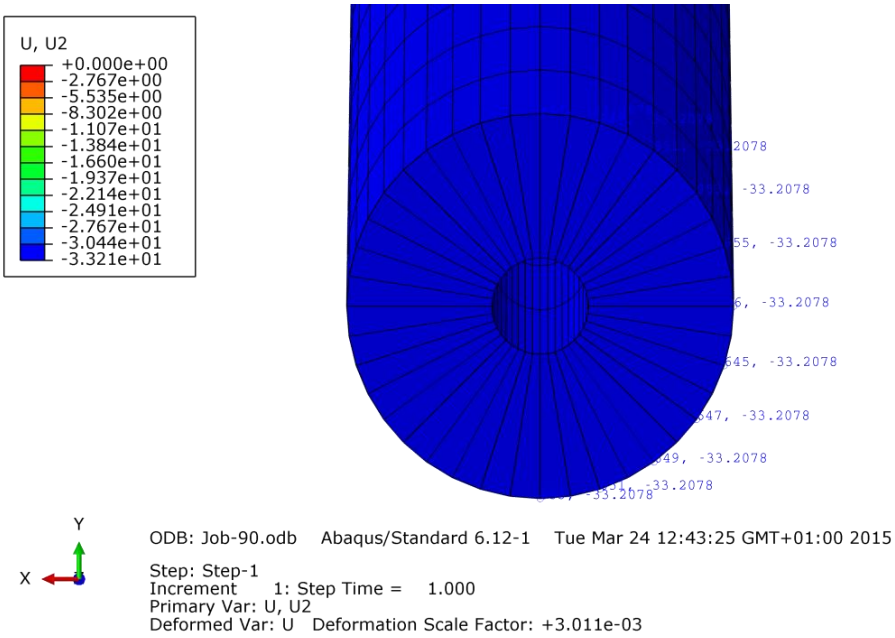


Figure 3.25: The detail of the end of the deformed beam with the values of the deflection

3.4.3 The Calculation Using the Volume Model

For the calculation using the volume model the full 3-D geometry is specified and each ply is created separately as a separate body. The input data were chosen according to the table 3.1 of the input data. The following is a description of the operations in Abaqus CAE to obtain a script in Python.

The following is a description of the operations in the particular modules of CAE:

Sketch

For modelling the geometry of a pipe as a volume model one used the ***Part manager →Create Part →Solid, Extrusion*** as by the modelling of the continuum shell. Then, a sketch is made by two circles for each ply as it is shown in the figure (3.26).

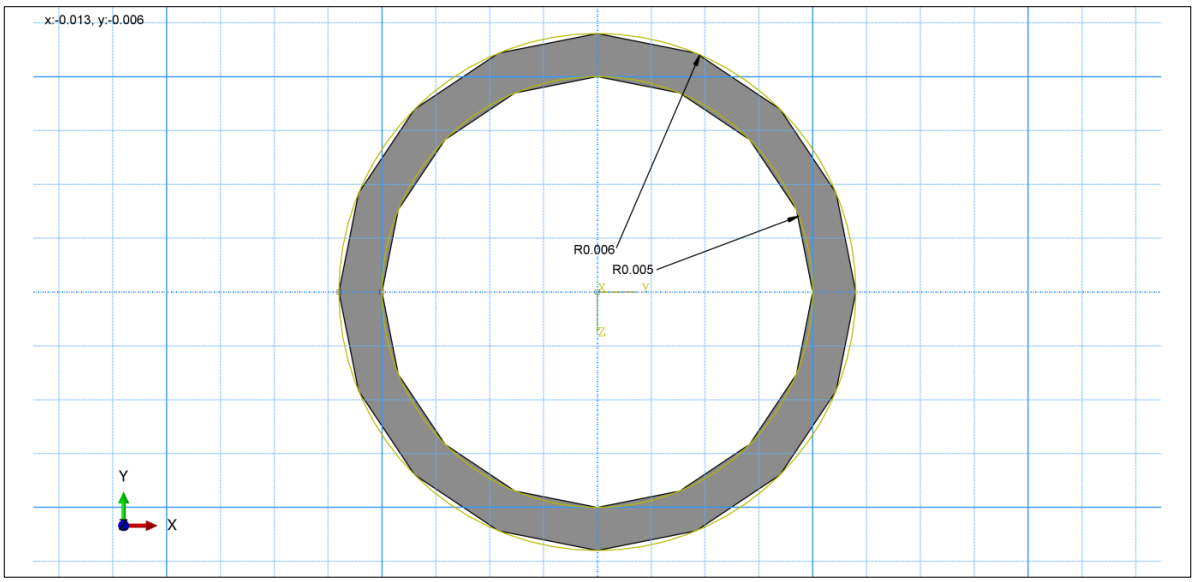


Figure 3.26: The sketch of the one layer for model of the pipe

Property

In this module the material and its orientation has to be determined. One used the ***Material manager → Create*** and set the density of a chosen material and its constants. For modeling a composite material the type ***Engineering Constants*** is used. Poisson's ratio in all three dimensions has to be determined as well as the module of elasticity and the shear module. Poison's ratio was defined the same in all directions

$$\nu_{12} = \nu_{23} = \nu_{13} = 0{,}328\ [-] \quad .$$

The modules of elasticity E_2 and E_3 in the direction of y-axis and z-axis were defined equal as the transversal modulus of elasticity E_T

$$E_2 = E_3 = E_T = 6{,}045\ GPa\ .$$

The shear modules ware defined as in previous models of composite beams according to the table of input values 3.1. The details are shown in the figure (3.27) below.

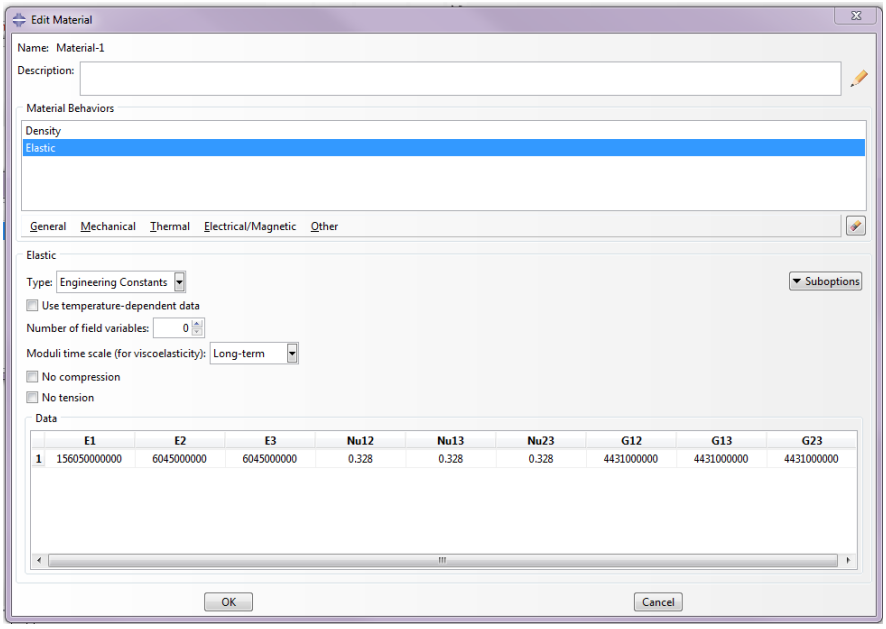


Figure 3.27: The window for editing the material

When the material is determined, the composite layup can be defined. For modelling the composite beam using the volume model each ply must be determinated separately, because we have a part for each ply. The material of each ply we determine using ***Section Manager → Create → Solid, Composite***, and then one can set the material of the ply, the rotation angle and the element relative thickness as it is shown in the figure (3.28 a). This operation is repeated with each layer.

The material orientation is defined by ***Assign Material Orientation***. It is important for the determination of the direction of fibres in the composite material. The direction is defined with respect to the coordinate system that has the x-axis coincident with the longitudinal axis of the pipe (the purple one in the figure (3.29)).

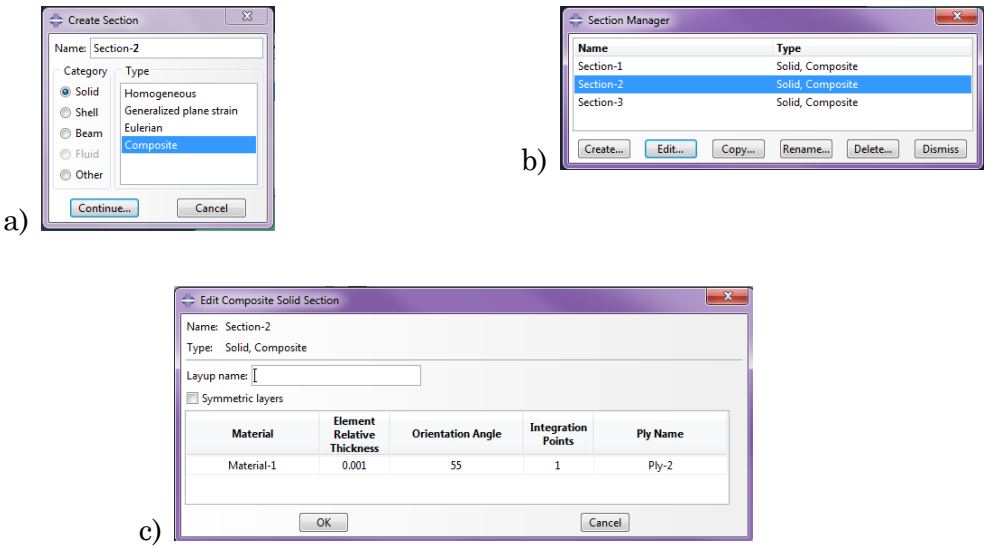


Figure 3.28: a) The window for creating the composite type of section
b) The window for choosing the section for editing its properties
c) The window for editing the composite layup

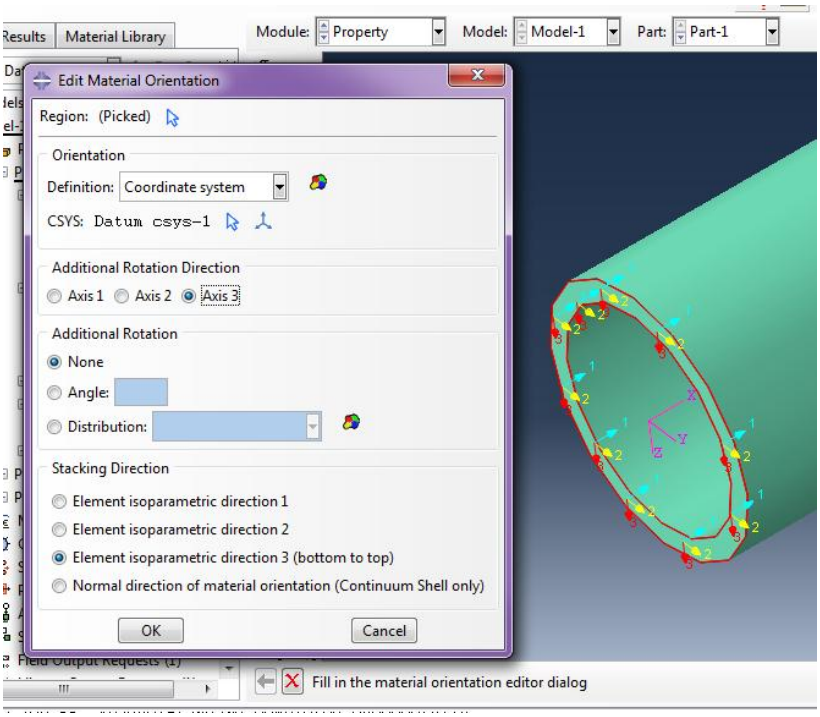


Figure 3.29: The window for specify the orientation of the material

Assembly

Parts have to be put into the assembly. In this case the assembly contains three parts – one part for each ply of the pipe. The coordinate system of the pipe is determinated as we can see in the figure (3.30). It is the coordinate system, which

is used for the determination of the direction of fibres. Three parts are put together to make a model of the wound composite pipe as one can see in the figure (3.30).

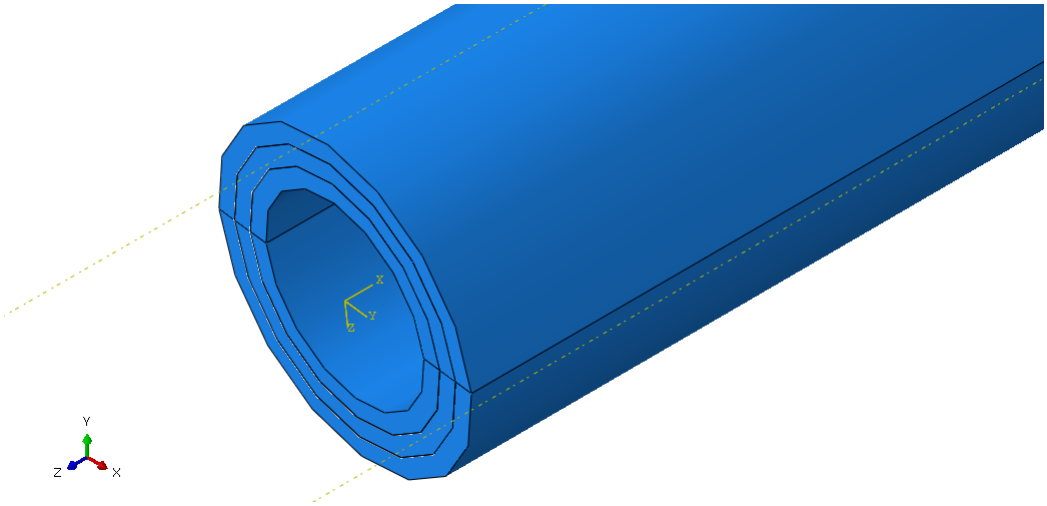


Figure 3.30: The assembly of the beam

Now the partition of the geometry into two parts of each ply is made. It is for better identification of the direction of the composite material and also for the better meshing of the parts. The partition is made by the order ***Partition Cell: Use Datum Plane***. First, one has to create the datum plane by using the order ***Create Datum Plane: 3 Points***.

