Constitutive modeling of nonlinear materials

with respect to application in biomechanics

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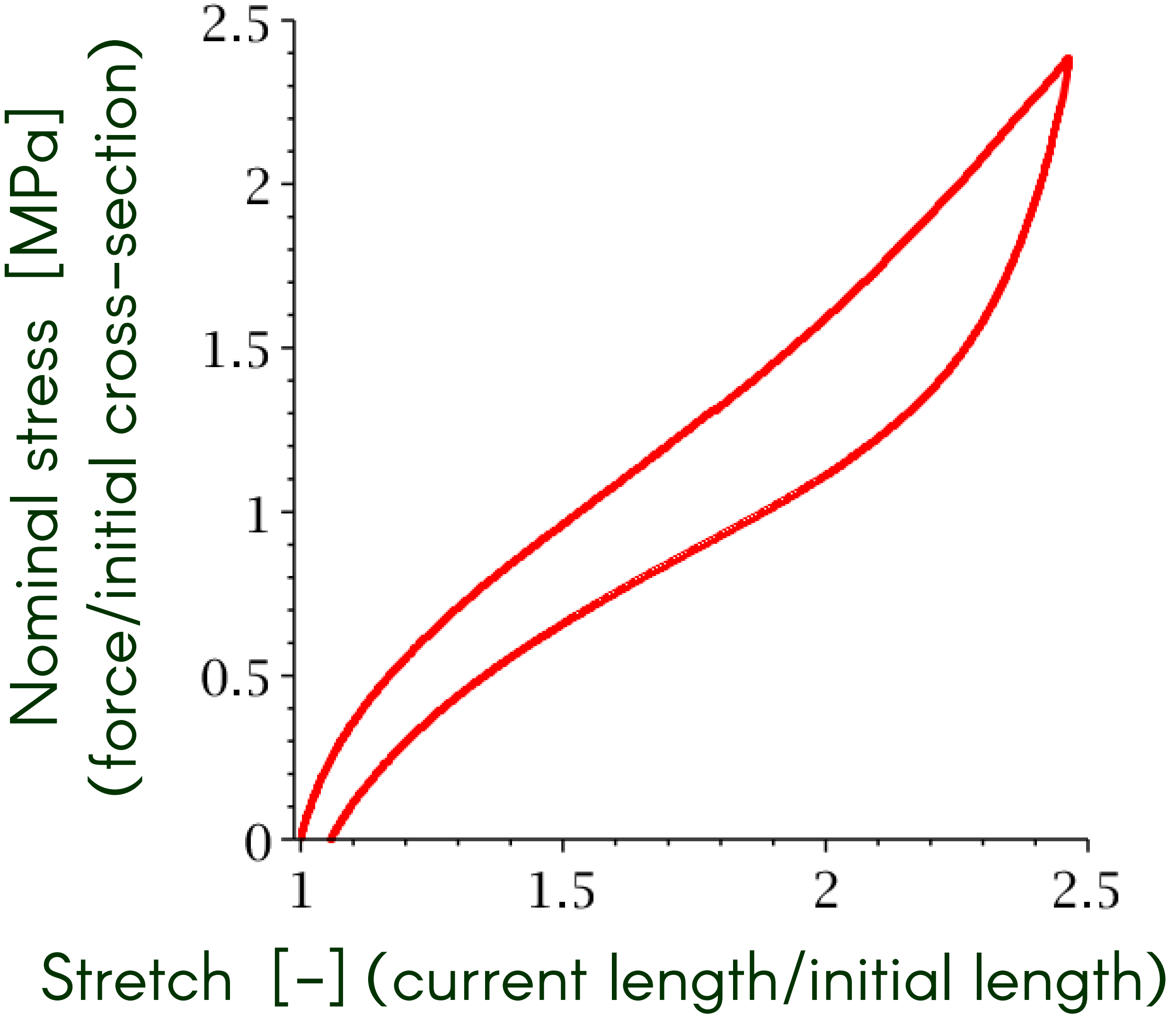
Objectives of the course

Nonlinear material response

Typical example: Uniaxial tensile test with vulcanized rubber



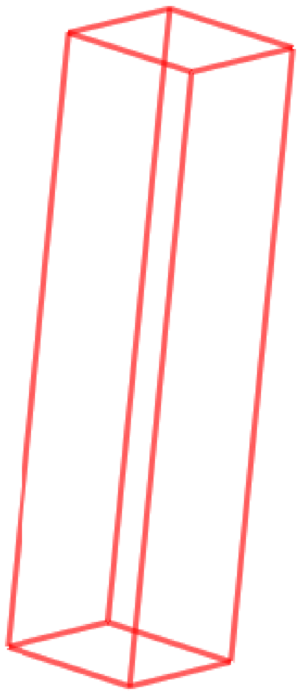
Objectives of the course

- Concepts of scalar, vector, and tensor quantities, basic operations and properties
- Kinematics of a deformation, deformation gradient and strain tensors
- Concept of stress, stress measures defined in the reference and deformed configuration
- Nonlinear behavior of materials found in experiments (elastomers, thermoplastics, resins, soft tissues)
- Hyperelasticity, strain energy density models

Preliminary mathematical notes

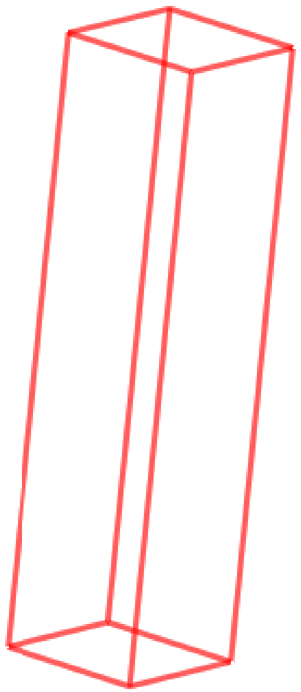
Scalars, vectors, tensors, mappings

Body



- We have a **body** $\equiv$ construction, machine, part,...
- We adopt phenomenological approach.
- The body is considered to be continuous set.
- Molecules and atoms are indistinguishable at our scale.
- The scale is continuous and thus there are no elementary particles like an atom or subatomic particles.

Body

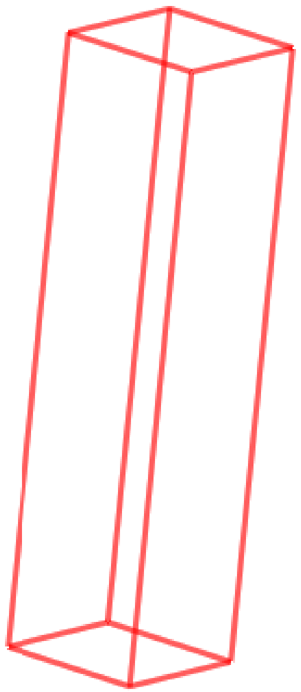


- The body is **embedded** in the mathematical (geometric) space
- Instead of elementary particle we have elementary mass dm distributed over elementary volume dV given by infinitesimal length elements

$$dm = \rho dV = \rho \cdot dx \cdot dy \cdot dz$$

- Here ρ is the mass density [kgm^{-3}]

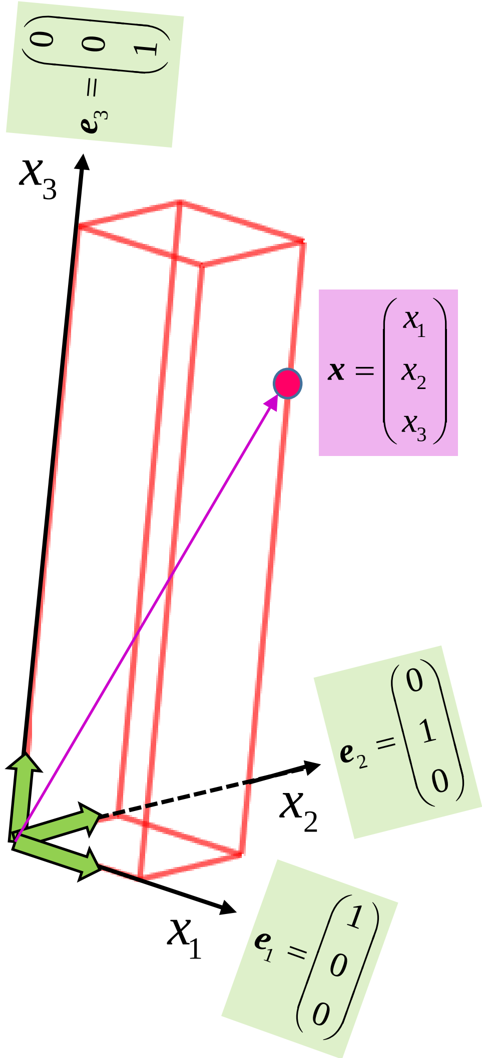
Body and geometric space



- We need a space where measurement of lengths is possible:
  (1) thus every point in the space will have a position vector with 3 components
  (2) thus we will have some origin and coordinate axes

Every point is located by a position vector expressed over some vector basis with respect to a give origin.

Body and geometric space



- Our space is usual $\mathbb{R}^3$ so-called **Euclidean space**
- Recall concept of
  - vector basis
  - algebraic operations
  - Coordinates
  - and transformations

Vectors

- Standard (Cartesian, orthonormal) **basis** in $\mathbb{R}^3$

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

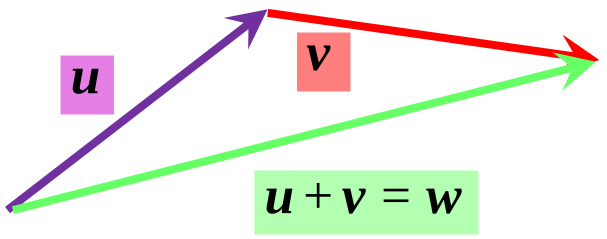
- **Components** of a vector
  in the basis

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3 = u_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + u_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Vectors

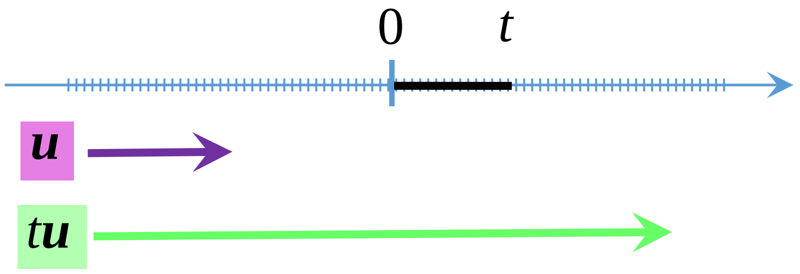
- Lets have a set of vectors $\mathcal{V}$ for $\boldsymbol{u},\boldsymbol{v}\in\mathcal{V}$ **vector addition** s defined as

$$\boldsymbol{u}+\boldsymbol{v}=\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}=\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}=\boldsymbol{w}\in\mathcal{V}$$



- Lets have $\boldsymbol{u}\in\mathcal{V}\wedge t\in\mathbb{R}$ multiplication by scalar (**scaling of a vector**)

$$t\boldsymbol{u}=t\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=\begin{pmatrix} t\cdot u_1 \\ t\cdot u_2 \\ t\cdot u_3 \end{pmatrix}=\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}=\boldsymbol{w}\in\mathcal{V}$$



Vectors

- Summation convention
  Always sum over the index which
  repeats one times in the expression

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3 = \sum_{i=1}^{3} u_i\boldsymbol{e}_i = \sum_{i=1}^{3} u_i\boldsymbol{e}_i = u_i\boldsymbol{e}_i$$

- Kronecker delta δ_{ij}
  replacement operator

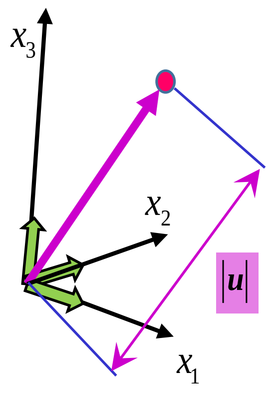
$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

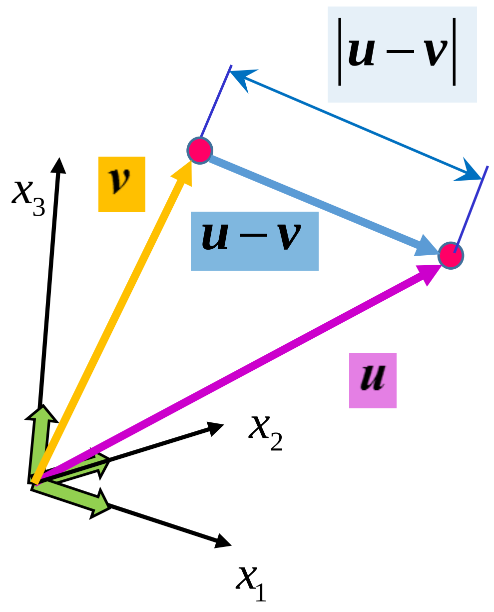
$$\delta_{ii} = 3 \qquad \delta_{ij}u_i = u_j \qquad \delta_{ij}\delta_{jk} = \delta_{ik}$$

Vectors

- Norm (length) of vector
  and metrics (distance between points two pints)
  …in Euclidean way

$$|\boldsymbol{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$



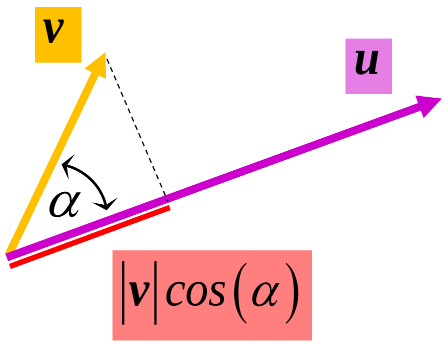


$$\rho(\boldsymbol{u},\boldsymbol{v}) = |\boldsymbol{u}-\boldsymbol{v}| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

