Analysis

of

Curved-Fibre Composites

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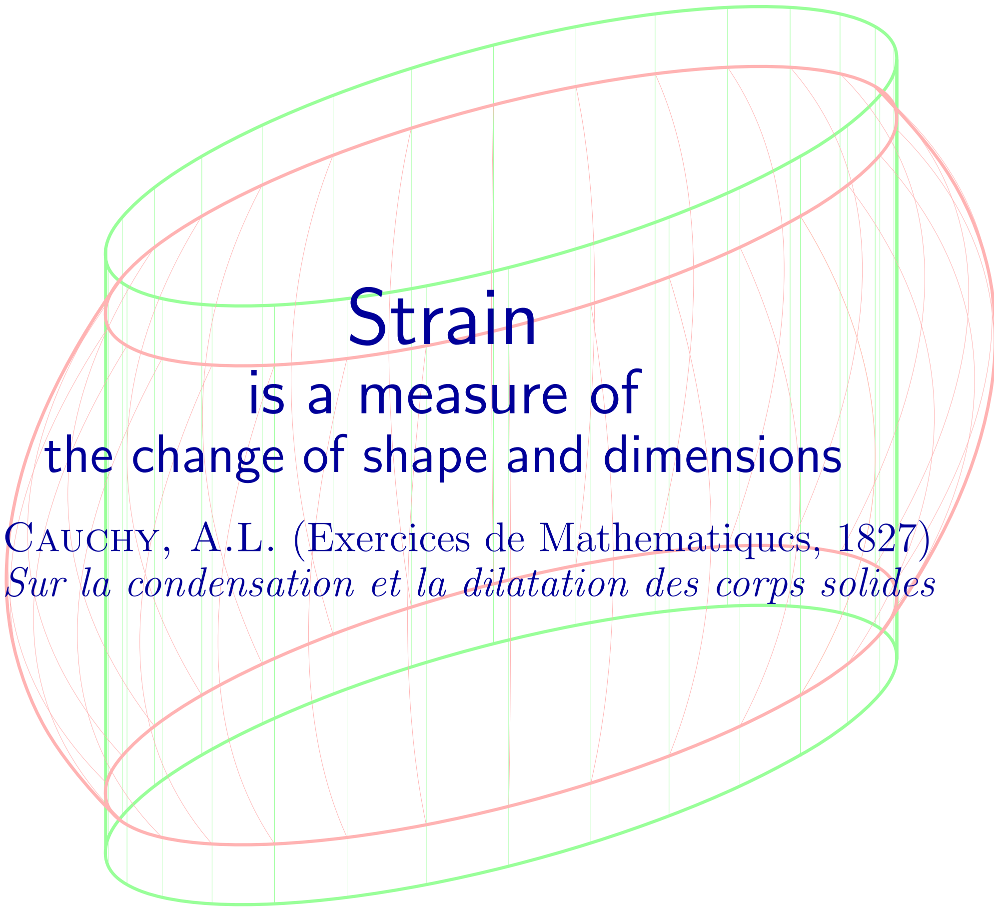
Contents:

Analysis of deformation

$$\text{Analysis} = A\nu\acute{\alpha}\lambda\upsilon\sigma\iota\varsigma$$

in Greek

The action of
taking something apart
in order to study it



Strain
is a measure of
the change of shape and dimensions

CAUCHY, A.L. (Exercices de Mathematiqucs, 1827)
Sur la condensation et la dilatation des corps solides

We are used to
connect deformation with an action of forces

SIR WILLIAM PETTY (London, 1674)
. . . a new Hypothesis of Springing or Elastique Motions

ROBERT HOOKE (London, 1678/1660)
De Potentiâ Restitutiva

This leads us to the
concept of Elastic body

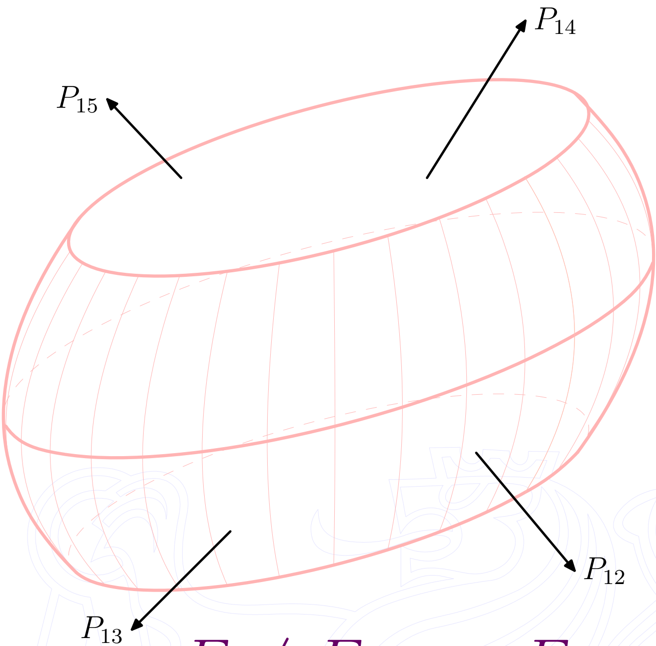
Stress

If we accept that

$$\text{deformation} = f(\text{acting forces})$$

we must look at the

description of these forces



$$F = \int_A \boldsymbol{t}\, \mathrm{d}A$$

where the traction

CAUCHY, A. L. (1823)

$$\boldsymbol{t} = \lim_{A \to 0} \frac{\boldsymbol{F}}{A}$$

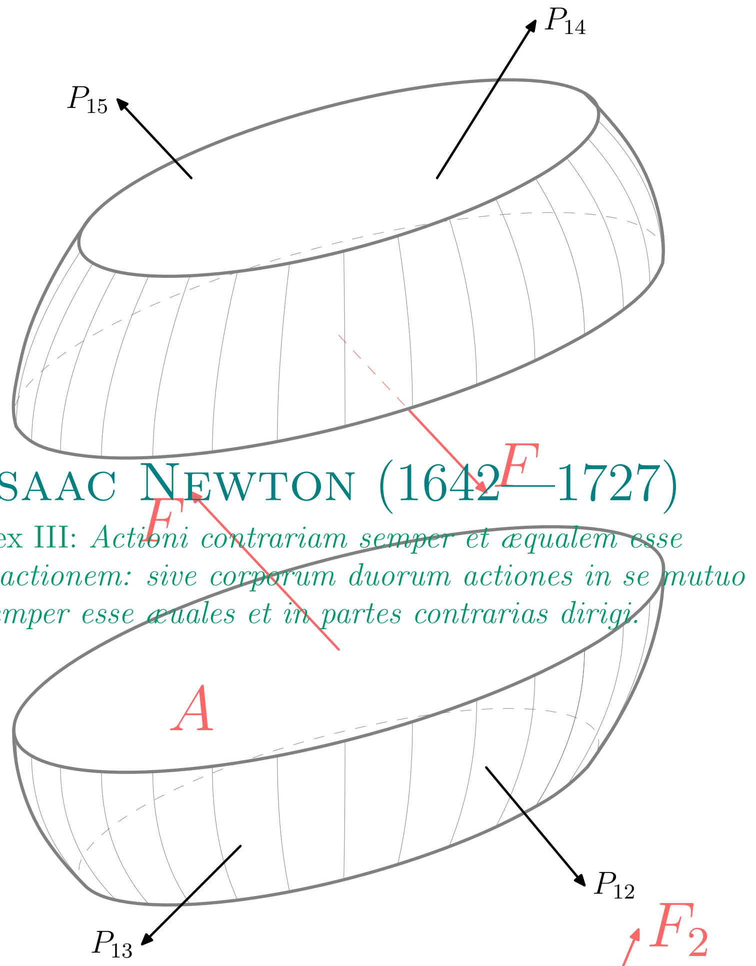
$$F \neq F_1 \;\Rightarrow\; F = f(A)$$
$$F \neq P \;\Rightarrow\; F = g(\boldsymbol{n})$$

$\boldsymbol{n}$ — outer normal

For $A \to 0$ we define stress σ as

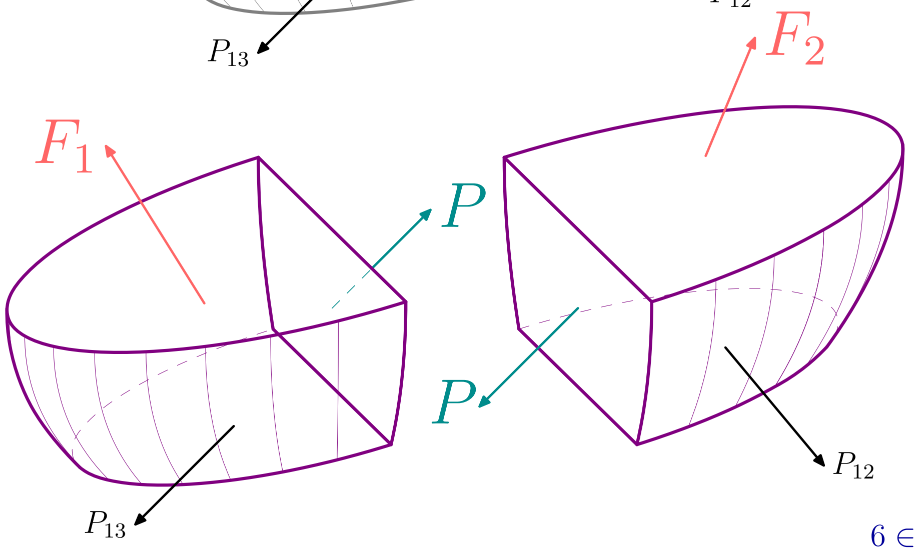
$$F = \boldsymbol{t}A = \sigma(\boldsymbol{A}),\; \boldsymbol{A} = A\boldsymbol{n}$$

it turns out that—



ISAAC NEWTON (1642—1727)

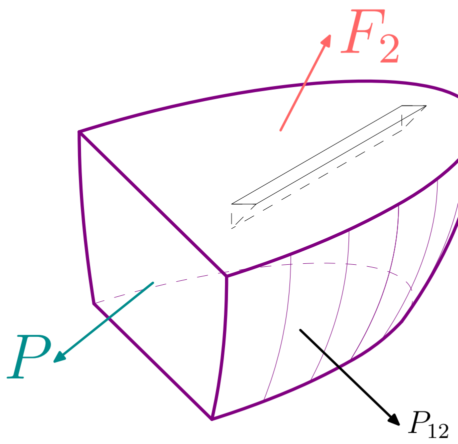
Lex III: *Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*

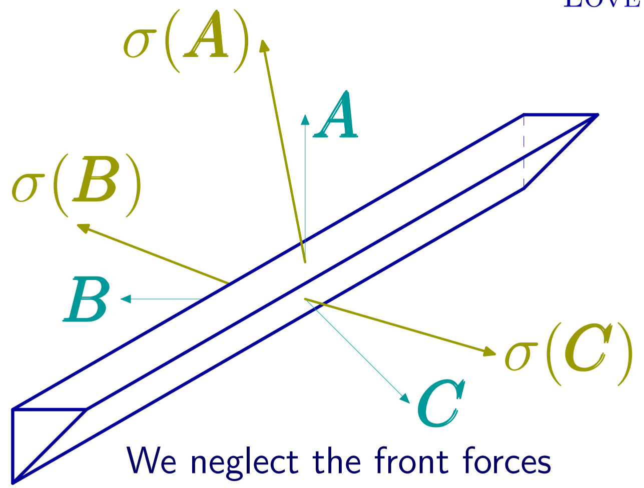


Stress tensor

LOVE, A. E. H. (1927)

Take out
a long element





The total force

$$\boldsymbol{F} = \sigma(\boldsymbol{A}) + \sigma(\boldsymbol{B}) + \sigma(\boldsymbol{C})$$

$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ — normals with length of the cross-section area

$\sigma(\boldsymbol{A})$ — the force on the $\boldsymbol{A} = A\boldsymbol{n}$, similarly $\sigma(\boldsymbol{B})$, $\sigma(\boldsymbol{C})$

$$\lim_{x\to 0} \frac{\mathrm{area}(x^2)}{\mathrm{volume}(x^3)} = \infty$$

$$\frac{\mathrm{area}}{\mathrm{volume}} \propto \frac{\boldsymbol{F}}{\mathrm{mass}} = \mathrm{acceleration}$$

$$(\boldsymbol{F} \neq \boldsymbol{0} \Rightarrow \mathrm{acceleration} \to \infty) \Rightarrow \boldsymbol{F} = \boldsymbol{0}$$

$$\sigma(\boldsymbol{A}) + \sigma(\boldsymbol{B}) + \sigma(\boldsymbol{C}) = 0$$

$$\sigma(-\boldsymbol{C}) = -\sigma(\boldsymbol{C}) = \sigma(\boldsymbol{A}) + \sigma(\boldsymbol{B})$$

From the definition: $\sigma(k\boldsymbol{A}) = k\sigma(\boldsymbol{A})$

Geometrie: $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = 0$

$$\sigma(\boldsymbol{A} + \boldsymbol{B}) = \sigma(\boldsymbol{A}) + \sigma(\boldsymbol{B})$$

The stress σ is a (linear) vector operator, *i.e.* tensor

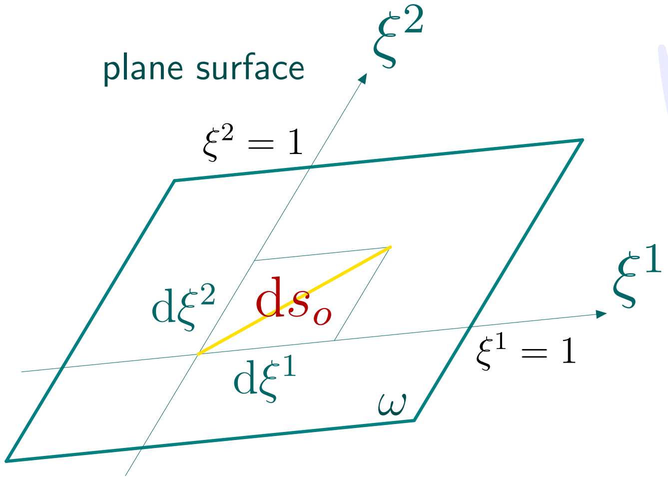
in abstract index notation:

$$F^a = t^a A = A\sigma^{ab} n_b \quad \text{tedy} \quad t^a = \sigma^{ab} n_b$$

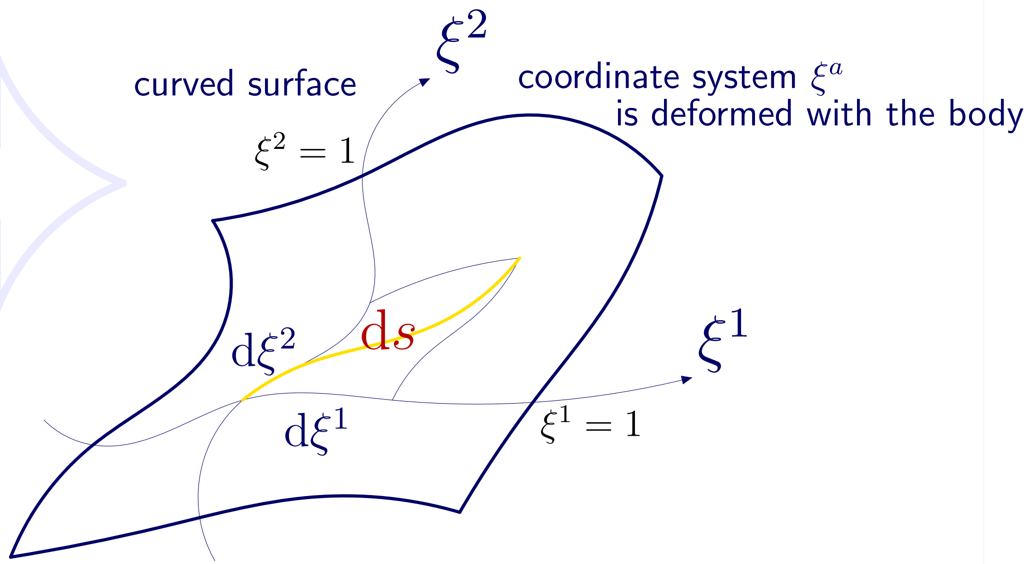
Tensor calculus

Distance of two points on a surface

HAMILTON, W. R. (1854, 1855)
WOLDEMAR VOIGT (1899)
GREGORIO RICCI-CURBASTRO (1890)
TULLIO LEVI-CIVITA (1900)
ALBERT EINSTEIN (1915)
SYNGE, J. L. and SCHILD, A. (1978)
LOVELOCK, D. and RUND, H. (1989)



$\theta : \omega \to {}^3$



$$\mathrm{d}s_o^2 = (\mathrm{d}\xi^1)^2 + (\mathrm{d}\xi^2)^2$$

$$\mathrm{d}s_o^2 = \begin{pmatrix} \mathrm{d}\xi^1 & \mathrm{d}\xi^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}\xi^1 \\ \mathrm{d}\xi^2 \end{pmatrix}$$

$\mathrm{d}s_o^2 = \delta_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b$ (Einstein summation)

δ_{ab} is Kronecker symbol

LEOPOLD KRONECKER (1823—1891)

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b$$

$$\mathrm{d}s_o \to \mathrm{d}s, \ \ \delta_{ab} \to g_{ab}$$

g_{ab} is metric tensor

How to determine metric tensor g_{ab}?

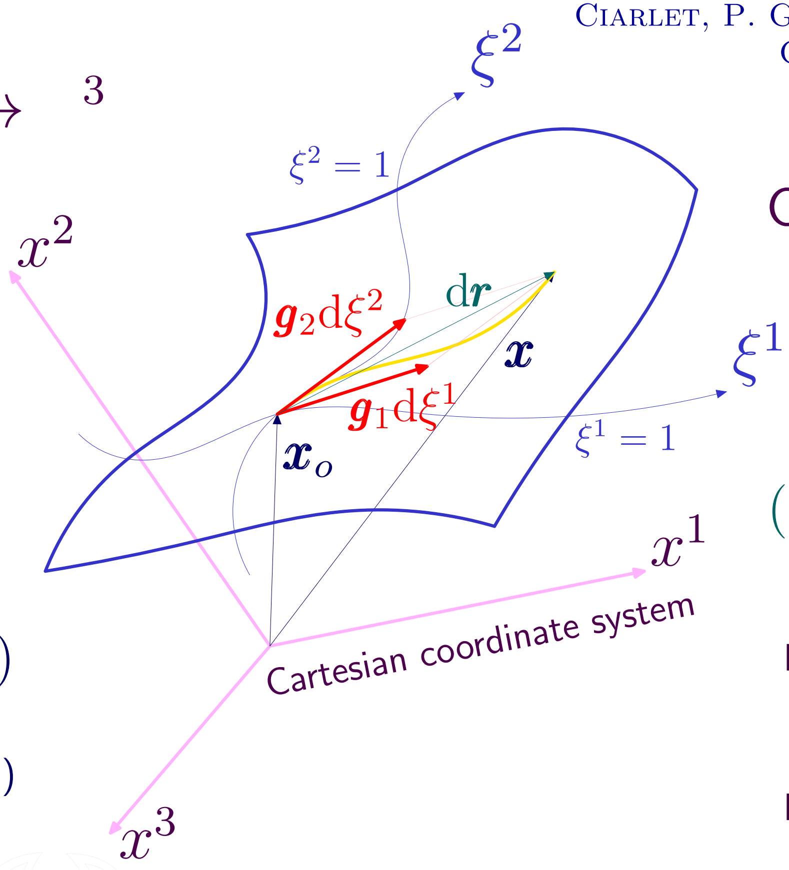
Just express the distance (metric) $\mathrm{d}s$ on the surface in ^{3}!

Metric, metric tensor

Mapping $\theta: \; {}^2 \ni \omega \to \; {}^3$

CIARLET, P. G., GRATIE, L., and MARDARE, C. (2006)
CIARLET, P. G. and LAURENT, F. (2003)
CIARLET, P. G. (2005)

$$\boldsymbol{x}_o = \theta(\xi^a)$$
$$\boldsymbol{x} = \theta(\xi^a + \mathrm{d}\xi^a)$$



vector algebra

$$(1) \quad \mathrm{d}\boldsymbol{r} = \boldsymbol{g}_1 \mathrm{d}\xi^1 + \boldsymbol{g}_2 \mathrm{d}\xi^2$$
$$\mathrm{d}\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_o$$
$$\mathrm{d}\boldsymbol{r} = \theta(\xi^a + \mathrm{d}\xi^a) - \theta(\xi^a)$$

vector analysis (pro $\mathrm{d}\xi^a \to 0$)

$$(2) \quad \mathrm{d}\boldsymbol{r} = \frac{\partial\theta}{\partial\xi^1}\mathrm{d}\xi^1 + \frac{\partial\theta}{\partial\xi^2}\mathrm{d}\xi^2$$

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} = \left(\frac{\partial\theta}{\partial\xi^1}\mathrm{d}\xi^1\right)^2 + 2\frac{\partial\theta}{\partial\xi^1}\frac{\partial\theta}{\partial\xi^2}\mathrm{d}\xi^1\mathrm{d}\xi^2 + \left(\frac{\partial\theta}{\partial\xi^2}\mathrm{d}\xi^2\right)^2$$

Using Einstein summation

$$(3) \quad \mathrm{d}s^2 = \frac{\partial\theta}{\partial\xi^a}\frac{\partial\theta}{\partial\xi^b}\mathrm{d}\xi^a\mathrm{d}\xi^b$$

Consequences:

$$(1) + (2): \quad \boldsymbol{g}_a = \frac{\partial\theta}{\partial\xi^a}$$

$$(3) + (4): \quad g_{ab} = \frac{\partial\theta}{\partial\xi^a}\frac{\partial\theta}{\partial\xi^a}$$

Metric tensor is symmetric

$$g_{ab} = g_{ba}$$

Base vectors are not orthogonal

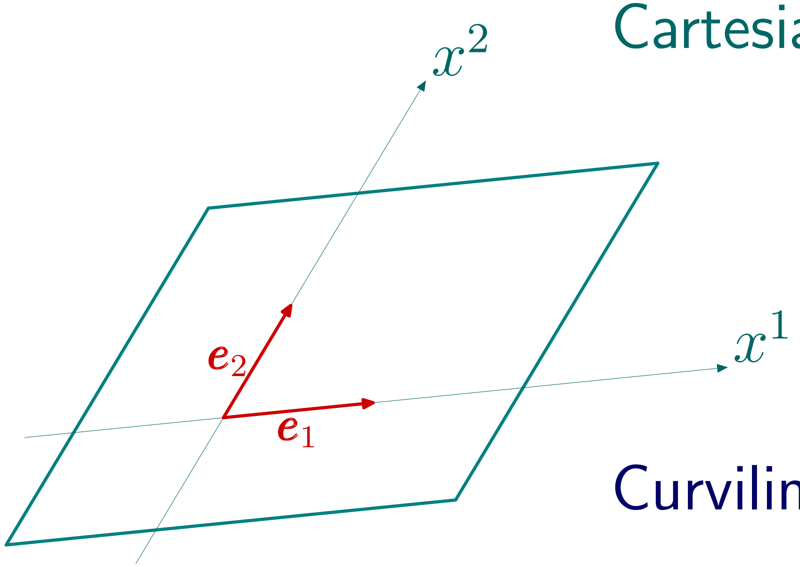
$$\boldsymbol{g}_a \cdot \boldsymbol{g}_b = g_{ab} \neq \delta_{ab}$$

From the definition

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b \quad (4)$$

Abstract index notation

Green, A. E. and Zerna, W. (1954)
Taber, L. A. (2004)



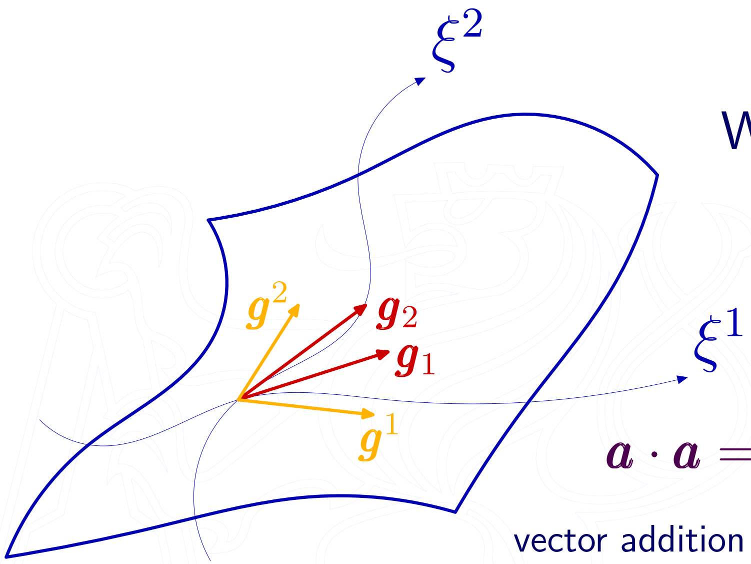
Cartesian coordinate system (Euclidean space)

Base vectors are orthonormal

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$$

Vector is linear combination of base vectors

$$\boldsymbol{a} = a^1 \boldsymbol{e}_1 + a^2 \boldsymbol{e}_2 = a^a \boldsymbol{e}_a$$



Curvilinear coordinates (Non-euclidean space)

Base vectors are not (generally) orthonormal

$$\boldsymbol{g}_a \cdot \boldsymbol{g}_b = g_{ab} \neq \delta_{ab}$$

We are not used to it ⇒ We introduce new base vectors: $\boldsymbol{g}^a$

such that

$$\boldsymbol{g}^a \cdot \boldsymbol{g}_b = \delta^a_b$$

We may write

$$\boldsymbol{a} = a_a \boldsymbol{g}^a = a^a \boldsymbol{g}_a$$

e.g. scalar product

$$\boldsymbol{a} \cdot \boldsymbol{a} = (a_a \boldsymbol{g}^a) \cdot (a_b \boldsymbol{g}^b) = (a_a \boldsymbol{g}^a) \cdot (a^b \boldsymbol{g}_b) = a_a a^b \boldsymbol{g}^a \cdot \boldsymbol{g}_b = a_a a^a$$

vector addition

$$\boldsymbol{a} + \boldsymbol{b} = a_a \boldsymbol{g}^a + b_a \boldsymbol{g}^a = (a_a + b_a)\, \boldsymbol{g}^a$$

Tensor equation

$$\boldsymbol{T} = \boldsymbol{a} \otimes \boldsymbol{b} \quad \Leftrightarrow \quad T^{ab} \boldsymbol{g}_a \otimes \boldsymbol{g}_b = (a^a \boldsymbol{g}_a) \otimes (b^b \boldsymbol{g}_b) = a^a a^b \boldsymbol{g}_a \otimes \boldsymbol{g}_b$$

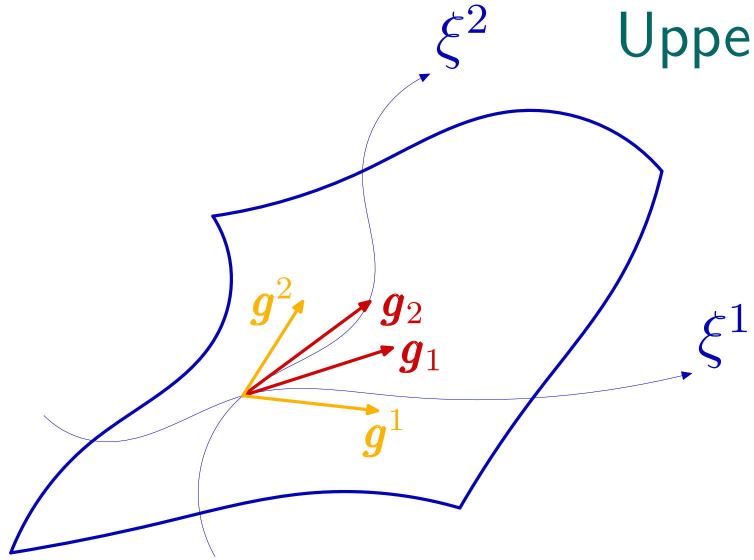
Leaving base vectors leads us to

Abstract index notation

$$a_a a^a = \boldsymbol{a} \cdot \boldsymbol{a}, \quad a^a + b^a \Leftrightarrow \boldsymbol{a} + \boldsymbol{b}, \quad a^a b^b \Leftrightarrow \boldsymbol{a} \otimes \boldsymbol{b}$$

Covariant derivative, transformation rules

SYNGE, J. L. and SCHILD, A. (1978)
LOVELOCK, D. and RUND, H. (1989)



Upper index

Definition: $g^{ab} = (g_{ab})^{-1}$

$a_a \boldsymbol{g}^a = a^a \boldsymbol{g}_a$

Multiplying by $\boldsymbol{g}_b$ leads at $a_a \boldsymbol{g}^a \cdot \boldsymbol{g}_b = a^a \boldsymbol{g}_a \cdot \boldsymbol{g}_b$

and $a_a \delta^a_b = a^a g_{ab}$

Multiply by g^{bd} and the definition $\quad g^{bd} a_b = a^d$

Diferencial operator

$\operatorname{grad} \varphi = \nabla_a \varphi \, \boldsymbol{g}^a = \partial_a \varphi \, \boldsymbol{g}^a$