Step

As it is written in previous sections 3.4.2, the procedure type ***Static, General*** is defined. The outputs are set the same way as in the previous case, where we choose the possibilities of displacement as it is shown in the figure (3.31).

Interaction

The end of the beam is prepared for the loading using ***Constraint Manager*** → ***Create*** → ***Coupling*** → ***Edit Constraint***. The reference point (***RP-1*** in the figure (3.32 a)) is set in the centre of the cross section as it is made in previous models. The coordinate system is selected according to the specification of the composite material. In this case all three parts have to be connected with the constraint as it is shown in the figure (3.32).

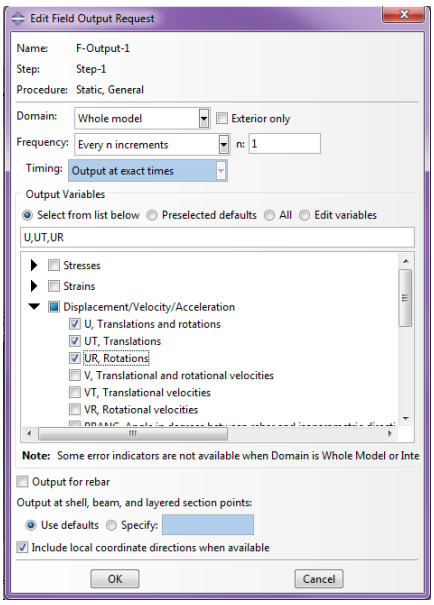


Figure 3.31: The window for choosing the outputs

a) b)

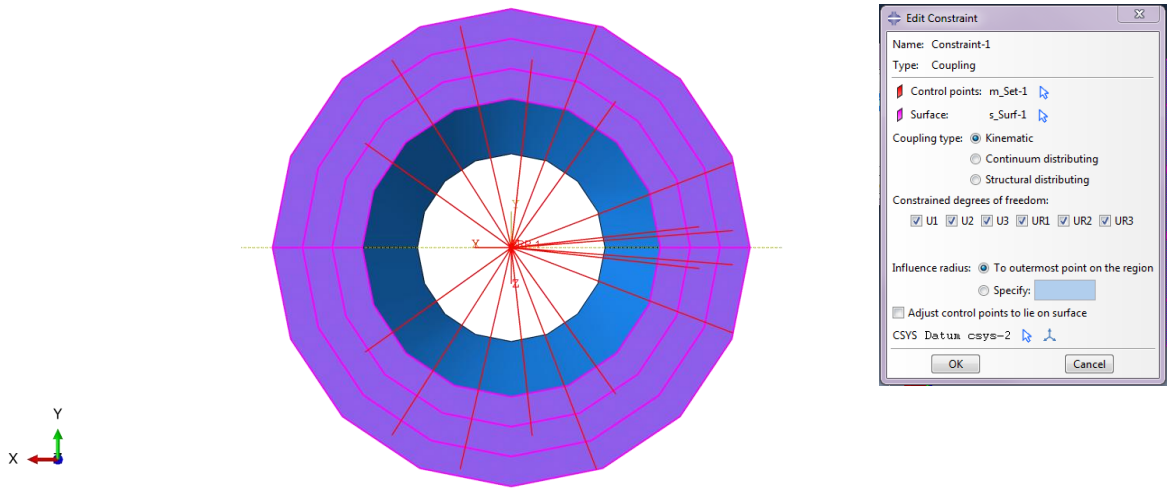


Figure 3.32: a) The pipe with the shown coupling properties
b) The window for editing the coupling properties

Load

The concentrated force is defined in ***Load Manager → Create → Concentrated Force → Edit Load***. We define a coordinate system that we used before (the red one in the figure (3.33 a)) and a concentrated force in the direction of the *z*-axis. This is shown in the figure (3.33 b).

a) b)

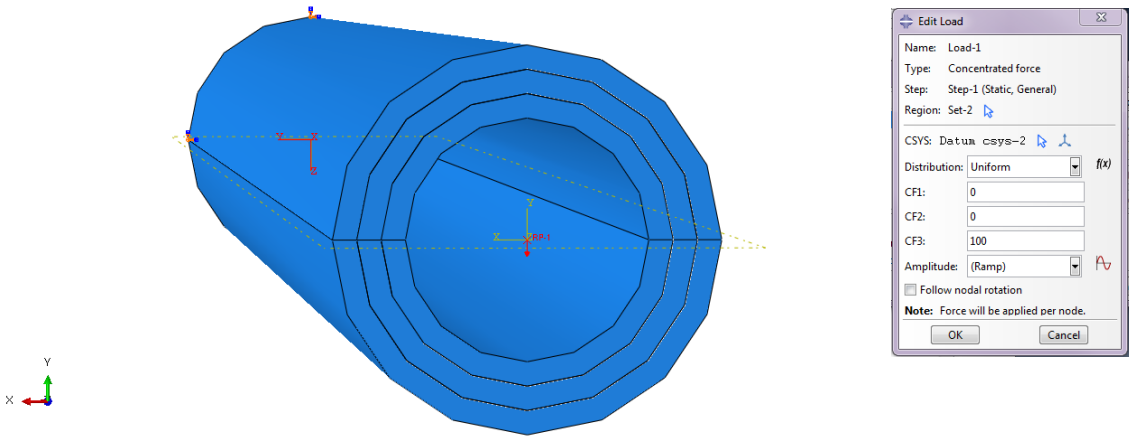


Figure 3.33: a) The pipe with the shown load and the fixation
b) The window for editing the load

The support of the beam is once again specified in ***Boundary Condition Manager → Create → Symmetry/Antisymmetry/Encastre → Encastre (U1 = U2 = U3 = UR1 = UR2 = UR3 = 0)***. The coordinate system has to be defined. The fixation is accomplished in the same way as in previous sections, but there have to be fixed all three parts. This is illustrated in the figure (3.34 a).

a) b)

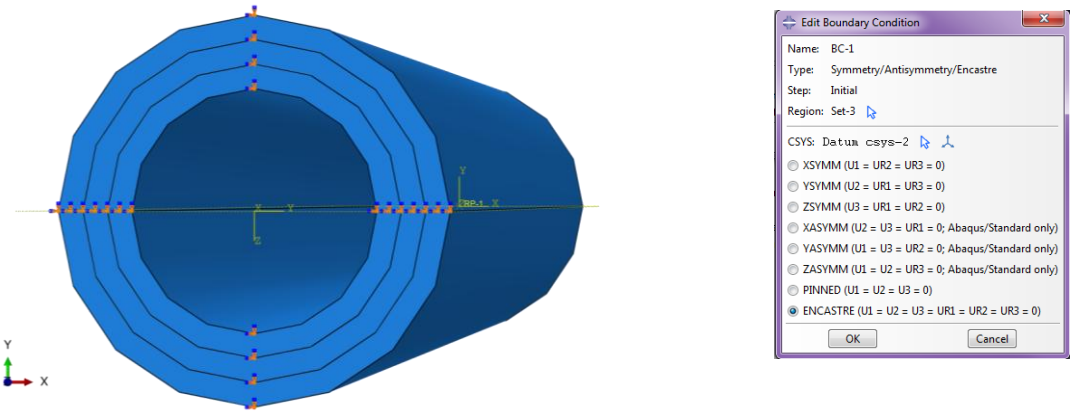


Figure 3.34: a) The beam with shown the fixation
b)The window for editing the boundary condition

Mesh

The element type C3D8R (three-dimensional hexahedral element) has been used for meshing the pipe. These elements are linear, reduced-integration elements. A good mesh of C3D8R elements usually provides a solution of accuracy at less cost. Quadrilateral and brick elements are preferred when such meshing is

reasonable. The size of elements has been chosen *0,005m* along the x-axis, quantity 40 elements along the perimeter as in the continuum shell model and five elements across each layer. It is not an optimal choice of size for this type of elements, because the aspect ratio should be less than three. But we do not review some local effects on the geometry, for calculating the deflection of the whole beam this selection of size of elements does not distort the calculation in the general scale.

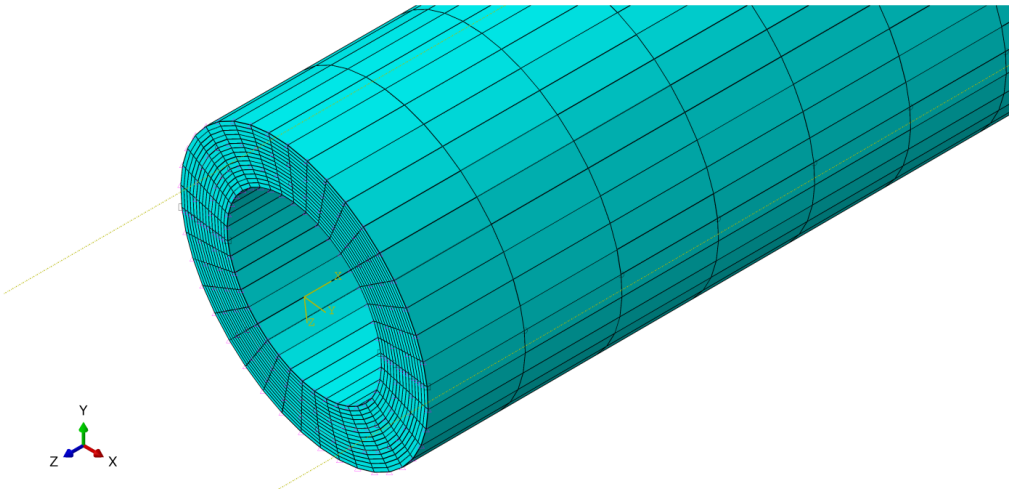


Figure 3.35: The meshed beam

Job

The calculation was carried out without errors. Details can be seen in the figure (3.36).

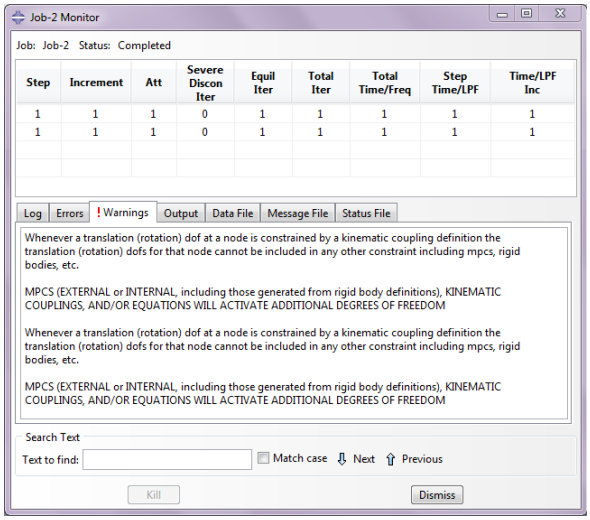


Figure 3.36: The listing of the calculation

Visualization

Deformation of the beam was most reflected as it had been expected in the direction of the concentrated force. It is evident from the figure (3.37 a).

The value of the deflection is different in individual nodes in the whole cross-section. This is shown in the figure (3.38). So the numerical size of the bending was calculated as the average of values of the deformation in the direction of the *y*-axis of individual nodes at the end of the beam.