Vectors

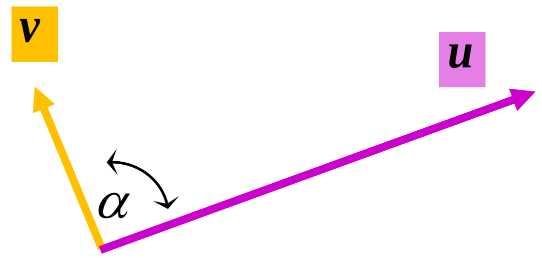
- Dot product (scalar or inner) of two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\boldsymbol{u}||\boldsymbol{v}| \cos(\alpha) \in \mathbb{R}$$



$$cos(\alpha) = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

- Orthogonality of vectors

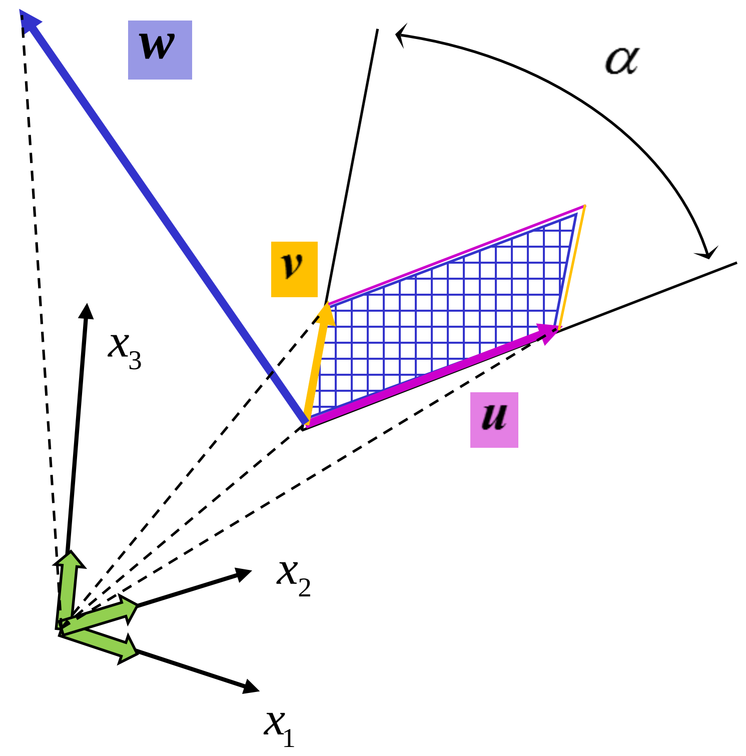


$$\boldsymbol{u} \perp \boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{u} \cdot \boldsymbol{v} = 0$$

Vectors

- Cross product (scalar or inner) of two vectors $\boldsymbol{u},\boldsymbol{v} \in \mathcal{V}$

$$\boldsymbol{u} \times \boldsymbol{v} = det \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \boldsymbol{w} \in \mathcal{V}$$



$$|\boldsymbol{w}| = |\boldsymbol{u}||\boldsymbol{v}| sin(\alpha)$$

$$\boldsymbol{u} \cdot \boldsymbol{w} = 0 \quad \wedge \quad \boldsymbol{v} \cdot \boldsymbol{w} = 0$$

Vectors

- **Levi-Civita** (permutation symbol) ε_{ijk}

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for even permutation of } i,j,k;\ (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{for odd permutation of } i,j,k;\ (i,j,k) \in \{(1,3,2),(2,1,3),(3,2,1)\} \\ 0 & \text{otherwise; repeating indices} \end{cases}$$

$$\boldsymbol{u} \times \boldsymbol{v} = \varepsilon_{ijk} u_i v_j \boldsymbol{e}_k =$$

$$= \varepsilon_{ij1} u_i v_j \boldsymbol{e}_1 + \varepsilon_{ij2} u_i v_j \boldsymbol{e}_2 + \varepsilon_{ij3} u_i v_j \boldsymbol{e}_3 =$$

$$= (\varepsilon_{111} u_1 v_1 + \varepsilon_{121} u_1 v_2 + \varepsilon_{131} u_1 v_3 + \varepsilon_{211} u_2 v_1 + \varepsilon_{221} u_2 v_2 + \varepsilon_{231} u_2 v_3 + \varepsilon_{311} u_3 v_1 + \varepsilon_{321} u_3 v_2 + \varepsilon_{331} u_3 v_3) \boldsymbol{e}_1 + (\ldots) \boldsymbol{e}_2 + (\ldots) \boldsymbol{e}_3 =$$

$$= (\underbrace{\varepsilon_{111}}_{0} u_1 v_1 + \underbrace{\varepsilon_{121}}_{0} u_1 v_2 + \underbrace{\varepsilon_{131}}_{0} u_1 v_3 + \underbrace{\varepsilon_{211}}_{0} u_2 v_1 + \underbrace{\varepsilon_{221}}_{0} u_2 v_2 + \underbrace{\varepsilon_{231}}_{1} u_2 v_3 + \underbrace{\varepsilon_{311}}_{0} u_3 v_1 + \underbrace{\varepsilon_{321}}_{-1} u_3 v_2 + \underbrace{\varepsilon_{331}}_{0} u_3 v_3) \boldsymbol{e}_1 + (\ldots) \boldsymbol{e}_2 + (\ldots) \boldsymbol{e}_3 =$$

$$= (u_2 v_3 - u_3 v_2) \boldsymbol{e}_1 + (u_3 v_1 - u_1 v_3) \boldsymbol{e}_1 + (u_1 v_2 - u_2 v_1) \boldsymbol{e}_3$$

Vector space $\mathcal{V} = \mathbb{R}^3$

- **Vector (Linear) space $\mathcal{V}$** is a set of vectors which by definition can be added together and scaled (multiplied by scalar)

set of vectors $\mathcal{V} = \mathbb{R}^3$ forms a commutative group

set of scalars $\mathbb{R}$ creates field

$$\mathcal{V} \equiv \left(\mathbb{R}^3, +\right)\left[\mathbb{R}, +, \cdot\right]$$

Linear mappings of vector spaces

- Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over real numbers.

  A map f: $\mathcal{V} \rightarrow \mathcal{W}$ is said to be **linear**
  if for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ and any $t \in \mathbb{R}$
  following conditions are satisfied:

  - Additivity $$f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$$
  - Homogeneity $$f(t\boldsymbol{u}) = t \cdot f(\boldsymbol{u})$$

- If $\mathcal{V} = \mathcal{W}$ and thus f: $\mathcal{V} \rightarrow \mathcal{V}$, f is said to be a **linear transformation** or **linear operator**

Linear mappings of vector spaces

- Since $\mathcal{V} = \mathbb{R}^3$ and $\mathcal{W} = \mathbb{R}^3$

  linear maps between vector spaces are **represented by matrices**

  Let $\boldsymbol{u} \in \mathcal{V}$

$$\mathbf{A} : \mathcal{V} \to \mathcal{V} \quad \Rightarrow \quad \mathbf{A}\boldsymbol{u} = \boldsymbol{w}$$

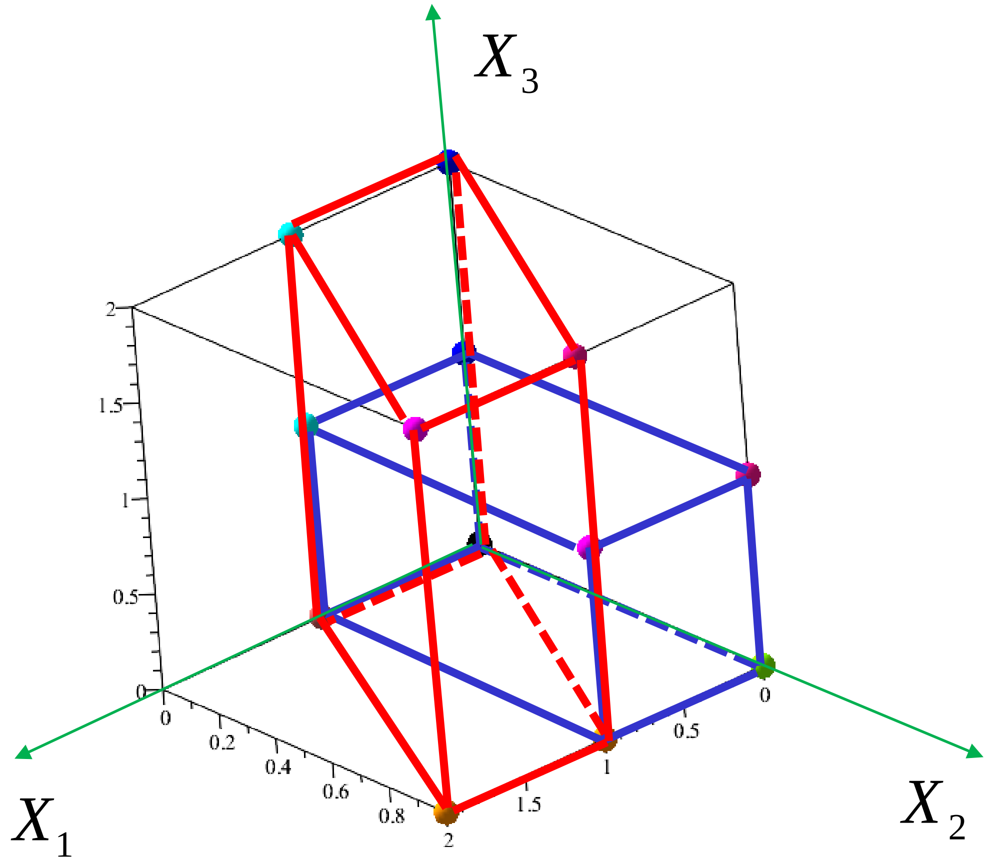
$$\mathbf{A}\boldsymbol{u} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \\ A_{21}u_1 + A_{22}u_2 + A_{23}u_3 \\ A_{31}u_1 + A_{32}u_2 + A_{33}u_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \boldsymbol{w} \in \mathcal{V}$$

Linear mappings of vector spaces

- Linear transformation $\mathbf{A}$
  Example

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \boldsymbol{u} = \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\boldsymbol{w} = \mathbf{A}\boldsymbol{u} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_1 + X_2 \\ X_2 \\ 2X_3 \end{pmatrix}$$



Composition of linear mappings = Matrix multiplication

$$\mathbf{C} = \mathbf{B}\,\mathbf{A} \quad \Leftrightarrow \quad C_{ij} = \sum_{k=1}^{3} B_{ik} A_{kj} = B_{ik} A_{kj}$$

$$C_{11} = B_{11}A_{11} + B_{12}A_{21} + B_{13}A_{31} = \sum_{k=1}^{3} B_{1k} A_{k1}$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$C_{23} = B_{21}A_{13} + B_{22}A_{23} + B_{23}A_{33} = \sum_{k=1}^{3} B_{2k} A_{k3}$$

$$C_{33} = B_{31}A_{13} + B_{32}A_{23} + B_{33}A_{33} = \sum_{k=1}^{3} B_{3k} A_{k3}$$

Composition of linear mappings = Matrix multiplication

- Dot product of matrices (composition of mappings) **is not commutative**

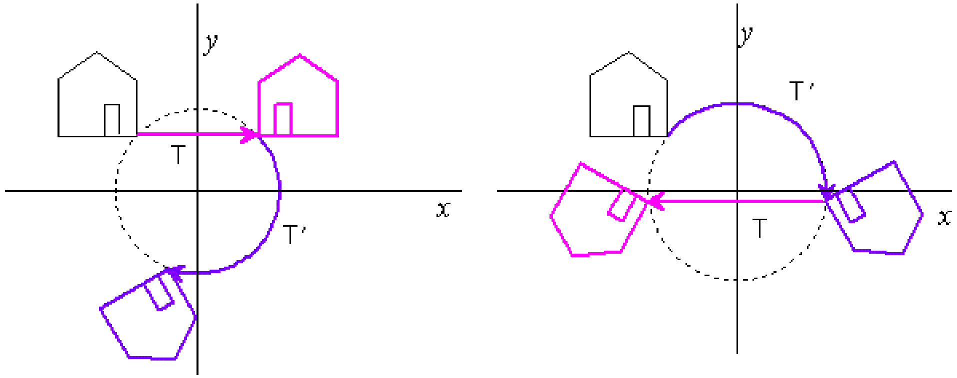
$$\mathbf{C} = \mathbf{B}\,\mathbf{A} \neq \mathbf{A}\mathbf{B} = \mathbf{C}'$$

$$\mathbf{C}(\boldsymbol{u}) = \mathbf{B} \circ \mathbf{A}(\boldsymbol{u}) = \mathbf{B}(\mathbf{A}(\boldsymbol{u})) = \mathbf{B}(\boldsymbol{w}) = \boldsymbol{z}$$

$\mathbf{A}\boldsymbol{u} = \boldsymbol{w}$ $\quad$ $\mathbf{B}\boldsymbol{w} = \boldsymbol{z}$

$$\mathbf{C}'(\boldsymbol{u}) = \mathbf{A} \circ \mathbf{B}(\boldsymbol{u}) = \mathbf{A}(\mathbf{B}(\boldsymbol{u})) = \mathbf{A}(\boldsymbol{w}') = \boldsymbol{z}'$$

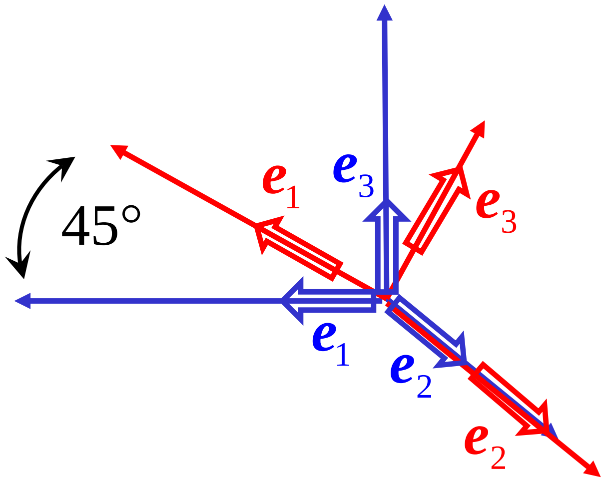
$\mathbf{B}\boldsymbol{u} = \boldsymbol{w}'$ $\quad$ $\mathbf{A}\boldsymbol{w}' = \boldsymbol{z}'$