$\operatorname{div} \boldsymbol{v} = \nabla \boldsymbol{v} = \nabla_a v^a$

$\operatorname{rot} \boldsymbol{A} = \nabla \times \boldsymbol{A} = \epsilon^{abc} \nabla_a A_b \, \boldsymbol{g}^c$

$\nabla^2 \varphi = \operatorname{div} \operatorname{grad} \varphi$

Covariant derivative

$$\frac{\partial \boldsymbol{a}}{\partial x^a} = \frac{\partial}{\partial x^a}(a^b \boldsymbol{g}_b) = (\partial_a a^b)\boldsymbol{g}_b + a^b \frac{\partial \boldsymbol{g_b}}{\partial x^a}$$

We seek linear combination of the base vectors

$$\frac{\partial \boldsymbol{a}}{\partial x^a} = (\partial_a a^b + \Gamma^b_{ac} a^c)\boldsymbol{g}_b \qquad \nabla_a a^b = \partial_a a^b + \Gamma^b_{ac} a^c$$

Sign it as $\quad \dfrac{\partial \boldsymbol{a}}{\partial x^a} = \nabla_a a^b \boldsymbol{g}_b$

The Γ^c_{ab} (Christoffel symbol of the 2^{nd} kind) exists:

$$\Gamma^d_{ab} = g^{dc} \frac{1}{2}(g_{ac,b} + g_{cb,a} - g_{ab,c})$$

Transformation rule for $x^a = x^a(\xi^b)$

Transformation rule for covectors (covariant)

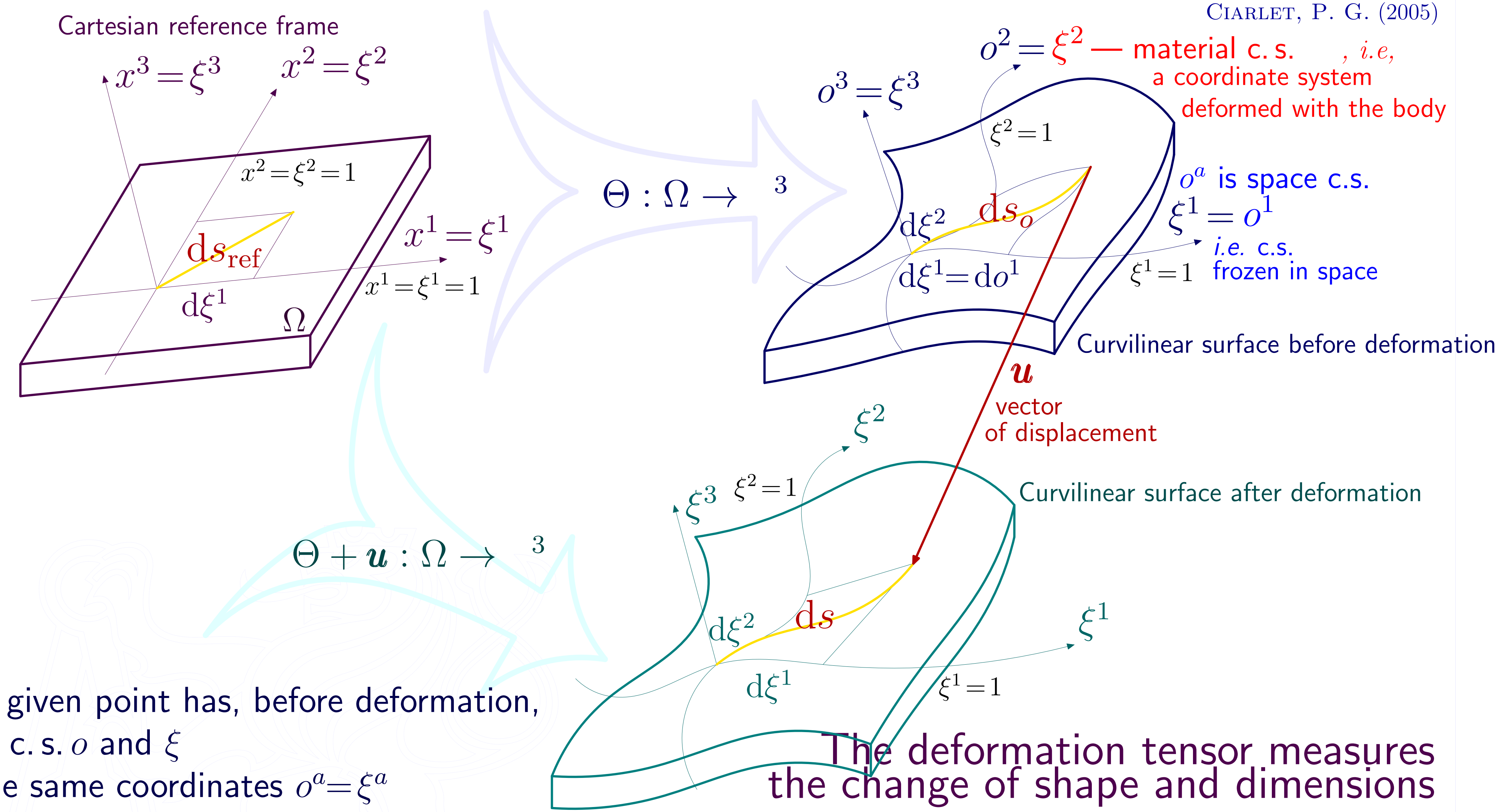
$$\frac{\partial \phi}{\partial x^a} = \frac{\partial \xi^b}{\partial x^a} \frac{\partial \phi}{\partial \xi^b}$$

Transformation rule for vectors (contravariant)

$$\mathrm{d}x^a = \frac{\partial x^a}{\partial \xi^b} \mathrm{d}\xi^b$$

A deformation tensor

Green, A. E. and Zerna, W. (1954)
Antman, S. S. (2005)
Ciarlet, P. G. (2005)



A given point has, before deformation,
in c. s. o and ξ
the same coordinates $o^a = \xi^a$

The same point has, after deformation
still the same coordinates at the c. s. ξ (ξ^a),
but at the c. s. o, the coordinates are not the same
and are given by (unknown) function $o^a = o^a(\xi^b)$

The deformation tensor measures
the change of shape and dimensions

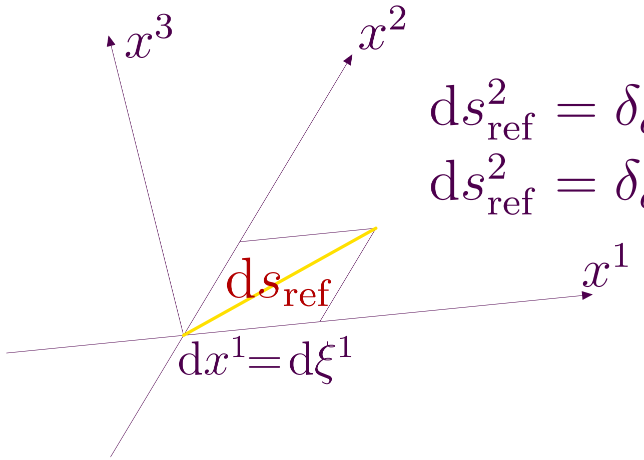
The shape and dimensions are described
via coordinate systems

And they are of various types...

Deformation tensor

Green, A. E. and Zerna, W. (1954)
Antman, S. S. (2005)
Ciarlet, P. G. (2005)

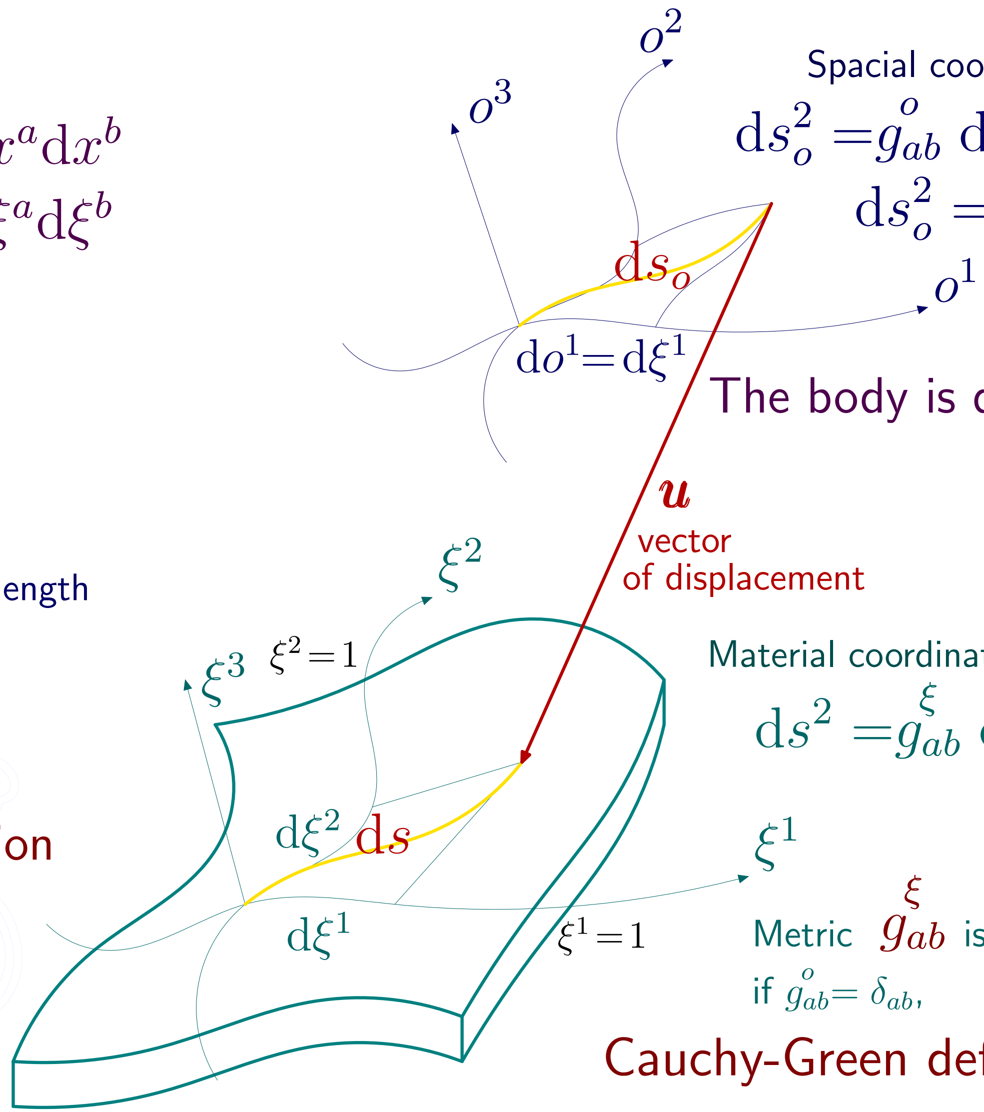
Cartesian (spacial) coordinate system



$$\mathrm{d}s^2_{\mathrm{ref}} = \delta_{ab}\mathrm{d}x^a\mathrm{d}x^b$$
$$\mathrm{d}s^2_{\mathrm{ref}} = \delta_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b$$

Spacial coordinate system

$$\mathrm{d}s_o^2 = \overset{o}{g}_{ab}\,\mathrm{d}o^a\mathrm{d}o^b$$
$$\mathrm{d}s_o^2 = \overset{o}{g}_{ab}\,\mathrm{d}\xi^a\mathrm{d}\xi^b$$



The body is displaced and deformed

Material coordinate system

$$\mathrm{d}s^2 = \overset{\xi}{g}_{ab}\,\mathrm{d}\xi^a\mathrm{d}\xi^b$$

Deformation is characterized by the change of a length
e.g. the element length, *i.e.* $(\mathrm{d}s^2 - \mathrm{d}s_o^2)$
$\mathrm{d}s^2 - \mathrm{d}s_o^2 = (\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab})\mathrm{d}\xi^a\mathrm{d}\xi^b$

The relation $\mathrm{d}s^2 - \mathrm{d}s_o^2 = 2\,\overset{\xi}{E}_{ab}\,\mathrm{d}\xi^a\mathrm{d}\xi^b$ defines

Green-Lagrange-St. Venant deformation

$$\overset{\xi}{E}_{ab} = \frac{1}{2}(\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab})$$

Metric $\overset{\xi}{g}_{ab}$ is known as,
if $\overset{o}{g}_{ab} = \delta_{ab}$,

Cauchy-Green deformation

If o, ξ^o is Cartesian c. s.
i.e. $\overset{o}{g}_{ab} = \delta_{ab}$ and $o^a = \xi^a + u^a$

$$\overset{\xi}{g}_{ab} = \frac{\partial a^c}{\partial \xi^a}\frac{\partial a^d}{\partial \xi^b}\,\overset{o}{g}_{cd} = (\delta_a^c + \partial_a u^c)(\delta_b^d + \partial_b u^d)\delta_{cd} = \delta_{ab} + \partial_a u_b + \partial_b u_a + \partial_a u^c \partial_b u_c$$

$$\overset{\xi}{E}_{ab} = \frac{1}{2}(\overset{\xi}{g}_{ab} - \delta_{ab}) = \frac{1}{2}(\partial_a u_b + \partial_b u_a + \partial_a u^c \partial_b u_c)$$ — the deformation tensor (lagrangian description)

Euler deformation (Almans deformation): $\overset{x}{E}_{ab}$

Small deformation tensor and energy minimum principles

Love, A. E. H. (1927)
Washizu, K. (1975)
Ciarlet, P. G. (2005)

Green-Lagrange-St. Venantova deformation is linearized

$$\overset{\xi}{E}_{ab} = \frac{1}{2}(\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab}) \Big| = \frac{1}{2}(\partial_a u_b + \partial_b u_a + \partial_a u^c \partial_b u_c)$$

in Cartesian coordinates (the term $\partial_a u^c \partial_b u_c$ is crossed out)

$\Rightarrow$ Small deformation tensor

$$\varepsilon_{ab} = \frac{1}{2}(\partial_a u_b + \partial_b u_a)$$

passing into curvilinear coordinates

$\partial_a \Rightarrow \nabla_a$ (Wald, R. M., 1984)

and check tensor transformation

Generally for small deformation

$$\varepsilon_{ab} = \frac{1}{2}(\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab})\Big|_{\text{lin.}} = \frac{1}{2}(\nabla_a u_b + \nabla_b u_a)$$

Principle of the total potential energy minimum $\Rightarrow$

(Maupertuis, 1746)
(Euler, 1744)
(Lagrange, 1788)

$$\hat{u}_a = \arg\min_{u_b \in} \Pi(u_c)$$

The real state of a deformed body, $\hat{u}_a$, minimizes the total potential energy (on a set of admissible states,)

$$\Pi(u_a) = a(u_a) - l(u_a)$$

The elastic strain energy

$$a(u_a) = \frac{1}{2}\int_\Omega E^{abcd}\varepsilon_{ab}(u_a)\varepsilon_{cd}(u_a)\mathrm{d}\Omega$$

The potential energy of the applied forces $p^a(\frac{\mathrm{N}}{\mathrm{mm}^3}), t^a(\frac{\mathrm{N}}{\mathrm{mm}^2}))$

$$l(u_a) = \int_\Omega p^a u_a \mathrm{d}\Omega + \int_{\partial_t\Omega} t^a u_a \mathrm{d}\Gamma$$

The min principle of complementary energy

$$\hat{\sigma}_{ab} = \arg\min_{\sigma^{ab}\in} \Pi_c(\sigma^{ab})$$

The equilibrium stress state, $\hat{\sigma}_{ab}$, minimizes the c. e. (on a set of admitted stress states)

$$= \{\sigma^{ab} \mid \nabla_a\sigma^{ab} + p^b = 0 \text{ na } \Omega, \; \sigma^{ab}\ell_b = t^a \text{ na } \partial_t\Omega\},$$

$$\Pi_c(\sigma^{ab}) = c(\sigma^{ab}) - l_u(\sigma^{ab})$$

The complementary energy

$$c(\sigma^{ab}) = \frac{1}{2}\int_\Omega C_{abcd}\sigma^{ab}\sigma^{cd}\mathrm{d}\Omega$$

The work done through kinematic boundary conditions

$$l_u(\sigma^{ab}) = \int_{\partial_u\Omega} \sigma^{ab}\tilde{u}_a\ell_b \mathrm{d}\Gamma$$

Elasticity tensor E^{abcd} and compliance tensor C_{abcd}

CIARLET, P. G. (2005)
MAREŠ, T. (2006)

Isotropic material (λ, μ — Lamé coefficients)

$$E^{abcd} = \lambda g^{ab} g^{cd} + \mu g^{ac} g^{bd} + \mu g^{ad} g^{bc}$$

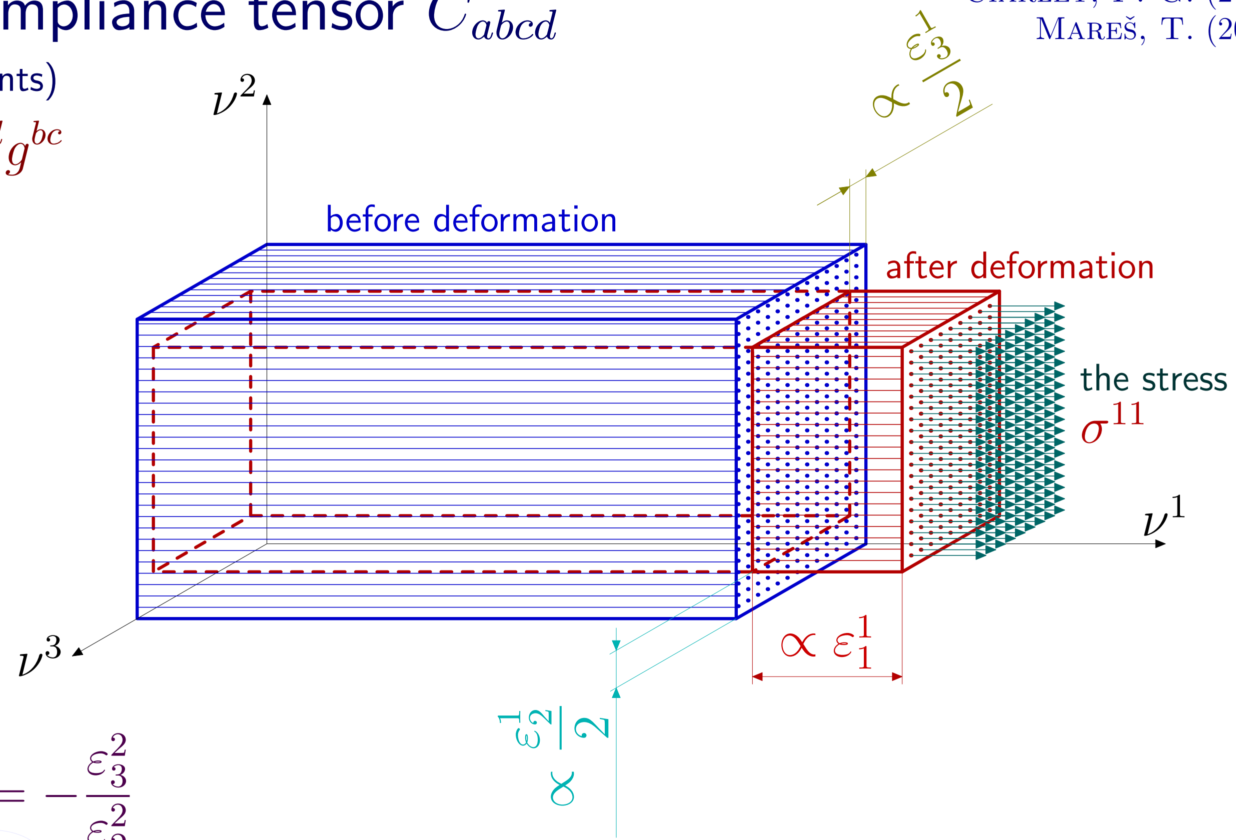
Orthotropic block

Young modulus in the direction ν^1

$$E_{11} = \frac{\sigma^{11}}{\varepsilon_1^1}$$

Poisson ratios

$$\nu_{12} = -\frac{\varepsilon_2^1}{\varepsilon_1^1}, \quad \nu_{13} = -\frac{\varepsilon_3^1}{\varepsilon_1^1}$$



Similarly in the direction of ν^2

$$E_{22} = \frac{\sigma^{22}}{\varepsilon_2^2}, \quad \nu_{21} = -\frac{\varepsilon_1^2}{\varepsilon_2^2}, \quad \nu_{23} = -\frac{\varepsilon_3^2}{\varepsilon_2^2}$$

and of ν^3

$$E_{33} = \frac{\sigma^{33}}{\varepsilon_3^3}, \quad \nu_{31} = -\frac{\varepsilon_1^3}{\varepsilon_3^3}, \quad \nu_{32} = -\frac{\varepsilon_2^3}{\varepsilon_3^3}$$

Strain in the ν^1 excited by all normal stresses

$$\varepsilon_{11} = \varepsilon_1^1 + \varepsilon_1^2 + \varepsilon_1^3$$

$$\varepsilon_{11} = \frac{\sigma^{11}}{E_{11}} - \nu_{21}\frac{\sigma^{22}}{E_{22}} - \nu_{31}\frac{\sigma^{33}}{E_{33}}$$

Similarly in the other directions (G_{23}, G_{31})

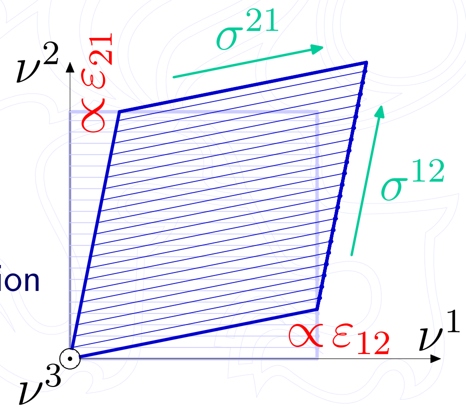
Pure shear

From the definition

$$\varepsilon_{12} = \varepsilon_{21}$$

the equilibrium equation

$$\sigma^{12} = \sigma^{21}$$



$$\sigma^{12} = \sigma^{12} = G_{12}(\varepsilon_{12} + \varepsilon_{21})$$

Compliance tensor C_{abcd}

in Cartesian coordinate system ν^a

alined with the principal material axes of the orthotropic material

CIARLET, P. G. (2005)
MAREŠ, T. (2006)

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{21}}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{31}}{E_{33}} \\ 0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\ 0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & 0 & 0 & 0 & \frac{1}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{32}}{E_{33}} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\ 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\ -\frac{\nu_{13}}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{23}}{E_{22}} & 0 & 0 & 0 & \frac{1}{E_{33}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix}$$

$$\Updownarrow$$

$$\overset{\nu}{\varepsilon}_{ab} = \overset{\nu}{C}_{abcd}\,\overset{\nu}{\sigma}{}^{cd}$$

$$C_{abcd} = C_{cdab} = C_{bacd}$$

$\nearrow$ Energy $\qquad$ $\nwarrow$ Equilibrium

Elasticity tensor E^{abcd}

in Cartesian coordinate system $\overset{\nu}{}{}^{a}$

by inversion of the previous expression

$$\overset{\nu}{\sigma}{}^{ij} = \overset{\nu}{E}{}^{ijkl} \overset{\nu}{\varepsilon}_{kl}$$

$$\left\{ \overset{\nu}{E}{}^{ijkl} \right\}_{\{ij\lceil kl\}} = \begin{pmatrix} \Phi_{1111} & 0 & 0 & 0 & \Phi_{1122} & 0 & 0 & 0 & \Phi_{1133} \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ \Phi_{2211} & 0 & 0 & 0 & \Phi_{2222} & 0 & 0 & 0 & \Phi_{2233} \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ \Phi_{3311} & 0 & 0 & 0 & \Phi_{3322} & 0 & 0 & 0 & \Phi_{3333} \end{pmatrix}$$

$$E^{abcd} = E^{bacd}$$

$$\Phi_{1111} = \frac{1-\nu_{23}\nu_{32}}{N}E_{11}, \quad \Phi_{1122} = \frac{\nu_{21}+\nu_{23}\nu_{31}}{N}E_{11}, \quad \Phi_{1133} = \frac{\nu_{31}+\nu_{32}\nu_{21}}{N}E_{11}$$

$$\Phi_{2211} = \frac{\nu_{12}+\nu_{13}\nu_{32}}{N}E_{22}, \quad \Phi_{2222} = \frac{1-\nu_{13}\nu_{31}}{N}E_{22}, \quad \Phi_{2233} = \frac{\nu_{32}+\nu_{31}\nu_{12}}{N}E_{22}$$

$$\Phi_{3311} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{N}E_{33}, \quad \Phi_{3322} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{N}E_{33}, \quad \Phi_{3333} = \frac{1-\nu_{12}\nu_{21}}{N}E_{33}$$

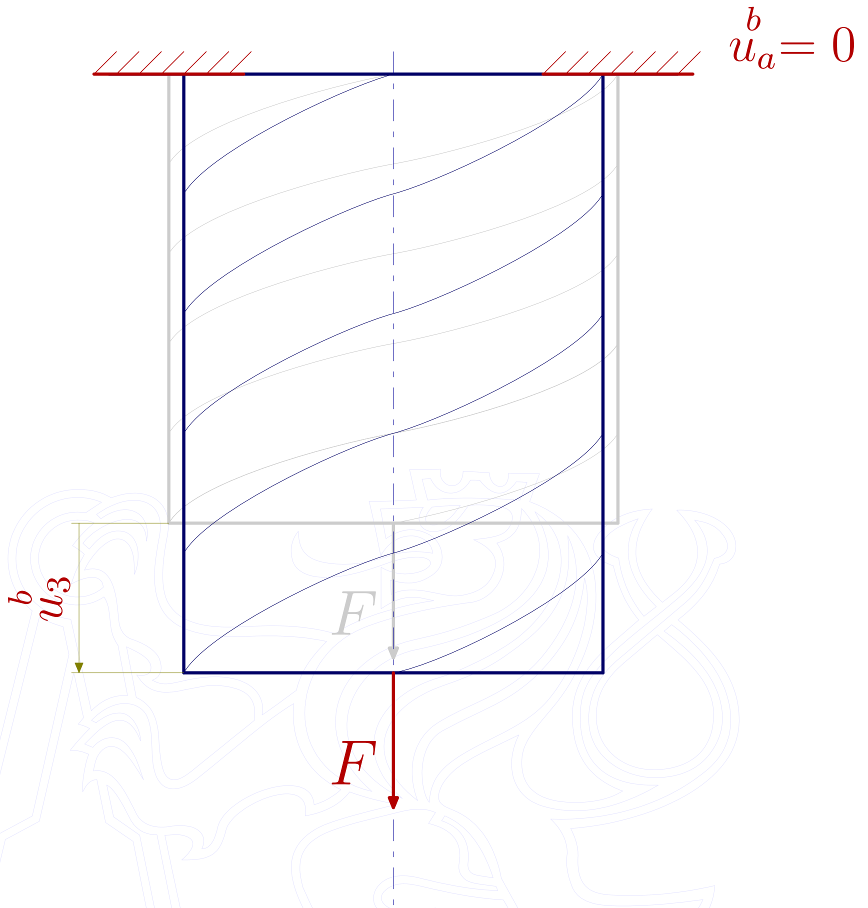
$$N = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{32}\nu_{21}$$

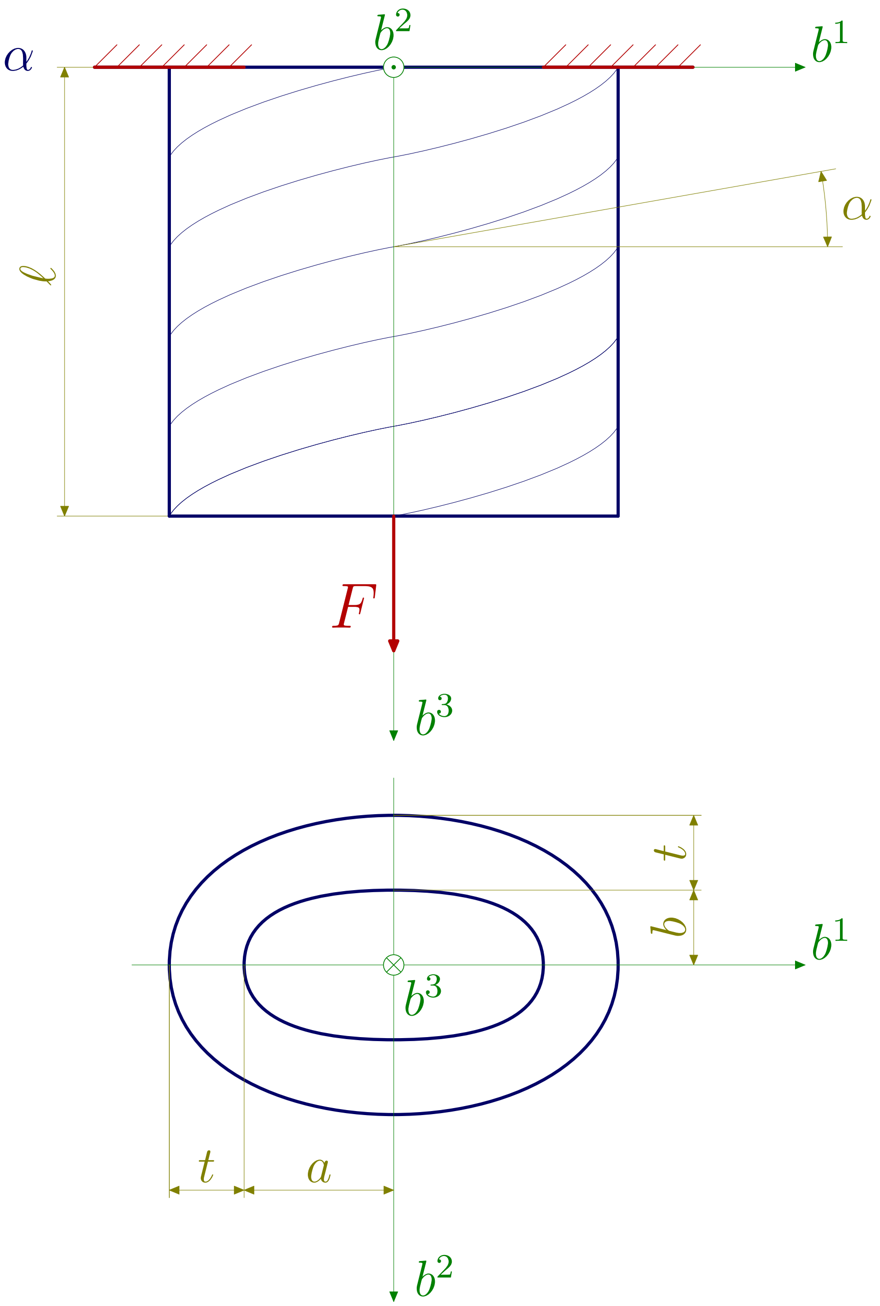
Energy $(E^{abcd} = E^{cdab}) \Rightarrow \Phi_{1122} = \Phi_{2211} \Rightarrow \nu_{21}E_{11} = \nu_{12}E_{22}$, *etc.*

The problem

The thick-walled elliptic tube coiled with an angle α

loaded with Force F and clamped as seen at Fig.