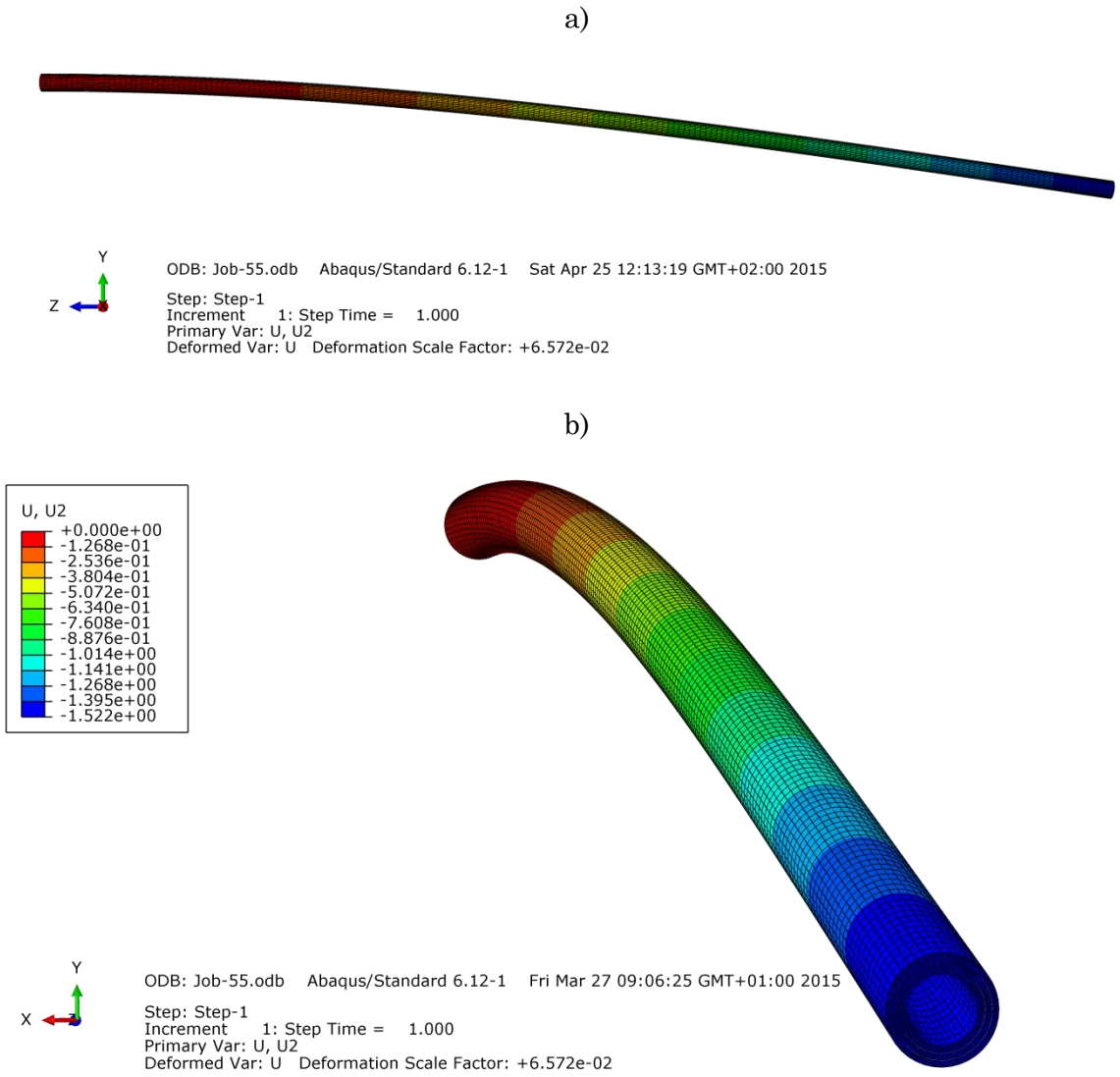


Figure 3.37: a) The deformed beam shown in the *yz* plane
b) The deformed beam in a general perspective with the scale

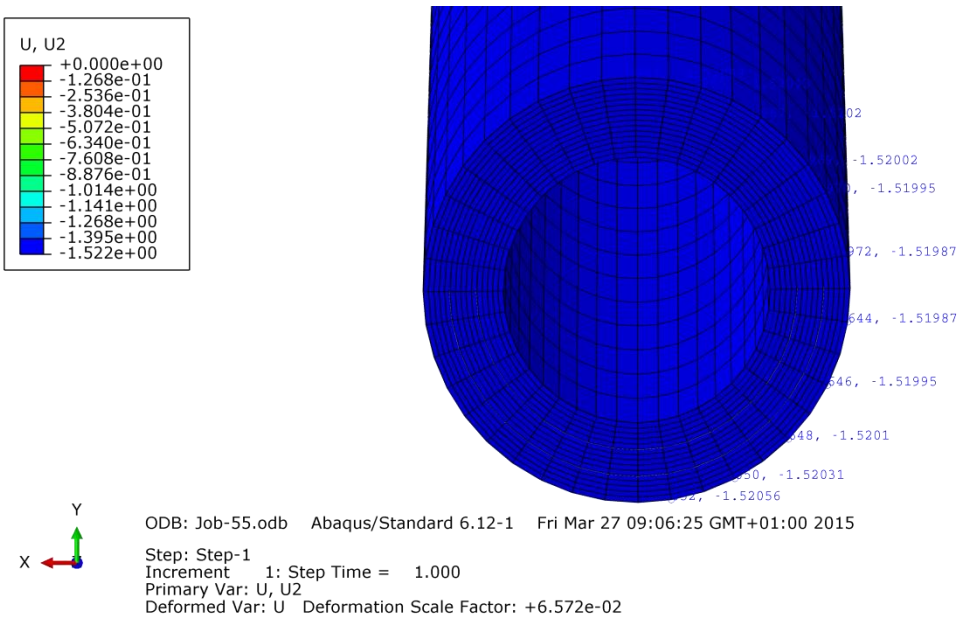


Figure 3.38: The detail of the end of the deformed beam with the values of the deflection

4 Results

The beam bending has been calculated by methods mentioned in the previous chapter. For all analyzed methods was used the same model of the beam and the same properties of a material, which they are specified in the table of Input data 3.1. All obtained values of the deflection are listed in the table 4.1 below.

Deflection (m)	Angles of fibers (°)										
d (mm)	0	5	15	25	35	45	55	65	75	85	90
2	1,13096	1,40923	3,53913	7,32757	12,00693	16,76582	20,95840	24,20021	26,34762	27,40277	27,53341
	1,13048	1,40875	3,53867	7,32714	12,00653	16,76543	20,95800	24,19978	26,34716	27,40230	27,53293
	3,25266	3,31299	3,69472	4,29963	5,10820	6,15034	7,51986	9,46172	12,94770	20,79490	22,20870
	1,46525	1,51444	1,86764	2,65495	4,28955	7,44343	12,89239	20,64141	28,39480	32,72279	33,20780
	1,14427	1,34350	2,90620	5,96703	10,29213	15,28243	20,06958	23,90649	26,45070	27,68670	27,83870
	1,56441	1,58724	1,79391	2,35681	3,68736	6,67597	12,49878	20,26116	25,64641	27,38751	27,53341
	1,56393	1,58676	1,79343	2,35633	3,68688	6,67549	12,49830	20,26068	25,64593	27,38703	27,53293
4	0,49799	0,62015	1,55082	3,18738	5,17623	7,16287	8,88379	10,19605	11,05663	11,47698	11,52891
	0,49765	0,61981	1,55049	3,18707	5,17594	7,16259	8,88350	10,19574	11,05630	11,47664	11,52857
	0,87850	0,89110	0,97658	1,11916	1,31141	1,55444	1,86219	2,27442	2,95759	4,28350	4,49091
	0,52150	0,61073	0,72800	1,00380	1,57880	2,71441	4,76785	7,82428	10,93090	12,61133	12,79300
	0,50392	0,58797	1,25272	2,55538	4,38779	6,48629	8,47878	10,05701	11,09211	11,59113	11,65230
	0,65519	0,66475	0,75129	0,98698	1,54411	2,79550	5,23362	8,48388	10,73878	11,46782	11,52891
	0,65485	0,66441	0,75094	0,98664	1,54377	2,79516	5,23328	8,48354	10,73844	11,46748	11,52857
6	0,26008	0,32371	0,80656	1,64748	2,65569	3,64805	4,49602	5,13549	5,55155	5,75384	5,77878
	0,25982	0,32345	0,80630	1,64724	2,65546	3,64783	4,49579	5,13525	5,55130	5,75357	5,77852
	0,36338	0,36733	0,39642	0,44870	0,52163	0,61385	0,72847	0,87695	1,11126	1,53335	1,59487
	0,29824	0,30421	0,35502	0,47876	0,73609	1,24947	2,20933	3,70184	5,25468	6,07535	6,16102
	0,26329	0,30615	0,64649	1,31198	2,24176	3,29670	4,28747	5,06371	5,56819	5,80997	5,83954
	0,32850	0,33329	0,37666	0,49480	0,77406	1,40129	2,62337	4,25251	5,38274	5,74816	5,77878
	0,32823	0,33302	0,37640	0,49454	0,77379	1,40103	2,62310	4,25224	5,38248	5,74790	5,77852
8	0,15213	0,18927	0,47015	0,95559	1,53139	2,09165	2,56538	2,91960	3,14870	3,25969	3,27336
	0,15192	0,18905	0,46994	0,95539	1,53121	2,09147	2,56519	2,91941	3,14849	3,25948	3,27315
	0,18496	0,18653	0,19888	0,22281	0,25752	0,30181	0,35645	0,42580	0,53167	0,71419	0,73967
	0,16994	0,17288	0,19891	0,26393	0,39897	0,66936	1,18597	2,01830	2,90580	3,36802	3,41484
	0,15411	0,17880	0,37532	0,75858	1,29067	1,88934	2,44636	2,87889	3,15801	3,29119	3,30745
	0,18614	0,18885	0,21342	0,28034	0,43852	0,79381	1,48603	2,40883	3,04903	3,25602	3,27336
	0,18592	0,18864	0,21320	0,28012	0,43830	0,79359	1,48581	2,40861	3,04882	3,25580	3,27315
10	0,09642	0,11991	0,29713	0,60153	0,95954	1,30475	1,59425	1,80932	1,94778	2,01468	2,02291
	0,09624	0,11973	0,29695	0,60136	0,95938	1,30459	1,59409	1,80916	1,94760	2,01450	2,02273
	0,10686	0,10759	0,11371	0,12634	0,14530	0,16976	0,19990	0,23770	0,29409	0,38860	0,40141
	0,10585	0,10750	0,12245	0,16051	0,23960	0,39822	0,70552	1,21414	1,76855	2,05482	2,08116
	0,09775	0,11324	0,23664	0,47675	0,80829	1,17870	1,52069	1,78437	1,95359	2,03406	2,04387
	0,11508	0,11676	0,13194	0,17329	0,27104	0,49060	0,91838	1,48865	1,88428	2,01220	2,02291
	0,11490	0,11657	0,13176	0,17311	0,27086	0,49042	0,91820	1,48847	1,88410	2,01201	2,02273

Bernoulli's method
Bernoulli's method without the shear
conventional shell
continuum shell
volume model
ABD matrix
ABD matrix without the shear

Table 4.1: The numerical results of the analysis of the methods for calculation the deflection

For both analytical methods (Bernoulli's method and the method using ABD matrix) the deflection was computed with the shear effect as it is written in the

chapter 2 and the bending without considering the shear effect. The values without consideration of the shear effect are in both cases lower but still very close to the values, where the shear is included. In the graphical results only the cases that they include the shear effect are mentioned.

All obtained data are put into the graphs in figures (4.1-5) to compare the individual methods of the deflection calculation. In the figures are compared the methods that are presented in this thesis with other existing methods used for calculation of the deflection.

The curves vC (vC) and vE (vE) are computed using the longitudinal modulus of elasticity E_L (for vE) and the transversal modulus of elasticityE_T (for vC) of composite material.[7] They operate with the highest and the lowest possible values of the modulus of elasticity so these methods give the upper and the lower bounds of the deflection in all cases. The assumption is that all other methods should give the results, which values will be between the mentioned curves. This assumption is the most fulfilled as is seen in the figures (4.1-5). The curves vSE ( ) and vK ( ) are computed in programs according the methods described in literature [7]. These are other methods to calculation of the deflection to comparison.

The results from the Bernoulli's method ( ) are very close to the calculations using the transversal modulus of elasticity E_T. This corresponds to the use of compliance matrix $\boldsymbol{C}$ to obtain the equivalent modulus of elasticity E_{eq} as it is written in the section 3.2. The results from calculation of bending using ABD matrix ( ) are fully consistent with the control calculation using the same method. Generally, this method is closer to the model with the higher values of equivalent modulus of elasticity, which corresponds to a lower deflection.

From the models using FEM gives the best results the classical volume model ( ). It gives in all cases the results that are close to the Bernoulli's method and the vC curve. The model of the continuum shell ( ) seems to approximate the calculation using ABD matrix; but this method, when it used the higher angle of direction of fibres, gives much greater values than all other methods. This is shown mainly in the cases with the smaller inner diameter. With the increasing inner diameter of the pipe this variation is narrowing. This is seen from the

comparing of the figures below. The deflection obtained using the conventional shell model (—▲—) gives the data, which can be considered unlikely. In the case of the pipe with 2mm inner diameter the data of the deflection are approximate to the other methods but in other cases this method gives much lower values of the deflection than the other methods. The cause of such unsatisfactory results should be subjected to the further research. But at present we do not believe that the reason is in the wrong compilation of the model.