Changes of the basis by a rotation

$$\mathbf{Q}=\begin{pmatrix} \boldsymbol{e}_1\cdot\boldsymbol{e}_1 & \boldsymbol{e}_1\cdot\boldsymbol{e}_2 & \boldsymbol{e}_1\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}_2\cdot\boldsymbol{e}_1 & \boldsymbol{e}_2\cdot\boldsymbol{e}_2 & \boldsymbol{e}_2\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}_3\cdot\boldsymbol{e}_1 & \boldsymbol{e}_3\cdot\boldsymbol{e}_2 & \boldsymbol{e}_3\cdot\boldsymbol{e}_3 \end{pmatrix}=\begin{pmatrix} cos(\boldsymbol{e}_1,\boldsymbol{e}_1) & cos(\boldsymbol{e}_1,\boldsymbol{e}_2) & cos(\boldsymbol{e}_1,\boldsymbol{e}_3) \\ cos(\boldsymbol{e}_2,\boldsymbol{e}_1) & cos(\boldsymbol{e}_2,\boldsymbol{e}_2) & cos(\boldsymbol{e}_2,\boldsymbol{e}_3) \\ cos(\boldsymbol{e}_3,\boldsymbol{e}_1) & cos(\boldsymbol{e}_3,\boldsymbol{e}_2) & cos(\boldsymbol{e}_3,\boldsymbol{e}_3) \end{pmatrix}$$

$$\boldsymbol{e}_i=\mathbf{Q}\boldsymbol{e}_i$$

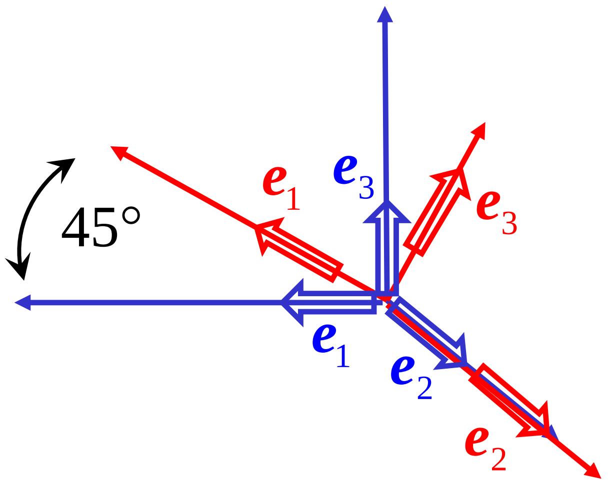


$$\mathbf{Q}=\begin{pmatrix} cos(45°) & cos(90°) & cos(135°) \\ cos(90°) & cos(0°) & cos(90°) \\ cos(45°) & cos(90°) & cos(45°) \end{pmatrix}=\begin{pmatrix} \tfrac{1}{2}\sqrt{2} & 0 & -\tfrac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \tfrac{1}{2}\sqrt{2} & 0 & \tfrac{1}{2}\sqrt{2} \end{pmatrix}$$

$$\boldsymbol{e}_1=\mathbf{Q}\boldsymbol{e}_1 \qquad \boldsymbol{e}_2=\mathbf{Q}\boldsymbol{e}_2 \qquad \boldsymbol{e}_3=\mathbf{Q}\boldsymbol{e}_3$$

$$\boldsymbol{e}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\longrightarrow\boldsymbol{e}_1=\begin{pmatrix}\tfrac{1}{2}\sqrt{2}\\0\\\tfrac{1}{2}\sqrt{2}\end{pmatrix} \qquad \boldsymbol{e}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}\longrightarrow\boldsymbol{e}_2=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \boldsymbol{e}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}\longrightarrow\boldsymbol{e}_3=\begin{pmatrix}-\tfrac{1}{2}\sqrt{2}\\0\\\tfrac{1}{2}\sqrt{2}\end{pmatrix}$$

Changes of the basis by a rotation



New basis $\quad$ Old basis

$$\mathbf{e}_i = \mathbf{Q}\mathbf{e}_i$$

- $\mathbf{Q}$ is orthogonal matrix (orthogonal transformation)

$$\mathbf{Q}^T = \mathbf{Q}^{-1}$$

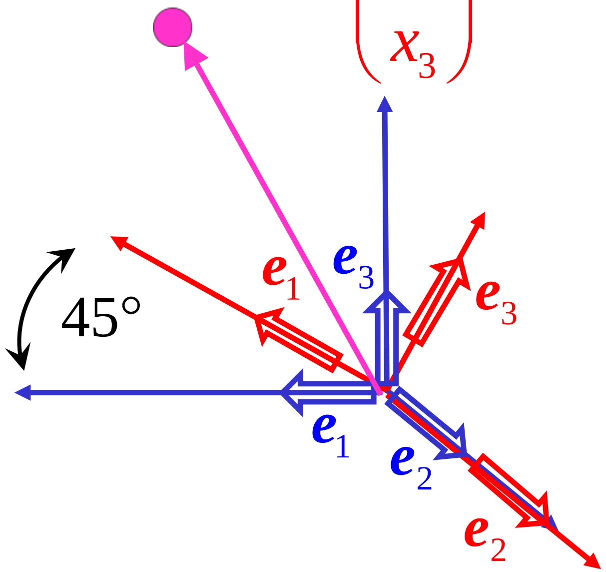
$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{-1} = \mathbf{I}$$

$$Q_{ji} = Q_{ij}^{-1}$$

Changes of coordinates by a rotation

- Vectors are transformed by $\mathbf{Q}^T$, that is transpose of $\mathbf{Q}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \boldsymbol{x} = \boldsymbol{x} = \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



New basis — Old basis

$$\boldsymbol{e}_i = \mathbf{Q}\boldsymbol{e}_i$$

New coordinates — Old coordinates

$$\boldsymbol{x} = \mathbf{Q}^T \boldsymbol{x}$$

$$x_i = Q_{ji} x_j$$

Note that $\mathbf{Q}^T\boldsymbol{u}$ gives $Q_{ji}u_j \equiv$ summation over non-neighboring indices

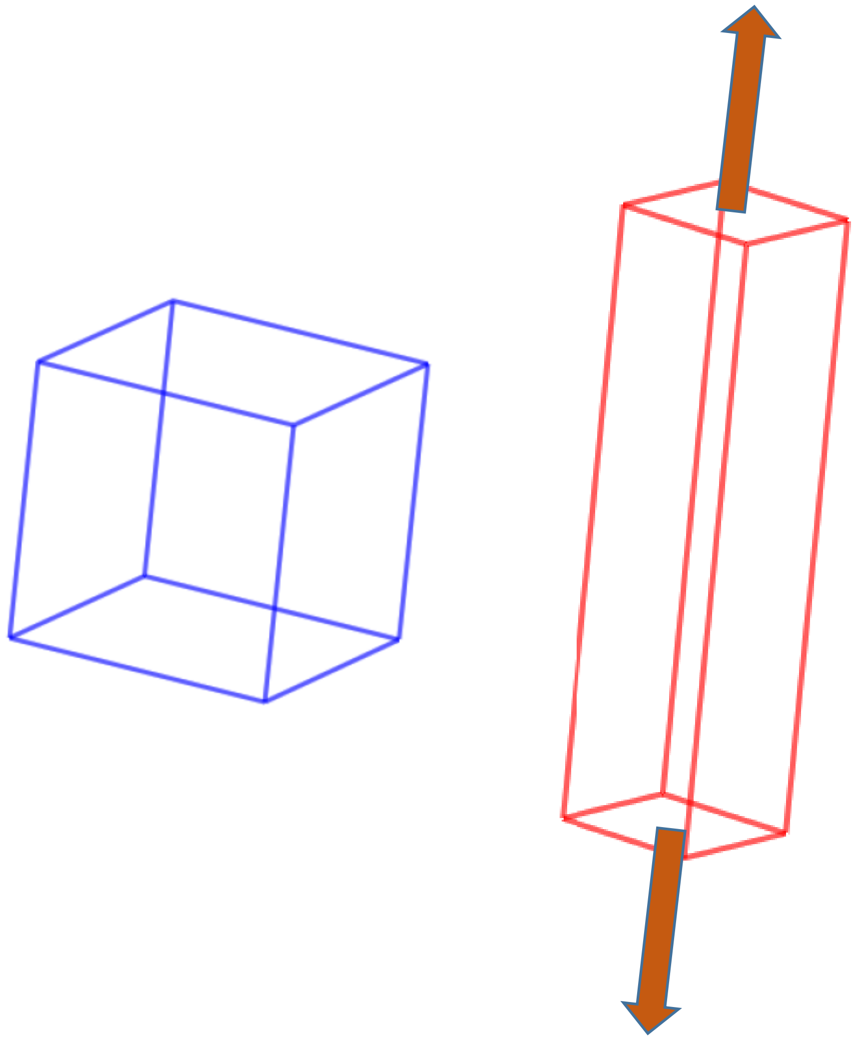
Mathematical intro

- Certainly, we did not explained all relevant concepts and definitions
- It was only brief introduction
- Other concepts like e.g. **tensors** will be explained ad hoc later

Kinematics and measures of deformation

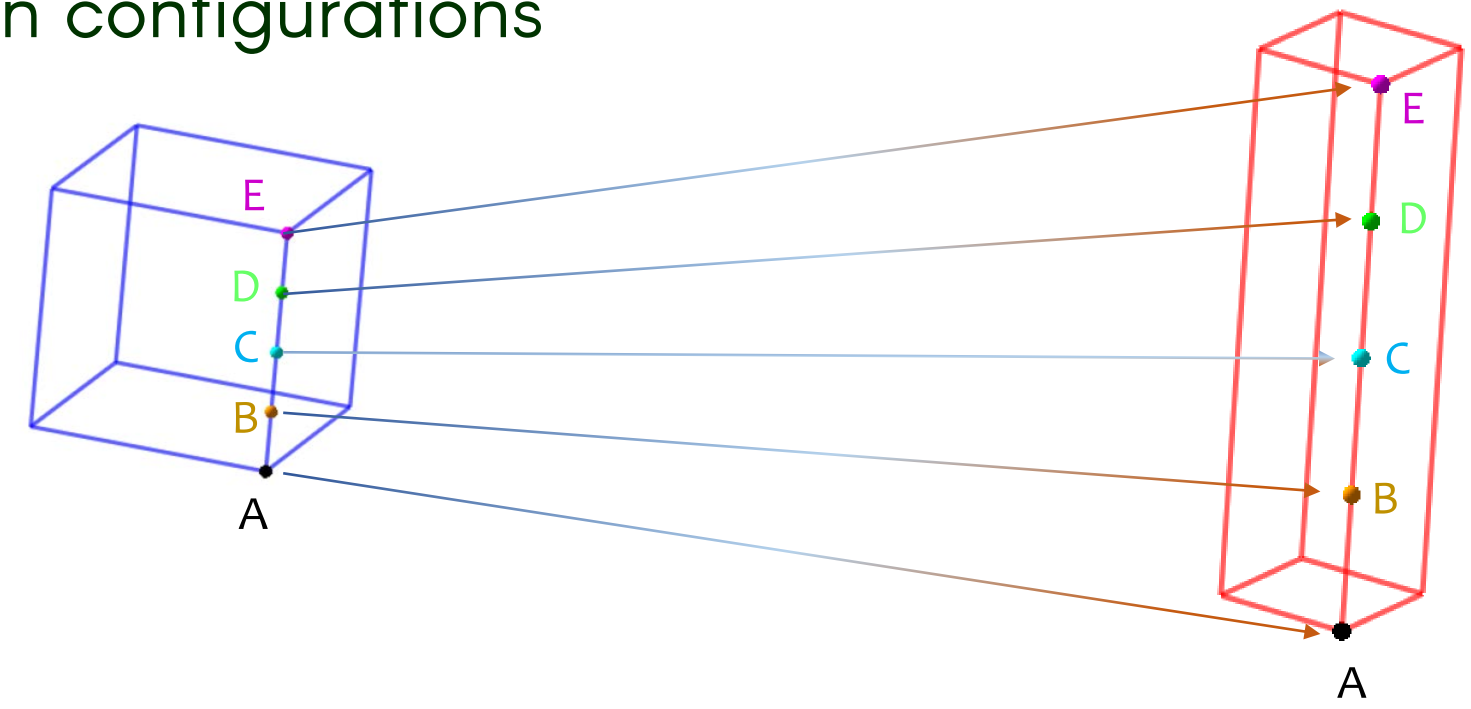
Motion and configuration of a body

- We have a body and forces act on the body and thus the body is in motion
- The motion means that the body occupies a continuous sequence of geometrical regions
- This regions are referred to as **configurations**



Motion and configuration of a body

- A motion of the body
  can be understand as a mapping
- Mapping between configurations