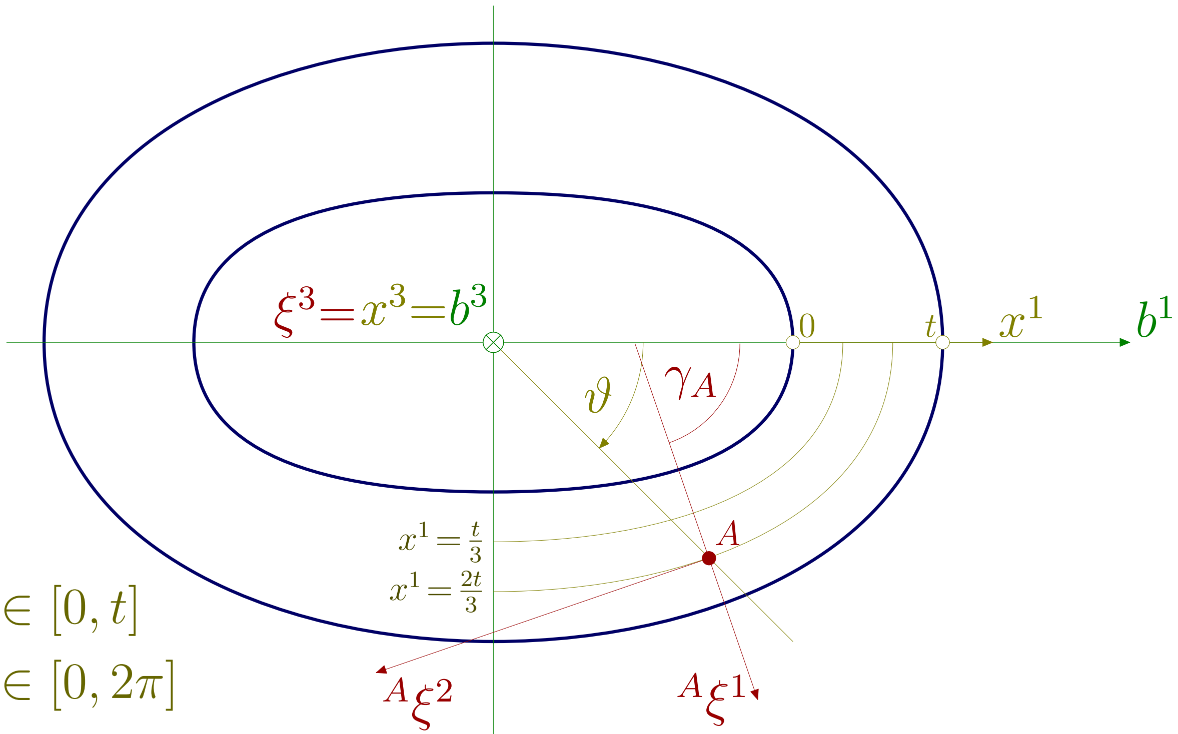
Used coordinate systems

b^a — Cartesian coordinate system

x^a — Elliptic coordinate system

ξ^a — Local Cartesian coordinate s.

ν^a — Main c. s. of the local orthotropy



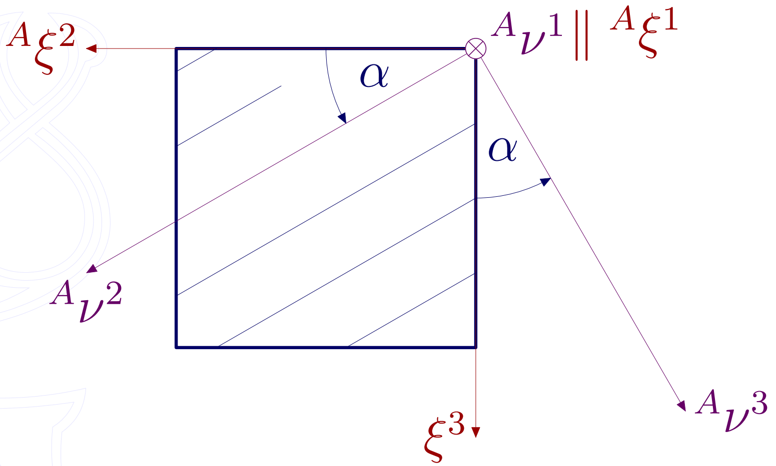
$\vartheta = \vartheta(x^1, x^2)$

$\gamma = \gamma(x^1, x^2)$

$x^1 \in [0, t]$

$x^2 \in [0, 2\pi]$

$x^3 \in [0, \ell]$



$\overset{b}{g}_{ab} = \delta_{ab}$

$\overset{\xi}{g}_{ab} = \delta_{ab}$

$\overset{\nu}{g}_{ab} = \delta_{ab}$

$b^1 = (a + x^1)\cos x^2$

$b^2 = (b + x^1)\sin x^2$

$b^3 = x^3$

Metric tensor for integration

$$\overset{x}{g}_{ab} = \frac{\partial b^c}{\partial x^a}\frac{\partial b^d}{\partial x^b}\delta_{cd} \qquad \frac{\partial b^a}{\partial x^b} = \begin{pmatrix} \cos x^2 & -(a+x^1)\sin x^2 & 0 \\ \sin x^2 & (b+x^1)\cos x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{a\lceil b}$$

$$\overset{x}{g}_{ab} = \begin{pmatrix} 1 & (b-a)\sin x^2\cos x^2 & 0 \\ (b-a)\sin x^2\cos x^2 & (a+x^1)^2\sin^2 x^2 + (b+x^1)^2\cos^2 x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Elasticity tensor transformation

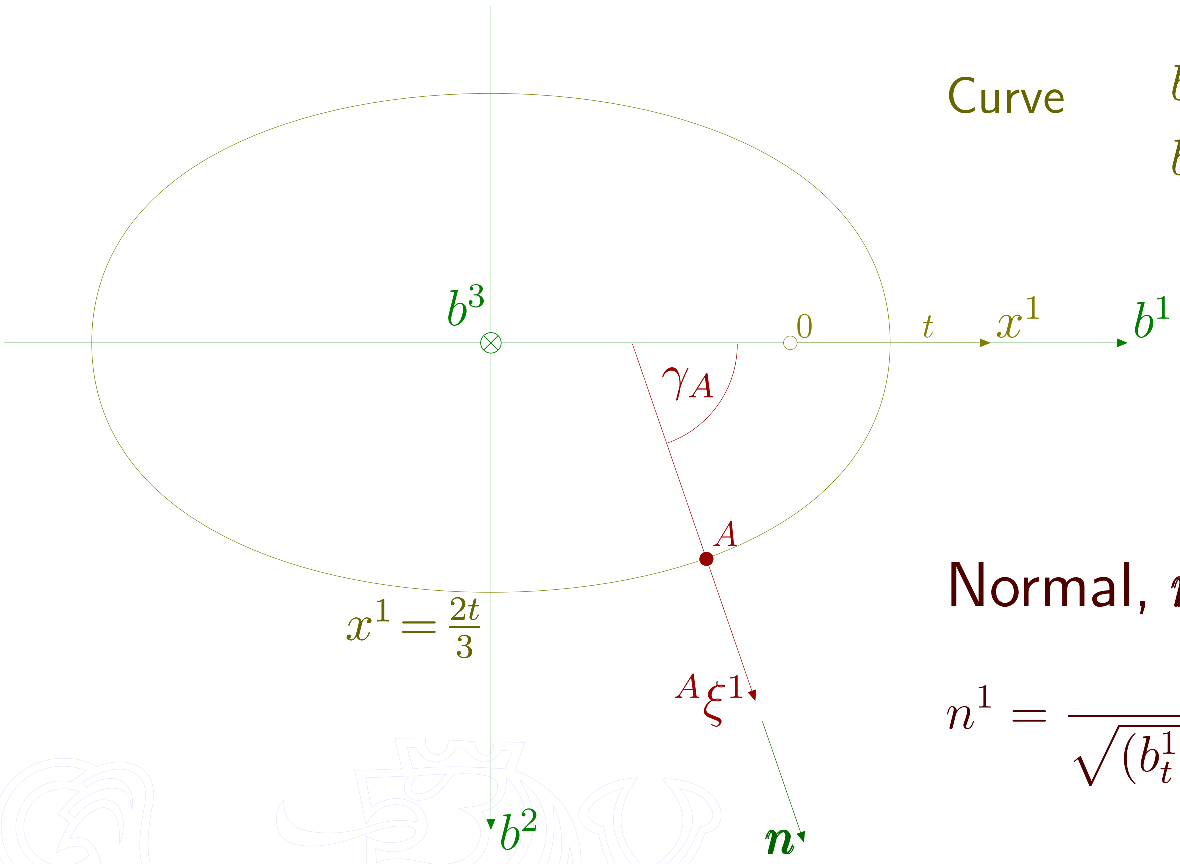
$$\overset{x}{E}^{abcd} = \frac{\partial x^a}{\partial \nu^i}\frac{\partial x^b}{\partial \nu^j}\frac{\partial x^c}{\partial \nu^k}\frac{\partial x^d}{\partial \nu^l}\overset{\nu}{E}^{ijkl}$$

$$\frac{\partial x^a}{\partial \nu^b} = \frac{\partial x^a}{\partial b^c}\frac{\partial b^c}{\partial \xi^d}\frac{\partial \xi^d}{\partial \nu^b} \qquad \frac{\partial x^a}{\partial b^b} = \left(\frac{\partial b^a}{\partial x^b}\right)^{-1} =$$

$$= \frac{1}{a\sin^2 x^2 + b\cos^2 x^2 + x^1}\begin{pmatrix} (b+x^1)\cos x^2 & (a+x^1)\sin x^2 & 0 \\ -\sin x^2 & \cos x^2 & 0 \\ 0 & 0 & a\sin^2 x^2 + b\cos^2 x^2 + x^1 \end{pmatrix}$$

$$\frac{\partial b^a}{\partial \xi^b} = \begin{pmatrix} \cos\gamma_A & -\sin\gamma_A & 0 \\ \sin\gamma_A & \cos\gamma_A & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \frac{\partial \xi^a}{\partial \nu^b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

Normal to the ellipse $\Rightarrow\ \gamma_A$



Curve
$$b^1 = (a + x^1)\cos x^2$$
$$b^2 = (b + x^1)\sin x^2$$

$$\cos\gamma_A = (1,0)\cdot\boldsymbol{n}$$
$$\sin\gamma_A = \pm|(1,0)\times\boldsymbol{n}|$$

$${}_t = \frac{\partial}{\partial x^2}$$

Normal, $\boldsymbol{n}$

$$n^1 = \frac{b_t^2}{\sqrt{(b_t^1)^2 + (b_t^2)^2}}, \qquad n^2 = \frac{-b_t^1}{\sqrt{(b_t^1)^2 + (b_t^2)^2}}$$

$$\boldsymbol{n} = \frac{1}{d}\begin{pmatrix}(b + x^1)\cos x^2\\ (a + x^1)\sin x^2\end{pmatrix}$$

$$d = \sqrt{(a + x^1)^2\sin^2 x^2 + (b + x^1)^2\cos^2 x^2}$$

Angle γ_A consequently

$$\cos\gamma_A = \frac{b + x^1}{d}\cos x^2$$
$$\sin\gamma_A = \frac{a + x^1}{d}\sin x^2$$

Total potential energy of the tube

SYNGE, J. L. and SCHILD, A. (1978)
LOVELOCK, D. and RUND, H. (1989)
CIARLET, P. G. (2005)
GNU MAXIMA gama.mac

Principle of the total potential energy minimum

$$\hat{u}_a = \arg \min_{u_b \in \mathbb{U}} \Pi(u_c)$$

The real state of a deformed body minimizes the total potential energy (on a set of admissible states, $\mathbb{U}$)

$$\Pi(u_a) = a(u_a) - l(u_a)$$

The elastic strain energy

$$a(u_a) = \frac{1}{2} \int_\Omega E^{abcd} \varepsilon_{ab}(u_a) \varepsilon_{cd}(u_a) \mathrm{d}\Omega$$

The potential energy of the applied forces $p^a(\frac{\mathrm{N}}{\mathrm{mm}^3}), t^a(\frac{\mathrm{N}}{\mathrm{mm}^2}))$

$$l(u_a) = \int_\Omega p^a u_a \mathrm{d}\Omega + \int_{\partial_t \Omega} t^a u_a \mathrm{d}\Gamma$$

$$\varepsilon_{ab} = \frac{1}{2}(\nabla_a u_b + \nabla_b u_a)$$

$$\nabla_a u_b = \partial_a u_b - \Gamma^c_{ab} u_c$$

Christoffel symbol of the 2^{nd} kind

$$\Gamma^d_{ab} = g^{dc} \frac{1}{2}(g_{ac,b} + g_{cb,a} - g_{ab,c})$$

In c. s. $\boldsymbol{x}$:

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{(a-b)\cos x^2 \sin x^2}{(b-a)\cos^2 x^2 + x^1 + a}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{(b-a)\cos^2 x^2 + x^1 + a}$$

$$\Gamma^1_{22} = -\frac{(x^1)^2 + x^1(a+b) + ab}{(b-a)\cos^2 x^2 + x^1 + a}$$

$$\Gamma^2_{22} = \frac{(a-b)\cos x^2 \sin x^2}{(b-a)\cos^2 x^2 + x^1 + a}$$

$$\overset{x}{\varepsilon}_{ab} = \frac{1}{2}(\partial_a \overset{x}{u}_b + \partial_b \overset{x}{u}_a - 2\, \Gamma^c_{ab} \overset{x}{u}_c)$$

$$\Gamma^c_{ab} \overset{x}{u}_c = \Gamma^1_{ab} \overset{x}{u}_1 + \Gamma^2_{ab} \overset{x}{u}_2 + \Gamma^3_{ab} \overset{x}{u}_3$$

$$\Gamma^1_{ab} = \frac{1}{J} \begin{pmatrix} 0 & (a-b)\cos x^2 \sin x^2 & 0 \\ (a-b)\cos x^2 \sin x^2 & -((x^1)^2 + x^1(a+b) + ab) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^2_{ab} = \frac{1}{J} \begin{pmatrix} 0 & 1 & 0 \\ 1 & (a-b)\cos x^2 \sin x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^3_{ab} = 0$$

$$J = (b-a)\cos^2 x^2 + x^1 + a$$

The solution is sought in the form of Fourier series

The solution is sought in the form of Fourier series

Boundary condition

$\overset{x}{x^3} = 0 : \quad \overset{x}{u}_1 = 0$

$\overset{x}{u}_2 = 0, \quad \overset{x}{u}_3 = 0$

$$\overset{x}{u}_1 = \sum_{j,k,m=-K}^{K} a_1^{jkm} x^3 e^{i\left(jx^1 \frac{2\pi}{t} + kx^2 + mx^3 \frac{2\pi}{\ell}\right)} \qquad K = \infty \ (3)$$

$$\overset{x}{u}_2 = \sum_{j,k,m=-K}^{K} a_2^{jkm} x^3 e^{i\left(jx^1 \frac{2\pi}{t} + kx^2 + mx^3 \frac{2\pi}{\ell}\right)} \qquad \overset{x}{u}_3 = \Sigma\, a_3\, \varphi$$

The potential energy of the F

$$l(u_a) = \int_S \frac{F}{S} \overset{x}{u}_3 \,\mathrm{d}S$$

$$\min_{\text{BC fulfilled}} (a - l)$$

$$\Updownarrow$$

$$\frac{\partial a}{\partial a_{1,2}^{jkm}} = 0, \quad \frac{\partial (a - l)}{\partial \bar{u}} = 0$$

Sign $\overset{x}{u}_{1,2} = \Sigma\, a_{1,2}\, \varphi \quad (\varphi = x^3 \phi)$, pak

$$\frac{\partial \overset{x}{u}_a}{\partial x^b} = \begin{pmatrix} \Sigma\, a_1\, \varphi\, ij \frac{2\pi}{t} & \Sigma\, a_1\, \varphi\, ik & \Sigma\, a_1 \left(\varphi\, im \frac{2\pi}{\ell} + \phi\right) \\ \Sigma\, a_2\, \varphi\, ij \frac{2\pi}{t} & \Sigma\, a_2\, \varphi\, ik & \Sigma\, a_2 \left(\varphi\, im \frac{2\pi}{\ell} + \phi\right) \\ \Sigma\, a_3\, \varphi\, ij \frac{2\pi}{t} & \Sigma\, a_3\, \varphi\, ik & \Sigma\, a_3 \left(\varphi\, im \frac{2\pi}{\ell} + \phi\right) \end{pmatrix}$$

$$a(u_a) = \frac{1}{2} \int_\Omega E^{abcd} \varepsilon_{ab}(u_a) \varepsilon_{cd}(u_a) \mathrm{d}\Omega$$

$$E^{abcd} = E^{bacd} \quad \Rightarrow \overset{x}{\varepsilon}_{ab} \to \partial_a \overset{x}{u}_b - \Gamma^c_{ab} \overset{x}{u}_c$$

$$a = \frac{1}{2} \int_\Omega \left(\partial_a \overset{x}{u}_b - \Gamma^p_{ab} \overset{x}{u}_p\right) \overset{x}{E}^{abcd} \left(\partial_c \overset{x}{u}_d - \Gamma^p_{cd} \overset{x}{u}_p\right) \left|\overset{x}{g}_{ab}\right|^{\frac{1}{2}} \mathrm{d}^3 x$$

Transformation

$$\overset{x}{E}^{abcd} = \frac{\partial x^a}{\partial \nu^i} \frac{\partial x^b}{\partial \nu^j} \frac{\partial x^c}{\partial \nu^k} \frac{\partial x^d}{\partial \nu^l} \overset{\nu}{E}^{ijkl}$$

is performed in GNU OCTAVE syntax simply

```
xnu=xb*bxi*xinu
Ex=kron(xnu,xnu)*Enu*kron(xnu',xnu')
```

Derivatives

$$\frac{\partial l}{\partial \bar{u}} = F \quad \frac{\partial a}{\partial u, a_{1,2}} = ?$$

We have chosen

$$\overset{x}{u}_{1,2} = \sum_{j,k,m=-K}^{K} a_{1,2}^{jkm} \varphi^{jkm}$$

$$\varphi^{jkm} = x^3 e^{ijx^1 \frac{2\pi}{t}} \cdot e^{ikx^2} \cdot e^{imx^3 \frac{2\pi}{\ell}}$$

v GNU OCTAVE

```
j=(-3:1:3); k=(-3:1:3); m=(-3:1:3);
phi=x3*kron(kron(exp(i*j*x1*2*pi/t),exp(i*k*x2)),exp(i*m*x3*2*pi/ell))
```

$$\begin{pmatrix} \overset{x}{u}_1 \\ \overset{x}{u}_2 \\ \overset{x}{u}_3 \end{pmatrix} = $$

```
ux=[phi,zeros(1,686);zeros(1,343),phi,zeros(1,343);zeros(1,686),phi]*A
```

A is a vector of coefficients

$$\left\{ \frac{\partial \overset{x}{u}_a}{\partial x^b} \right\}_{ab\lceil} = \texttt{B*A}$$

```
B=[i*2*pi/t*phi.*kron(kron(j,jedna),jedna),zeros(1,343),zeros(1,343);
i*phi.*kron(kron(jedna,k),jedna),zeros(1,343),zeros(1,343);
i*2*pi/ell*phi.*kron(kron(jedna,jedna),m)+ϕ*ones(1,343),zeros(1,686);
zeros(1,343),i*2*pi/t*phi.*kron(kron(j,jedna),jedna),zeros(1,343);
zeros(1,343),i*phi.*kron(kron(jedna,m),jedna),zeros(1,343);
zeros(1,343),i*2*pi/ell*phi.*kron(kron(je,je),m)+ϕ*ones(1,343),zeros(1,343);
zeros(1,343),zeros(1,343),i*2*pi/t*phi.*kron(kron(j,jedna),jedna);
zeros(1,343),zeros(1,343),i*phi.*kron(kron(jedna,k),jedna);
zeros(1,686),i*2*pi/ell*phi.*kron(kron(jedna,jedna),m)+ϕ*ones(1,343)]
```

Integration of the elastic energy

$$\Gamma^c_{ab} \overset{x}{u}_c = \Gamma^1_{ab} \overset{x}{u}_1 + \Gamma^2_{ab} \overset{x}{u}_2 + \Gamma^3_{ab} \overset{x}{u}_3$$

$$\left\{\Gamma^p_{ab} \overset{x}{u}_p\right\}_{ab\lceil} = \left\{\Gamma^1_{ab}\right\}_{ab\lceil} \texttt{*[phi,zeros(1,686)]*A} + \\ + \left\{\Gamma^2_{ab}\right\}_{ab\lceil} \texttt{*[zeros(1,343),phi,zeros(1,343)]*A}$$

$$\left(\partial_a \overset{x}{u}_b - \Gamma^p_{ab} \overset{x}{u}_p\right) = \texttt{(B-Gam)*A}$$

```
J=(b-a)*(cos(x2))**2+x1+a
G1=1/J*[0,(a-b)*cos(x2)*sin(x2),0;
      (a-b)*cos(x2)*sin(x2),-((x1)**2+x1*(a+b)+a*b),0;0,0,0]
G2=1/J*[0,1,0;1,(a-b)*cos(x2)*sin(x2),0;0,0,0]
Gam=vec(G1')*[phi,zeros(1,686)]+vec(G2')*[zeros(1,343),phi,zeros(1,343)]
```

Elasticity energy

$$a = \frac{1}{2} \texttt{A}^T K \texttt{A}$$

Stiffness matrix

$$K = \int_0^{\ell} \int_0^{2\pi} \int_0^{t} \texttt{(B-Gam)'*Ex*(B-Gam)*sqrt(det(gx))} \, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$$

```
gx=(xb**(-1))'*xb**(-1)
```

Integrand is expressed in GNU OCTAVE, integrate it numerically (energy.m)

The equation, right hand side and solution

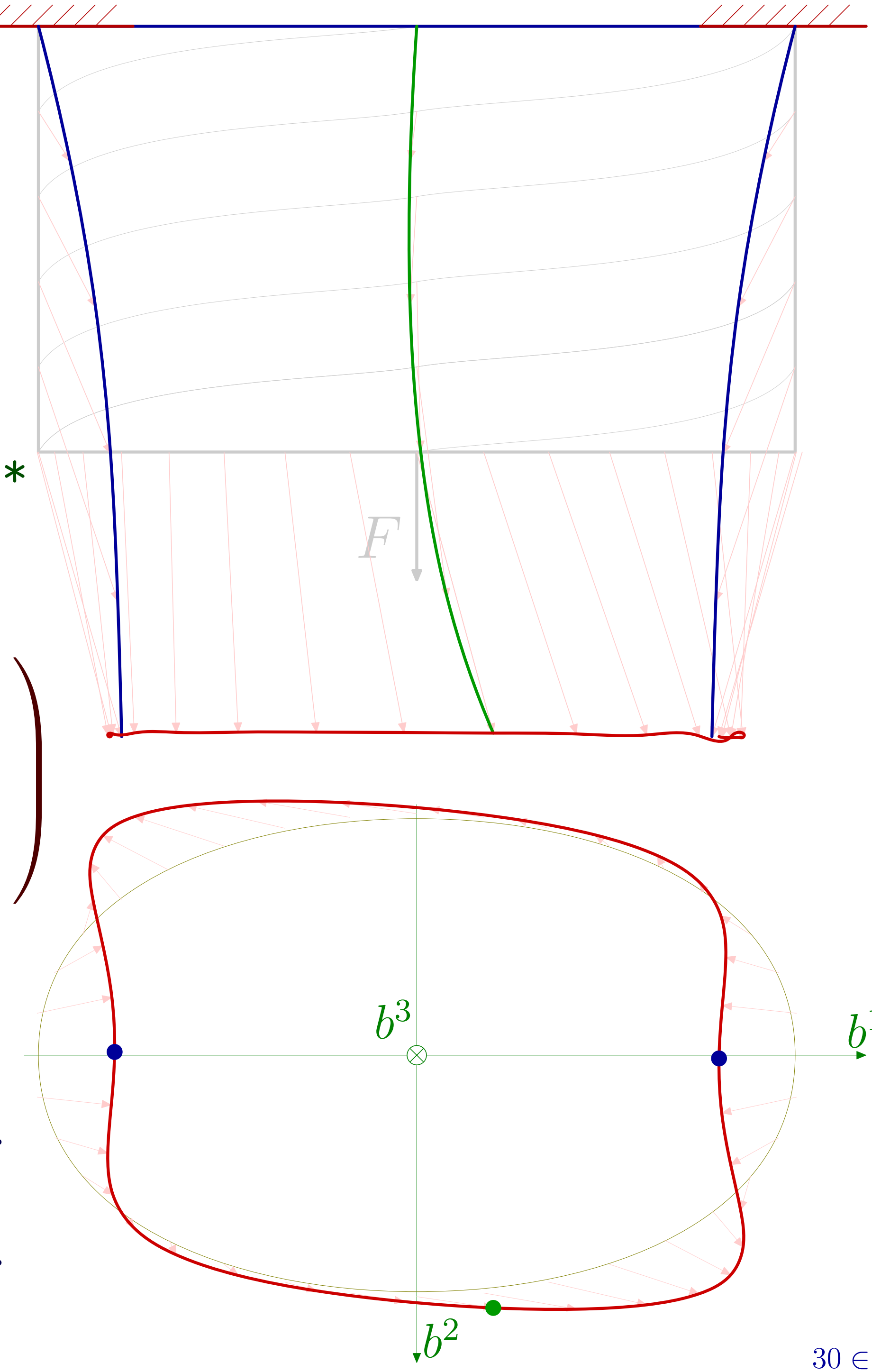
$$\frac{\partial a}{\partial A} = \frac{\partial l}{\partial A} \qquad \frac{\partial a}{\partial A} = KA \qquad KA = P$$

$$l = \int_S \frac{F}{S} \overset{x}{u}_3 \, \mathrm{d}S$$

$$l = \int_0^{2\pi}\int_0^{t} \frac{F}{S} \texttt{[zeros(1,343),zeros(1,343),phi]*} \texttt{*sqrt(det(gx))} \mathrm{d}x^1 \mathrm{d}x^2 \texttt{ *A}$$