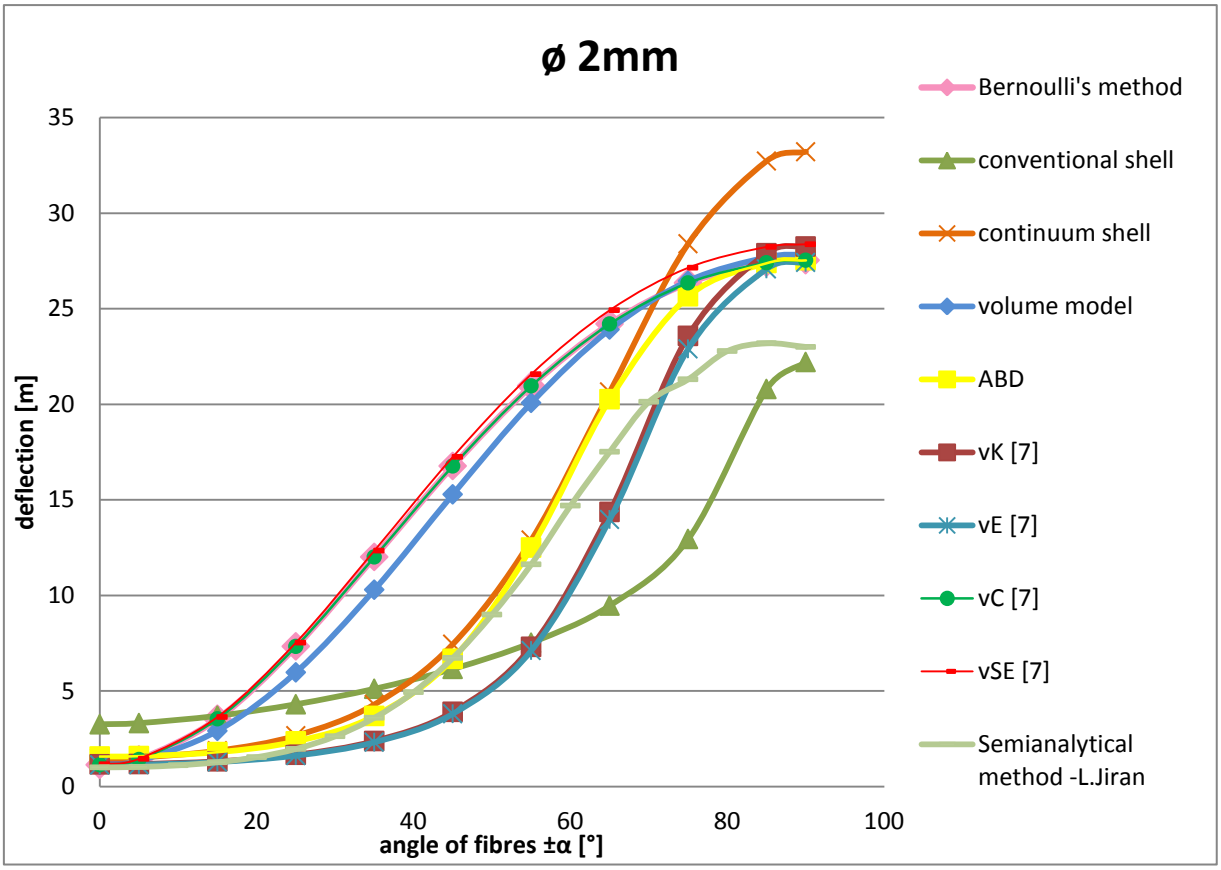


Figure 4.1: The graph of the deflection of the pipe with 2mm inner diameter

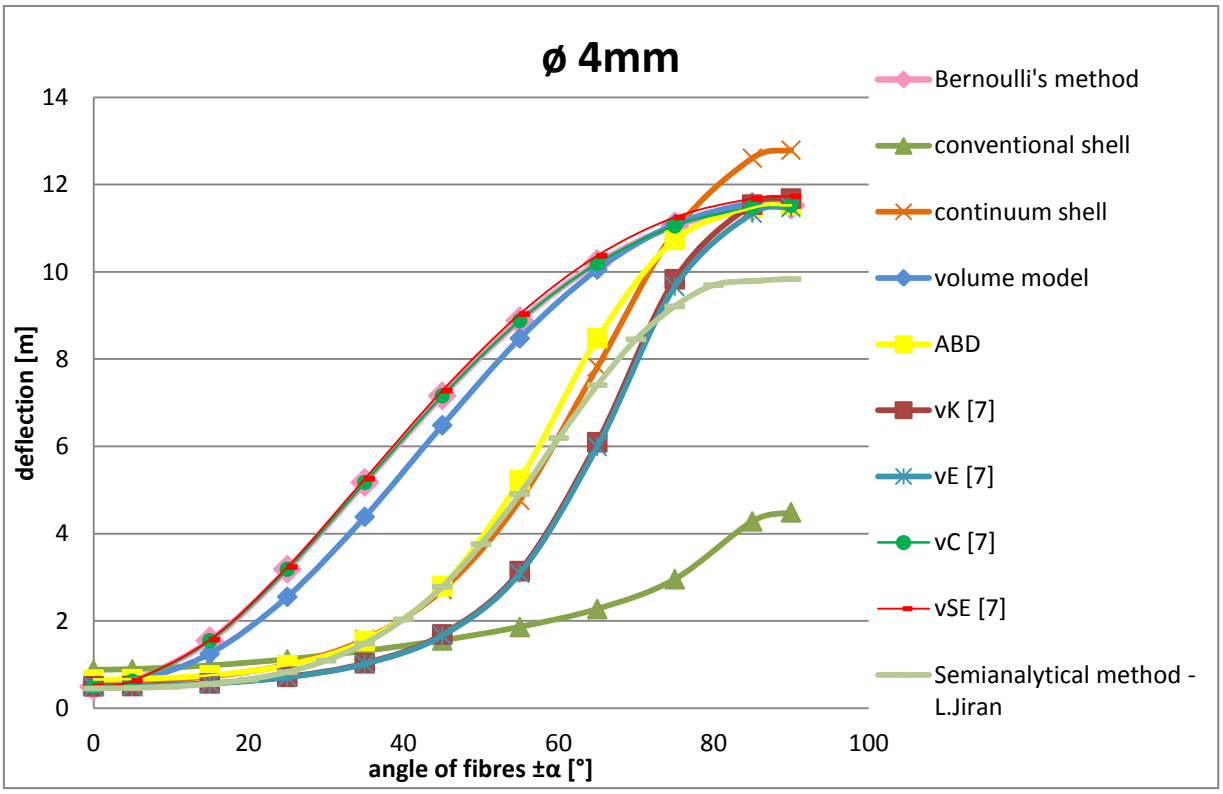


Figure 4.2: The graph of the deflection of the pipe with 4mm inner diameter

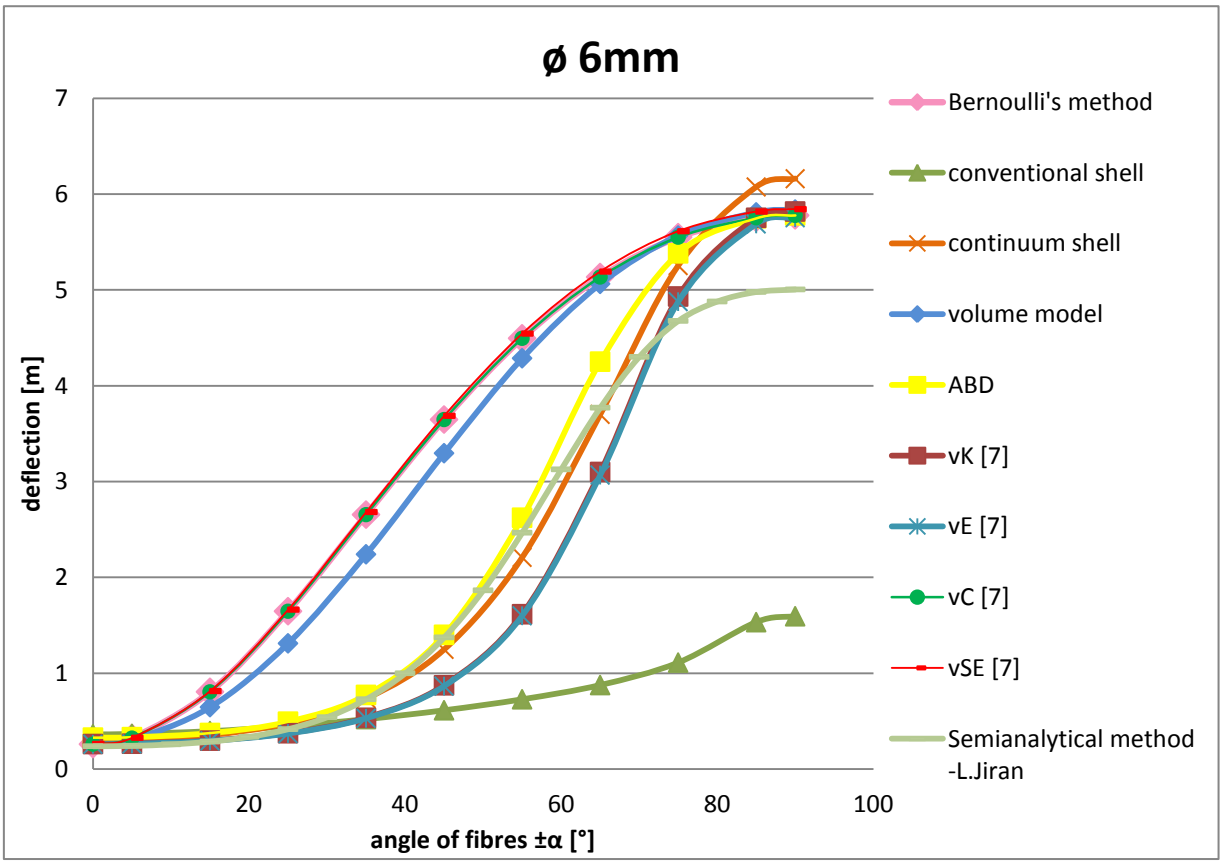


Figure 4.3: The graph of the deflection of the pipe with 6mm inner diameter

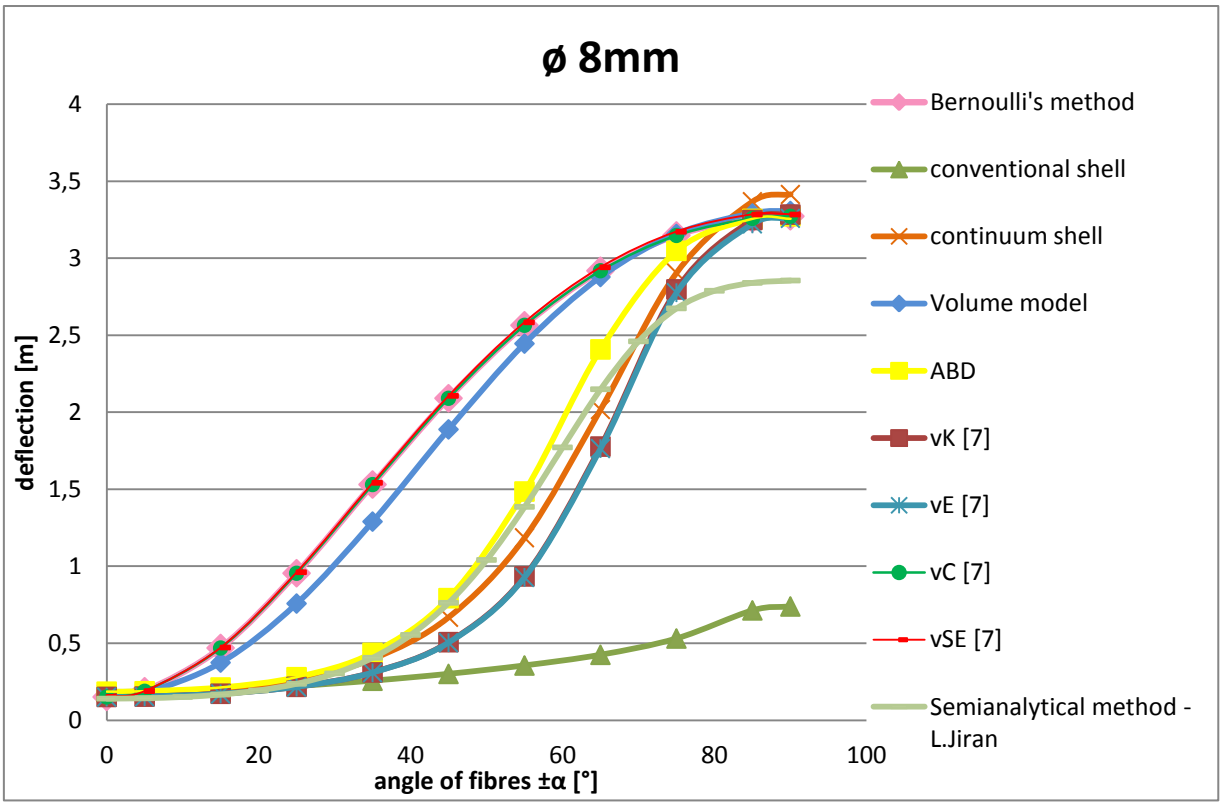


Figure 4.4: The graph of the deflection of the pipe with 8mm inner diameter

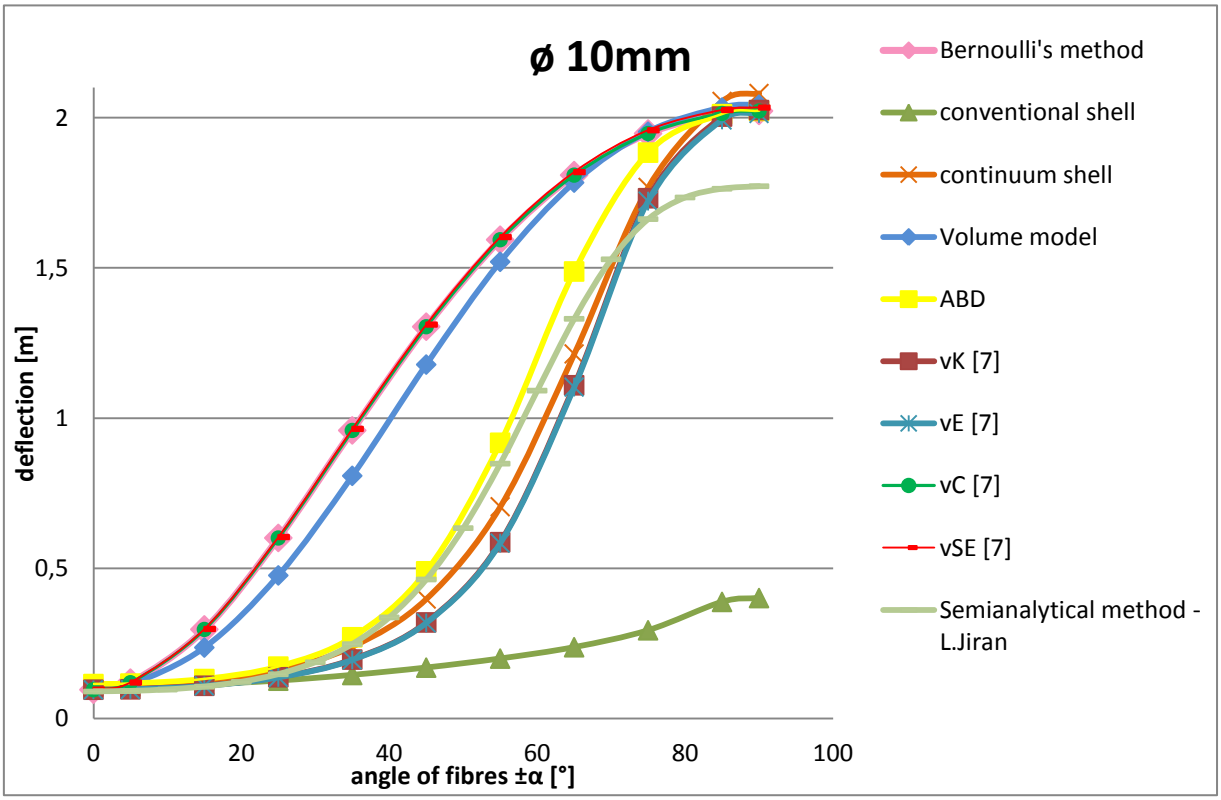


Figure 4.5: The graph of the deflection of the pipe with 10mm inner diameter

5 Conclusion

The aim of this study was to compare the analytical methods of calculation of the deflection of the beam with the results of the modelling of the same by using FEM. In this thesis were compared the Bernoulli's methods with the calculation using ABD matrix and with the FEM models of the conventional shell, the continuum shell and the volume model. The analytical models were performed in MATLAB® and the FEM models were implemented in Abaqus®.

All discussed methods were applied to several beams of a circular cross section with a different diameter and with the various layup of the composite material. The analytical methods give the results that were expected and show a material behaviour predicted by other theories. The FEM models give more interesting results. Results calculated using a volume model are the closest to the results of analytical methods. The continuum shell model gives the comparable results, but in some cases considerably deviates from the solution calculated by analytical methods. The conventional shell model provides the completely different outcomes from the other methods. It is recommended for a further research.

All results were presented both graphically and in the tabular form. The calculation methods with specific comments on the creation of individual programs and FEM models are described in detail in this thesis. The codes in MATLAB® and scripts of the models, which are created in Abaqus®, including the input files are attached in the annexes.

The analysis made in this thesis does not include all the existing methods for the calculation of the deflection of the composite materials; it would be appropriate the extension by another methods and the concrete comparison with an experiment would be appropriate.

List of Literature

[1] LAŠ, Vladislav. *Mechanika kompozitních materiálů*. 2. přeprac. vyd. Plzeň: Západočeská univerzita, 2008. 200s. ISBN 978-80-7043-689-9.

[2] MICHALEC, Jiří a kol. *Pružnost a pevnost I*. Vyd. 3. Praha: ČVUT, 2009. 308 s. ISBN 978-80-01-04224-3.

[3] ŘEZNÍČEK, Jan a Jitka ŘEZNÍČKOVÁ. *Pružnost a pevnost v technické praxi: příklady III*. Praha: Česká technika - nakladatelství ČVUT, 2008. 73 s. ISBN 978-80-01-03947-2.

[4] BARBERO, Ever J. *Introduction to composite materials design*. 2nd ed. Boca Raton: Taylor & Francis, ©2011. xxxix, 520 s. ISBN 978-1-4200-7915-9.

[5] GAY, Daniel. *Composite materials: design and applications*. Third ed. Boca Raton: CRC Press/Taylor & Francis, ©2015. xxiii, 611 s. ISBN 978-1-4665-8487-7.

[6] ZAVŘELOVÁ, Tereza. *Model ohybu kompozitního nosníku*. Praha, 2013. Bakalářská práce. České vysoké učení technické v Praze. Fakulta strojní. Vedoucí práce Karel DOUBRAVA.

[7] KULÍŠEK Viktor, Lukáš JIRAN, Tomáš MAREŠ, Milan RŮŽIČKA a kol. DV#7-1 (2012) Verifikované výpočetní modely pro predikci tuhosti a modálních vlastností komponent z nekonvenčních materiálů. Praha, 2012. Neveřejná výzkumná zpráva č. V-12-060. ČVUT v Praze. Fakulta strojní, Ústav výrobních strojů a zařízení.

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1.1 Program designed in MATLAB® using Bernoulli's method: DP_Trubka.m

```
clear all
close all
clc
%Vektor uhlu vlaken
Alfa=[0  5  15  25  35  45  55  65  75  85  90];

%Vektor prumeru
D=(1:20)*1e-3;

%Pocet promennych
K=size(D);
K=K(2);

Q=size(Alfa);
Q=Q(2);

%Pole pro vysledky
V_vysledky=zeros(K,Q);
W_vysledky=zeros(K,Q);

for k=1:K
   for q=1:Q


%VSTUPY
%sila F [N]
F=100;

%GEOMETRIE
%delka l [m]
l=1;
%polomery -  vnitrni, vnejsi [m]
r1=D(k)/2;
%r2=(30e-3)/2;

%EL, Et, Glt, vlt, alfa, t[m] v tabulce/  matici - poradi je zavazne
V=[156.05e9  6.045e9  4.431e9  0.328  90  1e-3;
   156.05e9  6.045e9  4.431e9  0.328  Alfa(q)  1e-3;
   156.05e9  6.045e9  4.431e9  0.328  -Alfa(q)  1e-3;]; %- V jako vstupy

%N- pocet vrstev
N=size(V,1);
%soucinitel beta
beta= 1 ;
```

```
%pole pro ulozeni dat pro jdn. vrstvy - mozna nadbytecne
E=zeros(3,3,N);
T=zeros(3,3,N);
%J=zeros(N,1);
%Ex=zeros(N,1);
Cxy=zeros(3,3,N);
Gxy=zeros(N,1);
A=zeros(N,1);

%vsechny potrebne prumery
d=zeros(N+1,1);
d(1)=2*r1;

for i=1:N

    %sestaveni Ci
    e11=V(i,1)/(1-(V(i,2)/V(i,1))*V(i,4)^2);
    e22=e11*(V(i,2)/V(i,1));
    e12=V(i,4)*e22;
    e21=e12;
    e66=V(i,3);

    E(:,:,i)=[e11 e12 0;