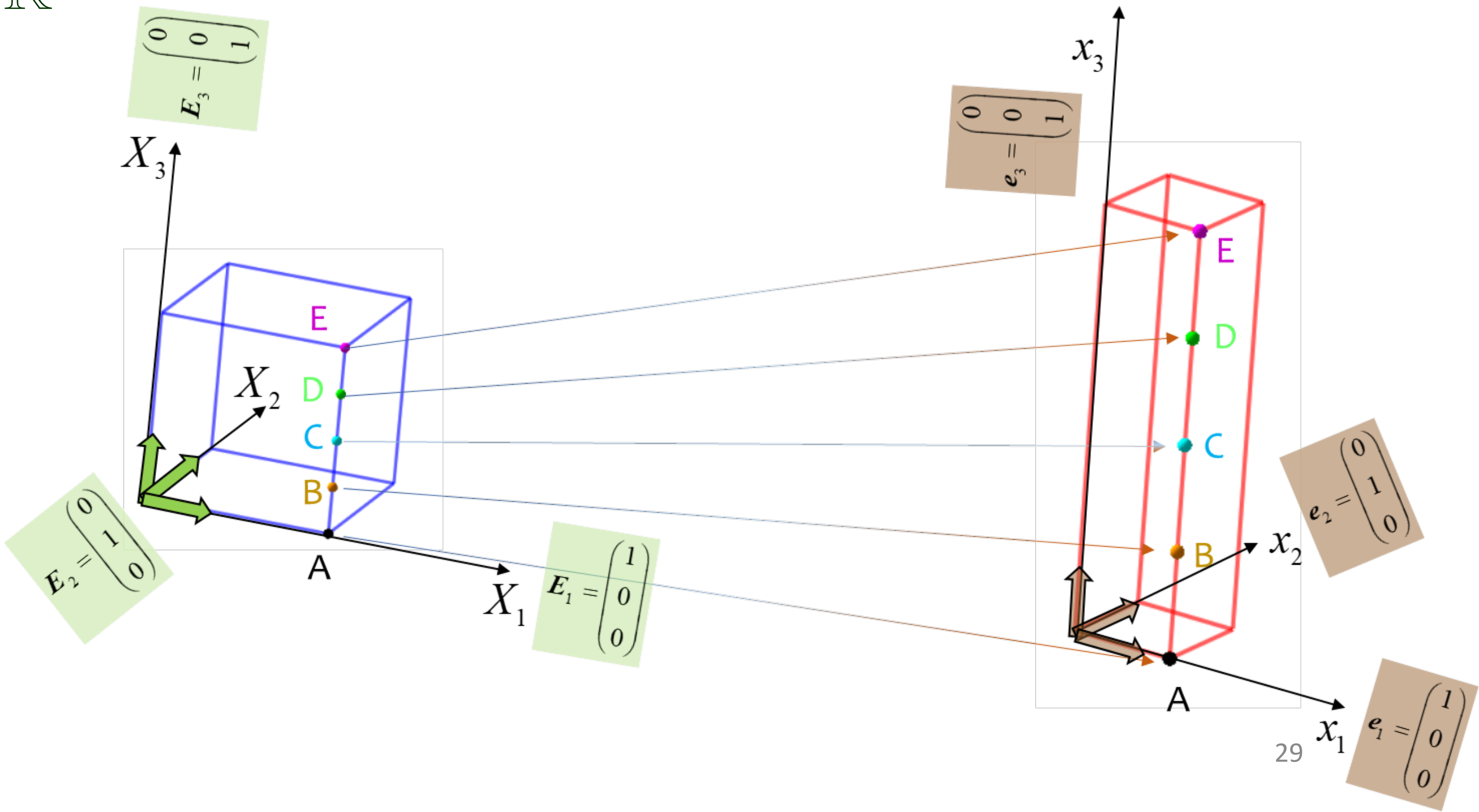


Non-deformed configuration

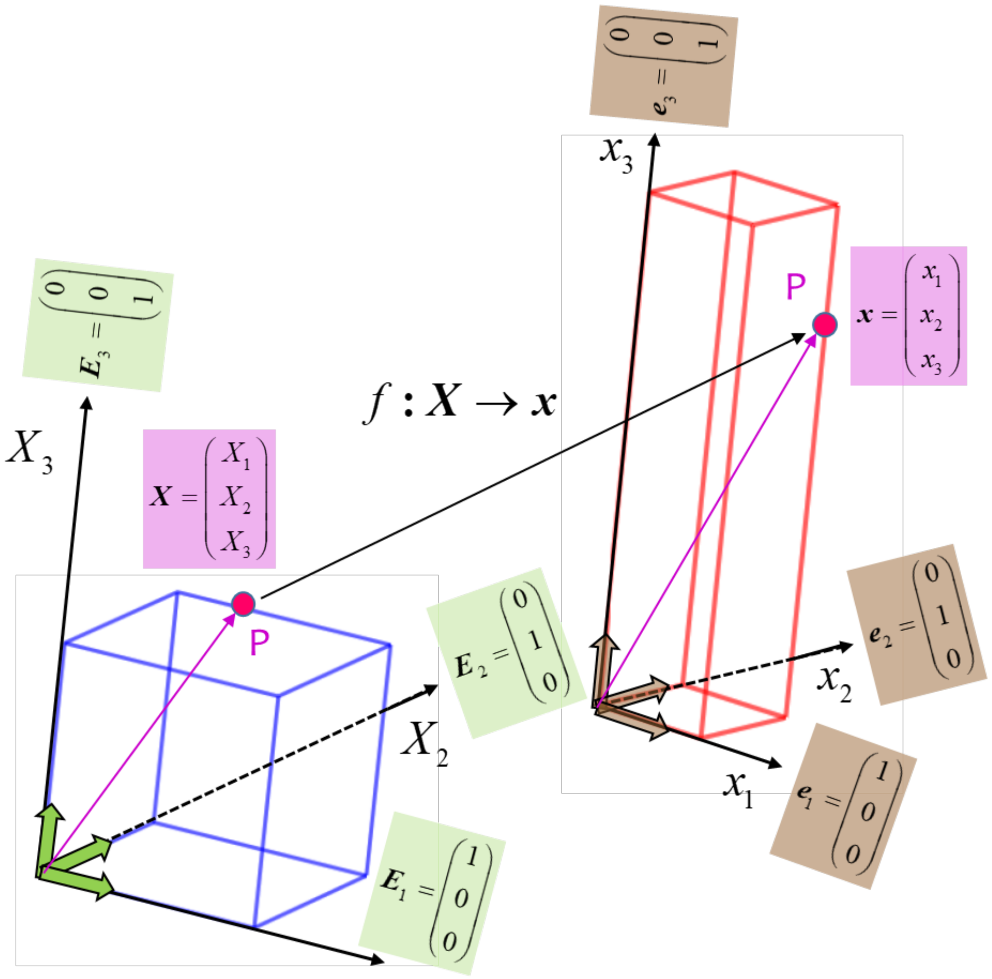
Deformed configuration

Motion and configuration of a body

- Describe motion ⇔ describe mapping between configurations
- Configurations are embedded in geometrical space $\mathbb{R}^3$



Motion equals to mapping



- Several ways to express that position vector $\boldsymbol{X}$ changes to $\boldsymbol{x}$ due to the motion f

$$f : \boldsymbol{X} \to \boldsymbol{x}$$

$$\boldsymbol{x} = f(\boldsymbol{X})$$

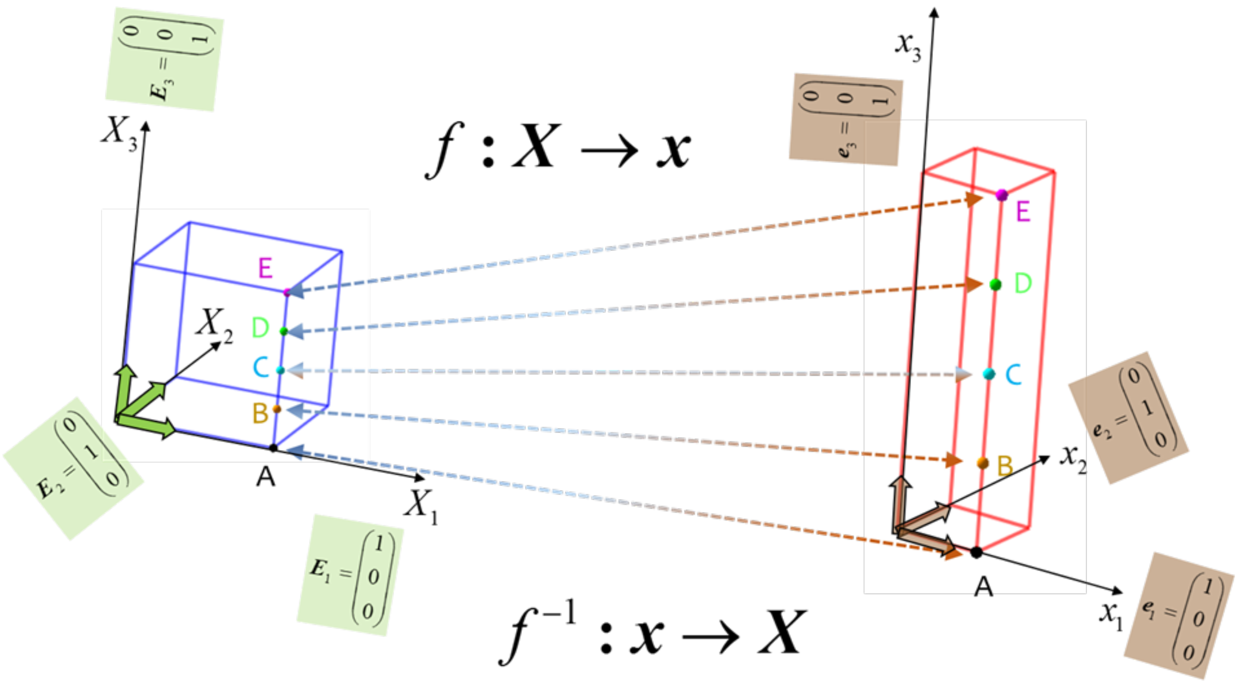
$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}f_1(X_1, X_2, X_3)\\f_2(X_1, X_2, X_3)\\f_3(X_1, X_2, X_3)\end{pmatrix} =$$

$$= f_1(X_1, X_2, X_3)\boldsymbol{e}_1 + f_2(X_1, X_2, X_3)\boldsymbol{e}_2 + f_3(X_1, X_2, X_3)\boldsymbol{e}_3$$

Motion equals to mapping

- A mapping expressing the motion has to satisfy several conditions

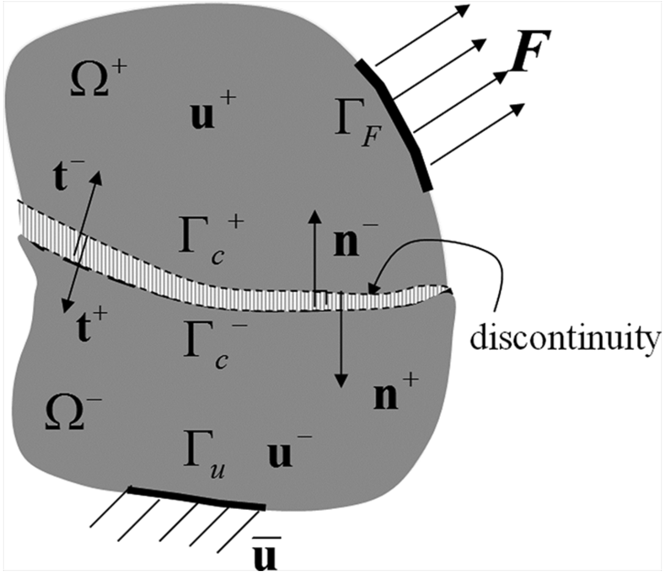
- f is bijection (one-to-one correspondence)
- f is continuous
- f^{-1} exists and is also continuous
- Derivatives of f and f^{-1} are also continuous



Motion equals to mapping

- We now clearly see that we are in continuum mechanics…
- It is a part of big family of continuum (field) theories
- But continuity may be so restrictive and
  the theory can be modified

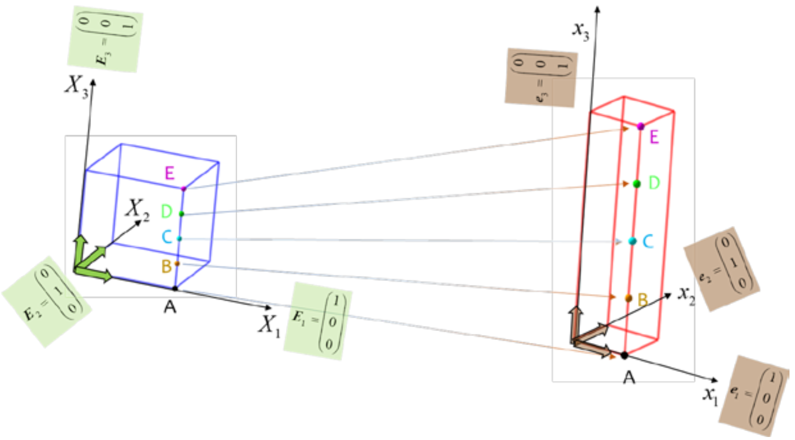
cracks
failure
shock waves



One motion but two descriptions

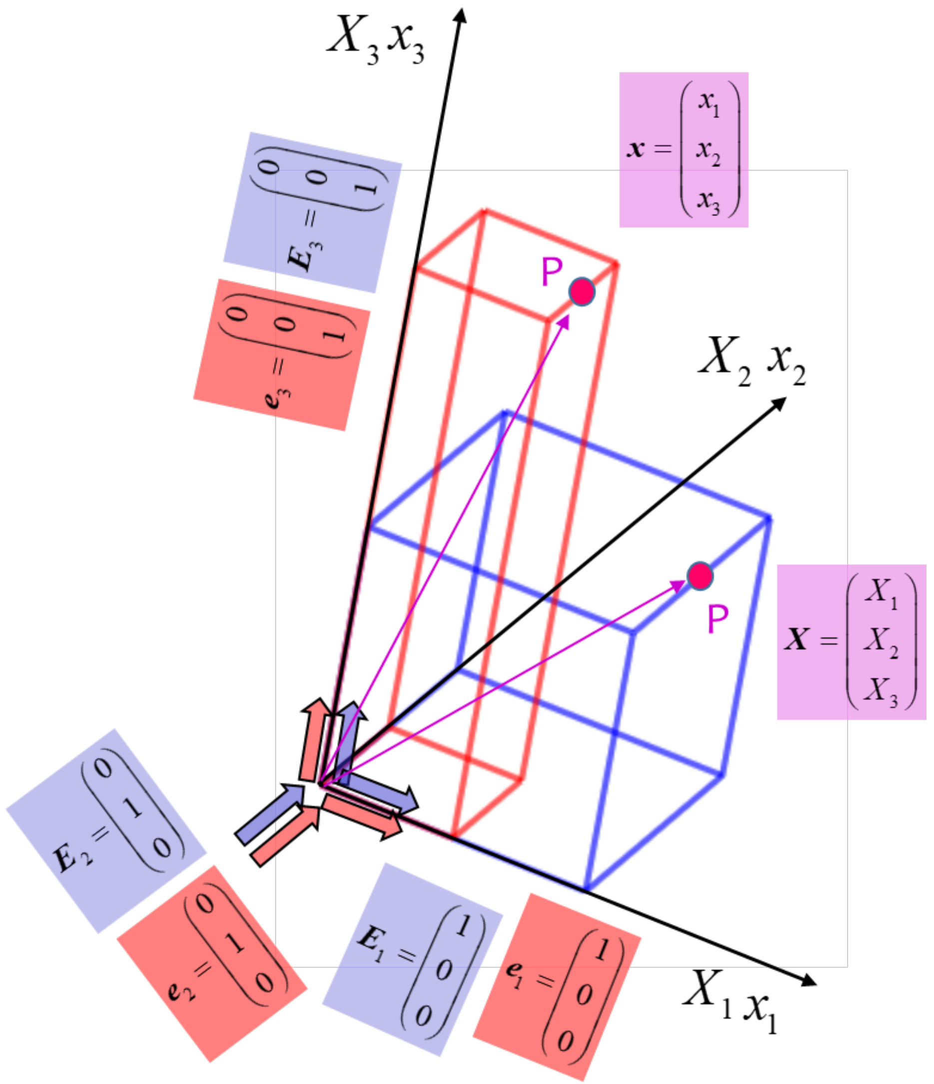
- Quantities describing a motion can be expressed by using **material** (reference, undeformed) coordinates or by **spatial** (current, deformed) coordinates

- Material description uses typical for solid mechanics $\{\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$
- Spatial description uses typical for fluid mechanics $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$