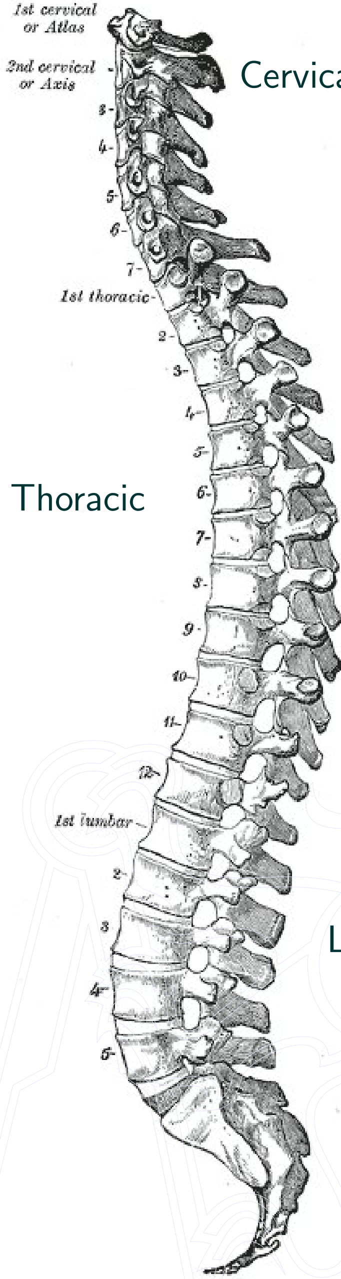
$$P = \frac{\partial l}{\partial A} = \begin{pmatrix} \texttt{zeros(363)} \\ \texttt{zeros(363)} \\ \int_0^{2\pi}\int_0^{t} \frac{F}{S} \texttt{phi'*sqrt(det(gx))} \mathrm{d}x^1 \mathrm{d}x^2 \end{pmatrix}$$

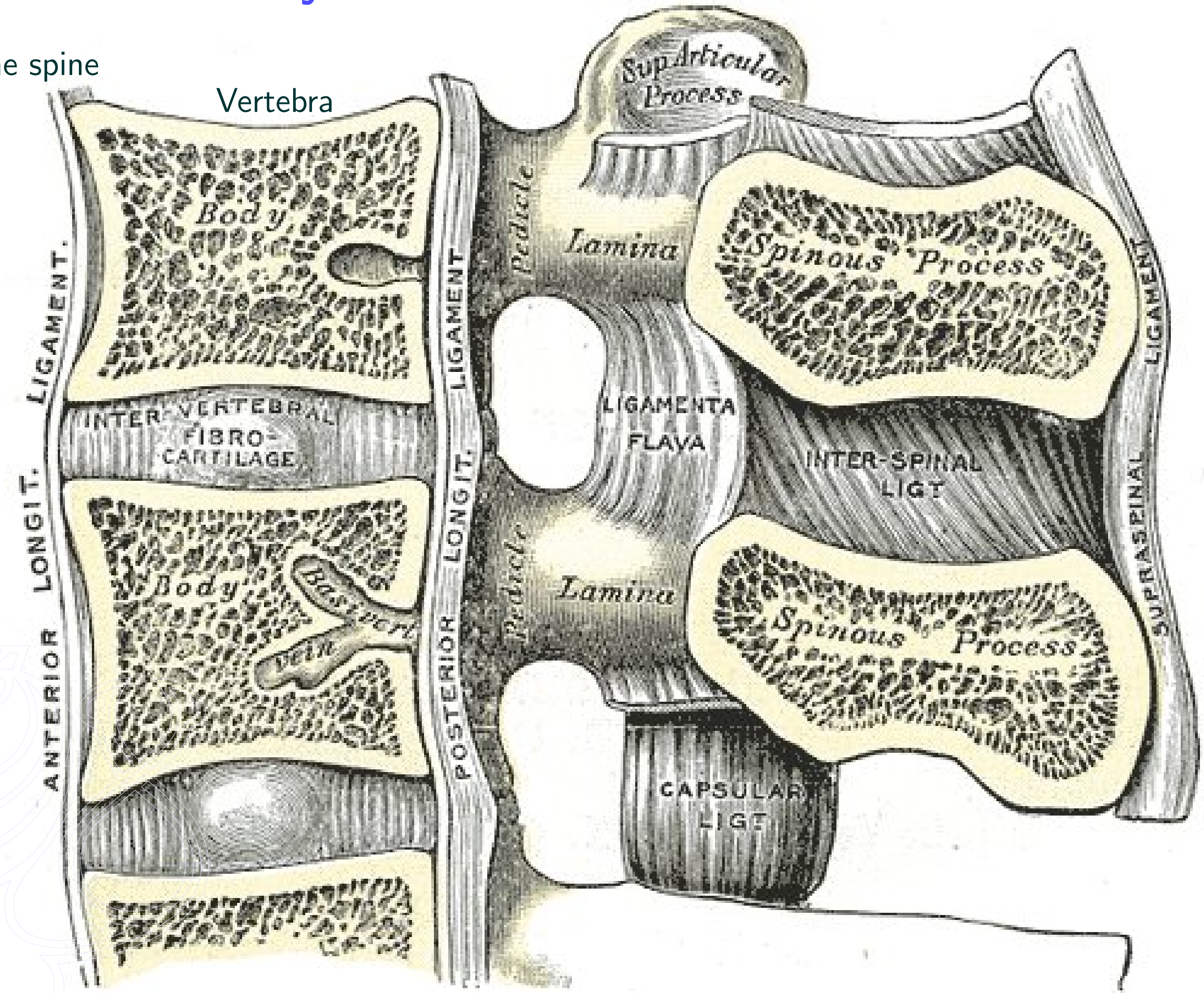
```
x1=...; x2=...; x3=...
A=K**(-1)*P
phi=x3*kron(kron(exp(i*j*x1*2*pi/t),...
ux=real([phi,zeros(1,siz),zeros(1,...
xb=1/(a*(sin(x2))**2+b*(cos(x2))**2+...
ub=xb*ux
```



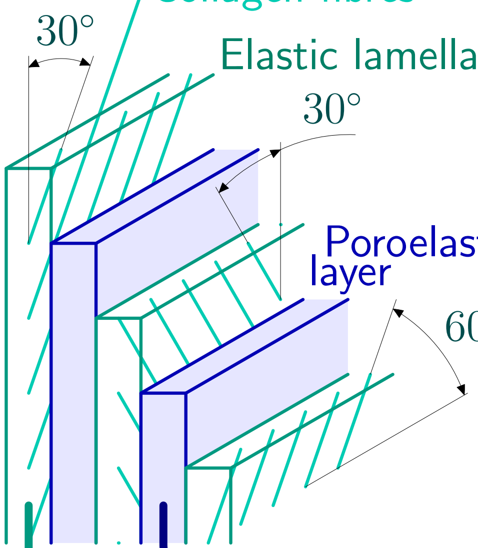
GRAY, H. (1918)

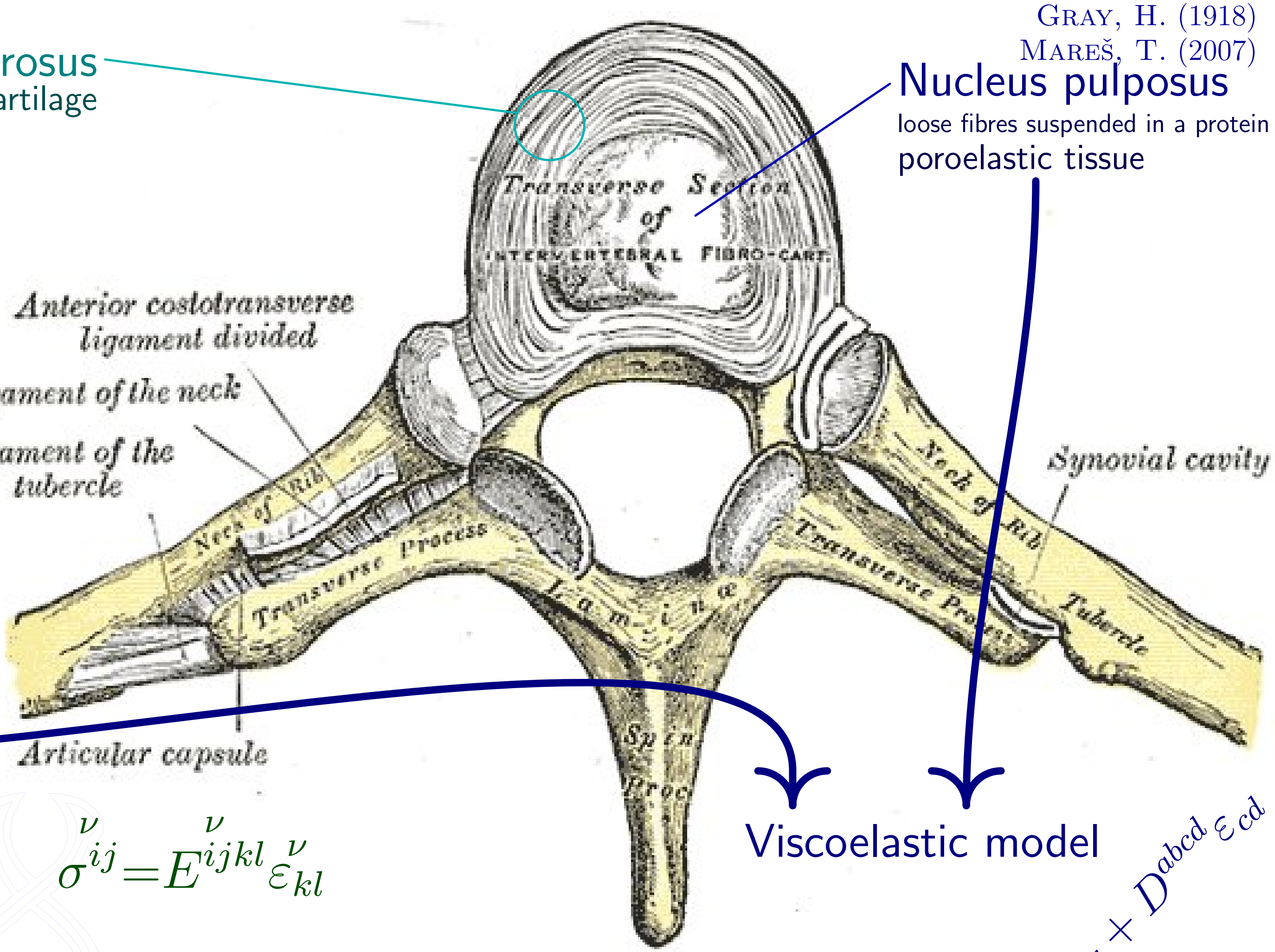
Deformation analysis of the Intervertebral disk





Gray, H. (1918)
Mareš, T. (2007)





Annulus fibrosus
layers of fibrocartilage

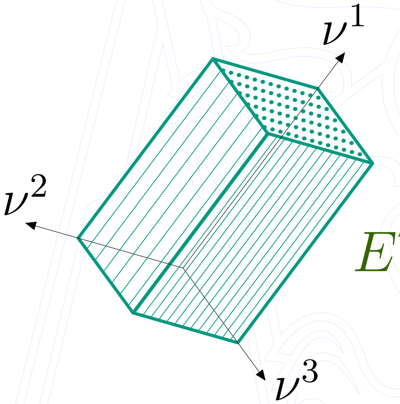
Nucleus pulposus
loose fibres suspended in a protein gel
poroelastic tissue

Viscoelastic model

$$\dot{\sigma}^{ab} + A^{ab}{}_{cd}\sigma^{cd} = B^{abcd}\dot{\varepsilon}_{cd} + D^{abcd}\varepsilon_{cd}$$

turn

Elastic locally orthotropic: $\overset{\nu}{\sigma}^{ij} = \overset{\nu}{E}^{ijkl}\overset{\nu}{\varepsilon}_{kl}$

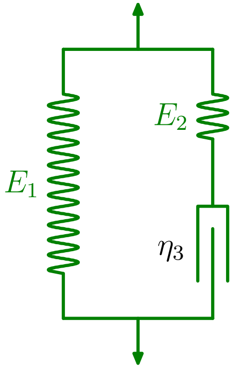


$$\overset{\nu}{E}^{ijkl} = \begin{pmatrix} \Phi_{11} & 0 & 0 & 0 & \Phi_{12} & 0 & 0 & 0 & \Phi_{13} \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ \Phi_{21} & 0 & 0 & 0 & \Phi_{22} & 0 & 0 & 0 & \Phi_{23} \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ \Phi_{31} & 0 & 0 & 0 & \Phi_{32} & 0 & 0 & 0 & \Phi_{33} \end{pmatrix}_{\{ij\lceil kl\}}$$

Viscoelastic model of the nucleus pulposus and the poroelastic layers

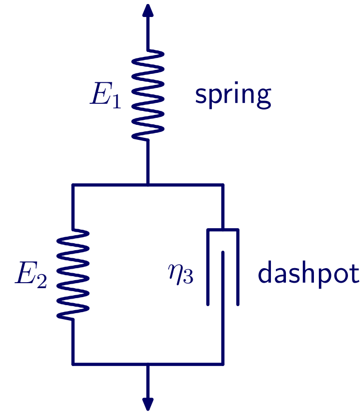
SOBOTKA, Z. (1981)
MAREŠ, T. (2007)

Three element Zener model



$$\dot{\sigma} + \frac{E_2}{\eta_3}\,\sigma = (E_1 + E_2)\,\dot{\varepsilon} + \frac{E_1 E_2}{\eta_3}\,\varepsilon$$

Poynting-Thompson model



$$\dot{\sigma} + \frac{E_1 + E_2}{\eta_3}\,\sigma = E_1\dot{\varepsilon} + \frac{E_1 E_2}{\eta_3}\,\varepsilon$$

a (t^{-1})

b (Pa)

d $(\mathrm{Pa}\cdot\mathrm{t}^{-1})$

$$\dot{\sigma} + a\,\sigma = b\dot{\varepsilon} + d\,\varepsilon$$

1D→3D

$$\dot{\sigma}^{ab} + A^{ab}{}_{cd}\sigma^{cd} = B^{abcd}\dot{\varepsilon}_{cd} + D^{abcd}\varepsilon_{cd}$$

c^{ab} – a constant tensor

$$\sigma^{ab} = U^{ab}{}_{cd}\Lambda^{cd}{}_{ij}\left(\int \mathcal{L}^{ij}{}_{kl}\mathcal{U}^{kl}{}_{mn}\left(B^{mnop}\dot{\varepsilon}_{op} + D^{mnop}\varepsilon_{op}\right)\mathrm{d}t + c^{ab}\right)$$

$$U^{ab}{}_{cd} = \left(\left\{M_1^{ab}\right\}_{ab\lceil}, \left\{M_2^{ab}\right\}_{ab\lceil}, \dots, \left\{M_9^{ab}\right\}_{ab\lceil}\right)_{ab\lceil cd}, \quad U^{ab}{}_{cd}\mathcal{U}^{cd}{}_{kl} = \delta^a_k\delta^b_l$$

λ, M^{ab} – eigenvalue problem: $\left(-A^{ab}{}_{cd} - \lambda\delta^a_c\delta^b_d\right)M^{cd} = 0$

OCTAVE, MAXIMA: eigenvalue and eigenvector of the matrix $\left\{-A^{ab}{}_{cd}\right\}_{ab\lceil cd}$

$$\Lambda^{ab}{}_{cd} = \begin{pmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_9 t} \end{pmatrix}_{ab\lceil cd}, \quad \mathcal{L}^{ab}{}_{cd} = \begin{pmatrix} e^{-\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-\lambda_9 t} \end{pmatrix}_{ab\lceil cd}$$

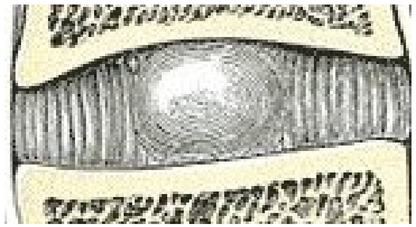
isotropic

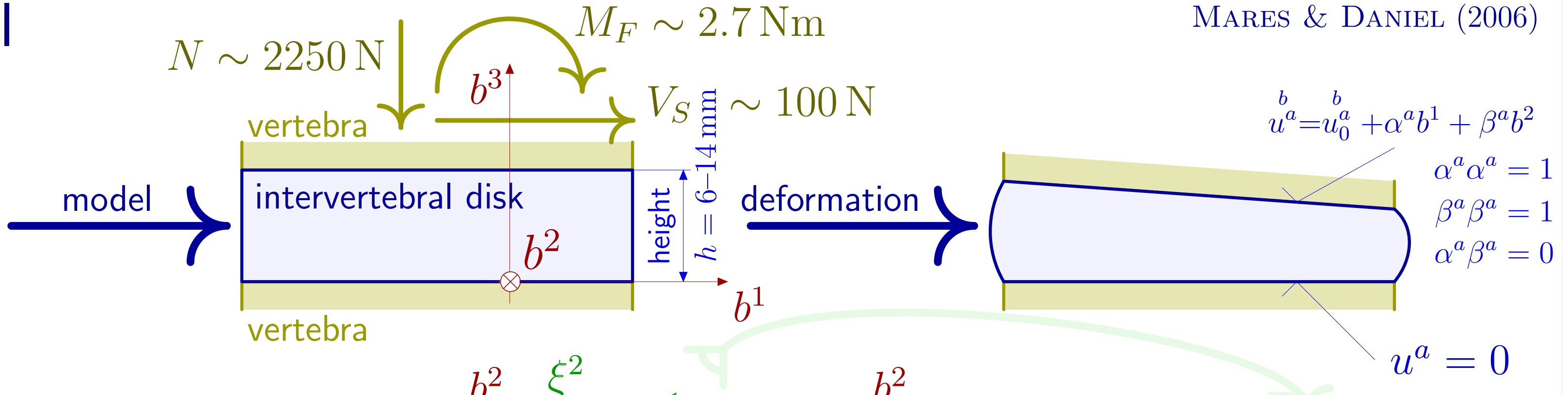
$$A^{ab}{}_{cd} = \alpha_1 g^{ab}g_{cd} + \alpha_2\left(\delta^a_c\delta^b_d + \delta^a_d\delta^b_c\right)$$

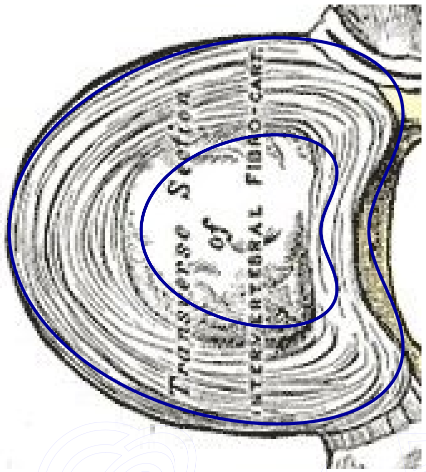
$$B^{abcd} = \beta_1 g^{ab}g^{cd} + \beta_2\left(g^{ac}g^{bd} + g^{ad}g^{bc}\right)$$

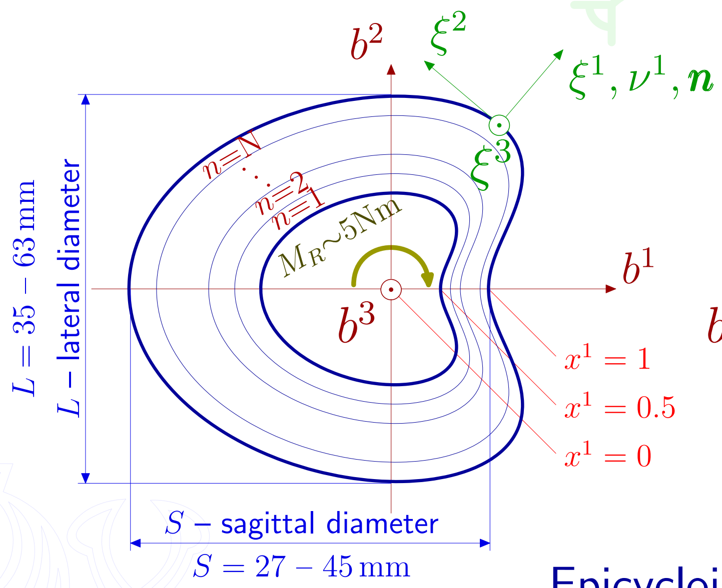
$$D^{abcd} = \gamma_1 g^{ab}g^{cd} + \gamma_2\left(g^{ac}g^{bd} + g^{ad}g^{bc}\right)$$

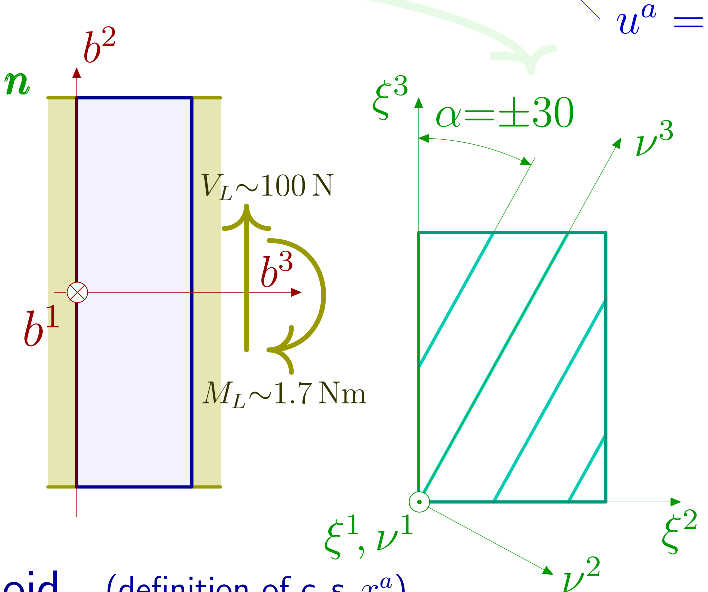
Geometrical model











Epicycloid (definition of c. s. x^a)

$$b^1 = 2r_x x^1 \cos x^2 - d_x x^1 \cos 2x^2$$
$$b^2 = 2r_y x^1 \sin x^2 - d_y x^1 \sin 2x^2$$
$$b^3 = x^3$$

$L, S+$ shape $\Rightarrow r_x, r_y, d_x, d_y$

$$0 \le x^1 \le 1$$
$$0 \le x^2 \le 2\pi$$
$$0 \le x^3 \le h$$

number of lamellas: 20

$\Delta x^1 = 0.05$

$$\frac{\partial b^a}{\partial x^b} = \begin{pmatrix} 2r_x \cos x^2 - d_x \cos 2x^2 & 2d_x x^1 \sin 2x^2 - 2r_x x^1 \sin x^2 & 0 \\ 2r_y \sin x^2 - d_y \sin 2x^2 & 2r_y x^1 \cos x^2 - 2d_y x^1 \cos 2x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathtt{bx}$$

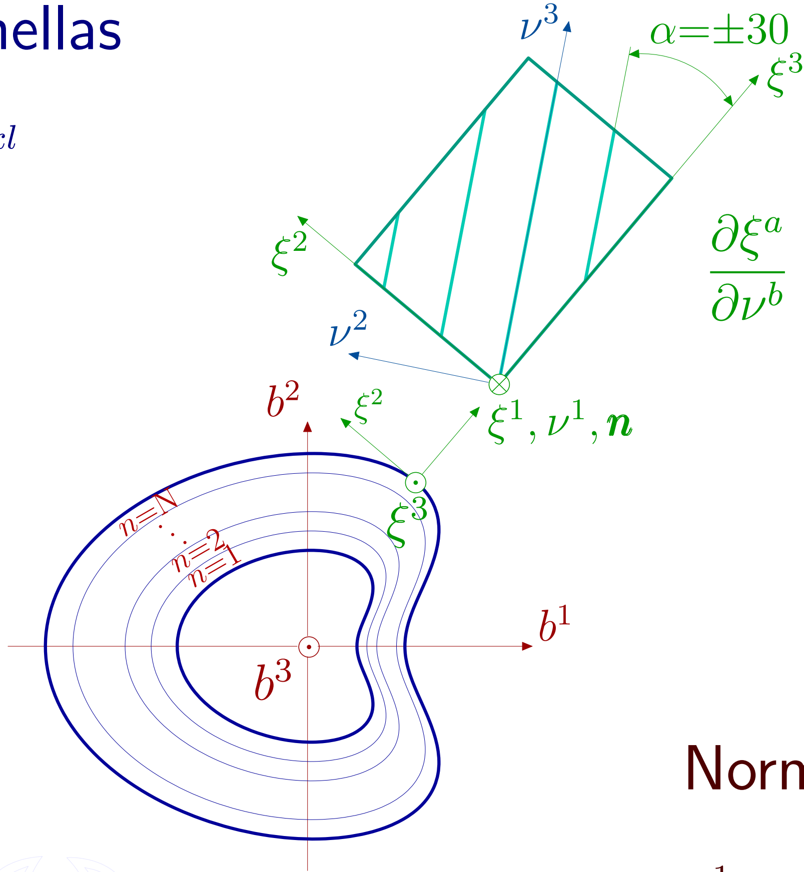
$$\frac{\partial x^a}{\partial b^b} = \left(\frac{\partial b^a}{\partial x^b}\right)^{-1} = \mathtt{xb} \qquad \overset{x}{g}_{ab} = \frac{\partial b^c}{\partial x^a}\frac{\partial b^d}{\partial x^b}\,\overset{b}{g}_{cd} = \mathtt{bx}'\mathtt{bx}$$

Elasticity tensor of the lamellas

$$\overset{x}{E}{}^{abcd} = \frac{\partial x^a}{\partial \nu^i}\frac{\partial x^b}{\partial \nu^j}\frac{\partial x^c}{\partial \nu^k}\frac{\partial x^d}{\partial \nu^l}\,\overset{\nu}{E}{}^{ijkl}$$

$$\frac{\partial x^a}{\partial \nu^i} = \frac{\partial x^a}{\partial b^b}\frac{\partial b^b}{\partial \xi^c}\frac{\partial \xi^c}{\partial \nu^i}$$

$$\frac{\partial x^a}{\partial b^b} = \left(\frac{\partial b^a}{\partial x^b}\right)^{-1}$$



$$\frac{\partial \xi^a}{\partial \nu^b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

GNU OCTAVE

```
xnu=xb*bxi*xinu
Ex=kron(xnu,xnu)*Enu*kron(xnu',xnu')
```

Normal $\boldsymbol{n} = (n^1, n^2, 0)$

$$n^1 = \frac{b_t^2}{\sqrt{(b_t^1)^2 + (b_t^2)^2}}$$

$$n^2 = \frac{-b_t^1}{\sqrt{(b_t^1)^2 + (b_t^2)^2}}$$

kde $_t = \dfrac{\partial}{\partial x^2}$

$$n^1 = \frac{N^1}{d},\quad N^1 = 2r_y x^1 \cos x^2 - 2d_y x^1 \cos 2x^2$$

$$n^2 = \frac{N^2}{d},\quad N^2 = 2r_x x^1 \sin x^2 - 2d_x x^1 \sin 2x^2$$

$$d = \sqrt{(N^1)^2 + (N^2)^2}$$

$$\frac{\partial \xi^a}{\partial b^b} = \begin{pmatrix} n^1 & n^2 & 0 \\ -n^2 & n^1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad \frac{\partial b^a}{\partial \xi^b} = \begin{pmatrix} n^1 & -n^2 & 0 \\ n^2 & n^1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Computational models

1. Small deformations, elastic nucleus pulposus

Isotropic elastic Nucleus pulposus (soft, incompressible)

$$E_{\text{polpus}}^{abcd} = \lambda g^{ab} g^{cd} + \mu g^{ac} g^{bd} + \mu g^{ad} g^{bc}$$

Minimum Π

2. Small deformations, viscoelastic nucleus pulposus

Isotropic viscoelastic Nucleus pulposus

$$\dot{\sigma}^{ab} + A^{ab}{}_{cd} \sigma^{cd} = B^{abcd} \dot{\varepsilon}_{cd} + D^{abcd} \varepsilon_{cd}$$

Galerkin method, base (BC)

$$\nabla_a \sigma^{ab} = 0, 2\varepsilon_{ab} = \nabla_a u_b + \nabla_b u_a$$

$$f(\sigma^{ab}, \dot{\sigma}^{ab}, \varepsilon^{ab}, \dot{\varepsilon}^{ab}) = 0$$

3. Large deformations, elastic nucleus pulposus

$$2E_{ab} = \nabla_a u_b + \nabla_b u_a + \nabla_a u^c \nabla_b u_c$$

$$E_{\text{polpus}}^{abcd} = \lambda g^{ab} g^{cd} + \mu g^{ac} g^{bd} + \mu g^{ad} g^{bc}$$

Minimum Π

4. Large deformations, viscoelastic nucleus pulposus

$$2E_{ab} = \nabla_a u_b + \nabla_b u_a + \nabla_a u^c \nabla_b u_c$$

Nucleus: $\dot{\Sigma}^{ab} + A^{ab}{}_{cd} \Sigma^{cd} = B^{abcd} \dot{E}_{cd} + D^{abcd} E_{cd}$

Galerkin method, base (BC)

$$\nabla_a \Sigma^{ab} = 0$$

The least complicated model: 1. Small deformations, elastic nucleus pulposus

1. Small deformations, elastic nucleus pulposus

SYNGE, J. L. and SCHILD, A. (1978)
LOVELOCK, D. and RUND, H. (1989)
CIARLET, P. G. (2005)
GNU MAXIMA `*.mac`

Principle of the total potential energy minimum

$$\hat{u}_a = \arg\min_{u_b \in} \Pi(u_c)$$

The real state of a deformed body minimizes the total potential energy (on a set of admissible states,)

$$\Pi(u_a) = a(u_a) - l(u_a)$$

The elastic strain energy

$$a(u_a) = \frac{1}{2} \int_{\Omega} E^{abcd} \varepsilon_{ab}(u_a) \varepsilon_{cd}(u_a) \mathrm{d}\Omega$$