              e21 e22 0;
              0   0   e66];

    %sestaveni Ti
    alfa=V(i,5)/180*pi;

    T(:,:,i)=[ (cos(alfa))^2              (sin(alfa))^2            -2*sin(alfa)*cos(alfa);
        (sin(alfa))^2              (cos(alfa))^2            2*sin(alfa)*cos(alfa);
        sin(alfa)*cos(alfa)   -sin(alfa)*cos(alfa)   (cos(alfa))^2-(sin(alfa))^2];


    %sestaveni Ji
    d(i+1)=d(i)+2*V(i,6);
    J(i)=(pi*(d(i+1)^4/64))*(1-(d(i)/d(i+1))^4);

    %transformace Ei do systemu xy
    Exy(:,:,i)=T(:,:,i)*E(:,:,i)*T(:,:,i)';

    %Matice poddajnosti
    Cxy(:,:,i)=inv(Exy(:,:,i));

    %Vektor hodnot modulu pruznosti v tahu Exi =[Ex1 Ex2 ....ExN]
    exi=1/Cxy(1,1,i);
    Ex(i)=exi;
```

```
     %plocha pro Gxy*A
    A(i)=(pi()/4)*(d(i+1)^2-d(i)^2);

    %modul pruznosti ve smyku
    gxyi=1/Cxy(3,3,i);
    Gxy(i)=gxyi;

    end

      %Ohybova tuhost
      EJ=Ex*J' ;%vychazim z toho, ze fce zeros generuje radkove vektory(?)

      %Soucin Gxy*A
      GxyA=Gxy'*A;

      %beta/Gxy*A
      betaGA=beta./GxyA;

      %PRUHYB
      x=0:l/10:l;

      %vypocet Mohr. integral
      v=(1/EJ)*((F*l^2*x)-(F*l*x.^2)-(F*x.^3)/3);

      %vypocet se zahrnutim smyku
      w=(1/EJ)*((F*l^2*x)-(F*l*x.^2)-(F*x.^3)/3)+betaGA*F*x;

      %Tabulky vysledku - matice

      V_vysledky(k,q)=v(1);
      W_vysledky(k,q)=w(1);
    end
end
%save ('DP_Trubka_uhly_prumery', 'V_vysledky', 'W_vysledky'  )
```

1.2 Program designed in MATLAB® using ABD matrices: DP_ABD_Trubka.m

```
clc
clear all
close all
%% Vstupy
 %Vektor uhlu vlaken
Alfa=[0  5  15  25  35  45  55  65  75  85  90];

%Vektor prumeru
P=(1:10)*1e-3;

%Pocet promennych
K=size(P);
K=K(2);

U=size(Alfa);
U=U(2);

%Pole pro vysledky
V_vysledky=zeros(K,U);
W_vysledky=zeros(K,U);

for s=1:K
    for t=1:U

%VSTUPZ
%Sila
F=100;

%GEOMETRIE
%delka l [m]
l=1;
%polomery -  vnitrni, vnejsi [m]
r1=P(s)/2;
%r2=(30e-3)/2;

% Matice vstupu pro jdntl. vrstvy
%    1     2      3      4        5           6
%   EL   ET   GLT   vLT   alfa(°)     t(m)
V=[156.05e9  6.045e9  4.431e9  0.328   90  1e-3;
     156.05e9  6.045e9  4.431e9  0.328  Alfa(t)   1e-3;
     156.05e9  6.045e9  4.431e9  0.328  -Alfa(t)  1e-3;]; %- V jako vstupy

%pocet vrstev k
xx=size(V);
k=xx(1);
```

```
%pole pro ukladani mezivypoctu
Tvx=zeros(6,6,k);
c=zeros(6,6,k);
C=zeros(3,3,k);
Cxy=zeros(3,3,k);
Q=zeros(3,3,k);
q=zeros(6,6,k);
A=zeros(3,3);
B=zeros(3,3);
D=zeros(3,3);

%% Matice tuhosti
for i=1:k

c(1,1,i)=V(i,1)/(1-((V(i,2)/V(i,1))*V(i,4)^2));
c(2,2,i)=V(i,2)/V(i,1)*c(1,1,i);
c(1,2,i)=V(i,4)*c(2,2,i);
c(2,1,i)=c(1,2,i);
c(6,6,i)=V(i,3);

C(:,:,i)=[c(1,1,i) c(1,2,i) 0;
          c(2,1,i) c(2,2,i) 0;
          0        0    c(6,6,i);];

%% Transformace matice tuhosti

% Transformacni matice ze systemu LTt do systemu xyz
   alfa=V(i,5)/180*pi;

   Ts(:,:,i)=[ (cos(alfa))^2              (sin(alfa))^2            2*sin(alfa)*cos(alfa);
       (sin(alfa))^2              (cos(alfa))^2            -2*sin(alfa)*cos(alfa);
       -sin(alfa)*cos(alfa)    sin(alfa)*cos(alfa)   (cos(alfa))^2-(sin(alfa))^2];

  Te(:,:,i)=[ (cos(alfa))^2              (sin(alfa))^2            sin(alfa)*cos(alfa);
       (sin(alfa))^2              (cos(alfa))^2            -sin(alfa)*cos(alfa);
       -2*sin(alfa)*cos(alfa)    2*sin(alfa)*cos(alfa)   (cos(alfa))^2-(sin(alfa))^2];
Cxy(:,:,i)=inv(Ts(:,:,i))*C(:,:,i)*Te(:,:,i);%(Tvx)^-1*C*Tvx'    ?????    \Tvx    !!!!!=Q    pro
rovinnou napjatost

end

%% Matice ABD
 h(1)=-sum(V(:,6))/2;%h0

for n=1:k;
      for m=1:k;
        for i=1:k;

            %Vsechna h
              h(i+1)=(h(1)+sum(V(1:i,6)));
```

```
            a(i)=(Cxy(n,m,i)*(h(i+1)-h(i)));
            b(i)=((1/2)*Cxy(n,m,i)*(h(i+1)^2-h(i)^2));
            d(i)=((1/3)*Cxy(n,m,i)*(h(i+1)^3-h(i)^3));
          end

A(n,m)=sum(a);
B(n,m)=sum(b);
D(n,m)=sum(d);
       end
end

%Modul pruznosti v tahu
E=(A(1,1)-A(1,2:3)*(A(2:3,2:3)^(-1)*A(2:3,1)))/sum(V(:,6));

%Modul pruznosti ve smyku
G=(A(3,3)-A(3,1:2)*(A(1:2,1:2)^(-1)*A(1:2,3)))/sum(V(:,6));

%vsechny potrebne prumery
dd=zeros(k+1,1);
dd(1)=2*r1;

%sestaveni Ji a GA
 for i=1:k
   dd(i+1)=dd(i)+2*V(i,6);
   j(i)=(pi*(dd(i+1)^4/64))*(1-(dd(i)/dd(i+1))^4);

   %plocha pro G*A
      S(i)=(pi/4)*(dd(i+1)^2-dd(i)^2);

   %modul pruznosti ve smyku
%     gxyi=1/Cxy(6,6,i);
%     Gxy(i)=gxyi;

 end

 J=sum(j);
S_celk=sum(S);

 %PRUHYB
    x=0:l/10:l;

    %vypocet Mohr. integral
    v=(1/(E*J))*((F*l^3)/3-(F*l^2*x)/2-(F*x.^3)/3+(F*x.^3)/2);

   %vypocet se zahrnutim smyku

    %soucinitel beta
     beta= 1 ;
```

```
      %Soucin Gxy*A
    GxyA=G*S_celk;

    %beta/Gxy*A
    betaGA=beta/GxyA;

 w=(1/(E*J))*((F*l^3)/3-(F*l^2*x)/2-(F*x.^3)/3+(F*x.^3)/2)+betaGA*F*(l-x);

    %Tabulky vysledku - matice
    V_vysledky_G(s,t)=v(1);
    W_vysledky_G(s,t)=w(1);

  end
end

save('DP_ABD_Trubka_uhly_prumery', 'V_vysledky_G', 'W_vysledky_G')
```