One motion but two descriptions

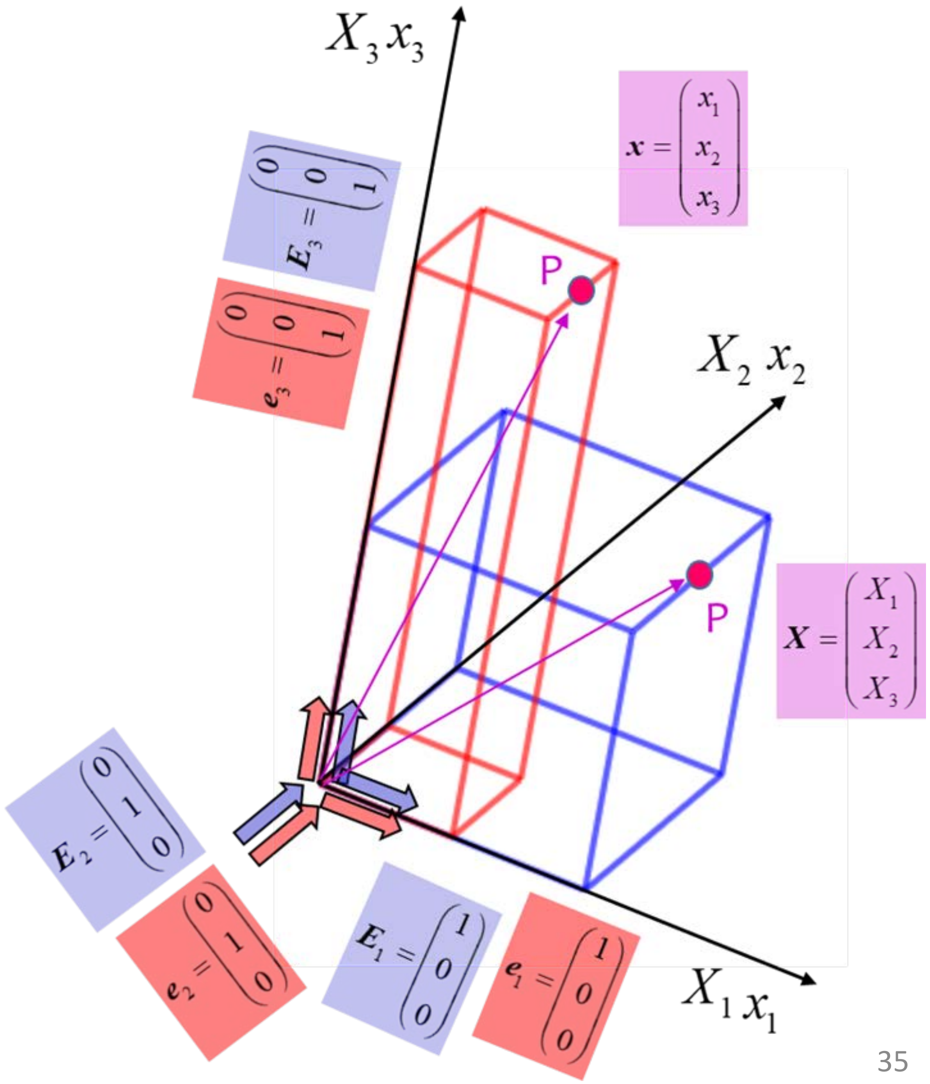
- Usually both referential and spatial coordinate axes have the same origin



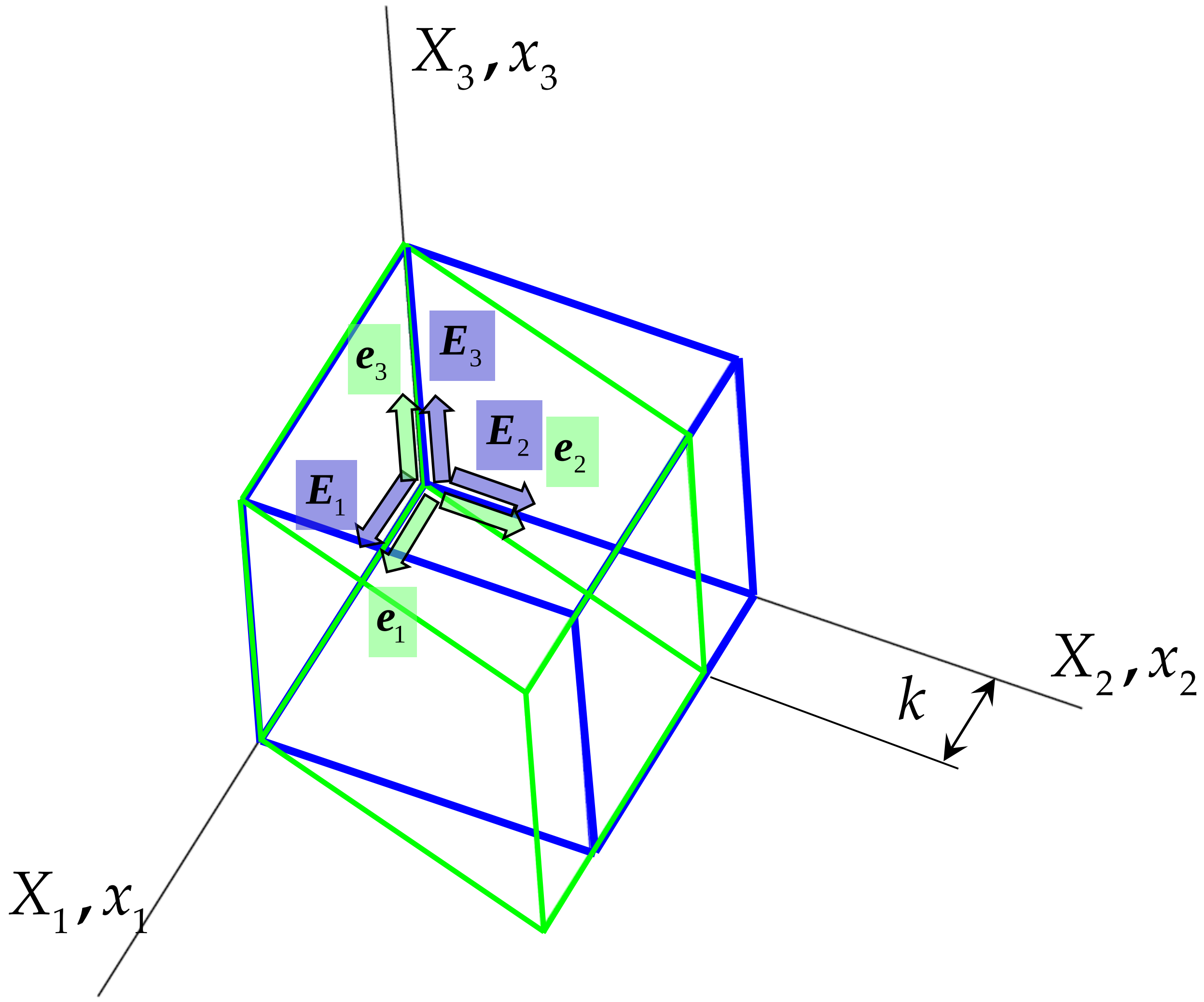
Example of the motion

$$f : \boldsymbol{X} \to \boldsymbol{x}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.5X_1 \\ 0.5X_2 \\ 2X_3 \end{pmatrix} = 0.5X_1\boldsymbol{e}_1 + 0.5X_2\boldsymbol{e}_2 + 2X_3\boldsymbol{e}_3$$



Example of the motion



$$f : \boldsymbol{X} \rightarrow \boldsymbol{x}$$

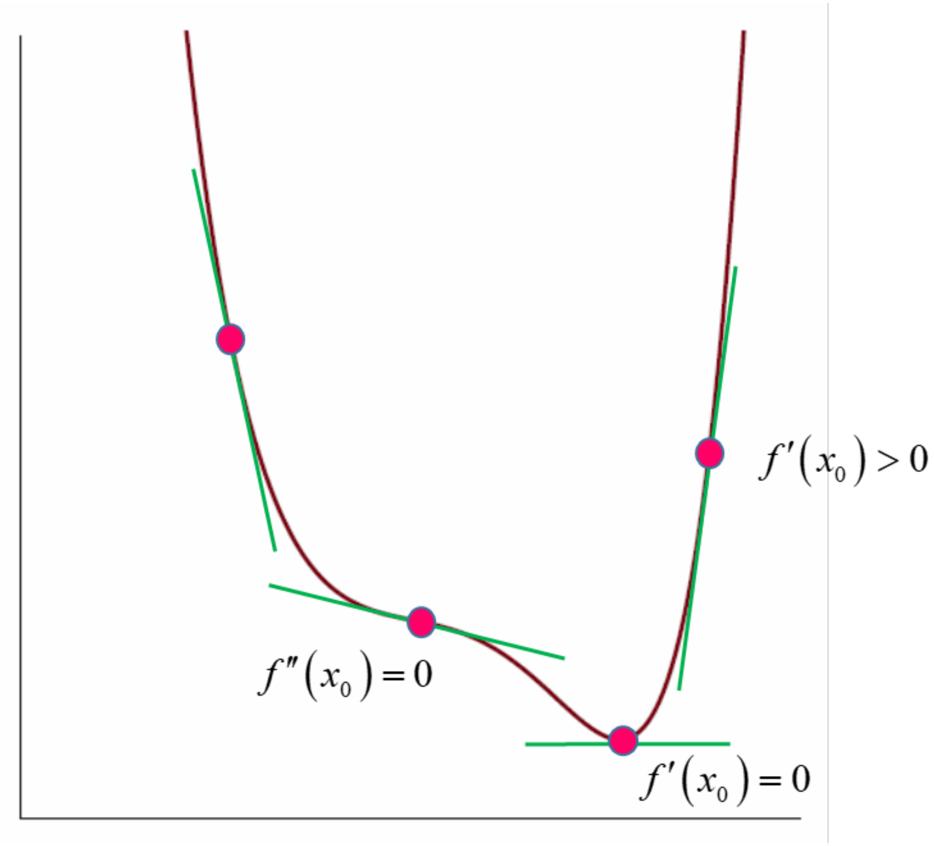
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 + kX_2 \\ X_2 \\ X_3 \end{pmatrix} = \left(X_1 + kX_2 \right) \boldsymbol{e}_1 + X_2 \boldsymbol{e}_2 + X_3 \boldsymbol{e}_3$$

Basic instrument used to treat functions

- Derivative can be used to find if a function is increasing/decreasing, if the function have extremum or flection points...

$$f:\mathbb{R}\rightarrow\mathbb{R}$$
$$x\rightarrow f(x)$$

$$f'(x_0)=\lim_{h\rightarrow 0}\frac{f(x_0+h)-f(x_0)}{h}$$



Basic instrument used in $f:\mathbb{R}^3 \to \mathbb{R}$

- In the multivariate calculus, the concept of derivative

$$f:\mathbb{R}\to\mathbb{R}$$

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

is extended to the form

of the directional or

$$\nabla_{\boldsymbol{v}} f(\boldsymbol{x}_0)=\lim_{h\to 0}\frac{f(\boldsymbol{x}_0+h\boldsymbol{v})-f(\boldsymbol{x}_0)}{h}=\nabla f(\boldsymbol{x}_0)\cdot\boldsymbol{v}$$

simple partial derivative

$$\nabla_{(1,0,0)} f(\boldsymbol{x}_0)=\lim_{h\to 0}\frac{f(\boldsymbol{x}_0+h(1,0,0))-f(\boldsymbol{x}_0)}{h}=\frac{\partial f(\boldsymbol{x}_0)}{\partial x_1}$$

and gradient

of the function

$$\nabla f(\boldsymbol{x}_0)=\begin{pmatrix}\dfrac{\partial f(\boldsymbol{x}_0)}{\partial x_1}\\ \dfrac{\partial f(\boldsymbol{x}_0)}{\partial x_2}\\ \dfrac{\partial f(\boldsymbol{x}_0)}{\partial x_3}\end{pmatrix}=\frac{\partial f(\boldsymbol{x}_0)}{\partial x_1}\boldsymbol{e}_1+\frac{\partial f(\boldsymbol{x}_0)}{\partial x_2}\boldsymbol{e}_2+\frac{\partial f(\boldsymbol{x}_0)}{\partial x_3}\boldsymbol{e}_3$$

But our motion is $f : \mathbb{R}^3 \to \mathbb{R}^3$

- Gradient of $\boldsymbol{x} = f(\boldsymbol{X})$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1(X_1, X_2, X_3) \\ f_2(X_1, X_2, X_3) \\ f_3(X_1, X_2, X_3) \end{pmatrix}$$

$$Grad(\boldsymbol{x}) = Grad(f(\boldsymbol{X})) = \begin{pmatrix} \frac{\partial f_1(X_1, X_2, X_3)}{\partial X_1} & \frac{\partial f_1(X_1, X_2, X_3)}{\partial X_2} & \frac{\partial f_1(X_1, X_2, X_3)}{\partial X_2} \\ \frac{\partial f_2(X_1, X_2, X_3)}{\partial X_1} & \frac{\partial f_2(X_1, X_2, X_3)}{\partial X_2} & \frac{\partial f_2(X_1, X_2, X_3)}{\partial X_3} \\ \frac{\partial f_3(X_1, X_2, X_3)}{\partial X_1} & \frac{\partial f_3(X_1, X_2, X_3)}{\partial X_2} & \frac{\partial f_3(X_1, X_2, X_3)}{\partial X_3} \end{pmatrix}$$

But our motion is $f : \mathbb{R}^3 \to \mathbb{R}^3$

- Gradient of $\boldsymbol{x} = f(\boldsymbol{X})$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1(X_1, X_2, X_3) \\ f_2(X_1, X_2, X_3) \\ f_3(X_1, X_2, X_3) \end{pmatrix}$$

$$Grad(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \end{pmatrix}$$

Local properties of a mapping

- Derivative, partial derivative, directional derivative, gradient, all these quantities are evaluated at a point X_0
  They give local information about a mapping f

Neighborhood in the domain of the mapping

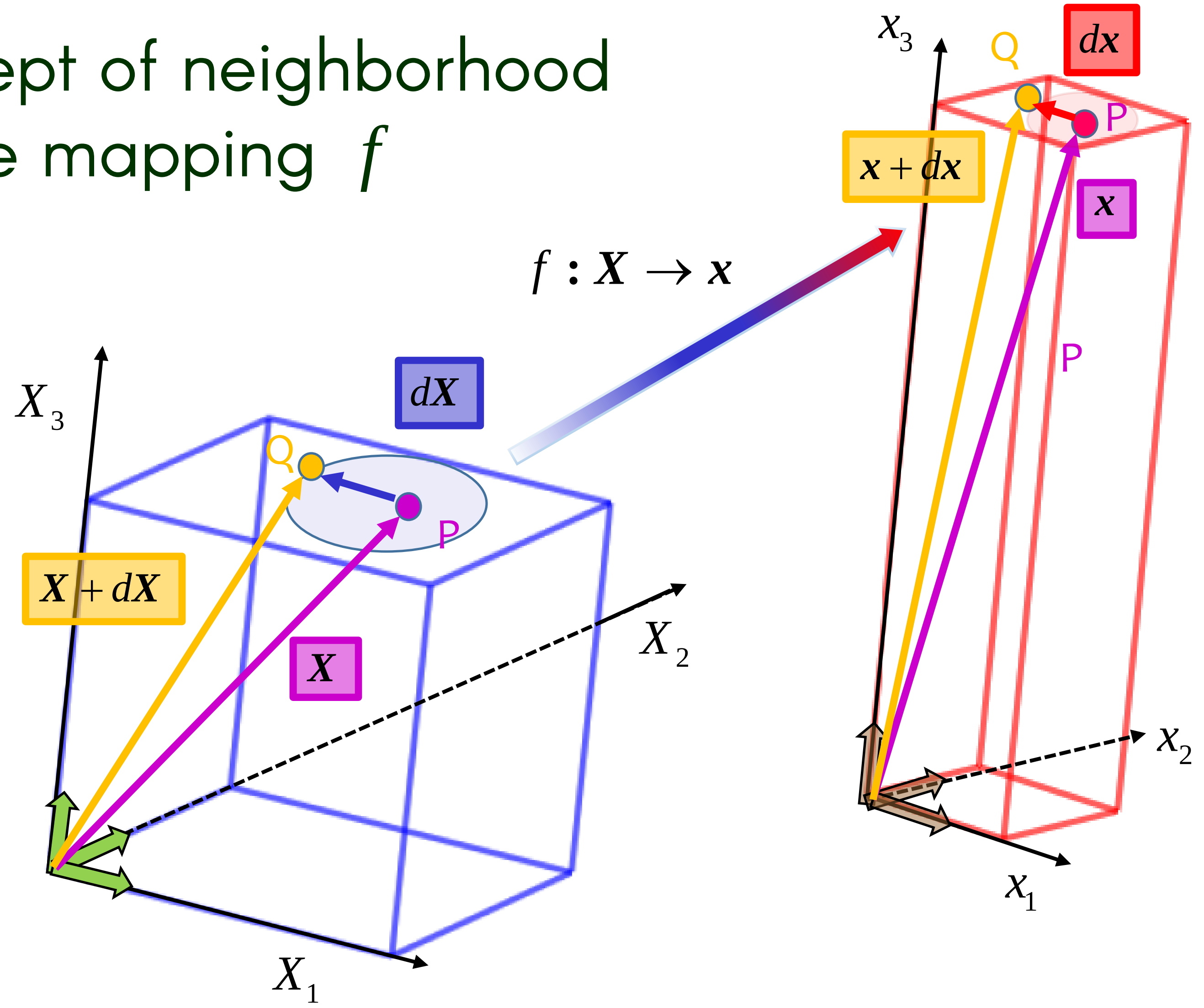
Neighborhood in the image of the mapping