The potential energy of the applied forces $p^a(\frac{\mathrm{N}}{\mathrm{mm}^3}), t^a(\frac{\mathrm{N}}{\mathrm{mm}^2}))$

$$l(u_a) = \int_{\Omega} p^a u_a \mathrm{d}\Omega + \int_{\partial_t \Omega} t^a u_a \mathrm{d}\Gamma$$

$$\varepsilon_{ab} = \frac{1}{2}(\nabla_a u_b + \nabla_b u_a)$$

$$\nabla_a u_b = \partial_a u_b - \Gamma^c_{ab} u_c$$

Christoffel symbol of the 2$^{\mathrm{nd}}$ kind

$$\Gamma^d_{ab} = g^{dc} \frac{1}{2}(g_{ac,b} + g_{cb,a} - g_{ab,c})$$

$$\overset{x}{\varepsilon}_{ab} = \frac{1}{2}(\partial_a \overset{x}{u}_b + \partial_b \overset{x}{u}_a - 2\, \Gamma^c_{ab} \overset{x}{u}_c)$$

$$\Gamma^c_{ab} \overset{x}{u}_c = \Gamma^1_{ab} \overset{x}{u}_1 + \Gamma^2_{ab} \overset{x}{u}_2 + \Gamma^3_{ab} \overset{x}{u}_3$$

In the coordinate system $\boldsymbol{x}$:

$$\Gamma^1_{22} = -\frac{(2d_x r_y + 4d_y r_x)\sin x^2 \sin 2x^2 + (4d_x r_y + 2d_y r_x)\cos x^2 \cos 2x^2 - 2r_x r_y - 4d_x d_y}{(2d_x r_y + d_y r_x)\sin x^2 \sin 2x^2 + (d_x r_y + 2d_y r_x)\cos x^2 \cos 2x^2 - 2r_x r_y - d_x d_y} x^1$$

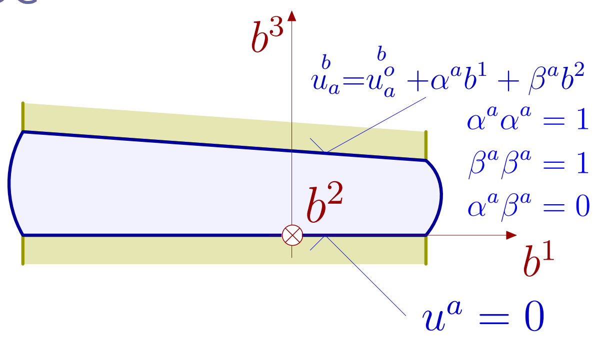
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{x^1}$$

$$\Gamma^2_{22} = -\frac{3d_y r_x \cos x^2 \sin 2x^2 - 3d_x r_y \sin x^2 \cos 2x^2}{(2d_x r_y + d_y r_x)\sin x^2 \sin 2x^2 + (d_x r_y + 2d_y r_x)\cos x^2 \cos 2x^2 - 2r_x r_y - d_x d_y}$$

$$\Gamma^1_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^2_{ab} = \begin{pmatrix} 0 & \Gamma^2_{12} & 0 \\ \Gamma^2_{21} & \Gamma^2_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^3_{ab} = 0$$

Fourier series expansion

BC



$$\overset{x}{u}_a = \sum_{k,l=-\infty}^{\infty} \sum_{m=1}^{\infty} U_a^{klm} \left(e^{i2\pi k x^1} e^{ilx^2} x^1 + 1 \right) \sin \frac{\pi m x^3}{h} + \overset{x}{u}{}^h_a \frac{x^3}{h}$$

where

$$\overset{x}{u}{}^h_a = \frac{\partial b^b}{\partial x^a} \overset{x}{u}{}^h_b$$

Small deformation: $\varphi_1, \varphi_2, \varphi_3 \longrightarrow 0$

The upper vertebra moves as a rigid body

$\overset{b}{u}{}^h_b$:

$$\overset{b}{u}{}^h_1 = \overset{b}{u}{}^o_1 - b^2 \varphi_3$$
$$\overset{b}{u}{}^h_2 = \overset{b}{u}{}^o_2 + b^1 \varphi_3$$
$$\overset{b}{u}{}^h_3 = \overset{b}{u}{}^o_3 - b^1 \varphi_2 + b^2 \varphi_1$$

$\overset{x}{u}_a =$ `ux=N*U`

`U` – unknown coefficients of the F. series

φ_1 – rotation around axis b^1

φ_2 – rotation around axis b^2

φ_3 – rotation around axis b^3

u^o_a – displacement of the point $b^{1,2} = 0, b^3 = h$

GNU OCTAVE

```
B=B(x1,x2,x3)
B=kron(kron(e.^(i*k*2*pi*x1),x1*e.^(i*l*x2)+1),sin(pi*m*x3/h))
bc=[0,0,-b(2); 0,0,b(1); b(2),-b(1),0];
N=[[B,zeros(1,K),zeros(1,K); zeros(1,K),B,zeros(1,K);
zeros(1,K),zeros(1,K),B],bx*x3/h,bx*bc*x3/h]; ## size(U)=3*K+6
```

$$\mathtt{U} = \begin{pmatrix} \bar{\mathtt{U}} \\ \overset{b}{u}{}^0_1 \\ \overset{b}{u}{}^0_2 \\ \overset{b}{u}{}^0_3 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

Elastic energy

CIARLET, P. G. (2005)
LOVELOCK, D. and RUND, H. (1989)
GNU OCTAVE *.m
GNU MAXIMA *.mac

$$a(u_a) = \frac{1}{2}\int_\Omega E^{abcd}\varepsilon_{ab}(u_a)\varepsilon_{cd}(u_a)\mathrm{d}\Omega$$

$$\varepsilon_{ab} = \frac{1}{2}(\nabla_a u_b + \nabla_b u_a)$$

$$\nabla_a u_b = \partial_a u_b - \Gamma^c_{ab} u_c$$

$$E^{abcd} = E^{bacd} \Rightarrow \overset{x}{\varepsilon}_{ab} \rightarrow \partial_a \overset{x}{u}_b - \Gamma^c_{ab}\overset{x}{u}_c$$

$$a = \frac{1}{2}\int_\Omega \left(\partial_a \overset{x}{u}_b - \Gamma^p_{ab}\overset{x}{u}_p\right) \overset{x}{E}{}^{abcd} \left(\partial_c \overset{x}{u}_d - \Gamma^p_{cd}\overset{x}{u}_p\right) \left|\overset{x}{g}_{ab}\right|^{\frac{1}{2}} \mathrm{d}^3 x$$

$$\left\{\partial_a \overset{x}{u}_b\right\}_{ab\lceil} = \texttt{DG*U}$$

DG... GNU OCTAVE *.m

$$a = \frac{1}{2}\,\texttt{U'*K*U}$$

$$\left\{\Gamma^p_{ab}\overset{x}{u}_p\right\}_{ab\lceil} = \texttt{Gamma*U}$$

`Gamma=vec(G1')*N(1,:)+vec(G2')*N(2,:)`

$\texttt{G1=}\left\{\Gamma^1_{ab}\right\}_{a\lceil b}$ $\texttt{G2=}\left\{\Gamma^2_{ab}\right\}_{a\lceil b}$

numerical integration — K

$$a = \frac{1}{2}\texttt{U}'*\int_0^1\int_0^{2\pi}\int_0^h (\texttt{DG(x)}' - \texttt{Gamma(x)}') * \texttt{Ex(x)} * (\texttt{DG(x)} - \texttt{Gamma(x)}) * \texttt{sqrt(det(gx(x)))}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\texttt{*U}$$

Right hand side

$$l(u_a) = \int\limits_{\Omega} p^a u_a \mathrm{d}\Omega + \int\limits_{\partial_t\Omega} t^a u_a \mathrm{d}\Gamma$$

$$l = \texttt{F*U}$$

```
F=[zeros(3*K,1);Vs;Vl;-N;Ml;Mf;Mr]
```

$$\texttt{U} = \begin{pmatrix} \bar{\texttt{U}} \\ \overset{b}{u}{}^0_1 \\ \overset{b}{u}{}^0_2 \\ \overset{b}{u}{}^0_3 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

Necessary condition of minimum

$$\frac{\partial \Pi}{\partial \texttt{U}} = 0$$

$$\Pi = \frac{1}{2}\texttt{U'*K*U-F*U}$$

```
K*U=F
```

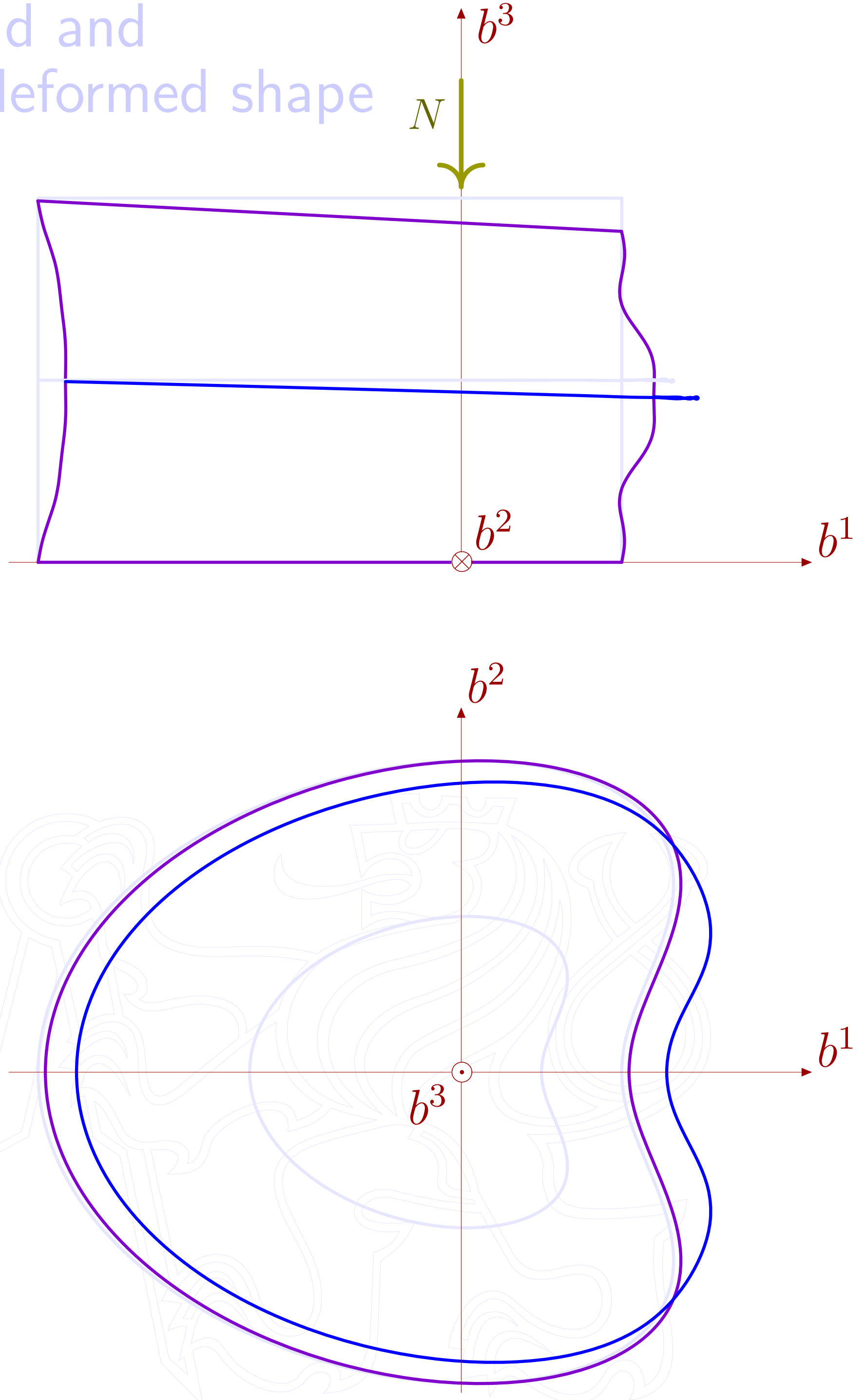
$$\texttt{U=K}^{-1}\texttt{*F}$$

$$\overset{x}{u}_a = \texttt{ux=real(N(x)*U)}$$

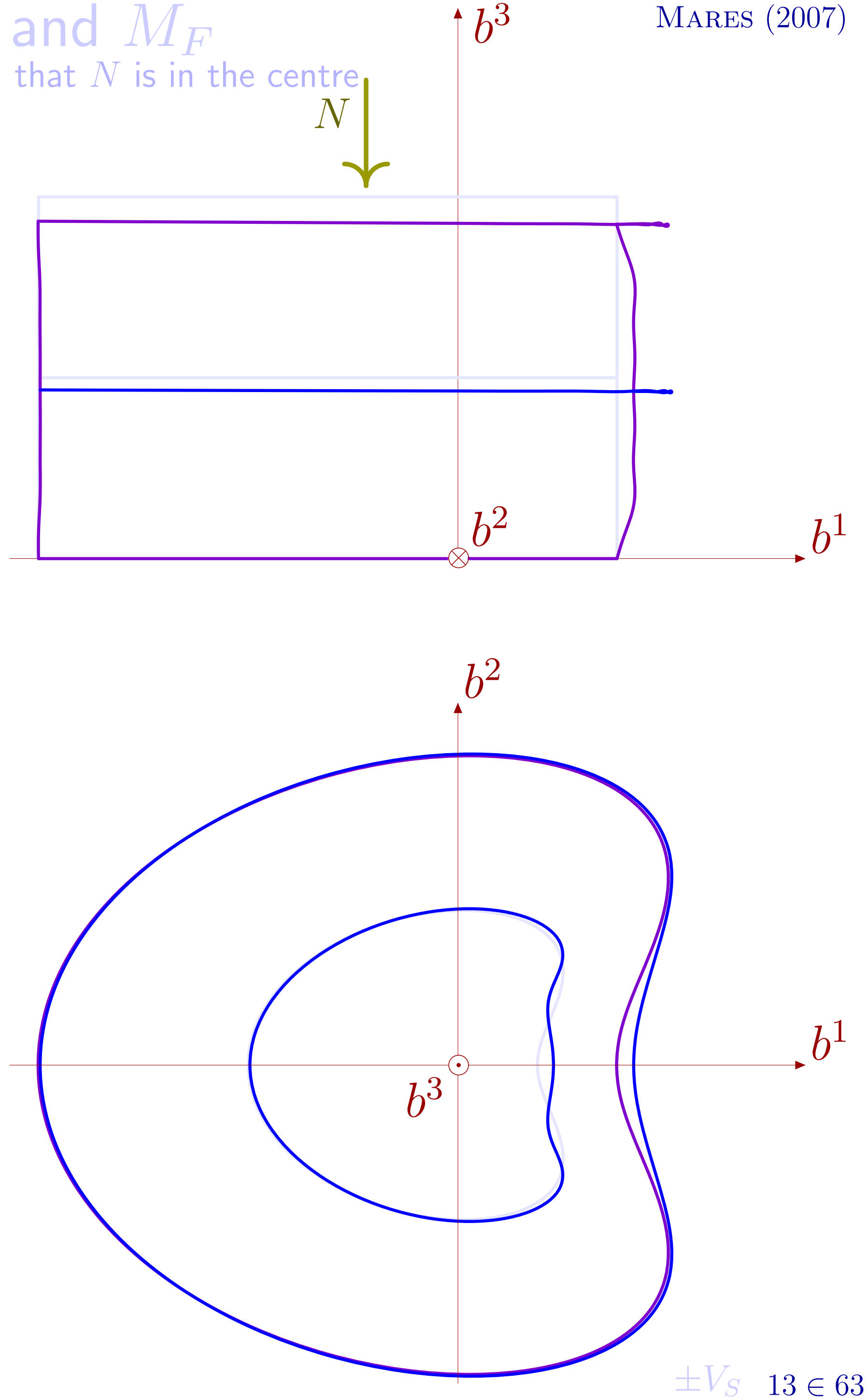
$$\overset{b}{u}_a = \frac{\partial x^b}{\partial b^a}\,\overset{x}{u}_b$$

```
ub=xb'*ux
```

Load and deformed shape



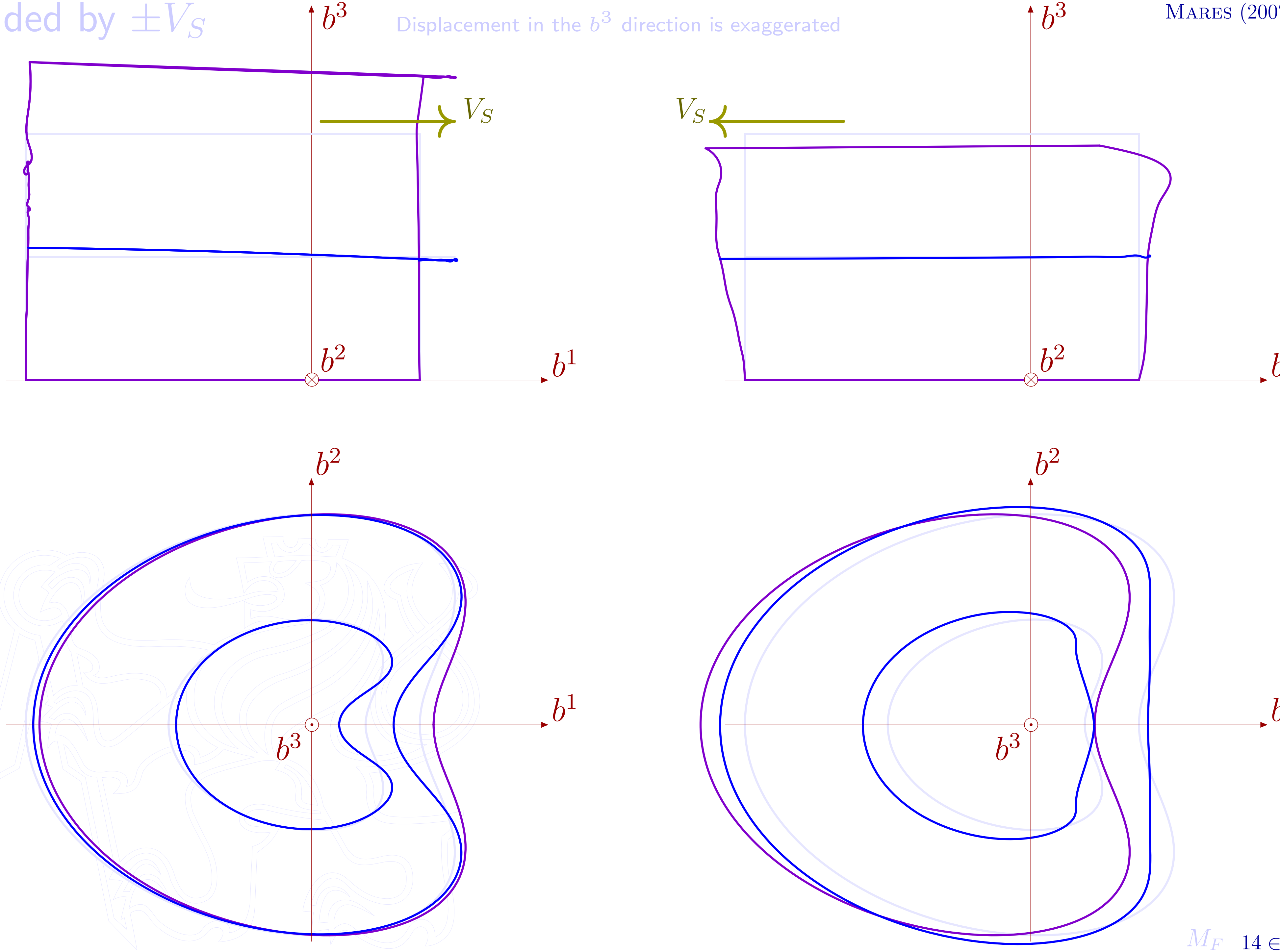
N and M_F such that N is in the centre



Loaded by $\pm V_S$

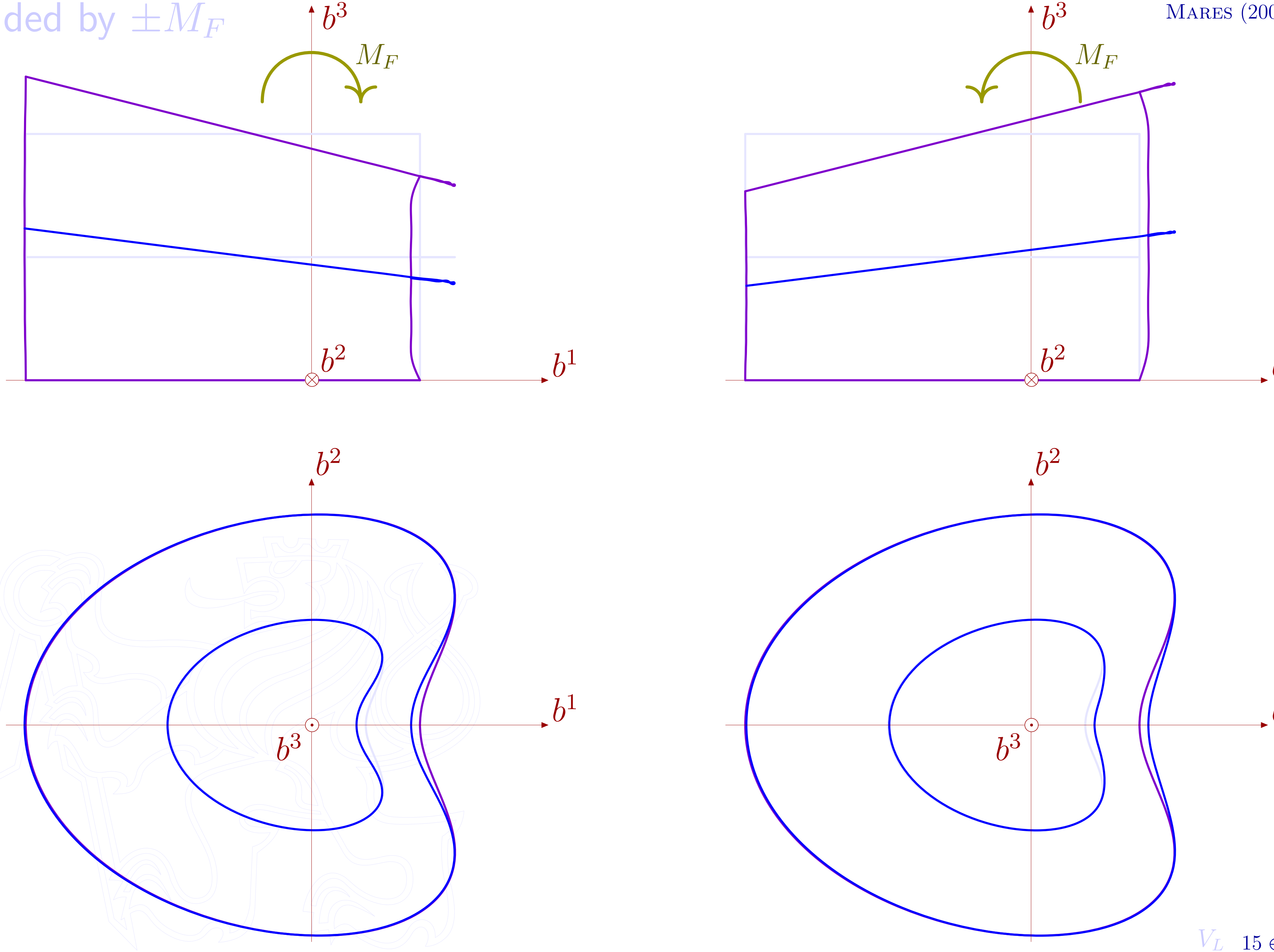
Displacement in the b^3 direction is exaggerated

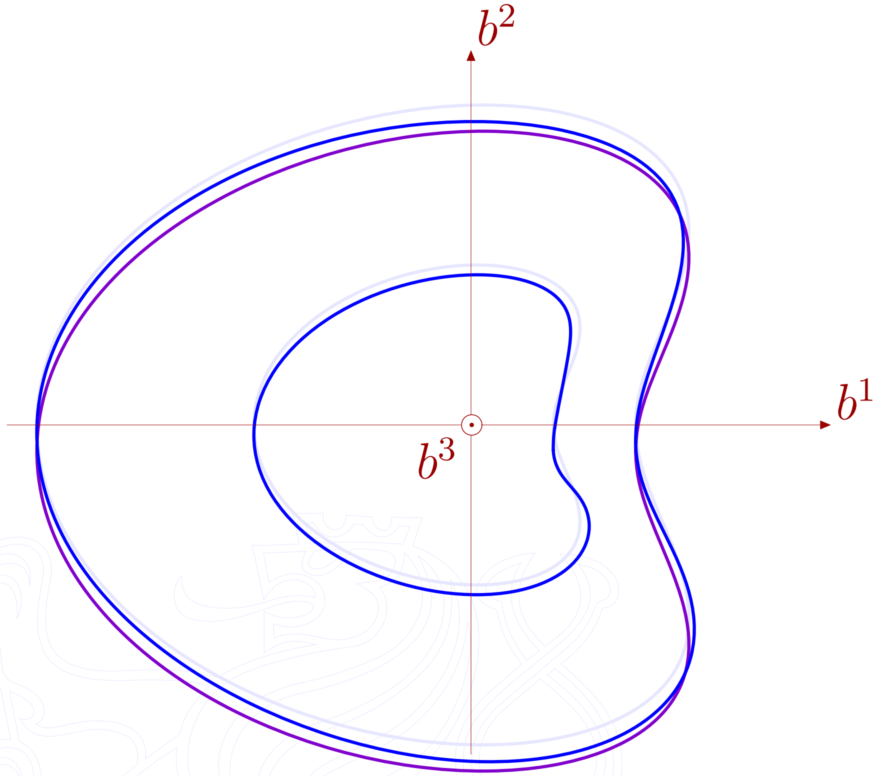
Mares (2007)

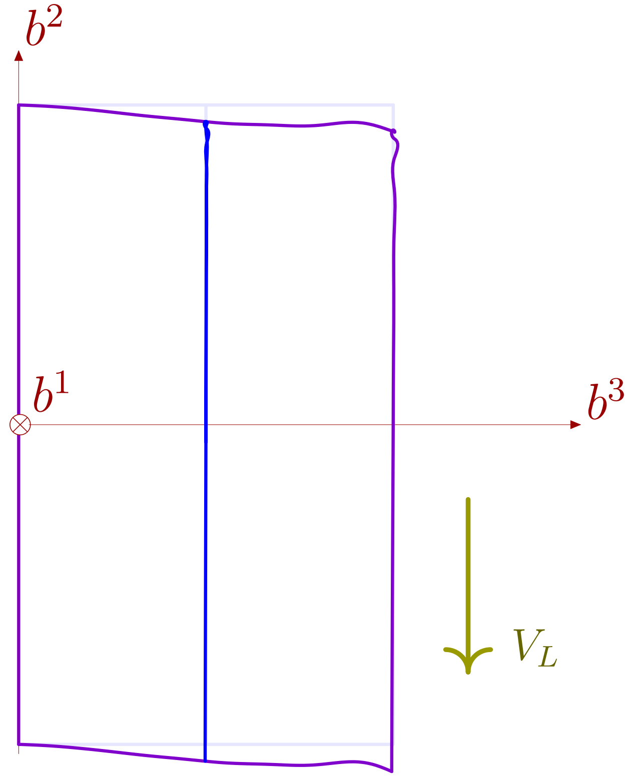


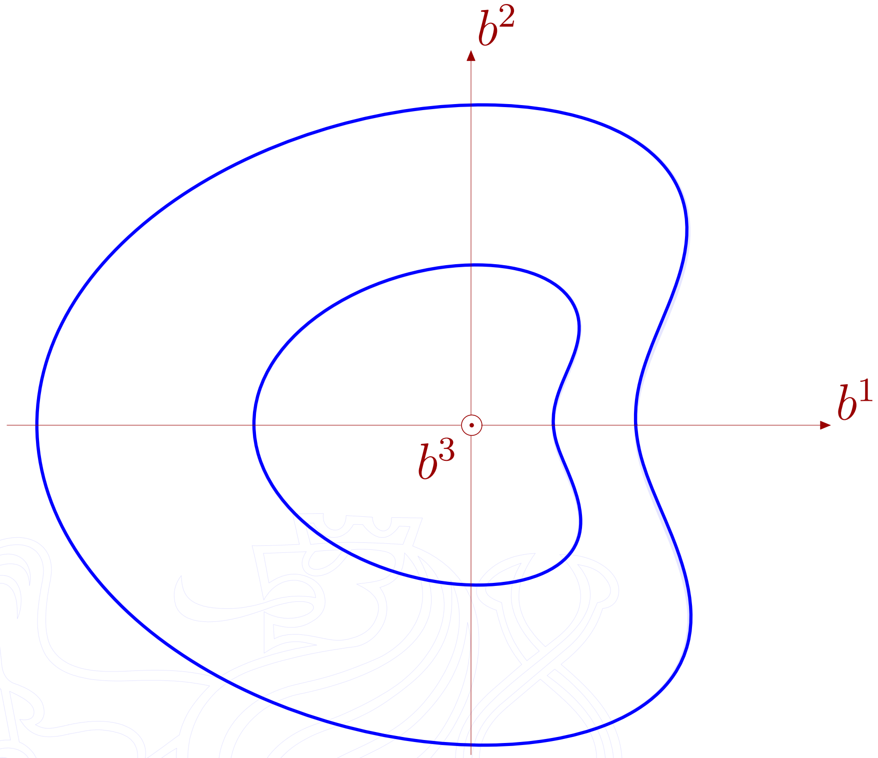
Loaded by $\pm M_F$

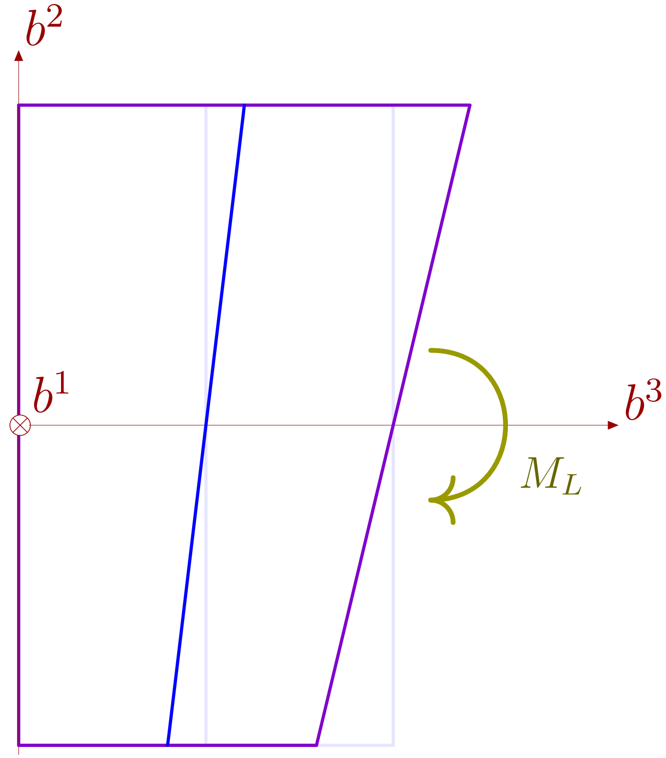
Mares (2007)