1.3 Script for FEM model using conventional shell

```
#prumer
d=0.008
x=d*0.1
r=d/2
#uhel
a=0

# -*- coding: mbcs -*-
from part import *
from material import *
from section import *
from assembly import *
from step import *
from interaction import *
from load import *
from mesh import *
from optimization import *
from job import *
from sketch import *
from visualization import *
from connectorBehavior import *

mdb.models['Model-1'].ConstrainedSketch(name='__profile__', sheetSize=2.0)
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-0.1, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[2], radius=0.005,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].sketches['__profile__'].delete(objectList=(
    mdb.models['Model-1'].sketches['__profile__'].dimensions[0], ))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[2], radius=r,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].Part(dimensionality=THREE_D, name='Part-1', type=
    DEFORMABLE_BODY)
mdb.models['Model-1'].parts['Part-1'].BaseShellExtrude(depth=1.0, sketch=
    mdb.models['Model-1'].sketches['__profile__'])
del mdb.models['Model-1'].sketches['__profile__']
mdb.models['Model-1'].parts['Part-1'].DatumCsysByThreePoints(coordSysType=
    CARTESIAN, name='Datum csys-1', origin=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[0], CENTER), point1=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[1], CENTER), point2=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[0], MIDDLE))
```

```
mdb.models['Model-1'].Material(name='Material-1')
mdb.models['Model-1'].materials['Material-1'].Density(table=((1474.0, ), ))
mdb.models['Model-1'].materials['Material-1'].Elastic(table=((156050000000.0,
    6045000000.0, 0.328, 4431000000.0, 4431000000.0, 4431000000.0), ), type=
    LAMINA)
mdb.models['Model-1'].parts['Part-1'].DatumPlaneByThreePoints(point1=
    mdb.models['Model-1'].parts['Part-1'].vertices[0], point2=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[0], MIDDLE), point3=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[1], MIDDLE))
mdb.models['Model-1'].parts['Part-1'].PartitionFaceByDatumPlane(datumPlane=
    mdb.models['Model-1'].parts['Part-1'].datums[3], faces=
    mdb.models['Model-1'].parts['Part-1'].faces.getSequenceFromMask(('[#1 ]',      ),
))
mdb.models['Model-1'].parts['Part-1'].CompositeLayup(description='',
    elementType=SHELL,                                  name='CompositeLayup-1',
offsetType=BOTTOM_SURFACE,
    symmetric=False, thicknessAssignment=FROM_SECTION)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].Section(
    integrationRule=SIMPSON, poissonDefinition=DEFAULT, preIntegrate=OFF,
    temperature=GRADIENT, thicknessType=UNIFORM, useDensity=OFF)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].ReferenceOrientation(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_2, fieldName='', localCsys=
    mdb.models['Model-1'].parts['Part-1'].datums[2], orientationType=SYSTEM)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',         additionalRotationType=ROTATION_ANGLE,
angle=
    90.0, axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[2], orientationType=CSYS,
    plyName='Ply-1', region=Region(
    faces=mdb.models['Model-1'].parts['Part-1'].faces.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=a
    , axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
```

```
    mdb.models['Model-1'].parts['Part-1'].datums[2], orientationType=CSYS,
    plyName='Ply-2', region=Region(
    faces=mdb.models['Model-1'].parts['Part-1'].faces.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='', additionalRotationType=ROTATION_NONE,
angle=-a
    , axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[2], orientationType=CSYS,
    plyName='Ply-3', region=Region(
    faces=mdb.models['Model-1'].parts['Part-1'].faces.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].rootAssembly.DatumCsysByDefault(CARTESIAN)
mdb.models['Model-1'].rootAssembly.Instance(dependent=OFF, name='Part-1-1',
    part=mdb.models['Model-1'].parts['Part-1'])
mdb.models['Model-1'].StaticStep(name='Step-1', previous='Initial')
mdb.models['Model-1'].rootAssembly.ReferencePoint(point=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].InterestingPoint(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].edges[3], CENTER))
mdb.models['Model-1'].rootAssembly.Set(name='m_Set-1', referencePoints=(
    mdb.models['Model-1'].rootAssembly.referencePoints[4], ))
mdb.models['Model-1'].rootAssembly.Set(edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#18 ]', ), ), name='s_Set-1')
mdb.models['Model-1'].Coupling(controlPoint=
    mdb.models['Model-1'].rootAssembly.sets['m_Set-1'],
couplingType=KINEMATIC,
    influenceRadius=WHOLE_SURFACE, localCsys=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].datums[2], name=
    'Constraint-1', surface=mdb.models['Model-1'].rootAssembly.sets['s_Set-1'],
    u1=ON, u2=ON, u3=ON, ur1=ON, ur2=ON, ur3=ON)
mdb.models['Model-1'].fieldOutputRequests['F-Output-1'].setValues(variables=(
    'U', 'UT', 'UR'))
mdb.models['Model-1'].rootAssembly.Set(name='Set-3', referencePoints=(
    mdb.models['Model-1'].rootAssembly.referencePoints[4], ))
mdb.models['Model-1'].ConcentratedForce(cf3=100.0, createStepName='Step-1',
    distributionType=UNIFORM, field='', localCsys=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].datums[2], name=
    'Load-1', region=mdb.models['Model-1'].rootAssembly.sets['Set-3'])
mdb.models['Model-1'].rootAssembly.Set(edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#22 ]', ), ), name='Set-4')
mdb.models['Model-1'].EncastreBC(createStepName='Step-1', localCsys=
```

```
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].datums[2], name=
    'BC-1', region=mdb.models['Model-1'].rootAssembly.sets['Set-4'])
mdb.models['Model-1'].rootAssembly.setElementType(elemTypes=(ElemType(
    elemCode=S4R, elemLibrary=STANDARD, secondOrderAccuracy=OFF,
    hourglassControl=DEFAULT), ElemType(elemCode=S3,
elemLibrary=STANDARD)),
    regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].faces.getSequenceFromMask(
    ('[#3 ]', ), ), ))
mdb.models['Model-1'].rootAssembly.setMeshControls(regions=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].faces.getSequenceFromMask(
    ('[#3 ]', ), ), technique=STRUCTURED)
mdb.models['Model-1'].rootAssembly.seedPartInstance(deviationFactor=0.1,
    minSizeFactor=0.1, regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'], ), size=0.005)
mdb.models['Model-1'].rootAssembly.seedEdgeBySize(constraint=FINER,
    deviationFactor=0.1, edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#3a ]', ), ), minSizeFactor=0.1, size=x)
mdb.models['Model-1'].rootAssembly.generateMesh(regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'], ))
mdb.Job(atTime=None, contactPrint=OFF, description='', echoPrint=OFF,
    explicitPrecision=SINGLE, getMemoryFromAnalysis=True, historyPrint=OFF,
    memory=50, memoryUnits=PERCENTAGE, model='Model-1',
modelPrint=OFF,
    multiprocessingMode=DEFAULT, name='Job-0',
nodalOutputPrecision=SINGLE,
    numCpus=1, numGPUs=0, queue=None, scratch='', type=ANALYSIS,
    userSubroutine='', waitHours=0, waitMinutes=0)
mdb.jobs['Job-0'].submit(consistencyChecking=OFF)
mdb.jobs['Job-0']._Message(STARTED, {'phase': BATCHPRE_PHASE,
    'clientHost': 'ntb-HPPB4310s', 'handle': 0, 'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(WARNING, {'phase': BATCHPRE_PHASE,
    'message': 'WHENEVER A TRANSLATION (ROTATION) DOF AT A NODE IS
CONSTRAINED BY A KINEMATIC COUPLING DEFINITION THE
TRANSLATION (ROTATION) DOFS FOR THAT NODE CANNOT BE
INCLUDED IN ANY OTHER CONSTRAINT INCLUDING MPCS, RIGID
BODIES, ETC.',
    'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(WARNING, {'phase': BATCHPRE_PHASE,
    'message': 'MPCS (EXTERNAL or INTERNAL, including those generated from
rigid body definitions), KINEMATIC COUPLINGS, AND/OR EQUATIONS WILL
ACTIVATE ADDITIONAL DEGREES OF FREEDOM',
    'jobName': 'Job-0'})
```

```
mdb.jobs['Job-0']._Message(ODB_FILE, {'phase': BATCHPRE_PHASE,
    'file': 'C:\\Temp\\Job-0.odb', 'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(COMPLETED, {'phase': BATCHPRE_PHASE,
    'message': 'Analysis phase complete', 'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(STARTED, {'phase': STANDARD_PHASE,
    'clientHost': 'ntb-HPPB4310s', 'handle': 5316, 'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(STEP, {'phase': STANDARD_PHASE, 'stepId': 1,
    'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(WARNING, {'phase': STANDARD_PHASE,
    'message': 'The 3-direction at one or more points in one or more layers in 3200
elements as defined in *ORIENTATION are in the opposite direction to the
element normals. Either the 1 or 2 and the 3-direction defined in
*ORIENTATION will be reversed. The elements have been identified in element
set WarnElem3DirOppElemNormalStep1Inc1.',
    'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(ODB_FRAME, {'phase': STANDARD_PHASE, 'step':
0,
    'frame': 0, 'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(STATUS, {'totalTime': 0.0, 'attempts': 0,
    'timeIncrement': 1.0, 'increment': 0, 'stepTime': 0.0, 'step': 1,
    'jobName': 'Job-0', 'severe': 0, 'iterations': 0, 'phase': STANDARD_PHASE,
    'equilibrium': 0})
mdb.jobs['Job-0']._Message(MEMORY_ESTIMATE, {'phase':
STANDARD_PHASE,
    'jobName': 'Job-0', 'memory': 132.121262550354})
mdb.jobs['Job-0']._Message(ODB_FRAME, {'phase': STANDARD_PHASE, 'step':
0,
    'frame': 1, 'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(STATUS, {'totalTime': 1.0, 'attempts': 1,
    'timeIncrement': 1.0, 'increment': 1, 'stepTime': 1.0, 'step': 1,
    'jobName': 'Job-0', 'severe': 0, 'iterations': 2, 'phase': STANDARD_PHASE,
    'equilibrium': 2})
mdb.jobs['Job-0']._Message(END_STEP, {'phase': STANDARD_PHASE, 'stepId':
1,
    'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(COMPLETED, {'phase': STANDARD_PHASE,
    'message': 'Analysis phase complete', 'jobName': 'Job-0'})
mdb.jobs['Job-0']._Message(JOB_COMPLETED, {'time': 'Thu Mar 12 16:21:06
2015',
    'jobName': 'Job-0'})
# Save by user on 2015_03_12-16.23.10; build 6.12-1 2012_03_13-20.44.39 119612
```