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

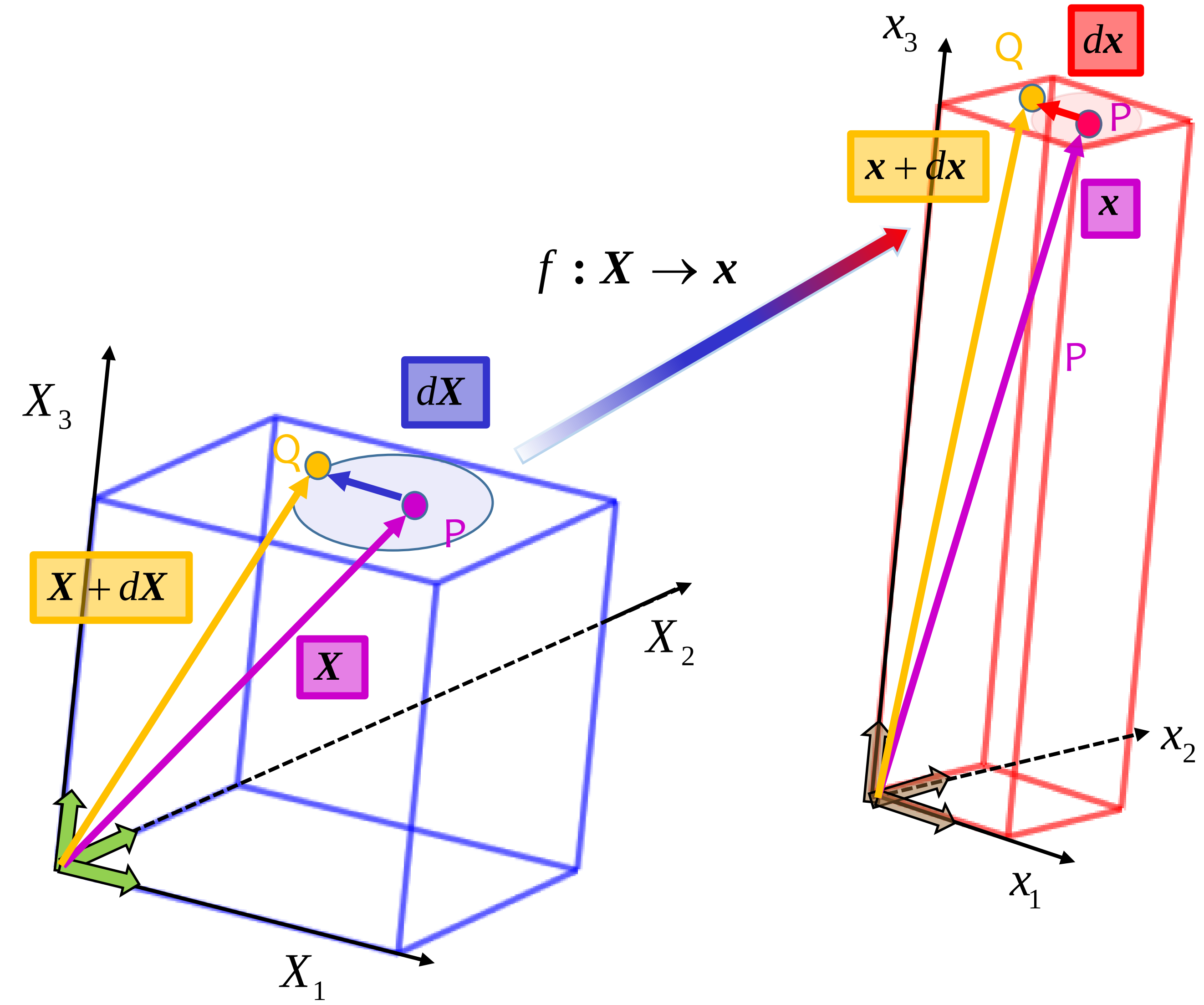
Point of interest

Deformation

- Apply the concept of neighborhood of a point on the mapping f from reference to deformed configuration



Deformation

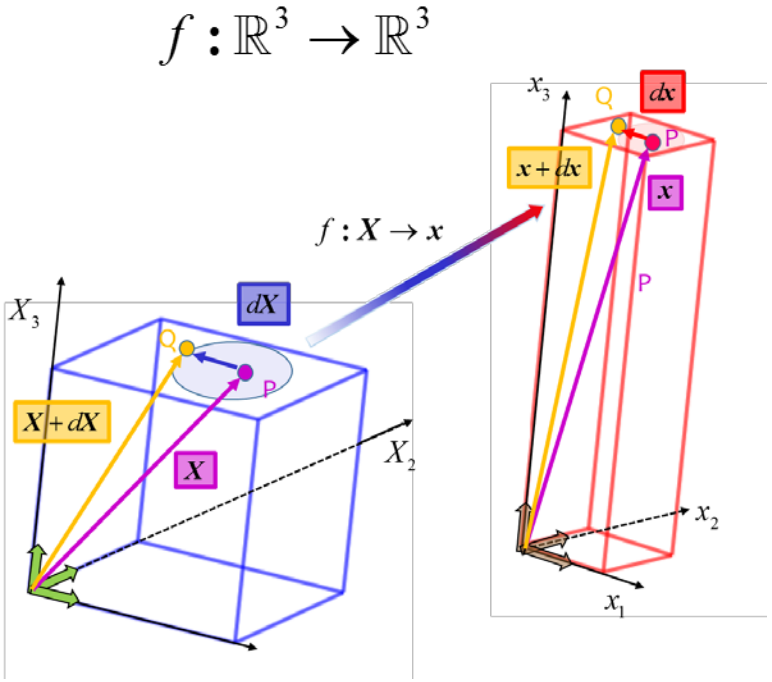


- Reference configuration point P is located by $\boldsymbol{X}$ and Q from infinitesimal neighborhood by $\boldsymbol{X} + d\boldsymbol{X}$

- Deformed configuration point P is located by $\boldsymbol{x}$ and Q from infinitesimal neighborhood by $\boldsymbol{x} + d\boldsymbol{x}$

Deformation gradient $\mathbf{F}$

- Basic and fundamental geometrical quantity to *measure* deformation



- Deformation gradient $\mathbf{F}$ describes infinitesimal change in f by considering infinitesimal vectors $d\boldsymbol{X}$ and $d\boldsymbol{x}$

$$\mathbf{F} : d\boldsymbol{X} \to d\boldsymbol{x}$$

$$d\boldsymbol{x} = \mathbf{F}\, d\boldsymbol{X}$$

Deformation gradient $\mathbf{F}$

$$\mathbf{F} = Grad\left(\boldsymbol{x}\right) = \frac{d\boldsymbol{x}}{d\boldsymbol{X}}$$

$$dx_i \quad i \in \{1,2,3\}$$

$$dX_K \quad K \in \{1,2,3\}$$

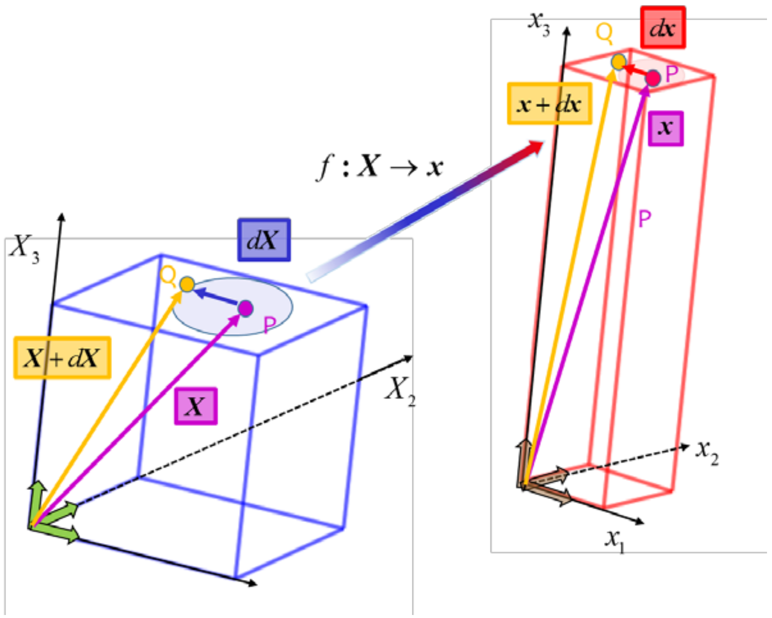
$$\mathbf{F} = Grad\left(\boldsymbol{x}\right) = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \end{pmatrix}$$

$$F_{iK} = \frac{\partial x_i}{\partial X_K} \quad i,K \in \{1,2,3\}$$

Deformation gradient $\mathbf{F}$

$$\mathbf{F} : d\mathbf{X} \rightarrow d\mathbf{x}$$

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$$



$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} \begin{pmatrix} dX_1 \\ dX_2 \\ dX_3 \end{pmatrix}$$

Second-order tensor

$$\mathbf{F} : d\boldsymbol{X} \rightarrow d\boldsymbol{x}$$

$$dx_1\boldsymbol{e}_1 + dx_2\boldsymbol{e}_2 + dx_3\boldsymbol{e}_3 = \mathbf{F}\left\{dX_1\boldsymbol{E}_1 + dX_2\boldsymbol{E}_2 + dX_3\boldsymbol{E}_3\right\}$$

$$d\boldsymbol{x} = \mathbf{F}d\boldsymbol{X}$$

- $\mathbf{F}$ is linear mapping between two vector spaces
- Such a quantity is referred to as **second-order tensor**
- Second-order tensors are represented by **matrices**
- $\mathbf{F}$ is so-called **mixed tensor** because vectors spaces for $\{d\boldsymbol{X}\}$ and $\{d\boldsymbol{x}\}$ has to be mutually distinguished

Consider basis vectors

$$\mathbf{F} : d\boldsymbol{X} \rightarrow d\boldsymbol{x}$$

$$\begin{pmatrix} dx_1\boldsymbol{e}_1 \\ dx_2\boldsymbol{e}_2 \\ dx_3\boldsymbol{e}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1\boldsymbol{e}_1}{\partial X_1\boldsymbol{E}_1} & \frac{\partial x_1\boldsymbol{e}_1}{\partial X_2\boldsymbol{E}_2} & \frac{\partial x_1\boldsymbol{e}_1}{\partial X_3\boldsymbol{E}_3} \\ \frac{\partial x_2\boldsymbol{e}_2}{\partial X_1\boldsymbol{E}_1} & \frac{\partial x_2\boldsymbol{e}_2}{\partial X_2\boldsymbol{E}_2} & \frac{\partial x_2\boldsymbol{e}_2}{\partial X_3\boldsymbol{E}_3} \\ \frac{\partial x_3\boldsymbol{e}_3}{\partial X_1\boldsymbol{E}_1} & \frac{\partial x_3\boldsymbol{e}_3}{\partial X_2\boldsymbol{E}_2} & \frac{\partial x_3\boldsymbol{e}_3}{\partial X_3\boldsymbol{E}_3} \end{pmatrix} \begin{pmatrix} dX_1\boldsymbol{E}_1 \\ dX_2\boldsymbol{E}_2 \\ dX_3\boldsymbol{E}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1\boldsymbol{e}_1}{\partial X_1\boldsymbol{E}_1} dX_1\boldsymbol{E}_1 + \frac{\partial x_1\boldsymbol{e}_1}{\partial X_2\boldsymbol{E}_2} dX_2\boldsymbol{E}_2 + \frac{\partial x_1\boldsymbol{e}_1}{\partial X_3\boldsymbol{E}_3} dX_3\boldsymbol{E}_3 \\ \frac{\partial x_2\boldsymbol{e}_2}{\partial X_1\boldsymbol{E}_1} dX_1\boldsymbol{E}_1 + \frac{\partial x_2\boldsymbol{e}_2}{\partial X_2\boldsymbol{E}_2} dX_2\boldsymbol{E}_2 + \frac{\partial x_2\boldsymbol{e}_2}{\partial X_3\boldsymbol{E}_3} dX_3\boldsymbol{E}_3 \\ \frac{\partial x_3\boldsymbol{e}_3}{\partial X_1\boldsymbol{E}_1} dX_1\boldsymbol{E}_1 + \frac{\partial x_3\boldsymbol{e}_3}{\partial X_2\boldsymbol{E}_2} dX_2\boldsymbol{E}_2 + \frac{\partial x_3\boldsymbol{e}_3}{\partial X_3\boldsymbol{E}_3} dX_3\boldsymbol{E}_3 \end{pmatrix} =$$