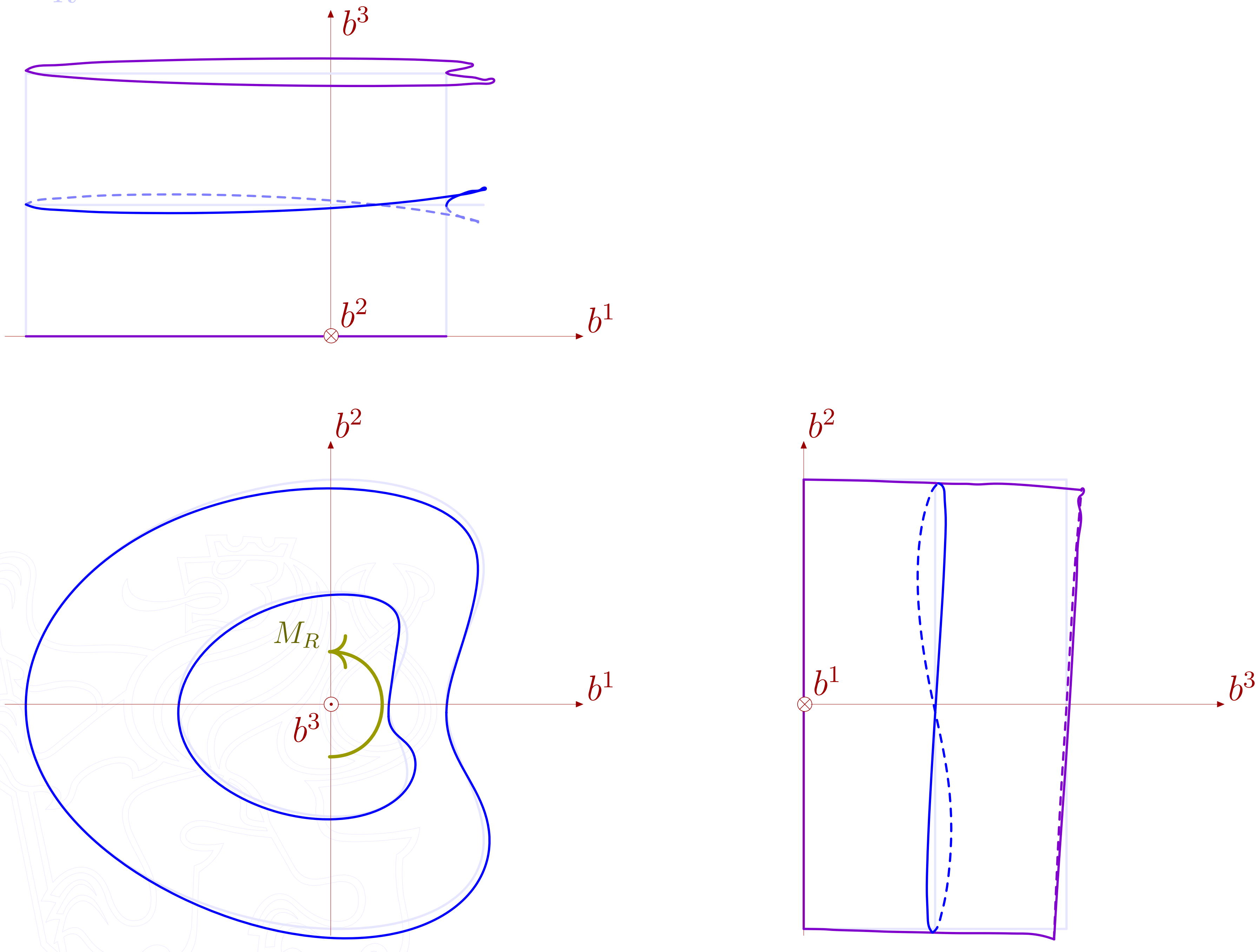

b^2
b^1
b^3

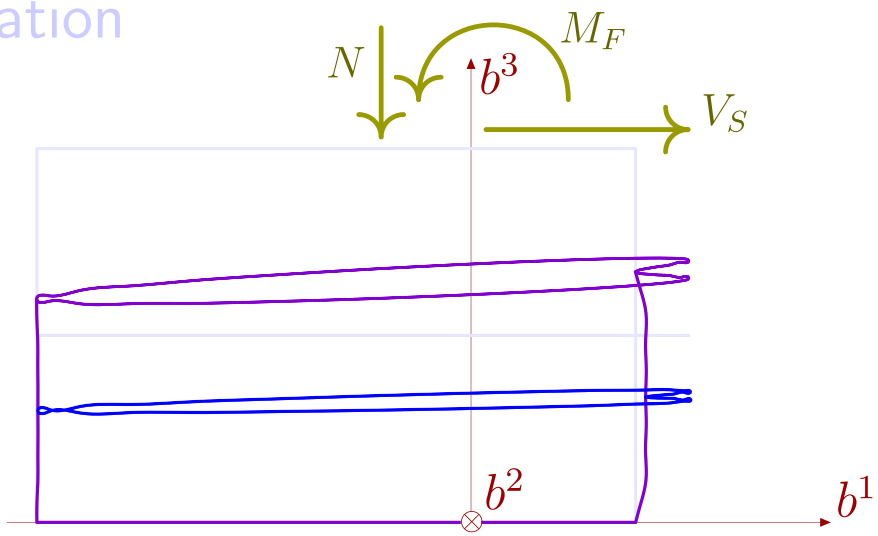

b^2
b^1
b^3
V_L


b^2
b^1
b^3


b^2
b^1
b^3
M_L

Torsion M_R





$N = 2250\,\mathrm{N}$

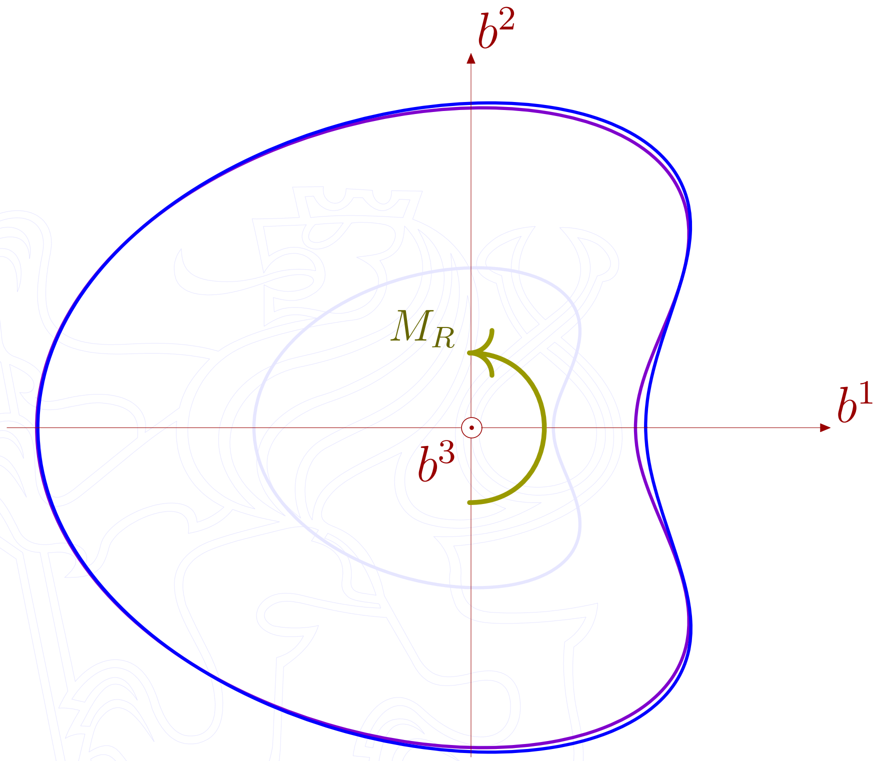
$M_F = 2700\,\mathrm{Nm}$

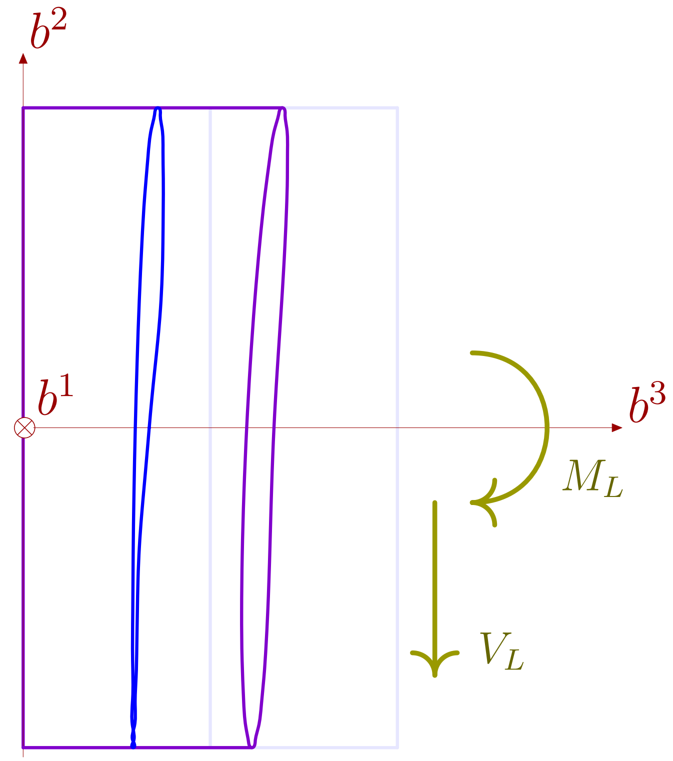
$V_S = 100\,\mathrm{N}$

$M_R = 0\,\mathrm{Nm}$

$V_L = 100\mathrm{N}$

$M_L = 1700\,\mathrm{Nm}$



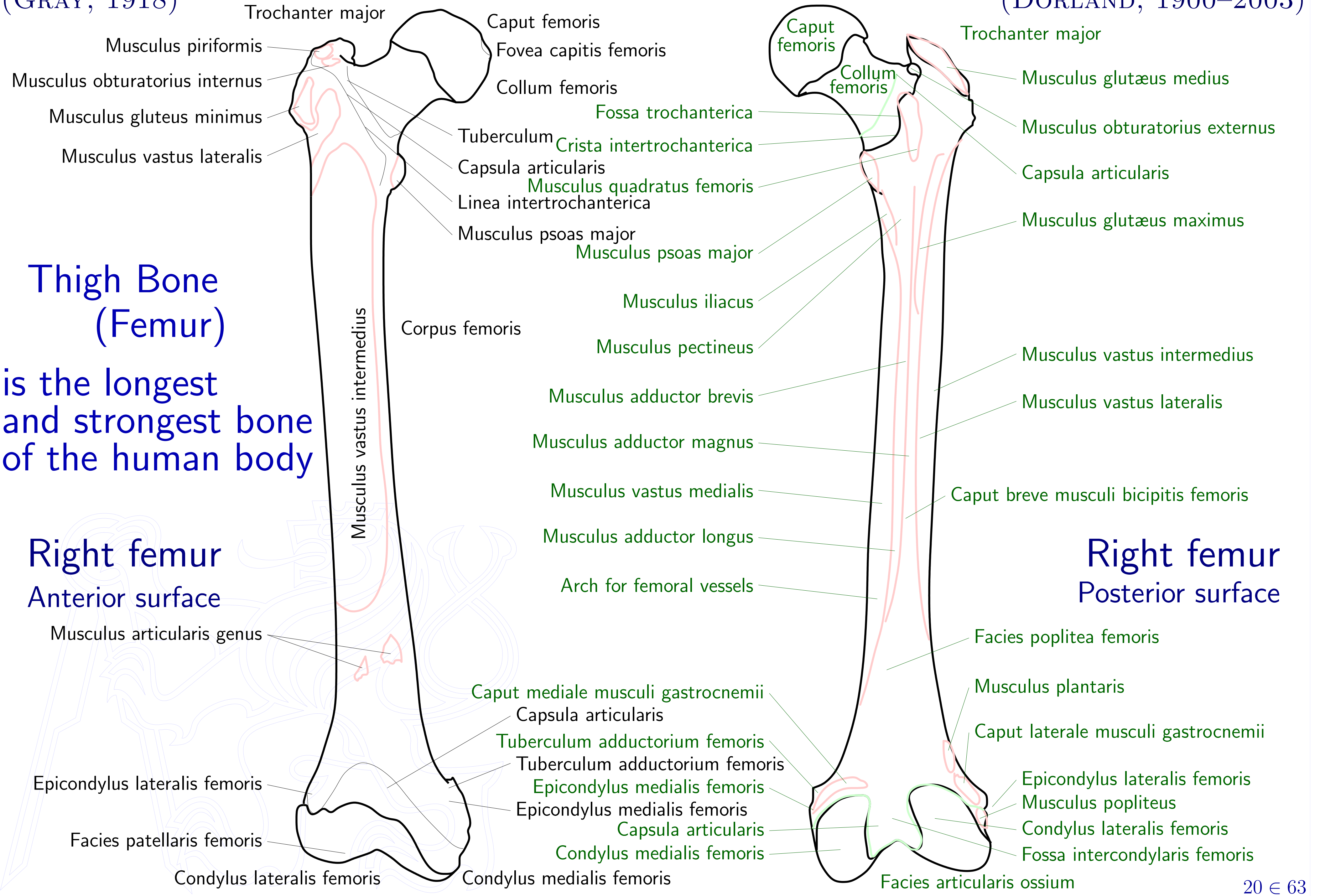


Cortical bone (compact bone), the shaft of a long bone, as a fibre composite

(Gray, 1918) (Dorland, 1900–2003)

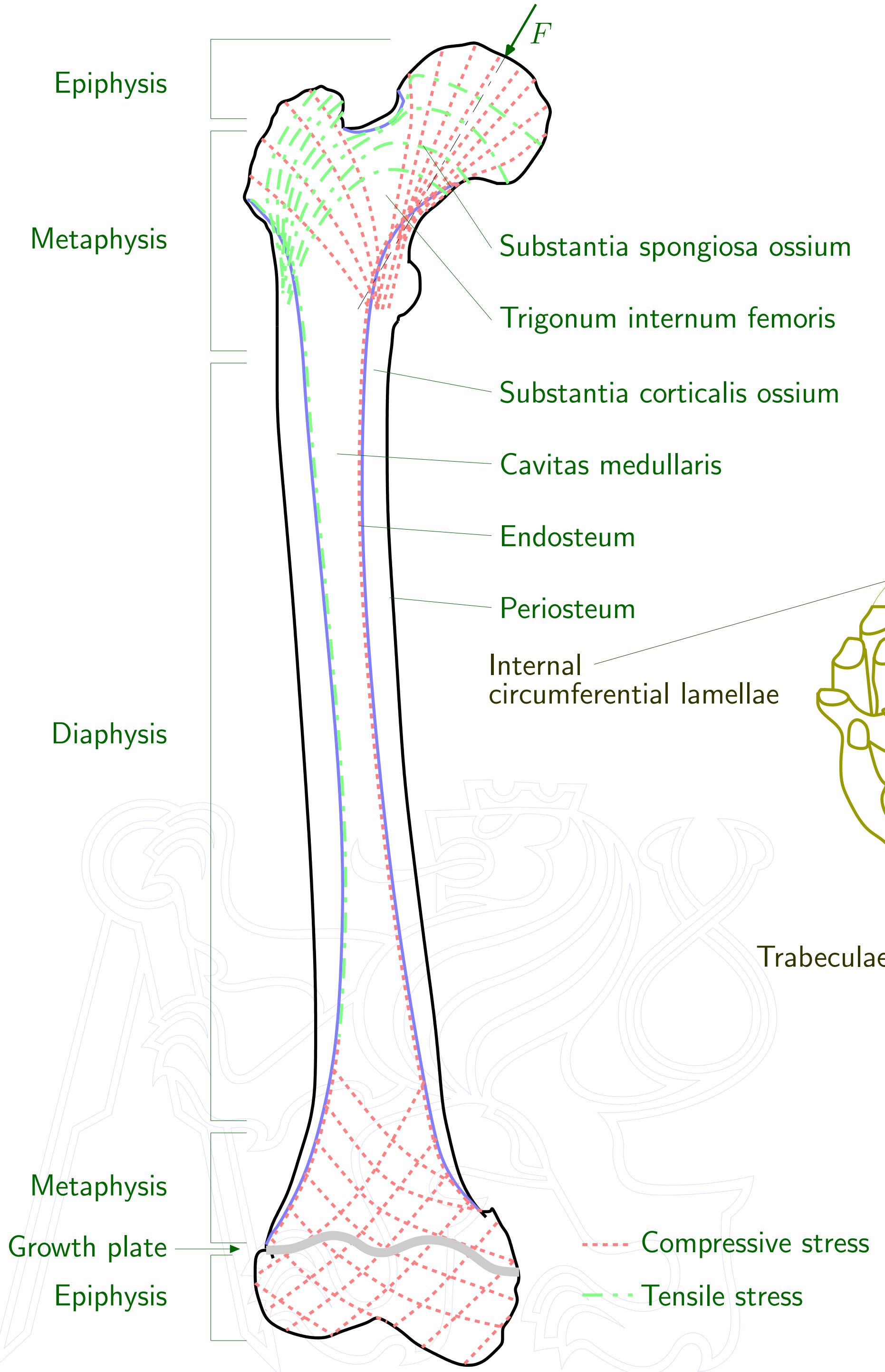
Thigh Bone (Femur)

is the longest and strongest bone of the human body

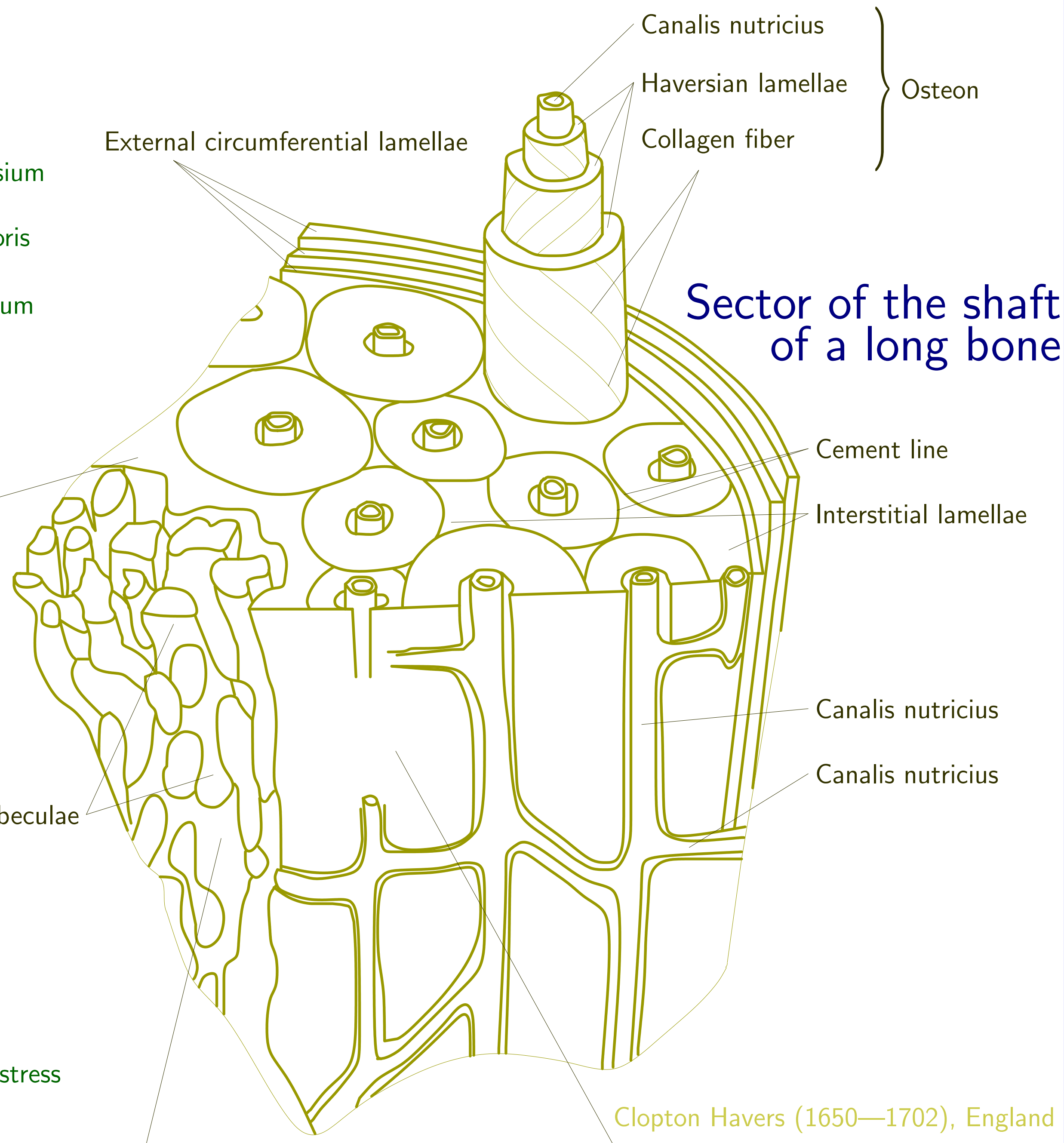


Internal structure of the right femur

(Gray, 1918), (Dorland, 1900–2003)



Sector of the shaft of a long bone

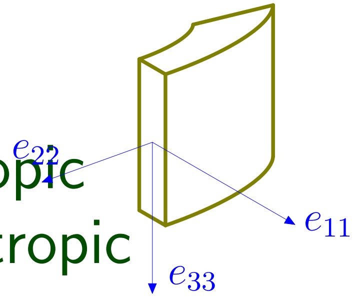


Clopton Havers (1650—1702), England

Alfred Wilhelm Volkmann (1800—1877), Germany

There are several models of the cortical bone amongst them the

- homogeneous isotropic model
- homogeneous transversely isotropic
- homogeneous cylindrically othotropic
  (axis, r, t)



$$\boldsymbol{E} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 \\ e_{12} & e_{22} & e_{23} & 0 & 0 & 0 \\ e_{13} & e_{23} & e_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{66} \end{pmatrix}$$

[1] – (Goldmann, 2006)
[2] – (Orías, 2005)
[3] – (Yoon, Katz, 1976)
[4] – (Katz *et al.*, 1984)
[5] – (Ashman *et al.*, 1984)
[6] – (Rho, 1996)
[7] – (Taylor *et al.*, 2002)
[8] – (Buskirk *et al.*, 1981)
[9] – (Maharidge, 1984)
[0] – (Lang, 1970)

$$\left\{ \overset{z}{\mathcal{E}}{}^{ij}{}_{kl} \right\}_{\{ij\lceil kl\}} = \begin{pmatrix} e_{11} & 0 & 0 & 0 & e_{12} & 0 & 0 & 0 & e_{13} \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ e_{12} & 0 & 0 & 0 & e_{22} & 0 & 0 & 0 & e_{23} \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ e_{13} & 0 & 0 & 0 & e_{23} & 0 & 0 & 0 & e_{33} \end{pmatrix}$$

The average material characteristics of these models are determined experimentally (mechanical, using acoustic waves)
(Orías, 2005), (Goldmann, 2006)

The entries of the stiffness tensor [GPa] (dry/fresh bovine/human)

[GPa]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[0]
e_{11}	27.4±1.6	16.75±2.27	23.4±0.0031	21.2±0.5	18.0	19.4±1.3	24.89	14.1	22.4	19.7
e_{22}	30.3±2.8	19.66±2.09	24.1±0.0035	21.0±1.4	20.2	20.0±1.4	26.16	18.4	25.0	19.7
e_{33}	34.1±1.7	27.33±1.64	32.5±0.0044	29.0±1.0	27.6	30.9±1.9	33.20	25.0	35.0	32.0
e_{44}	9.3± 0.9	6.22±0.31	8.7±0.0013	6.3±0.4	6.23	5.7±0.5	7.11	7.0	8.2	5.4
e_{55}	7.0± 0.4	5.65±0.53	6.9±0.0012	6.3±0.2	5.6	5.2±0.6	6.58	6.3	7.1	5.4
e_{66}	6.9± 0.5	4.64±0.43	7.2±0.0011	5.4±0.2	4.5	4.1±0.5	5.71	5.28	6.1	3.8
e_{12}	9.1		9.1±0.0038	11.7±0.7	10.0	11.3±0.1	11.18	6.34	14.0	12.1
e_{13}	8.3±5.3		9.1±0.0055	11.1±0.8	10.1	12.5±0.1	13.59	4.84	15.8	12.6
e_{23}	8.5		9.2±0.0055	12.7±0.8	10.7	12.6±0.1	13.84	6.94	13.6	12.6

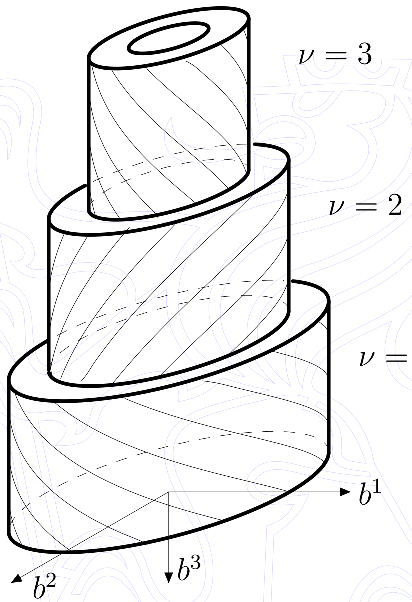
Let us build up
a methodology of another model of cortical bone, say
Heterogeneous locally orthotropic model of cortical bone
(because of the locality of the othotropy the model is in essence anisotropic)

The basic unit of compact bone is known as the osteon

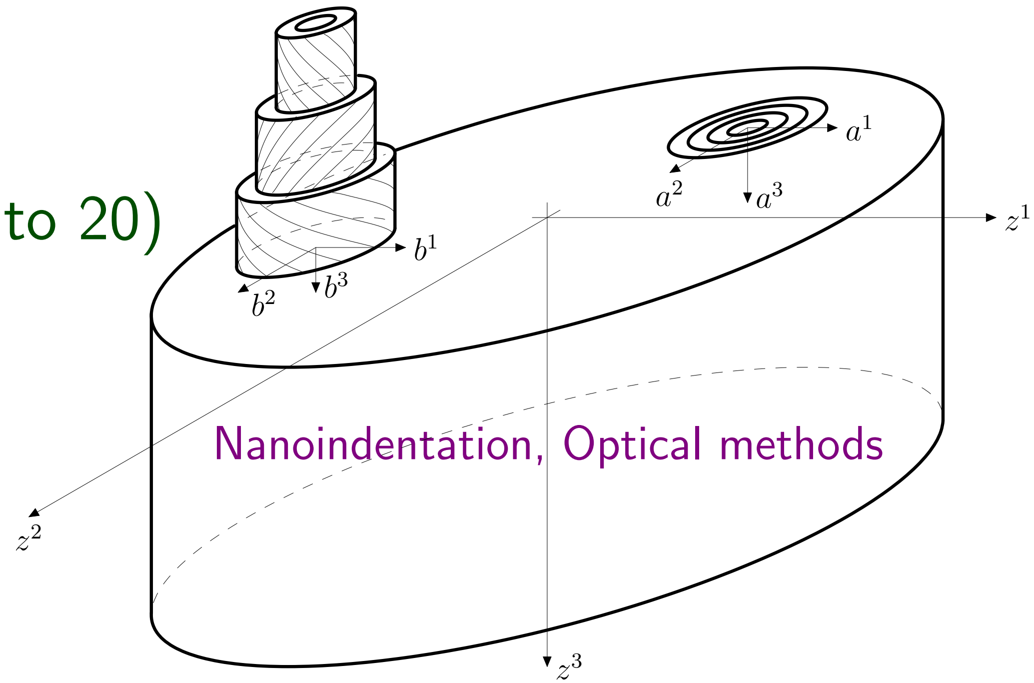
The osteon consist of a number (4 to 20) Haversian lamellae