1.4 Script for FEM model using continuum shell

```
#prumer
d=0.002
r=d/2
k=r+0.003
#uhel
a=90

# Save by user on 2015_03_11-14.24.50; build 6.12-1 2012_03_13-20.44.39 119612
from part import *
from material import *
from section import *
from optimization import *
from assembly import *
from step import *
from interaction import *
from load import *
from mesh import *
from job import *
from sketch import *
from visualization import *
from connectorBehavior import *

mdb.models['Model-1'].ConstrainedSketch(name='__profile__', sheetSize=2.0)
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-2.5, 0.0))
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-1.25, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[2], radius=r,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[3], radius=k,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].Part(dimensionality=THREE_D, name='Part-1', type=
    DEFORMABLE_BODY)
mdb.models['Model-1'].parts['Part-1'].BaseSolidExtrude(depth=1.0, sketch=
    mdb.models['Model-1'].sketches['__profile__'])
del mdb.models['Model-1'].sketches['__profile__']
mdb.models['Model-1'].Material(name='Material-1')
mdb.models['Model-1'].materials['Material-1'].Density(table=((1474.0, ), ))
mdb.models['Model-1'].materials['Material-1'].Elastic(table=((156050000000.0,
    6045000000.0, 0.328, 4431000000.0, 4431000000.0, 4431000000.0), ), type=
    LAMINA)
```

```
mdb.models['Model-1'].parts['Part-1'].DatumPlaneByThreePoints(point1=
    mdb.models['Model-1'].parts['Part-1'].vertices[2], point2=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[2], MIDDLE), point3=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[3], MIDDLE))
mdb.models['Model-1'].parts['Part-1'].PartitionCellByDatumPlane(cells=
    mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(('[#1 ]',
    ), ), datumPlane=mdb.models['Model-1'].parts['Part-1'].datums[2])
mdb.models['Model-1'].parts['Part-1'].DatumCsysByThreePoints(coordSysType=
    CARTESIAN, name='Datum csys-1', origin=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[10], CENTER), point1=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[8], CENTER), point2=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[11], MIDDLE))
mdb.models['Model-1'].parts['Part-1'].CompositeLayup(description='',
    elementType=CONTINUUM_SHELL,                name='CompositeLayup-1',
symmetric=False)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].Section(
    integrationRule=SIMPSON, poissonDefinition=DEFAULT, preIntegrate=OFF,
    temperature=GRADIENT, thicknessModulus=None, useDensity=OFF)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].ReferenceOrientation(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_2, fieldName='', localCsys=
    mdb.models['Model-1'].parts['Part-1'].datums[4], orientationType=SYSTEM,
    stackDirection=STACK_3)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[4], orientationType=CSYS,
    plyName='Ply-1', region=Region(
    cells=mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
```

```
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[4], orientationType=CSYS,
    plyName='Ply-2', region=Region(
    cells=mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[4], orientationType=CSYS,
    plyName='Ply-3', region=Region(
    cells=mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].rootAssembly.DatumCsysByDefault(CARTESIAN)
mdb.models['Model-1'].rootAssembly.Instance(dependent=OFF, name='Part-1-1',
    part=mdb.models['Model-1'].parts['Part-1'])
mdb.models['Model-1'].StaticStep(name='Step-1', previous='Initial')
mdb.models['Model-1'].fieldOutputRequests['F-Output-1'].setValues(variables=(
    'U', 'UT', 'UR', 'RF', 'CF'))
mdb.models['Model-1'].rootAssembly.ReferencePoint(point=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].InterestingPoint(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].edges[8], CENTER))
mdb.models['Model-1'].rootAssembly.Set(name='m_Set-1', referencePoints=(
    mdb.models['Model-1'].rootAssembly.referencePoints[4], ))
mdb.models['Model-1'].rootAssembly.Surface(name='s_Surf-1', side1Faces=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].faces.getSequenceFromMask(
    ('[#204 ]', ), ))
mdb.models['Model-1'].Coupling(controlPoint=
    mdb.models['Model-1'].rootAssembly.sets['m_Set-1'],
couplingType=KINEMATIC,
    influenceRadius=WHOLE_SURFACE, localCsys=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].datums[4], name=
    'Constraint-1', surface=
    mdb.models['Model-1'].rootAssembly.surfaces['s_Surf-1'], u1=ON, u2=ON, u3=
    ON, ur1=ON, ur2=ON, ur3=ON)
mdb.models['Model-1'].rootAssembly.Set(faces=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].faces.getSequenceFromMask(
    ('[#110 ]', ), ), name='Set-2')
```

```
mdb.models['Model-1'].EncastreBC(createStepName='Step-1', localCsys=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].datums[4], name=
    'BC-1', region=mdb.models['Model-1'].rootAssembly.sets['Set-2'])
mdb.models['Model-1'].rootAssembly.Set(name='Set-3', referencePoints=(
    mdb.models['Model-1'].rootAssembly.referencePoints[4], ))
mdb.models['Model-1'].ConcentratedForce(cf2=100.0, createStepName='Step-1',
    distributionType=UNIFORM, field='', localCsys=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].datums[4], name=
    'Load-1', region=mdb.models['Model-1'].rootAssembly.sets['Set-3'])
mdb.models['Model-1'].rootAssembly.setElementType(elemTypes=(ElemType(
    elemCode=SC8R, elemLibrary=STANDARD, secondOrderAccuracy=OFF,
    hourglassControl=DEFAULT), ElemType(elemCode=SC6R,
elemLibrary=STANDARD),
    ElemType(elemCode=UNKNOWN_TET, elemLibrary=STANDARD)),
regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].cells.getSequenceFromMask(
    ('[#3 ]', ), ), ))
mdb.models['Model-1'].rootAssembly.seedPartInstance(deviationFactor=0.1,
    minSizeFactor=0.1, regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'], ), size=0.05)
mdb.models['Model-1'].rootAssembly.seedPartInstance(deviationFactor=0.1,
    minSizeFactor=0.1, regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'], ), size=0.005)
mdb.models['Model-1'].rootAssembly.seedEdgeBySize(constraint=FINER,
    deviationFactor=0.1, edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#3ff00 ]', ), ), minSizeFactor=0.1, size=0.001)
mdb.models['Model-1'].rootAssembly.seedEdgeBySize(constraint=FINER,
    deviationFactor=0.1, edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#3ff00 ]', ), ), minSizeFactor=0.1, size=0.01)
mdb.models['Model-1'].rootAssembly.seedEdgeBySize(constraint=FINER,
    deviationFactor=0.1, edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#3ff00 ]', ), ), minSizeFactor=0.1, size=0.005)
mdb.models['Model-1'].rootAssembly.seedEdgeBySize(constraint=FINER,
    deviationFactor=0.1, edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#3ff00 ]', ), ), minSizeFactor=0.1, size=0.0005)
mdb.models['Model-1'].rootAssembly.seedEdgeBySize(constraint=FINER,
    deviationFactor=0.1, edges=
```

```
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#3ff00 ]', ), ), minSizeFactor=0.1, size=0.001)
mdb.models['Model-1'].rootAssembly.generateMesh(regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'], ))
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].deletePlies(    )
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',         additionalRotationType=ROTATION_ANGLE,
angle=
    90.0, axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[4], orientationType=CSYS,
    plyName='Ply-1', region=Region(
    cells=mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',         additionalRotationType=ROTATION_NONE,
angle=a
    , axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[4], orientationType=CSYS,
    plyName='Ply-2', region=Region(
    cells=mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].parts['Part-1'].compositeLayups['CompositeLayup-
1'].CompositePly(
    additionalRotationField='',         additionalRotationType=ROTATION_NONE,
angle=-a
    , axis=AXIS_2, material='Material-1', numIntPoints=3, orientation=
    mdb.models['Model-1'].parts['Part-1'].datums[4], orientationType=CSYS,
    plyName='Ply-3', region=Region(
    cells=mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(
    mask=('[#3 ]', ), )), suppressed=False, thickness=0.001, thicknessType=
    SPECIFY_THICKNESS)
mdb.models['Model-1'].rootAssembly.regenerate()
mdb.Job(atTime=None, contactPrint=OFF, description='', echoPrint=OFF,
    explicitPrecision=SINGLE, getMemoryFromAnalysis=True, historyPrint=OFF,
    memory=50,          memoryUnits=PERCENTAGE,          model='Model-1',
modelPrint=OFF,
    multiprocessingMode=DEFAULT,                                  name='Job-e',
nodalOutputPrecision=SINGLE,
    numCpus=1, numGPUs=0, queue=None, scratch='', type=ANALYSIS,
    userSubroutine='', waitHours=0, waitMinutes=0)
mdb.jobs['Job-e'].submit(consistencyChecking=OFF)
mdb.jobs['Job-e']._Message(STARTED, {'phase': BATCHPRE_PHASE,
```

'clientHost': 'ntb-HPPB4310s', 'handle': 0, 'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(WARNING, {'phase': BATCHPRE_PHASE,
'message': 'WHENEVER A TRANSLATION (ROTATION) DOF AT A NODE IS CONSTRAINED BY A KINEMATIC COUPLING DEFINITION THE TRANSLATION (ROTATION) DOFS FOR THAT NODE CANNOT BE INCLUDED IN ANY OTHER CONSTRAINT INCLUDING MPCS, RIGID BODIES, ETC.',
'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(WARNING, {'phase': BATCHPRE_PHASE,
'message': 'MPCS (EXTERNAL or INTERNAL, including those generated from rigid body definitions), KINEMATIC COUPLINGS, AND/OR EQUATIONS WILL ACTIVATE ADDITIONAL DEGREES OF FREEDOM',
'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(ODB_FILE, {'phase': BATCHPRE_PHASE,
'file': 'C:\\Temp\\Job-e.odb', 'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(COMPLETED, {'phase': BATCHPRE_PHASE,
'message': 'Analysis phase complete', 'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(STARTED, {'phase': STANDARD_PHASE,
'clientHost': 'ntb-HPPB4310s', 'handle': 3840, 'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(STEP, {'phase': STANDARD_PHASE, 'stepId': 1,
'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(WARNING, {'phase': STANDARD_PHASE,
'message': 'The 3-direction at one or more points in one or more layers in 8200 elements as defined in *ORIENTATION are in the opposite direction to the element normals. Either the 1 or 2 and the 3-direction defined in *ORIENTATION will be reversed. The elements have been identified in element set WarnElem3DirOppElemNormalStep1Inc1.',
'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(ODB_FRAME, {'phase': STANDARD_PHASE, 'step': 0,
'frame': 0, 'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(STATUS, {'totalTime': 0.0, 'attempts': 0,
'timeIncrement': 1.0, 'increment': 0, 'stepTime': 0.0, 'step': 1,
'jobName': 'Job-e', 'severe': 0, 'iterations': 0, 'phase': STANDARD_PHASE,
'equilibrium': 0})
mdb.jobs['Job-e']._Message(MEMORY_ESTIMATE, {'phase': STANDARD_PHASE,
'jobName': 'Job-e', 'memory': 348.168928146362})
mdb.jobs['Job-e']._Message(ODB_FRAME, {'phase': STANDARD_PHASE, 'step': 0,
'frame': 1, 'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(STATUS, {'totalTime': 1.0, 'attempts': 1,
'timeIncrement': 1.0, 'increment': 1, 'stepTime': 1.0, 'step': 1,
'jobName': 'Job-e', 'severe': 0, 'iterations': 1, 'phase': STANDARD_PHASE,
'equilibrium': 1})
mdb.jobs['Job-e']._Message(END_STEP, {'phase': STANDARD_PHASE, 'stepId': 1,
'jobName': 'Job-e'})

```
mdb.jobs['Job-e']._Message(COMPLETED, {'phase': STANDARD_PHASE,
    'message': 'Analysis phase complete', 'jobName': 'Job-e'})
mdb.jobs['Job-e']._Message(JOB_COMPLETED, {'time': 'Wed Mar 11 14:48:04
2015',
    'jobName': 'Job-e'})
# Save by user on 2015_03_11-14.50.17; build 6.12-1 2012_03_13-20.44.39 119612
```