Consider basis vectors

$$
=\begin{pmatrix}
\frac{\partial x_1 \mathbf{e}_1}{\partial X_1 \cancel{\mathbf{E}_1}} dX_1 \cancel{\mathbf{E}_1} + \frac{\partial x_1 \mathbf{e}_1}{\partial X_2 \cancel{\mathbf{E}_2}} dX_2 \cancel{\mathbf{E}_2} + \frac{\partial x_1 \mathbf{e}_1}{\partial X_3 \cancel{\mathbf{E}_3}} dX_3 \cancel{\mathbf{E}_3} \\
\frac{\partial x_2 \mathbf{e}_2}{\partial X_1 \cancel{\mathbf{E}_1}} dX_1 \cancel{\mathbf{E}_1} + \frac{\partial x_2 \mathbf{e}_2}{\partial X_2 \cancel{\mathbf{E}_2}} dX_2 \cancel{\mathbf{E}_2} + \frac{\partial x_2 \mathbf{e}_2}{\partial X_3 \cancel{\mathbf{E}_3}} dX_3 \cancel{\mathbf{E}_3} \\
\frac{\partial x_3 \mathbf{e}_3}{\partial X_1 \cancel{\mathbf{E}_1}} dX_1 \cancel{\mathbf{E}_1} + \frac{\partial x_3 \mathbf{e}_3}{\partial X_2 \cancel{\mathbf{E}_2}} dX_2 \cancel{\mathbf{E}_2} + \frac{\partial x_3 \mathbf{e}_3}{\partial X_3 \cancel{\mathbf{E}_3}} dX_3 \cancel{\mathbf{E}_3}
\end{pmatrix}
=\begin{pmatrix}
\frac{\partial x_1 \mathbf{e}_1}{\partial X_1} dX_1 + \frac{\partial x_1 \mathbf{e}_1}{\partial X_2} dX_2 + \frac{\partial x_1 \mathbf{e}_1}{\partial X_3} dX_3 \\
\frac{\partial x_2 \mathbf{e}_2}{\partial X_1} dX_1 + \frac{\partial x_2 \mathbf{e}_2}{\partial X_2} dX_2 + \frac{\partial x_2 \mathbf{e}_2}{\partial X_3} dX_3 \\
\frac{\partial x_3 \mathbf{e}_3}{\partial X_1} dX_1 + \frac{\partial x_3 \mathbf{e}_3}{\partial X_2} dX_2 + \frac{\partial x_3 \mathbf{e}_3}{\partial X_3} dX_3
\end{pmatrix} =
$$

$$
=\begin{pmatrix}
\frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3 \\
\frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3 \\
\frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3
\end{pmatrix}
=\left(\frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3\right)\mathbf{e}_1 + \left(\frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3\right)\mathbf{e}_2 + \left(\frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3\right)\mathbf{e}_3
$$

Consider basis vectors

$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_1 \boldsymbol{E}_1} & \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_2 \boldsymbol{E}_2} & \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_3 \boldsymbol{E}_3} \\ \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_1 \boldsymbol{E}_1} & \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_2 \boldsymbol{E}_2} & \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_3 \boldsymbol{E}_3} \\ \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_1 \boldsymbol{E}_1} & \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_2 \boldsymbol{E}_2} & \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_3 \boldsymbol{E}_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} \boldsymbol{e}_1 \boldsymbol{E}_1 & \frac{\partial x_1}{\partial X_2} \boldsymbol{e}_1 \boldsymbol{E}_2 & \frac{\partial x_1}{\partial X_3} \boldsymbol{e}_1 \boldsymbol{E}_3 \\ \frac{\partial x_2}{\partial X_1} \boldsymbol{e}_2 \boldsymbol{E}_1 & \frac{\partial x_2}{\partial X_2} \boldsymbol{e}_2 \boldsymbol{E}_2 & \frac{\partial x_2}{\partial X_3} \boldsymbol{e}_2 \boldsymbol{E}_3 \\ \frac{\partial x_3}{\partial X_1} \boldsymbol{e}_3 \boldsymbol{E}_1 & \frac{\partial x_3}{\partial X_2} \boldsymbol{e}_3 \boldsymbol{E}_2 & \frac{\partial x_3}{\partial X_3} \boldsymbol{e}_3 \boldsymbol{E}_3 \end{pmatrix} =$$

$$= \frac{\partial x_1}{\partial X_1}\left(\boldsymbol{e}_1 \otimes \boldsymbol{E}_1\right) + \frac{\partial x_1}{\partial X_2}\left(\boldsymbol{e}_1 \otimes \boldsymbol{E}_2\right) + \frac{\partial x_1}{\partial X_3}\left(\boldsymbol{e}_1 \otimes \boldsymbol{E}_3\right) + \frac{\partial x_2}{\partial X_1}\left(\boldsymbol{e}_2 \otimes \boldsymbol{E}_1\right) + \frac{\partial x_2}{\partial X_2}\left(\boldsymbol{e}_2 \otimes \boldsymbol{E}_2\right) +$$

$$+ \frac{\partial x_2}{\partial X_3}\left(\boldsymbol{e}_2 \otimes \boldsymbol{E}_3\right) + \frac{\partial x_3}{\partial X_1}\left(\boldsymbol{e}_3 \otimes \boldsymbol{E}_1\right) + \frac{\partial x_3}{\partial X_2}\left(\boldsymbol{e}_3 \otimes \boldsymbol{E}_2\right) + \frac{\partial x_3}{\partial X_3}\left(\boldsymbol{e}_3 \otimes \boldsymbol{E}_3\right)$$

Consider basis vectors

$$\mathbf{F} = \frac{\partial x_1}{\partial X_1}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_1) + \frac{\partial x_1}{\partial X_2}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_2) + \frac{\partial x_1}{\partial X_3}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_3) + \frac{\partial x_2}{\partial X_1}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_1) + \frac{\partial x_2}{\partial X_2}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_2) +$$

$$+ \frac{\partial x_2}{\partial X_3}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_3) + \frac{\partial x_3}{\partial X_1}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_1) + \frac{\partial x_3}{\partial X_2}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_2) + \frac{\partial x_3}{\partial X_3}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_3)$$

$$\mathbf{F} = F_{11}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_1) + F_{12}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_2) + F_{13}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_3) + F_{21}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_1) + F_{22}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_2) +$$

$$+ F_{23}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_3) + F_{31}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_1) + F_{32}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_2) + F_{33}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_3)$$

Once more some notes on tensors

- Second-order tensors are linear mappings of vector spaces
- Can be mixed or simple/regular

$$\mathbf{A} : \mathcal{V}\left(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\right) \rightarrow \mathcal{V}\left(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\right) \qquad \mathbf{A} = A_{IK} \boldsymbol{E}_I \otimes \boldsymbol{E}_K$$

$$\mathbf{F} : \mathcal{V}\left(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\right) \rightarrow \mathcal{W}\left(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\right) \qquad \mathbf{F} = F_{iK} \boldsymbol{E}_i \otimes \boldsymbol{E}_K$$

Once more some notes on tensors

All properties arise from vector properties...
...are given „by components"

- Tensor (dyadic) product of two vectors is the dyad $\boldsymbol{u}\otimes\boldsymbol{v}$

$$(\boldsymbol{u}\otimes\boldsymbol{v})_{ij} = u_i v_j \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix} \qquad A_{ij} = u_i v_j$$

Once more some notes on tensors

- Dyads of basis vectors can define tensor basis

$$\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = A_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \cdots + A_{23}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + A_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 =$$

$$= A_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A_{12}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + A_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + A_{33}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Once more some notes on tensors

- Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be second-order tensors and $t \in \mathbb{R}$ then

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \qquad A_{ik} + B_{ik} = C_{ik}$$

$$t \cdot \mathbf{A} = \mathbf{C} \qquad t \cdot A_{ik} = C_{ik}$$

$$\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{3} A_{ij} B_{ij} = A_{ij} B_{ij}$$

Inner product (double dot product, analog to scalar product of vectors)

Once more some notes on tensors

- Let **A**, **B**, **C** be second-order tensors, the dot product of tensors or composition of tensors (composition of linear transformations) is defined as ...similarly to matrix multiplication

$$\mathbf{A}\mathbf{B} = \mathbf{C}$$

$$C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} = A_{ik} B_{kj}$$

$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{x} \otimes \boldsymbol{y}) = (\boldsymbol{v} \cdot \boldsymbol{x})(\boldsymbol{u} \otimes \boldsymbol{y}) = (\boldsymbol{u} \otimes \boldsymbol{y})(\boldsymbol{v} \cdot \boldsymbol{x})$$

$$\boldsymbol{v} \cdot \boldsymbol{x} = \sum_{k=1}^{3} v_k x_k = v_k x_k$$

Once more some notes on tensors

- Inverse tensor $\mathbf{A}^{-1}$ and unit tensor $\mathbf{I}$

$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ $\quad I_{ij} = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$

- Trace $tr(\mathbf{A})$ of a tensor $\mathbf{A}$ is a function given by

$$tr(\mathbf{A}) = \sum_{i=1}^{3} A_{ii} = A_{ii}$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

- Transpose $\mathbf{A}^T$ of $\mathbf{A}$ satisfies relations

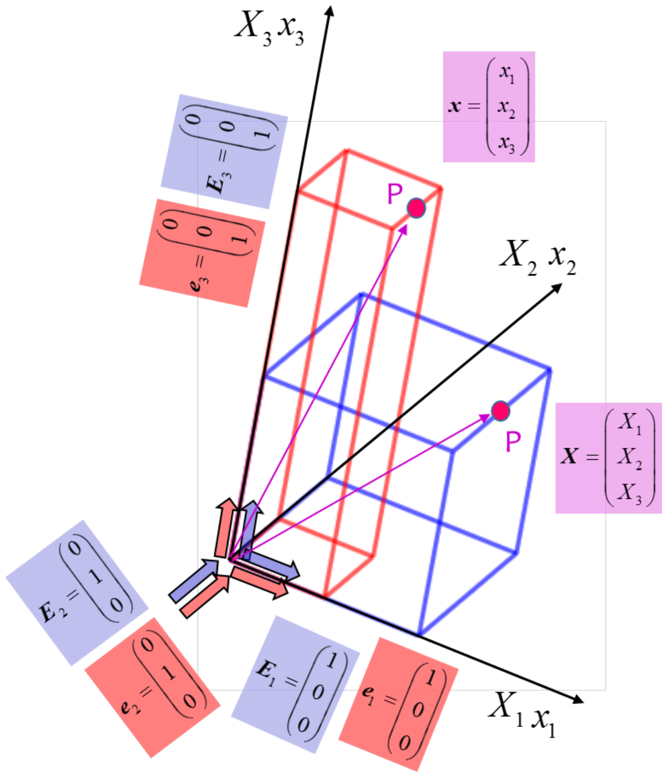
$$(\boldsymbol{u} \otimes \boldsymbol{v})^T = \boldsymbol{v} \otimes \boldsymbol{u} \qquad \boldsymbol{v} \cdot (\mathbf{A}\,\boldsymbol{u}) = \boldsymbol{u} \cdot (\mathbf{A}^T \boldsymbol{v}) \qquad \mathbf{A}\,\boldsymbol{u} = \boldsymbol{u}\,\mathbf{A}^T$$

Back to examples of a motion

$$f : \boldsymbol{X} \rightarrow \boldsymbol{x}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.5X_1 \\ 0.5X_2 \\ 2X_3 \end{pmatrix} = 0.5X_1\boldsymbol{e}_1 + 0.5X_2\boldsymbol{e}_2 + 2X_3\boldsymbol{e}_3$$

$$\mathbf{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} \qquad F_{iK} = \frac{\partial x_i}{\partial X_K}$$



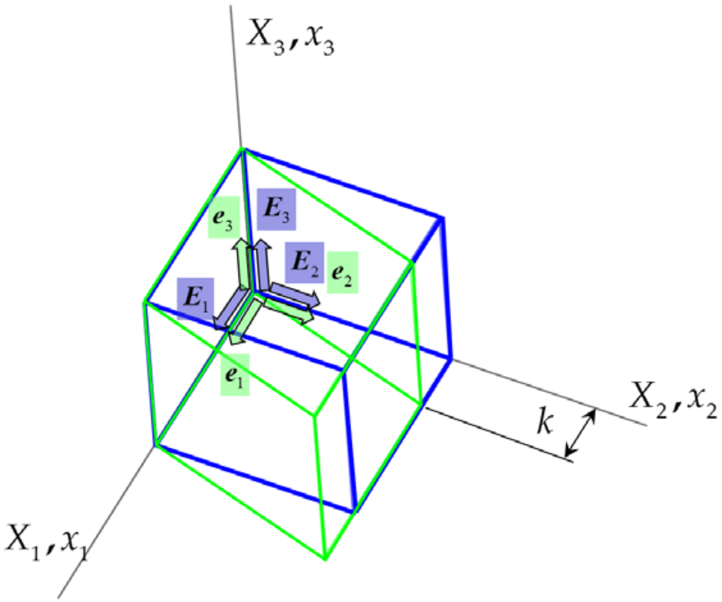
$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Back to examples of a motion

$$f : \boldsymbol{X} \rightarrow \boldsymbol{x}$$

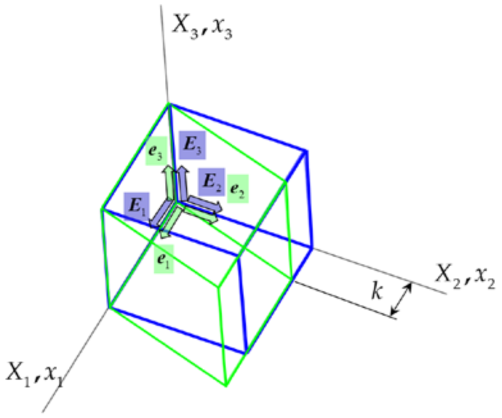
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 + kX_2 \\ X_2 \\ X_3 \end{pmatrix} = \left(X_1 + kX_2 \right)\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3$$

$$\mathbf{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} \qquad F_{iK} = \frac{\partial x_i}{\partial X_K}$$