Each of these lamellae is 3 to 7 microns thick

Clopton Havers (1650—1702), England



Osteon, the global coordinate systyem



Cortical bone and two of the osteons

The lamellae are composed of collagen fibers that are winded under an angle α changing from one lamella to another lamella

Elasticity tensor E^{abcd} in the c.s. of the local orthotropy

in Cartesian coordinate system ν^a

$$\overset{\nu}{\sigma}{}^{ij} = \overset{\nu}{E}{}^{ijkl} \overset{\nu}{\varepsilon}_{kl}$$

$$\left\{ \overset{\nu}{E}{}^{ijkl} \right\}_{\{ij\lceil kl\}} = \begin{pmatrix} \Phi_{1111} & 0 & 0 & 0 & \Phi_{1122} & 0 & 0 & 0 & \Phi_{1133} \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & G_{12} & 0 & G_{12} & 0 & 0 & 0 & 0 & 0 \\ \Phi_{2211} & 0 & 0 & 0 & \Phi_{2222} & 0 & 0 & 0 & \Phi_{2233} \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ 0 & 0 & G_{13} & 0 & 0 & 0 & G_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{23} & 0 & G_{23} & 0 \\ \Phi_{3311} & 0 & 0 & 0 & \Phi_{3322} & 0 & 0 & 0 & \Phi_{3333} \end{pmatrix}$$

$$E^{abcd} = E^{bacd}$$

$$\Phi_{1111} = \frac{1 - \nu_{23}\nu_{32}}{N} E_{11}, \quad \Phi_{1122} = \frac{\nu_{21} + \nu_{23}\nu_{31}}{N} E_{11}, \quad \Phi_{1133} = \frac{\nu_{31} + \nu_{32}\nu_{21}}{N} E_{11}$$

$$\Phi_{2211} = \frac{\nu_{12} + \nu_{13}\nu_{32}}{N} E_{22}, \quad \Phi_{2222} = \frac{1 - \nu_{13}\nu_{31}}{N} E_{22}, \quad \Phi_{2233} = \frac{\nu_{32} + \nu_{31}\nu_{12}}{N} E_{22}$$

$$\Phi_{3311} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{N} E_{33}, \quad \Phi_{3322} = \frac{\nu_{23} + \nu_{21}\nu_{13}}{N} E_{33}, \quad \Phi_{3333} = \frac{1 - \nu_{12}\nu_{21}}{N} E_{33}$$

$$N = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{32}\nu_{21}$$

Energy $(E^{abcd} = E^{cdab}) \Rightarrow \Phi_{1122} = \Phi_{2211} \Rightarrow \nu_{21}E_{11} = \nu_{12}E_{22}$, *etc.*

To build up the potential energy we will use the concept of locally orthotropic material

Everything follows from the used coordinate systems

1. Local coordinate system of the orthotropy (ν^a)
2. Global coordinate system (z^a) that is common for the whole model
3. A sequence of working coordinates

The global transformation rule is given by the sequence of transformation rules

$$\overset{z}{E}{}^{abcd} = \frac{\partial z^a}{\partial \nu^i}\frac{\partial z^b}{\partial \nu^j}\ \overset{\nu}{E}{}^{ijkl}\ \frac{\partial z^c}{\partial \nu^k}\frac{\partial z^d}{\partial \nu^l}$$

The principle of minimum total potential energy

$$\hat{\boldsymbol{u}} = \arg \min_{\boldsymbol{u} \in} \Pi(\boldsymbol{u}),\ \text{kde}$$

$$\Pi(\boldsymbol{u}) = a(\boldsymbol{u}, \boldsymbol{u}) - l(\boldsymbol{u})$$

$$a = \int_\Omega \varepsilon_{ab}\varepsilon_{cd}E^{abcd}\,\mathrm{d}\Omega,\ \ l(\boldsymbol{u}) = \int_\Omega p^i u_i\,\mathrm{d}\Omega + \int_{\partial_t\Omega} t^i u_i\,\mathrm{d}\Gamma$$

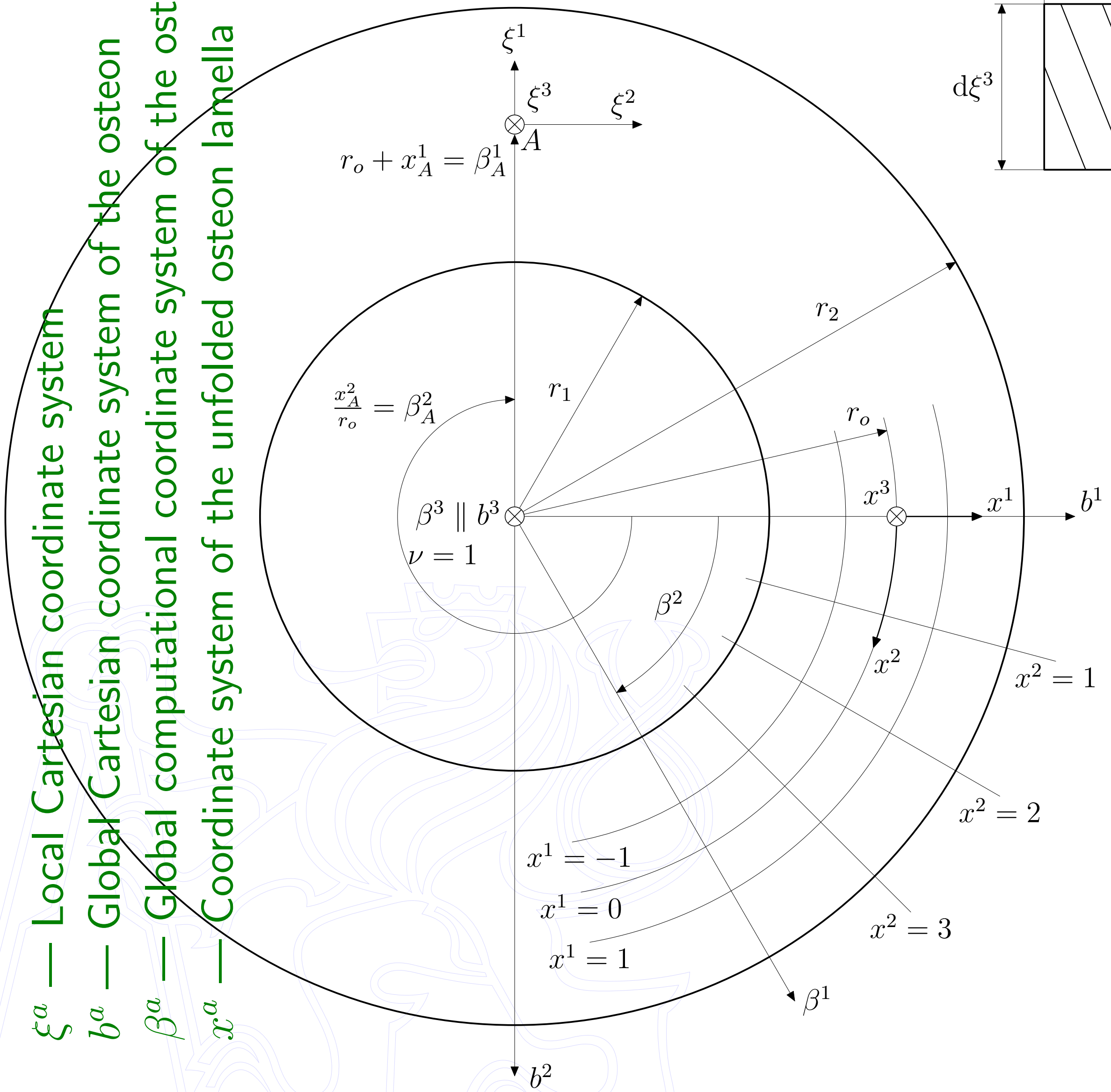
$$\mathrm{d}\Omega = \left|\overset{\beta}{g}_{ab}\right|^{\frac{1}{2}} \mathrm{d}^3\beta,\ \ \mathrm{d}\Gamma = \left|\overset{\phi}{h}_{\alpha\beta}\right|^{\frac{1}{2}} \mathrm{d}^2\phi$$

The concept of local orthotropy is very suitable for the detailed description of the bone behaviour

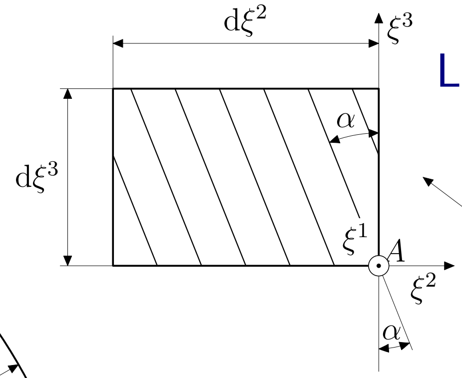
(Lovelock, 1989), (Synge, 1978), and (Mareš, 2005)

The computational frames of one osteon lamella

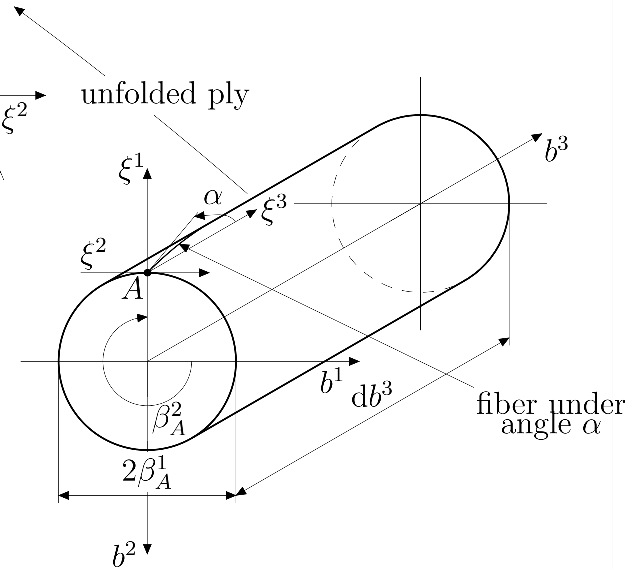
ξ^a — Local Cartesian coordinate system
b^a — Global Cartesian coordinate system of the osteon
β^a — Global computational coordinate system of the osteon
x^a — Coordinate system of the unfolded osteon lamella

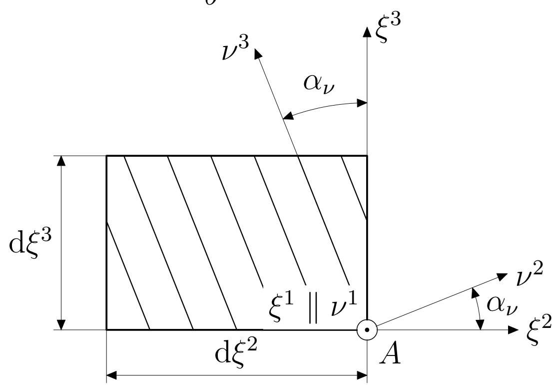


Cortical bone osteon lamella coordinate system



Local c.s. ξ of an infinitesimal part of the lamella





Main material coordinate system ν^a of an unrolled infinitesimal part of the lamella

Via derivative of the relations between the coordinate systems we obtain a range of transformation matrices for the components

of the tensors and range of matrics

(LOVELOCK, 1989)
(SYNGE, 1978)

$$\frac{\partial b^a}{\partial x^b} = \frac{\partial b^a}{\partial \beta^c}\frac{\partial \beta^c}{\partial x^b} = \begin{pmatrix} \cos\beta^2 & -\frac{\beta^1}{r_o}\sin\beta^2 & 0 \\ \sin\beta^2 & \frac{\beta^1}{r_o}\cos\beta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{\partial b^a}{\partial \beta^b} = \begin{pmatrix} \cos\beta^2 & -\beta^1\sin\beta^2 & 0 \\ \sin\beta^2 & \beta^1\cos\beta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \frac{\partial x^a}{\partial \beta^b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_o & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

metrics $\quad \mathrm{d}s^2 = \overset{x}{g}_{ab}\,\mathrm{d}x^a\mathrm{d}x^b = \overset{\beta}{g}_{ab}\,\mathrm{d}\beta^a\mathrm{d}\beta^b = \overset{b}{g}_{ab}\,\mathrm{d}b^a\mathrm{d}b^b = \delta_{ab}\mathrm{d}b^a\mathrm{d}b^b$

$$\delta_{ab} = \overset{\xi}{g}_{ab} = \frac{\partial x^c}{\partial \xi^a}\frac{\partial x^d}{\partial \xi^b}\,\overset{x}{g}_{cd}$$

$$\Rightarrow \frac{\partial x^a}{\partial \xi^b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{r_o}{\beta^1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inversely, from a known metric we can obtain the transformation rule

$$\overset{x}{g}_{ab} = \frac{\partial b^c}{\partial x^a}\frac{\partial b^c}{\partial x^d}\delta_{cd} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{\beta^1}{r_o}\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\beta}{g}_{ab} = \frac{\partial b^c}{\partial \beta^a}\frac{\partial b^d}{\partial \beta^b}\delta_{cd} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\beta^1\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{\partial \nu^a}{\partial \xi^b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_\nu & \sin\alpha_\nu \\ 0 & -\sin\alpha_\nu & \cos\alpha_\nu \end{pmatrix}$$

For the principal directions of an locally orthotropic block

Analysis of a cortical bone means analysis of the assemblage of the osteons embedded in isotropic interstitial matrix

to subtract
to add

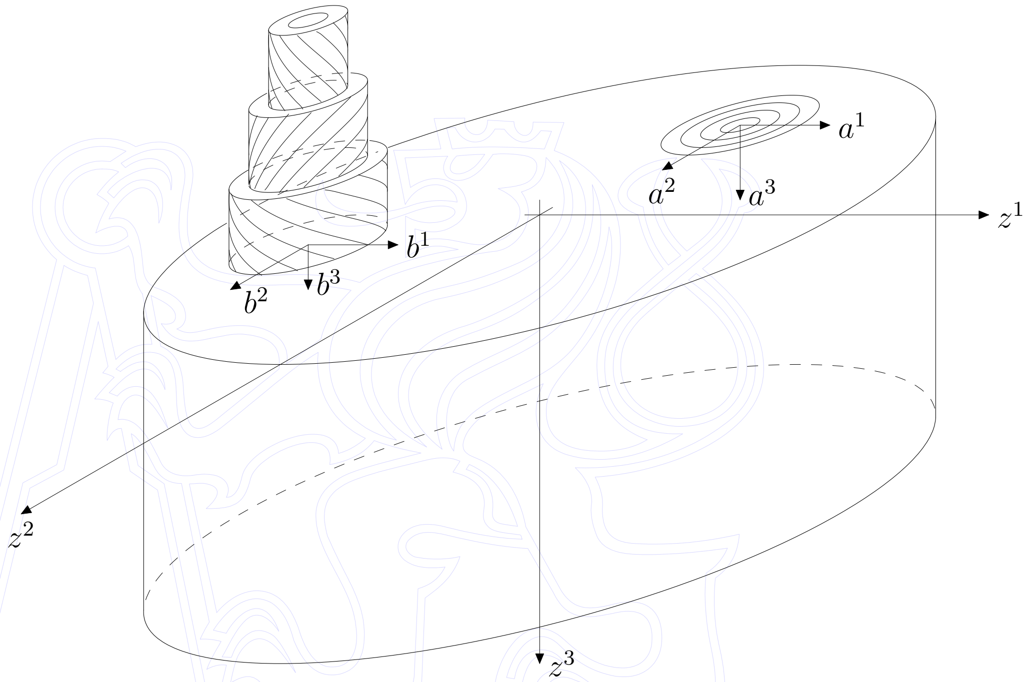
$$\hat{\boldsymbol{u}} = \arg \min_{\boldsymbol{u} \in} \Pi(\boldsymbol{u}), \text{ kde } \Pi(\boldsymbol{u}) = a(\boldsymbol{u}, \boldsymbol{u}) - l(\boldsymbol{u})$$

$$a = \int_{\Omega} \varepsilon_{ab} \overset{z}{\varepsilon^{cd}} \mathcal{E}^{ab}{}_{cd} \, \mathrm{d}\Omega + \sum_{\ell=1}^{n} \int_{\Omega_\ell} \varepsilon_{ab} \overset{\beta}{\varepsilon^{cd}} E^{ab}{}_{cd} \, \mathrm{d}\Omega - \sum_{\ell=1}^{n} \int_{\Omega_\ell} \varepsilon_{ab} \overset{\beta}{\varepsilon^{cd}} \mathcal{E}^{ab}{}_{cd} \, \mathrm{d}\Omega$$

(Washizu, 1975)

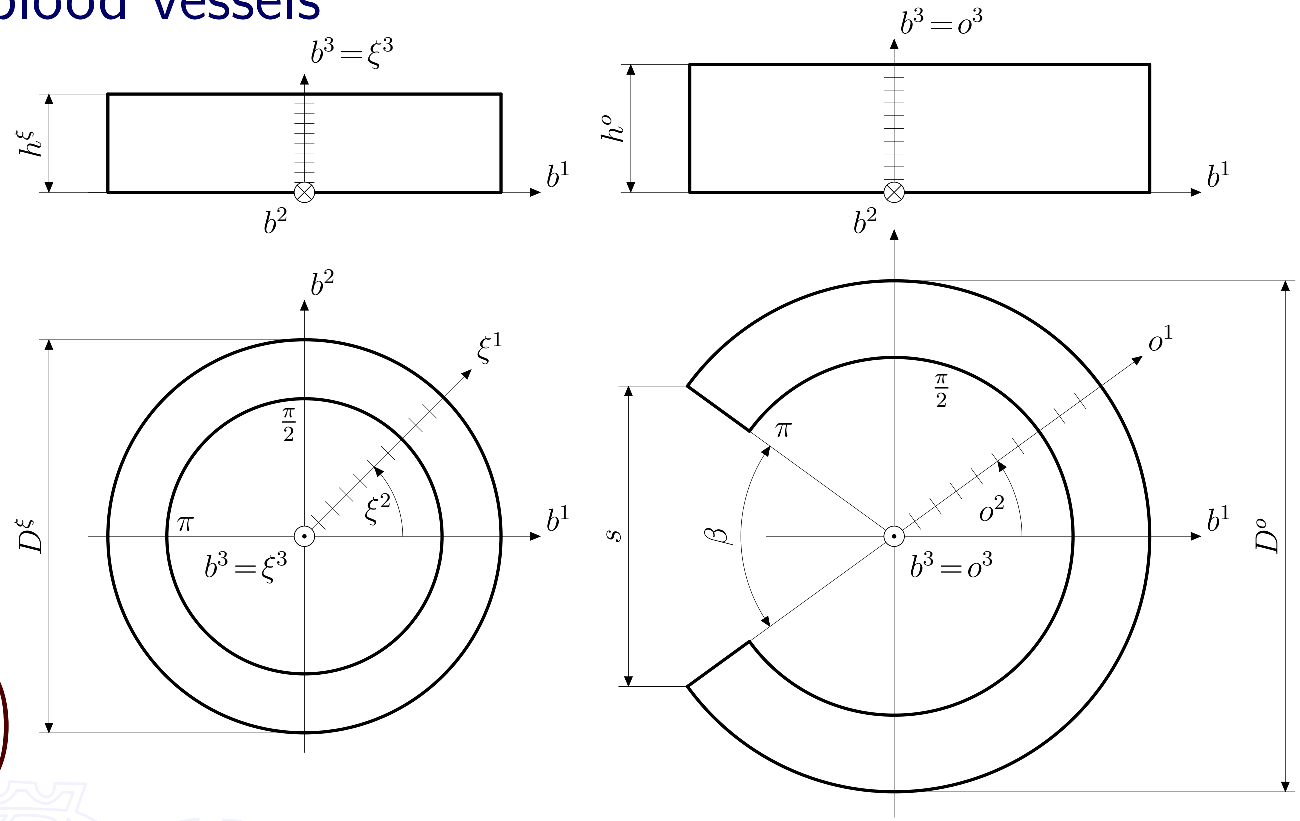
$$l(\boldsymbol{u}) = \int_{\Omega} p^i u_i \, \mathrm{d}\Omega + \int_{\partial_t \Omega} t^i u_i \, \mathrm{d}\Gamma$$

$$\mathrm{d}\Omega = \left| \overset{\beta}{g_{ab}} \right|^{\frac{1}{2}} \mathrm{d}^3\beta, \; \mathrm{d}\Gamma = \left| \overset{\phi}{h_{\alpha\beta}} \right|^{\frac{1}{2}} \mathrm{d}^2\phi$$



Then the procedure is a similar one as in the previous examples

Residual stresses in blood vessels



$$\xi^1 = \delta o^1$$

$$\xi^2 = \gamma o^2$$

$$\xi^3 = \eta o^3$$

$$\delta = \frac{D^o}{D^\xi}, \; \gamma = \frac{\pi - \frac{\beta}{2}}{\pi}, \; \eta = \frac{h^o}{h^\xi}$$

$$\overset{o}{E}_{ab} = \frac{1}{2}\left(\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab}\right)$$

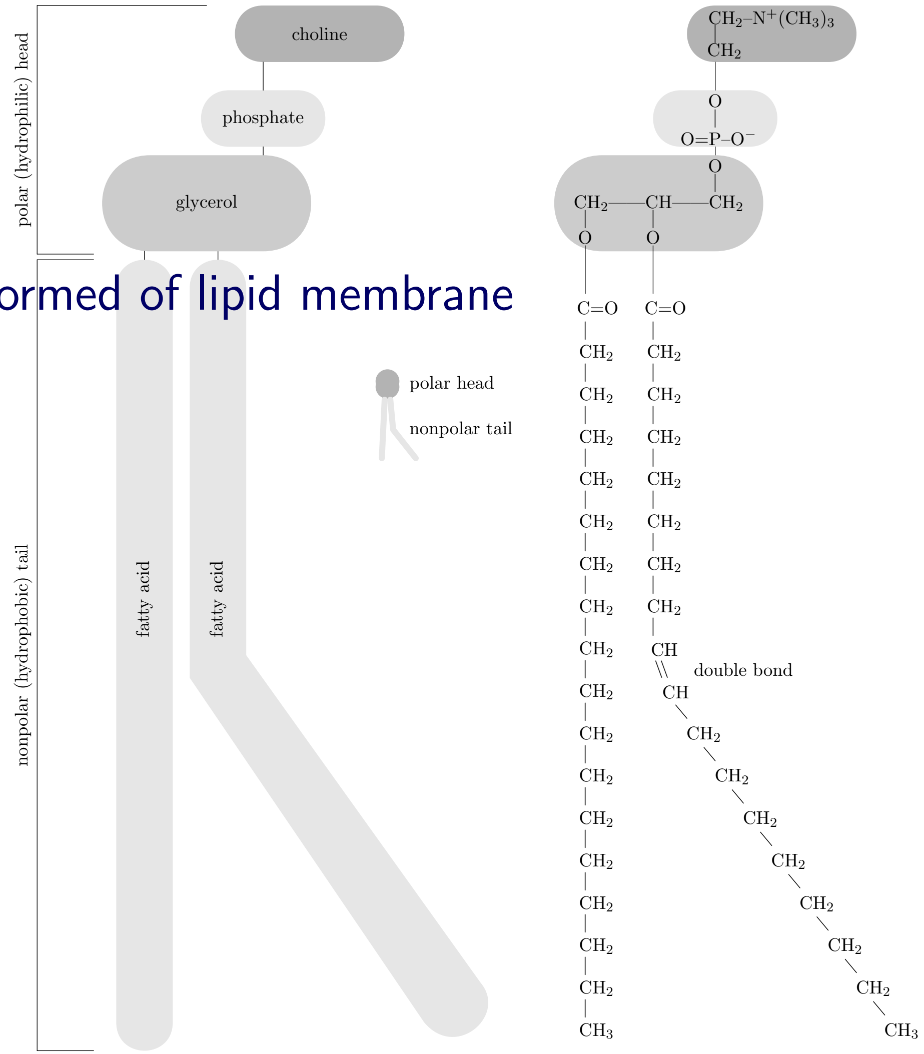
$$\overset{o}{E}_{ab} = \frac{1}{2}\left(\overset{\xi}{g}_{ab} - \overset{o}{g}_{ab}\right) = \frac{1}{2}\begin{pmatrix} 1-\delta^2 & 0 & 0 \\ 0 & (1-\gamma^2)\,(\xi^1)^2 & 0 \\ 0 & 0 & 1-\eta^2 \end{pmatrix}$$

$$\overset{\xi}{E}_{ab} = \frac{\partial o^c}{\partial \xi^a}\frac{\partial o^d}{\partial \xi^b}\,\overset{o}{E}_{cd} = \frac{1}{2}\begin{pmatrix} \frac{1-\delta^2}{\delta^2} & 0 & 0 \\ 0 & \frac{1-\gamma^2}{\gamma^2}\,(\xi^1)^2 & 0 \\ 0 & 0 & \frac{1-\eta^2}{\eta^2} \end{pmatrix}$$

$$\overset{\xi}{S}{}^{ab} = \overset{\xi}{E}{}^{abcd}\,\overset{\xi}{E}_{cd} = \left(\lambda\,\overset{\xi}{g}{}^{ab}\overset{\xi}{g}{}^{cd} + \mu\,\overset{\xi}{g}{}^{ac}\overset{\xi}{g}{}^{bd} + \mu\,\overset{\xi}{g}{}^{ad}\overset{\xi}{g}{}^{bc}\right)\overset{\xi}{E}_{cd}$$

$$\overset{t}{S}{}^{ab} = \frac{\partial t^a}{\partial \xi^c}\frac{\partial t^b}{\partial \xi^d}\,\overset{\xi}{S}{}^{cd}$$

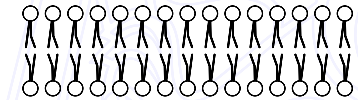
Shape analysis of lipid membranes with intrinsic (anisotropic) curvature



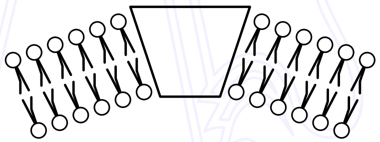
Here I am interested in shape analysis of vesicles formed of lipid membrane

Two approaches were used:

6. Statistical mechanical approach
7. Classical mechanical approach



Lipid membrane without inclusion



Lipid membrane with inclusion

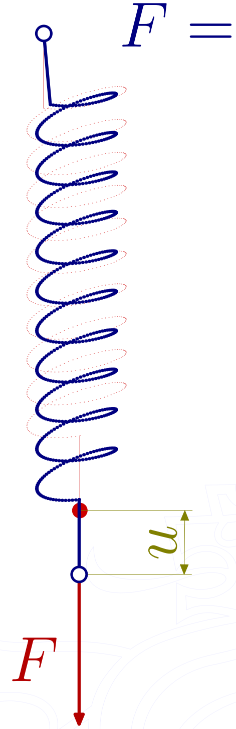
Stiffness maximization

Bendsøe, M. P. (2003)
Mareš, T. (2006)
Allaire, G. (2002)
Mareš, T. (2005)

Minimum work of external forces ⇔ Maximum stiffness

$$\min F \cdot u \Leftrightarrow \min \frac{F^2}{k} \Leftrightarrow \max k$$

$$F = k \cdot u$$



$$\min \Pi = \frac{1}{2}(A\hat{u}, \hat{u}) - (f, \hat{u}) \ \wedge \ (A\hat{u}, \hat{u}) = (f, \hat{u})$$

$$\Rightarrow \quad \min \Pi = -\frac{1}{2}(f, \hat{u})$$

$$\Rightarrow \quad \arg\min (f, \hat{u}) = \max \min \Pi$$

$$\{\hat{\boldsymbol{E}}, \hat{\boldsymbol{u}}\} = \arg \max_{\boldsymbol{E} \in} \min_{\boldsymbol{u} \in} \Pi(\boldsymbol{u}, \boldsymbol{E})$$

$$\Pi = \Pi(w, \boldsymbol{\alpha})$$

Necessary condition $\dfrac{\delta \Pi}{\delta w} = 0, \quad \dfrac{\partial \Pi}{\partial \boldsymbol{\alpha}} = 0$

Alternative fulfilment of the necessary condition

etc. until convergence

0. Choose $\boldsymbol{\alpha}^0$
1. The elasticity problem $\dfrac{\partial \Pi(\boldsymbol{w}, \boldsymbol{\alpha}^k)}{\partial \boldsymbol{w}} = 0 \Rightarrow \boldsymbol{w}^{k+1} \rightarrow$ 2. The opt. condition $\dfrac{\partial \Pi(\boldsymbol{w}^{k+1})}{\partial \boldsymbol{\alpha}} = 0 \Rightarrow \boldsymbol{\alpha}^{k+1}$

$\boldsymbol{w}$ — Fourier series coefficients

Saddle point implies convergence

Stress variant converges always...