1.5 Script for FEM model using volume model

```
#prumer
d=0.01

#uhel
a=55

#polomery
r1=d/2
r2=r1+0.001
r3=r2+0.001
r4=r3+0.001

x1=d*0.1

# -*- coding: mbcs -*-
from part import *
from material import *
from section import *
from assembly import *
from step import *
from interaction import *
from load import *
from mesh import *
from optimization import *
from job import *
from sketch import *
from visualization import *
from connectorBehavior import *

mdb.models['Model-1'].ConstrainedSketch(name='__profile__', sheetSize=0.2)
mdb.models['Model-1'].sketches['__profile__'].sketchOptions.setValues(
    decimalPlaces=3)
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-0.005, 0.0))
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-0.01, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[2], radius=r1,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[3], radius=r2,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].Part(dimensionality=THREE_D, name='Part-1', type=
    DEFORMABLE_BODY)
mdb.models['Model-1'].parts['Part-1'].BaseSolidExtrude(depth=1.0, sketch=
    mdb.models['Model-1'].sketches['__profile__'])
```

```
del mdb.models['Model-1'].sketches['__profile__']
mdb.models['Model-1'].ConstrainedSketch(name='__profile__', sheetSize=0.2)
mdb.models['Model-1'].sketches['__profile__'].sketchOptions.setValues(
    decimalPlaces=3)
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-0.005, 0.0))
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-0.01, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[2], radius=r2,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[3], radius=r3,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].Part(dimensionality=THREE_D, name='Part-2', type=
    DEFORMABLE_BODY)
mdb.models['Model-1'].parts['Part-2'].BaseSolidExtrude(depth=1.0, sketch=
    mdb.models['Model-1'].sketches['__profile__'])
del mdb.models['Model-1'].sketches['__profile__']
mdb.models['Model-1'].ConstrainedSketch(name='__profile__', sheetSize=0.2)
mdb.models['Model-1'].sketches['__profile__'].sketchOptions.setValues(
    decimalPlaces=3)
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-0.005, 0.0))
mdb.models['Model-1'].sketches['__profile__'].CircleByCenterPerimeter(center=(
    0.0, 0.0), point1=(-0.01, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[2], radius=r3,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].sketches['__profile__'].RadialDimension(curve=
    mdb.models['Model-1'].sketches['__profile__'].geometry[3], radius=r4,
    textPoint=(0.0, 0.0))
mdb.models['Model-1'].Part(dimensionality=THREE_D, name='Part-3', type=
    DEFORMABLE_BODY)
mdb.models['Model-1'].parts['Part-3'].BaseSolidExtrude(depth=1.0, sketch=
    mdb.models['Model-1'].sketches['__profile__'])
del mdb.models['Model-1'].sketches['__profile__']
mdb.models['Model-1'].Material(name='Material-1')
mdb.models['Model-1'].materials['Material-1'].Density(table=((1474.0, ), ))
mdb.models['Model-1'].materials['Material-1'].Elastic(table=((156050000000.0,
    6045000000.0, 6045000000.0, 0.328, 0.328, 0.328, 4431000000.0,
    4431000000.0, 4431000000.0), ), type=ENGINEERING_CONSTANTS)
mdb.models['Model-1'].CompositeSolidSection(layup=(SectionLayer(
    thickness=0.001, orientAngle=90.0, numIntPts=1, material='Material-1',
    plyName='Ply-1'), ), layupName='', name='Section-1', symmetric=False)
mdb.models['Model-1'].CompositeSolidSection(layup=(SectionLayer(
    thickness=0.001, orientAngle=a, numIntPts=1, material='Material-1',
    plyName='Ply-2'), ), layupName='', name='Section-2', symmetric=False)
```

```
mdb.models['Model-1'].CompositeSolidSection(layup=(SectionLayer(
    thickness=0.001, orientAngle=-a, numIntPts=1, material='Material-1',
    plyName='Ply-3'), ), layupName='', name='Section-3', symmetric=False)
mdb.models['Model-1'].parts['Part-1'].DatumCsysByThreePoints(coordSysType=
    CARTESIAN, name='Datum csys-1', origin=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[0], CENTER), point1=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[1], CENTER), point2=
    mdb.models['Model-1'].parts['Part-1'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-1'].edges[0], MIDDLE))
mdb.models['Model-1'].parts['Part-2'].DatumCsysByThreePoints(coordSysType=
    CARTESIAN, name='Datum csys-1', origin=
    mdb.models['Model-1'].parts['Part-2'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-2'].edges[0], CENTER), point1=
    mdb.models['Model-1'].parts['Part-2'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-2'].edges[1], CENTER), point2=
    mdb.models['Model-1'].parts['Part-2'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-2'].edges[0], MIDDLE))
mdb.models['Model-1'].parts['Part-3'].DatumCsysByThreePoints(coordSysType=
    CARTESIAN, name='Datum csys-1', origin=
    mdb.models['Model-1'].parts['Part-3'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-3'].edges[0], CENTER), point1=
    mdb.models['Model-1'].parts['Part-3'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-3'].edges[1], CENTER), point2=
    mdb.models['Model-1'].parts['Part-3'].InterestingPoint(
    mdb.models['Model-1'].parts['Part-3'].edges[0], MIDDLE))
mdb.models['Model-1'].parts['Part-1'].MaterialOrientation(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_3, fieldName='', localCsys=
    mdb.models['Model-1'].parts['Part-1'].datums[2], orientationType=SYSTEM,
    region=Region(
    cells=mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(
    mask=('[#1 ]', ), )), stackDirection=STACK_3)
mdb.models['Model-1'].parts['Part-2'].MaterialOrientation(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_3, fieldName='', localCsys=
    mdb.models['Model-1'].parts['Part-2'].datums[2], orientationType=SYSTEM,
    region=Region(
    cells=mdb.models['Model-1'].parts['Part-2'].cells.getSequenceFromMask(
    mask=('[#1 ]', ), )), stackDirection=STACK_3)
mdb.models['Model-1'].parts['Part-3'].MaterialOrientation(
    additionalRotationField='',          additionalRotationType=ROTATION_NONE,
angle=0.0
    , axis=AXIS_3, fieldName='', localCsys=
```

```
    mdb.models['Model-1'].parts['Part-3'].datums[2], orientationType=SYSTEM,
    region=Region(
    cells=mdb.models['Model-1'].parts['Part-3'].cells.getSequenceFromMask(
    mask=('[#1 ]', ), )), stackDirection=STACK_3)
mdb.models['Model-1'].parts['Part-1'].Set(cells=
    mdb.models['Model-1'].parts['Part-1'].cells.getSequenceFromMask(('[#1 ]',
    ), ), name='Set-2')
mdb.models['Model-1'].parts['Part-1'].SectionAssignment(offset=0.0,
    offsetField='', offsetType=MIDDLE_SURFACE, region=
    mdb.models['Model-1'].parts['Part-1'].sets['Set-2'], sectionName=
    'Section-1', thicknessAssignment=FROM_SECTION)
mdb.models['Model-1'].parts['Part-2'].Set(cells=
    mdb.models['Model-1'].parts['Part-2'].cells.getSequenceFromMask(('[#1 ]',
    ), ), name='Set-2')
mdb.models['Model-1'].parts['Part-2'].SectionAssignment(offset=0.0,
    offsetField='', offsetType=MIDDLE_SURFACE, region=
    mdb.models['Model-1'].parts['Part-2'].sets['Set-2'], sectionName=
    'Section-2', thicknessAssignment=FROM_SECTION)
mdb.models['Model-1'].parts['Part-3'].Set(cells=
    mdb.models['Model-1'].parts['Part-3'].cells.getSequenceFromMask(('[#1 ]',
    ), ), name='Set-2')
mdb.models['Model-1'].parts['Part-3'].SectionAssignment(offset=0.0,
    offsetField='', offsetType=MIDDLE_SURFACE, region=
    mdb.models['Model-1'].parts['Part-3'].sets['Set-2'], sectionName=
    'Section-3', thicknessAssignment=FROM_SECTION)
mdb.models['Model-1'].rootAssembly.DatumCsysByDefault(CARTESIAN)
mdb.models['Model-1'].rootAssembly.Instance(dependent=OFF, name='Part-1-1',
    part=mdb.models['Model-1'].parts['Part-1'])
mdb.models['Model-1'].rootAssembly.Instance(dependent=OFF, name='Part-2-1',
    part=mdb.models['Model-1'].parts['Part-2'])
mdb.models['Model-1'].rootAssembly.Instance(dependent=OFF, name='Part-3-1',
    part=mdb.models['Model-1'].parts['Part-3'])
mdb.models['Model-1'].rootAssembly.Coaxial(fixedAxis=
    mdb.models['Model-1'].rootAssembly.instances['Part-2-1'].faces[1], flip=OFF
    , movableAxis=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].faces[1])
mdb.models['Model-1'].rootAssembly.Coaxial(fixedAxis=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].faces[1], flip=OFF
    , movableAxis=
    mdb.models['Model-1'].rootAssembly.instances['Part-2-1'].faces[1])
mdb.models['Model-1'].rootAssembly.Coaxial(fixedAxis=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].faces[0], flip=OFF
    , movableAxis=
    mdb.models['Model-1'].rootAssembly.instances['Part-2-1'].faces[1])
mdb.models['Model-1'].rootAssembly.ParallelFace(fixedPlane=
    mdb.models['Model-1'].rootAssembly.instances['Part-2-1'].faces[2], flip=OFF
    , movablePlane=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].faces[2])
```

```
mdb.models['Model-1'].rootAssembly.ParallelFace(fixedPlane=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].faces[2], flip=OFF
    , movablePlane=
    mdb.models['Model-1'].rootAssembly.instances['Part-2-1'].faces[2])
mdb.models['Model-1'].rootAssembly.DatumPlaneByThreePoints(point1=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].vertices[0],
    point2=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].InterestingPoint(
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].edges[0], MIDDLE),
    point3=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].InterestingPoint(
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].edges[1], MIDDLE))
mdb.models['Model-1'].rootAssembly.PartitionCellByDatumPlane(cells=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-
1'].cells.getSequenceFromMask(
    mask=('[#1 ]', ), )+\
    mdb.models['Model-1'].rootAssembly.instances['Part-2-
1'].cells.getSequenceFromMask(
    mask=('[#1 ]', ), )+\
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].cells.getSequenceFromMask(
    mask=('[#1 ]', ), ), datumPlane=
    mdb.models['Model-1'].rootAssembly.datums[13])
mdb.models['Model-1'].rootAssembly.DatumCsysByThreePoints(coordSysType=
    CARTESIAN, name='Datum csys-2', origin=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].InterestingPoint(
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].edges[10], CENTER)
    , point1=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].InterestingPoint(
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].edges[8], CENTER),
    point2=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'].vertices[0])
mdb.models['Model-1'].StaticStep(name='Step-1', previous='Initial')
mdb.models['Model-1'].fieldOutputRequests['F-Output-1'].setValues(variables=(
    'U', ))
mdb.models['Model-1'].rootAssembly.ReferencePoint(point=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].InterestingPoint(
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1'].edges[8], CENTER))
mdb.models['Model-1'].rootAssembly.Set(name='m_Set-1', referencePoints=(
    mdb.models['Model-1'].rootAssembly.referencePoints[16], ))
mdb.models['Model-1'].rootAssembly.Surface(name='s_Surf-1', side1Faces=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].faces.getSequenceFromMask(
    mask=('[#204 ]', ), )+\
    mdb.models['Model-1'].rootAssembly.instances['Part-2-
1'].faces.getSequenceFromMask(
    mask=('[#204 ]', ), )+\
```

```
    mdb.models['Model-1'].rootAssembly.instances['Part-3-
1'].faces.getSequenceFromMask(
    mask=('[#204 ]', ), ))
mdb.models['Model-1'].Coupling(controlPoint=
    mdb.models['Model-1'].rootAssembly.sets['m_Set-1'],
couplingType=KINEMATIC,
    influenceRadius=WHOLE_SURFACE, localCsys=
    mdb.models['Model-1'].rootAssembly.datums[15], name='Constraint-1',
    surface=mdb.models['Model-1'].rootAssembly.surfaces['s_Surf-1'], u1=ON, u2=
    ON, u3=ON, ur1=ON, ur2=ON, ur3=ON)
mdb.models['Model-1'].rootAssembly.Set(name='Set-2', referencePoints=(
    mdb.models['Model-1'].rootAssembly.referencePoints[16], ))
mdb.models['Model-1'].ConcentratedForce(cf3=100.0, createStepName='Step-1',
    distributionType=UNIFORM, field='', localCsys=
    mdb.models['Model-1'].rootAssembly.datums[15], name='Load-1', region=
    mdb.models['Model-1'].rootAssembly.sets['Set-2'])
mdb.models['Model-1'].rootAssembly.Set(faces=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].faces.getSequenceFromMask(
    mask=('[#110 ]', ), )+\
    mdb.models['Model-1'].rootAssembly.instances['Part-2-
1'].faces.getSequenceFromMask(
    mask=('[#110 ]', ), )+\
    mdb.models['Model-1'].rootAssembly.instances['Part-3-
1'].faces.getSequenceFromMask(
    mask=('[#110 ]', ), ), name='Set-3')
mdb.models['Model-1'].EncastreBC(createStepName='Initial', localCsys=
    mdb.models['Model-1'].rootAssembly.datums[15], name='BC-1', region=
    mdb.models['Model-1'].rootAssembly.sets['Set-3'])
mdb.models['Model-1'].rootAssembly.seedPartInstance(deviationFactor=0.1,
    minSizeFactor=0.1, regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'],
    mdb.models['Model-1'].rootAssembly.instances['Part-2-1'],
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1']), size=0.005)
mdb.models['Model-1'].rootAssembly.seedEdgeBySize(constraint=FINER,
    deviationFactor=0.1, edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-
1'].edges.getSequenceFromMask(
    mask=('[#aa ]', ), )+\
    mdb.models['Model-1'].rootAssembly.instances['Part-2-
1'].edges.getSequenceFromMask(
    mask=('[#aa ]', ), )+\
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    mask=('[#aa ]', ), ), minSizeFactor=0.1, size=0.0002)
```

```
mdb.models['Model-1'].rootAssembly.seedEdgeByNumber(constraint=FINER,
edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-1-
1'].edges.getSequenceFromMask(
    ('[#a900 ]', ), ), number=20)
mdb.models['Model-1'].rootAssembly.seedEdgeByNumber(constraint=FINER,
edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-2-
1'].edges.getSequenceFromMask(
    ('[#a900 ]', ), ), number=20)
mdb.models['Model-1'].rootAssembly.seedEdgeByNumber(constraint=FINER,
edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-
1'].edges.getSequenceFromMask(
    ('[#a900 ]', ), ), number=20)
mdb.models['Model-1'].rootAssembly.seedEdgeByNumber(constraint=FINER,
edges=
    mdb.models['Model-1'].rootAssembly.instances['Part-3-
1'].edges.getSequenceFromMask(
    ('[#5600 ]', ), ), number=20)
mdb.models['Model-1'].rootAssembly.generateMesh(regions=(
    mdb.models['Model-1'].rootAssembly.instances['Part-3-1'],
    mdb.models['Model-1'].rootAssembly.instances['Part-2-1'],
    mdb.models['Model-1'].rootAssembly.instances['Part-1-1']))
mdb.Job(atTime=None, contactPrint=OFF, description='', echoPrint=OFF,
    explicitPrecision=SINGLE, getMemoryFromAnalysis=True, historyPrint=OFF,
    memory=50, memoryUnits=PERCENTAGE, model='Model-1',
modelPrint=OFF,
    multiprocessingMode=DEFAULT, name='Job-55',
nodalOutputPrecision=SINGLE,
    numCpus=1, numGPUs=0, queue=None, scratch='', type=ANALYSIS,
    userSubroutine='', waitHours=0, waitMinutes=0)
mdb.jobs['Job-55'].submit(consistencyChecking=OFF)
```