Stress variant of the stiffness maximization problem

BENDSØE, M. P. (2003)
MAREŠ, T. (2006)
ALLAIRE, G. (2002)

$$\{\hat{\boldsymbol{E}}, \hat{\boldsymbol{u}}\} = \arg \max_{\boldsymbol{E} \in} \min_{\boldsymbol{u} \in} \Pi(\boldsymbol{u}, \boldsymbol{E})$$

$$\Pi = \frac{1}{2} \int_{\Omega} E_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, \mathrm{d}\Omega - \int_{\Omega} p_i u_i \, \mathrm{d}\Omega - \int_{\partial_t \Omega} t_i u_i \, \mathrm{d}S$$

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\{\hat{\boldsymbol{C}}, \hat{\boldsymbol{\sigma}}\} = \arg \min_{\boldsymbol{C} \in} \min_{\boldsymbol{\sigma} \in} \frac{1}{2} \int_{\Omega} C_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}\Omega$$

$$= \{\sigma_{ij} \mid \sigma_{ij,i} + p_j = 0 \text{ na } \Omega \ \wedge \ \sigma_{ij} \ell_j = t_i \text{ na } \partial_t \Omega\}$$

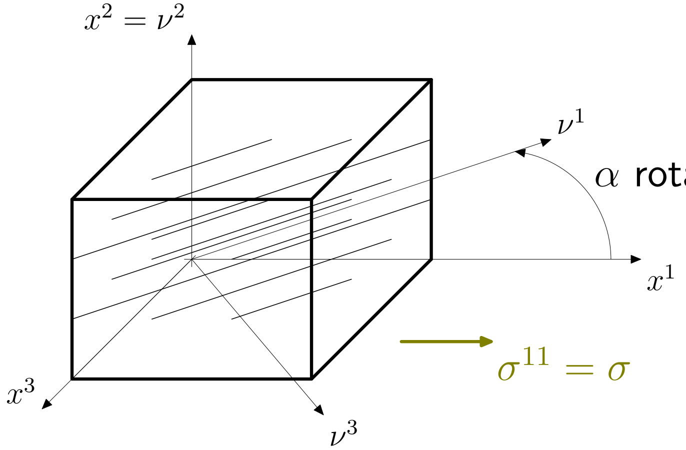
Maximization under uncertain conditions

$$\{\hat{\boldsymbol{C}}, \hat{\boldsymbol{t}}, \hat{\boldsymbol{\sigma}}\} = \arg \min_{\boldsymbol{C} \in} \max_{\boldsymbol{t} \in} \min_{\boldsymbol{\sigma} \in} \int_{\Omega} C_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}\Omega$$

— the set of possible loading states

The simplest (illustrating) problem of fibre composite stiffness maximization

Mareš, T. (2009)



$$\overset{x}{\sigma}_{ab} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\nu}{\varepsilon}_{ab} = \overset{\nu}{C}_{abcd} \overset{\nu}{\sigma}^{cd}$$

$$\left\{ \overset{\nu}{C}_{abcd} \right\}_{\{ab\lceil cd\}} = \begin{pmatrix}
\frac{1}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{21}}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{31}}{E_{33}} \\
0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\
0 & \frac{1}{4G_{12}} & 0 & \frac{1}{4G_{12}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_{11}} & 0 & 0 & 0 & \frac{1}{E_{22}} & 0 & 0 & 0 & -\frac{\nu_{32}}{E_{33}} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\
0 & 0 & \frac{1}{4G_{13}} & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & \frac{1}{4G_{23}} & 0 \\
-\frac{\nu_{13}}{E_{11}} & 0 & 0 & 0 & -\frac{\nu_{23}}{E_{22}} & 0 & 0 & 0 & \frac{1}{E_{33}}
\end{pmatrix}$$

$$\min_{\alpha} \int_V c \, \mathrm{d}V$$

$$c = \overset{x}{\sigma}^{ab} \overset{x}{C}_{abcd} \overset{x}{\sigma}^{cd} = \sigma \; \overset{x}{C}_{1111} \; \sigma$$

The transformation of the compliance tensor and results

$$\overset{x}{C}_{abcd} = \frac{\partial \nu^i}{\partial x^a}\frac{\partial \nu^j}{\partial x^b}\frac{\partial \nu^k}{\partial x^c}\frac{\partial \nu^l}{\partial x^d}\overset{\nu}{C}_{ijkl}$$

$$\overset{x}{C}_{1111} = \frac{\partial \nu^i}{\partial x^1}\frac{\partial \nu^j}{\partial x^1}\frac{\partial \nu^k}{\partial x^1}\frac{\partial \nu^l}{\partial x^1}\overset{\nu}{C}_{ijkl}$$

$$\frac{\partial \nu^i}{\partial x^a} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

$$c = \overset{x}{\sigma}{}^{ab}\overset{x}{C}_{abcd}\overset{x}{\sigma}{}^{cd} = \sigma\, \overset{x}{C}_{1111}\, \sigma$$

$$\overset{x}{C}_{1111} = \begin{pmatrix} \cos^2\alpha & 0 & \cos\alpha\sin\alpha & 0 & 0 & 0 & \sin\alpha\cos\alpha & 0 & \sin^2\alpha \end{pmatrix} \left\{\overset{\nu}{C}_{abcd}\right\}_{\{ab\lceil cd\}} \begin{pmatrix} \cos^2\alpha \\ 0 \\ \cos\alpha\sin\alpha \\ 0 \\ 0 \\ 0 \\ \sin\alpha\cos\alpha \\ 0 \\ \sin^2\alpha \end{pmatrix}$$

$$c = \sigma^2\left(\cos^4\alpha\frac{1}{E_{11}} + \cos^2\alpha\sin^2\alpha\left(\frac{1}{G_{13}} - \frac{\nu_{31}}{E_{11}} - \frac{\nu_{13}}{E_{33}}\right) + \sin^4\alpha\frac{1}{E_{33}}\right)$$

$$\frac{\partial c}{\partial \alpha} = 0$$

$$\cos^3\alpha\sin\alpha A_1 + \cos\alpha\sin^3\alpha A_2 = 0$$

$$A_1 = \frac{1}{G_{13}} - \frac{\nu_{31}}{E_{11}} - \frac{\nu_{13}}{E_{33}} - \frac{2}{E_{11}}$$

$$A_2 = \frac{2}{E_{22}} + \frac{\nu_{31}}{E_{11}} + \frac{\nu_{13}}{E_{33}} - \frac{1}{G_{13}}$$

$$\hat{\alpha}_1 = \pm\frac{\pi}{2} \text{ with } c_1 = \frac{\sigma^2}{E_{33}}$$

$$\hat{\alpha}_2 = 0, \pi \text{ with } c_2 = \frac{\sigma^2}{E_{11}}$$

$$\hat{\alpha}_{3,4} = \arctan\left(\pm\sqrt{-\frac{A_1}{A_2}}\right)$$

Stiffness maximization of plates

The elasticity problem

$$P^{abcd} w_{cd} = q^{ab}$$

The necessary condition of optimum

$$w_{ab} w_{cd} R^{abcd}(\alpha_\nu) = 0$$

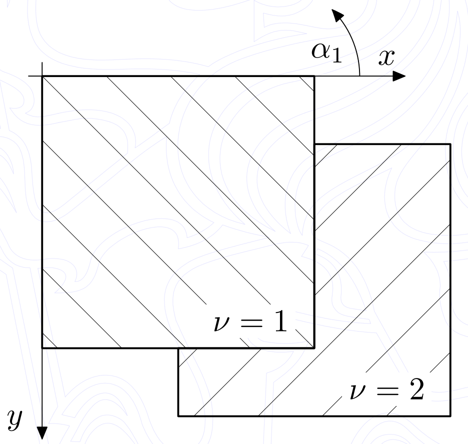
Laminated multilayer Kirchhoff plates of symetric layout

$R^{abcd}(\alpha_\nu)$ functions of the design parameters

α_ν stands for the layer orientation

w_{ab} Fourier series expansion coefficients of the perpendicular displacement

q^{ab} Fourier series expansion coefficients of the load



The alternative fulfilment of the necessary condition

$\alpha_1 = -45°$

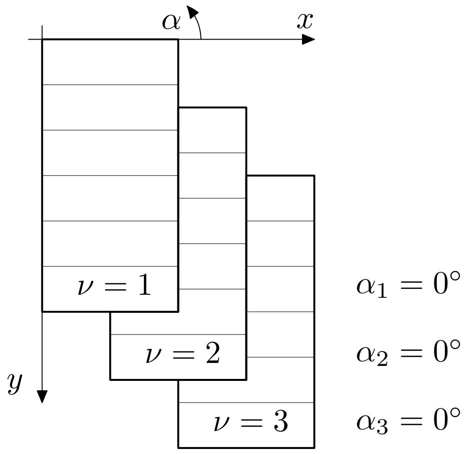
$\alpha_2 = 45°$

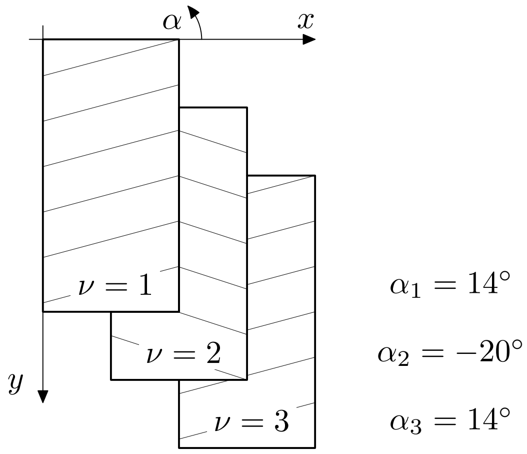
Any permutation of the layout is possible

Square plate of four layers loaded by $q = q_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$

The layout maximizing the stiffness

Rectangular plate (1:2) of six layers loaded by $q = q_o xy$

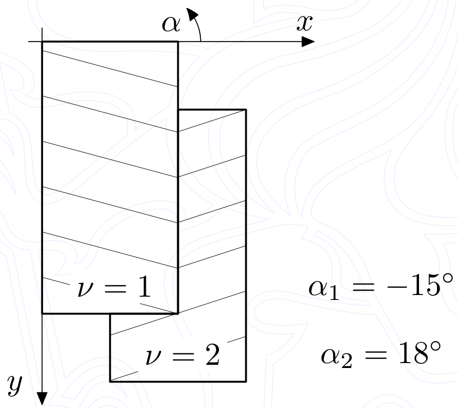




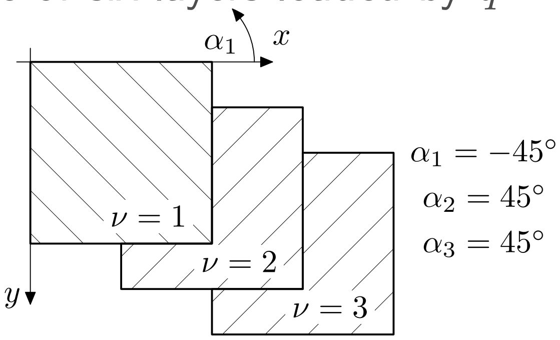
Rectangular plate (1:2) of six layers loaded by $q = q_o \sin \dfrac{\pi x}{a} \sin \dfrac{\pi y}{b}$

Any permutation of the layout is possible

Rectangular plate (1:2) of four layers loaded by $q = q_o xy$



Square plate of six layers loaded by $q = q_o \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{b}$



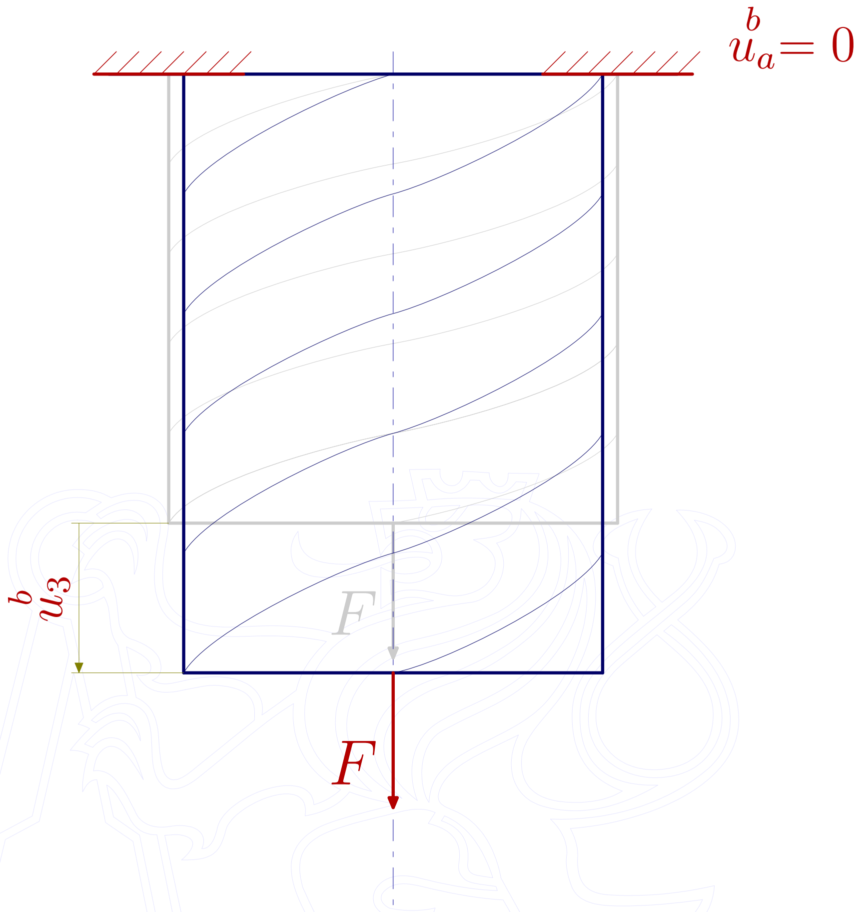
The alternative fulfilment of the necessary condition

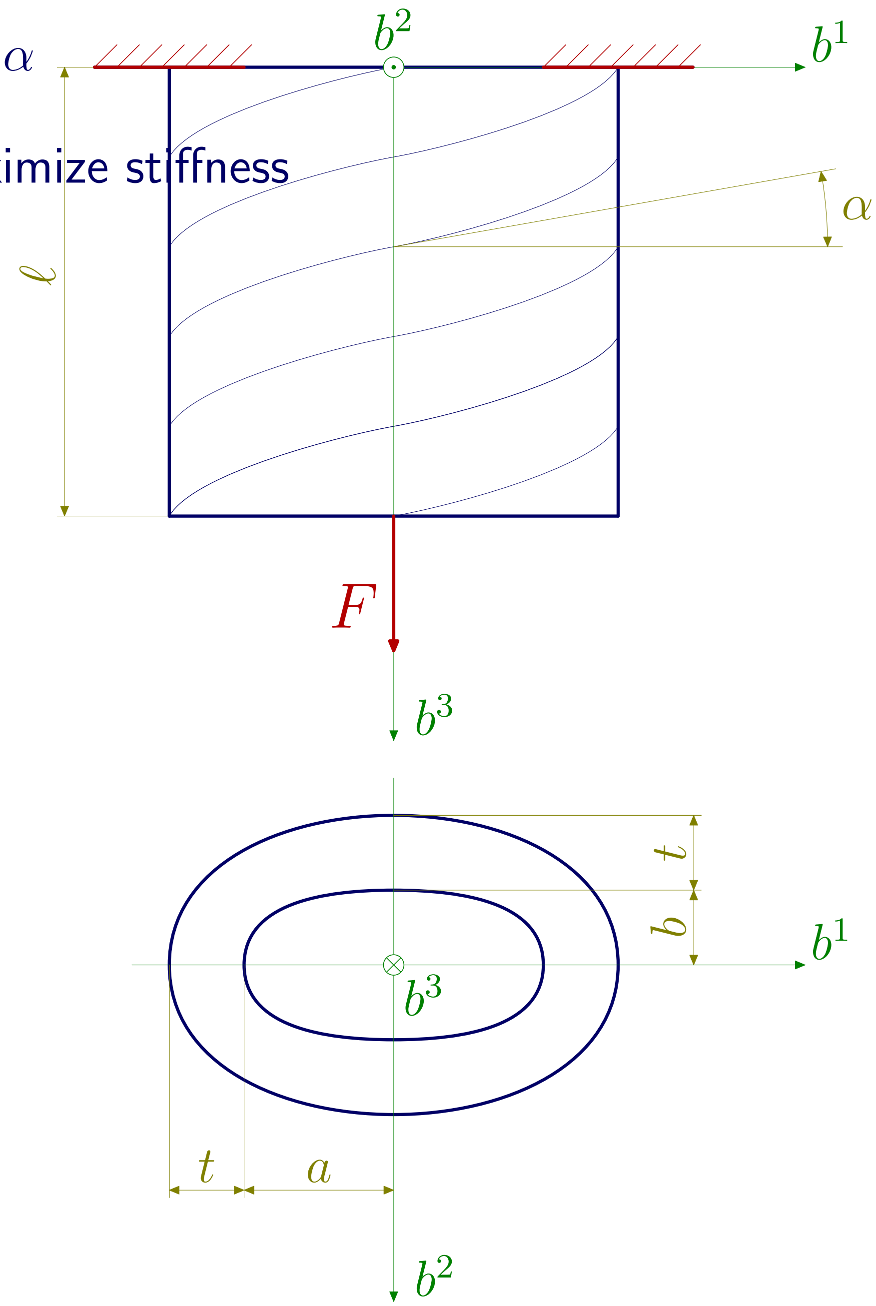
The problem

The thick-walled elliptic tube coiled with an angle α

loaded with Force F and clamped as seen at Fig.

The question is how to choose the angle α to maximize stiffness





The equation, right hand side and solution

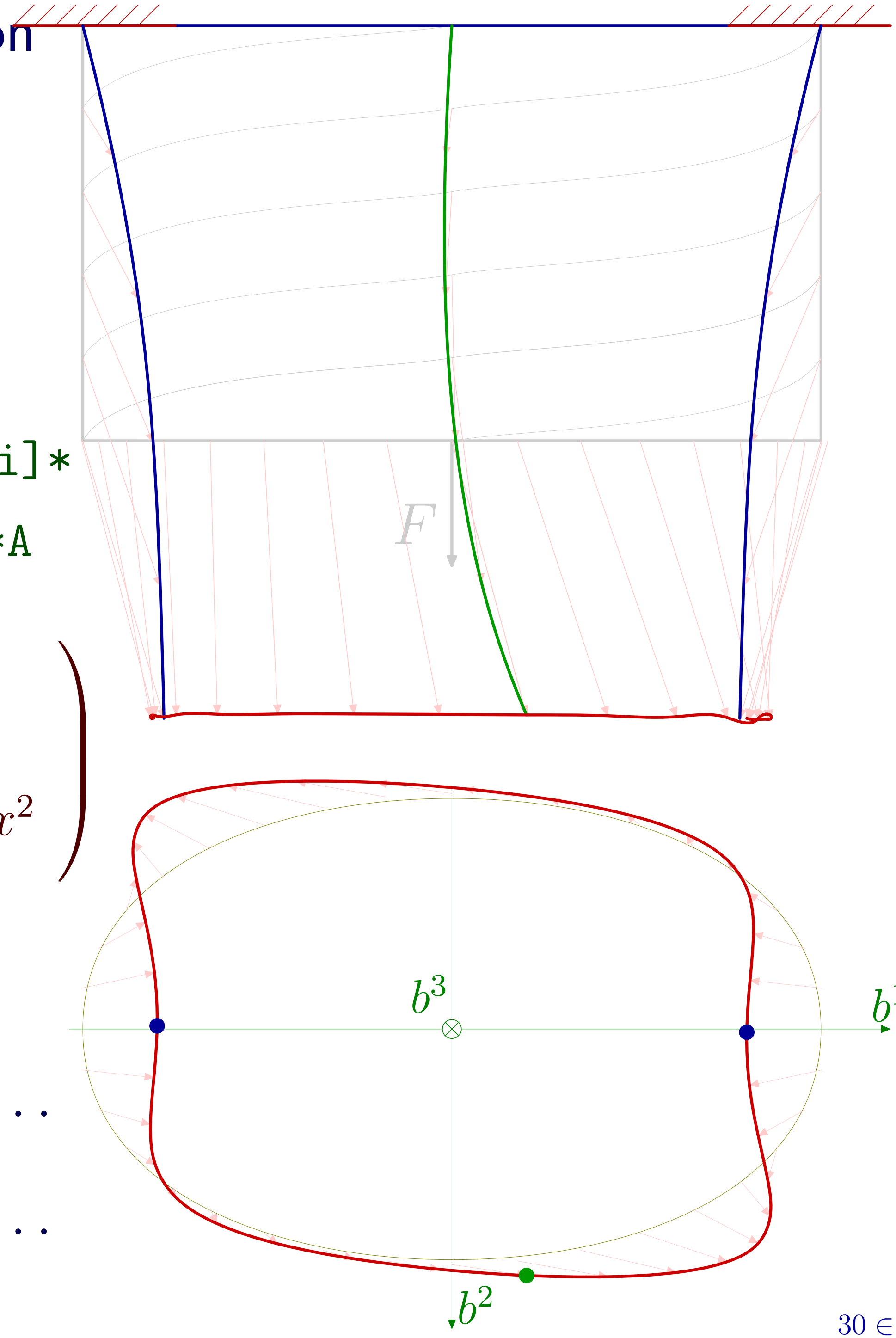
$$\frac{\partial a}{\partial A} = \frac{\partial l}{\partial A} \qquad \frac{\partial a}{\partial A} = KA \qquad KA = P$$

$$l = \int_S \frac{F}{S} \overset{x}{u}_3 \, \mathrm{d}S$$

$$l = \int_0^{2\pi}\int_0^{t} \frac{F}{S} \texttt{[zeros(1,343),zeros(1,343),phi]*} \texttt{*sqrt(det(gx))} \mathrm{d}x^1 \mathrm{d}x^2 \texttt{ *A}$$

$$P = \frac{\partial l}{\partial A} = \begin{pmatrix} \texttt{zeros(363)} \\ \texttt{zeros(363)} \\ \int_0^{2\pi}\int_0^{t} \frac{F}{S}\texttt{phi'*sqrt(det(gx))}\mathrm{d}x^1\mathrm{d}x^2 \end{pmatrix}$$

```
x1=...; x2=...; x3=...
A=K**(-1)*P
phi=x3*kron(kron(exp(i*j*x1*2*pi/t),...
ux=real([phi,zeros(1,siz),zeros(1,...
xb=1/(a*(sin(x2))**2+b*(cos(x2))**2+...
ub=xb*ux
```



Alternative fulfilment of the necessary condition of SM

Bendsøe, M. P. (2003)
Mareš, T. (2006)
Allaire, G. (2002)

0. Choose an angle α
1. The problem of elaticity as already solved

$$\mathtt{A} = \mathtt{K}^{-1}\mathtt{P}$$

2. The stiffness maximum condition,

$$\frac{\partial \Pi}{\partial \alpha} = 0, \text{ i.e., } \frac{1}{2}\mathtt{A}^T \frac{\partial \mathtt{K}}{\partial \alpha}\mathtt{A} = 0$$

$$\frac{\partial \mathtt{K}}{\partial \alpha} = \int_0^{\ell}\int_0^{2\pi}\int_0^{t} \mathtt{(B-Gam)'*}\frac{\partial \mathtt{Ex}}{\partial \alpha}\mathtt{*(B-Gam)*sqrt(det(gx))}\,\mathrm{d}^3 x$$

$$\frac{\partial \mathtt{Ex}}{\partial \alpha} = \left\{ \frac{\partial \overset{x}{E}{}^{abcd}}{\partial \alpha} \right\}_{ab\lceil cd}$$

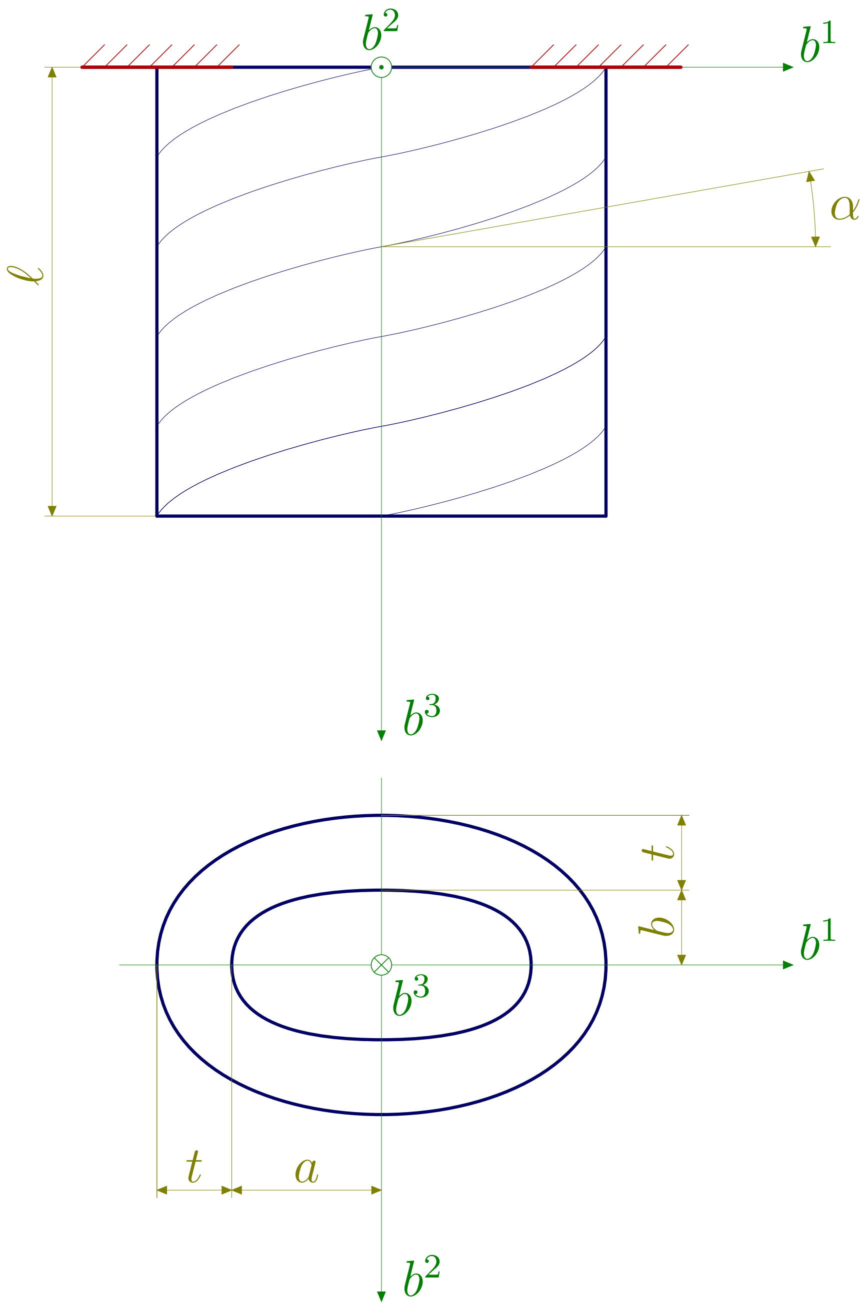
$$\frac{\partial \overset{x}{E}{}^{abcd}}{\partial \alpha} = \left(\alpha^a_i \frac{\partial x^b}{\partial \nu^j}\frac{\partial x^c}{\partial \nu^k}\frac{\partial x^d}{\partial \nu^l} + \frac{\partial x^a}{\partial \nu^i}\alpha^b_j \frac{\partial x^c}{\partial \nu^k}\frac{\partial x^d}{\partial \nu^l} + \frac{\partial x^a}{\partial \nu^i}\frac{\partial x^b}{\partial \nu^j}\alpha^c_k \frac{\partial x^d}{\partial \nu^l} + \frac{\partial x^a}{\partial \nu^i}\frac{\partial x^b}{\partial \nu^j}\frac{\partial x^c}{\partial \nu^k}\alpha^d_l \right) \overset{\nu}{E}{}^{ijkl}$$

$$\alpha^a_b = \frac{\partial}{\partial \alpha}\left[\frac{\partial x^a}{\partial \nu^b}\right] = \frac{\partial x^a}{\partial b^c}\frac{\partial b^c}{\partial \xi^d}\frac{\partial}{\partial \alpha}\left[\frac{\partial \xi^d}{\partial \nu^b}\right]$$

3. go to item 1

The results of the problem

The optimum angle α for given loadings



For F_3 $\alpha_{\text{opt}} = 90^\circ$

For T_1 $\alpha_{\text{opt}} = 0^\circ$

For T_2 $\alpha_{\text{opt}} = 45^\circ$

Thank you for your attention!

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