$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{F}$ is not symmetric



$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Polar decomposition theorem …such $\mathbf{R}$, $\mathbf{U}$, $\mathbf{v}$ always exist
- They are unique
- $\mathbf{U}$ and $\mathbf{v}$ are symmetric positive definite
- $\mathbf{R}$ is proper orthogonal (rotation)

$$\mathbf{F} = \mathbf{RU} = \mathbf{vR}$$

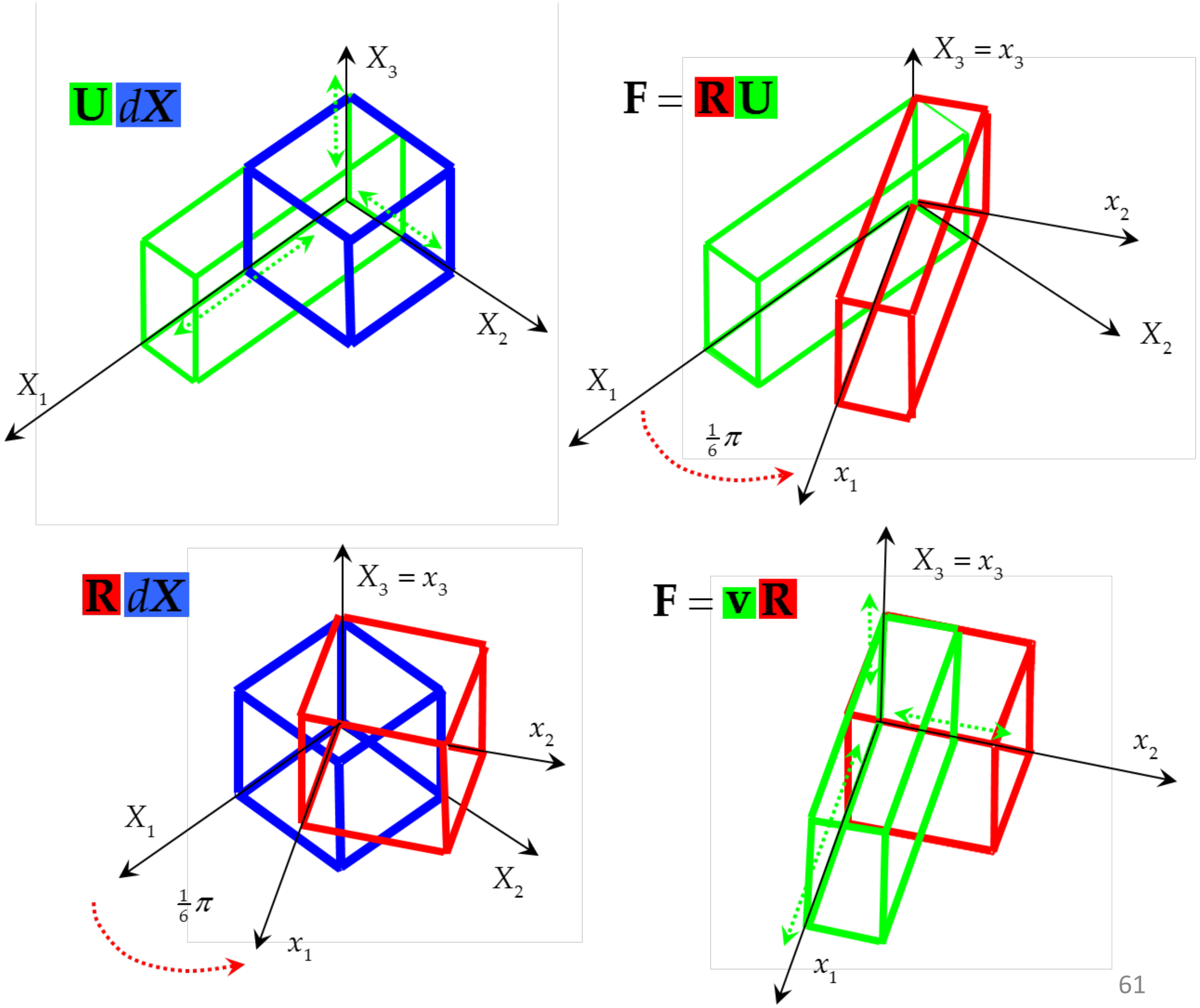
$$F_{iK} = R_{iJ} U_{JK} = v_{im} R_{mK}$$

Polar decomposition: route to deformation

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}$$

$$\underbrace{\begin{pmatrix} \sqrt{3} & -\frac{1}{4} & 0 \\ 1 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{F}} = \underbrace{\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{U}}$$

$$\underbrace{\begin{pmatrix} \sqrt{3} & -\frac{1}{4} & 0 \\ 1 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{F}} = \underbrace{\begin{pmatrix} \frac{13}{8} & \sqrt{3}\frac{3}{8} & 0 \\ \sqrt{3}\frac{3}{8} & \frac{7}{8} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{v}} \underbrace{\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{R}}$$



Polar decomposition: route to deformation

- Right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$

$$\mathbf{C} = \mathbf{F}^T\,\mathbf{F} = \left(\mathbf{RU}\right)^T\,\mathbf{RU} = \mathbf{U}^T\,\mathbf{R}^T\,\mathbf{RU} = \mathbf{U}\,\mathbf{R}^{-1}\,\mathbf{RU} = \mathbf{UIU} = \mathbf{U}^2$$

- Left Cauchy-Green deformation tensor $\mathbf{b} = \mathbf{F}\,\mathbf{F}^T = \mathbf{v}^2$

- Green (Lagrange) deformation tensor $\mathbf{E} = \tfrac{1}{2}\left(\mathbf{C} - \mathbf{I}\right)$

- Almansi (Euler) deformation tensor $\mathbf{e} = \tfrac{1}{2}\left(\mathbf{I} - \mathbf{b}^{-1}\right)$

- Logarithmic (Hencky) deformation tensor $ln\left(\mathbf{U}\right)$

$$\mathbf{C}=\mathbf{F}^T\,\mathbf{F}=\begin{pmatrix} F_{11}\boldsymbol{E}_1\otimes\boldsymbol{e}_1 & F_{21}\boldsymbol{E}_1\otimes\boldsymbol{e}_2 & F_{31}\boldsymbol{E}_1\otimes\boldsymbol{e}_3 \\ F_{12}\boldsymbol{E}_2\otimes\boldsymbol{e}_1 & F_{22}\boldsymbol{E}_2\otimes\boldsymbol{e}_2 & F_{32}\boldsymbol{E}_2\otimes\boldsymbol{e}_3 \\ F_{13}\boldsymbol{E}_3\otimes\boldsymbol{e}_1 & F_{23}\boldsymbol{E}_3\otimes\boldsymbol{e}_2 & F_{33}\boldsymbol{E}_3\otimes\boldsymbol{e}_3 \end{pmatrix}\begin{pmatrix} F_{11}\boldsymbol{e}_1\otimes\boldsymbol{E}_1 & F_{12}\boldsymbol{e}_1\otimes\boldsymbol{E}_2 & F_{13}\boldsymbol{e}_1\otimes\boldsymbol{E}_3 \\ F_{21}\boldsymbol{e}_2\otimes\boldsymbol{E}_1 & F_{22}\boldsymbol{e}_2\otimes\boldsymbol{E}_2 & F_{23}\boldsymbol{e}_2\otimes\boldsymbol{E}_3 \\ F_{31}\boldsymbol{e}_3\otimes\boldsymbol{E}_1 & F_{32}\boldsymbol{e}_3\otimes\boldsymbol{E}_2 & F_{33}\boldsymbol{e}_3\otimes\boldsymbol{E}_3 \end{pmatrix}=$$

- **C is symmetric**

$$C_{11}\sim \begin{aligned} & F_{11}F_{11}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_1\right)\cdot\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_1\right)+F_{21}F_{21}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_2\right)\cdot\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_1\right)+F_{31}F_{31}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_3\right)\cdot\left(\boldsymbol{e}_3\otimes\boldsymbol{E}_1\right)= \\ &=F_{11}F_{11}\boldsymbol{E}_1\otimes\boldsymbol{E}_1+F_{21}F_{21}\boldsymbol{E}_1\otimes\boldsymbol{E}_1+F_{31}F_{31}\boldsymbol{E}_1\otimes\boldsymbol{E}_1=\left(F_{11}F_{11}+F_{21}F_{21}+F_{31}F_{31}\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_1 \end{aligned}$$

$$C_{12}\sim \begin{aligned} & F_{11}F_{12}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_1\right)\cdot\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_2\right)+F_{21}F_{22}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_2\right)\cdot\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_2\right)+F_{31}F_{32}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_3\right)\cdot\left(\boldsymbol{e}_3\otimes\boldsymbol{E}_2\right)= \\ &=F_{11}F_{12}\boldsymbol{E}_1\otimes\boldsymbol{E}_2+F_{21}F_{22}\boldsymbol{E}_1\otimes\boldsymbol{E}_2+F_{31}F_{32}\boldsymbol{E}_1\otimes\boldsymbol{E}_2=\left(F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32}\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_2 \end{aligned}$$

$$C_{21}\sim \begin{aligned} & F_{12}F_{11}\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_1\right)\cdot\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_1\right)+F_{22}F_{21}\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_2\right)\cdot\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_1\right)+F_{32}F_{31}\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_3\right)\cdot\left(\boldsymbol{e}_3\otimes\boldsymbol{E}_1\right)= \\ &=F_{12}F_{11}\boldsymbol{E}_2\otimes\boldsymbol{E}_1+F_{22}F_{21}\boldsymbol{E}_2\otimes\boldsymbol{E}_1+F_{32}F_{31}\boldsymbol{E}_2\otimes\boldsymbol{E}_1=\left(F_{12}F_{11}+F_{22}F_{21}+F_{32}F_{31}\right)\boldsymbol{E}_2\otimes\boldsymbol{E}_1 \end{aligned}$$

$$=\begin{pmatrix} F_{11}^2+F_{21}^2+F_{31}^2 & F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32} & F_{11}F_{13}+F_{21}F_{23}+F_{31}F_{33} \\ F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32} & F_{12}^2+F_{22}^2+F_{32}^2 & F_{12}F_{13}+F_{22}F_{23}+F_{32}F_{33} \\ F_{11}F_{13}+F_{21}F_{23}+F_{31}F_{33} & F_{12}F_{13}+F_{22}F_{23}+F_{32}F_{33} & F_{13}^2+F_{23}^2+F_{33}^2 \end{pmatrix}=$$

$$\begin{aligned} &=\left(F_{11}^2+F_{21}^2+F_{31}^2\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_1+\left(F_{12}^2+F_{22}^2+F_{32}^2\right)\boldsymbol{E}_2\otimes\boldsymbol{E}_2+ \\ &+\left(F_{13}^2+F_{23}^2+F_{33}^2\right)\boldsymbol{E}_3\otimes\boldsymbol{E}_3+2\left(F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32}\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_2+ \\ &+2\left(F_{12}F_{13}+F_{22}F_{23}+F_{32}F_{33}\right)\boldsymbol{E}_2\otimes\boldsymbol{E}_3+2\left(F_{11}F_{13}+F_{21}F_{23}+F_{31}F_{33}\right)\boldsymbol{E}_3\otimes\boldsymbol{E}_1=\mathbf{C} \end{aligned}$$

- **b** is symmetric

$$\mathbf{b}=\mathbf{F}\,\mathbf{F}^{T}=\begin{pmatrix} F_{11}\boldsymbol{e}_1\otimes\boldsymbol{E}_1 & F_{12}\boldsymbol{e}_1\otimes\boldsymbol{E}_2 & F_{13}\boldsymbol{e}_1\otimes\boldsymbol{E}_3\\ F_{21}\boldsymbol{e}_2\otimes\boldsymbol{E}_1 & F_{22}\boldsymbol{e}_2\otimes\boldsymbol{E}_2 & F_{23}\boldsymbol{e}_2\otimes\boldsymbol{E}_3\\ F_{31}\boldsymbol{e}_3\otimes\boldsymbol{E}_1 & F_{32}\boldsymbol{e}_3\otimes\boldsymbol{E}_2 & F_{33}\boldsymbol{e}_3\otimes\boldsymbol{E}_3 \end{pmatrix}\begin{pmatrix} F_{11}\boldsymbol{E}_1\otimes\boldsymbol{e}_1 & F_{21}\boldsymbol{E}_1\otimes\boldsymbol{e}_2 & F_{31}\boldsymbol{E}_1\otimes\boldsymbol{e}_3\\ F_{12}\boldsymbol{E}_2\otimes\boldsymbol{e}_1 & F_{22}\boldsymbol{E}_2\otimes\boldsymbol{e}_2 & F_{32}\boldsymbol{E}_2\otimes\boldsymbol{e}_3\\ F_{13}\boldsymbol{E}_3\otimes\boldsymbol{e}_1 & F_{23}\boldsymbol{E}_3\otimes\boldsymbol{e}_2 & F_{33}\boldsymbol{E}_3\otimes\boldsymbol{e}_3 \end{pmatrix}$$

$$b_{11}=F_{11}F_{11}\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_1\right)\cdot\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_1\right)+F_{12}F_{12}\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_2\right)\cdot\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_1\right)+F_{13}F_{13}\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_3\right)\cdot\left(\boldsymbol{E}_3\otimes\boldsymbol{e}_1\right)=$$

$$=F_{11}F_{11}\boldsymbol{e}_1\otimes\boldsymbol{e}_1+F_{12}F_{12}\boldsymbol{e}_1\otimes\boldsymbol{e}_1+F_{13}F_{13}\boldsymbol{e}_1\otimes\boldsymbol{e}_1=\left(F_{11}F_{11}+F_{12}F_{12}+F_{13}F_{13}\right)\boldsymbol{e}_1\otimes\boldsymbol{e}_1$$

$$b_{12}=F_{11}F_{21}\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_1\right)\cdot\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_2\right)+F_{12}F_{22}\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_2\right)\cdot\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_2\right)+F_{13}F_{23}\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_3\right)\cdot\left(\boldsymbol{E}_3\otimes\boldsymbol{e}_2\right)=$$

$$=F_{11}F_{21}\boldsymbol{e}_1\otimes\boldsymbol{e}_2+F_{12}F_{22}\boldsymbol{e}_1\otimes\boldsymbol{e}_2+F_{13}F_{23}\boldsymbol{e}_1\otimes\boldsymbol{e}_2=\left(F_{11}F_{21}+F_{12}F_{22}+F_{13}F_{23}\right)\boldsymbol{e}_1\otimes\boldsymbol{e}_2$$

$$b_{21}=F_{21}F_{11}\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_1\right)\cdot\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_1\right)+F_{22}F_{12}\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_2\right)\cdot\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_1\right)+F_{23}F_{13}\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_3\right)\cdot\left(\boldsymbol{E}_3\otimes\boldsymbol{e}_1\right)=$$

$$=F_{21}F_{11}\boldsymbol{e}_2\otimes\boldsymbol{e}_1+F_{22}F_{12}\boldsymbol{e}_2\otimes\boldsymbol{e}_1+F_{23}F_{13}\boldsymbol{e}_2\otimes\boldsymbol{e}_1=\left(F_{21}F_{11}+F_{12}F_{22}+F_{13}F_{23}\right)\boldsymbol{e}_2\otimes\boldsymbol{e}_1$$

$$=\left(F_{11}^2+F_{12}^2+F_{13}^2\right)\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\left(F_{21}^2+F_{22}^2+F_{23}^2\right)\boldsymbol{e}_2\otimes\boldsymbol{e}_2+\left(F_{31}^2+F_{32}^2+F_{33}^2\right)\boldsymbol{e}_3\otimes\boldsymbol{e}_3+2\left(F_{11}F_{21}+F_{12}F_{22}+F_{13}F_{23}\right)\boldsymbol{e}_1\otimes\boldsymbol{e}_2$$

$$+2\left(F_{21}F_{31}+F_{22}F_{32}+F_{23}F_{33}\right)\boldsymbol{e}_2\otimes\boldsymbol{e}_3+2\left(F_{11}F_{31}+F_{12}F_{32}+F_{13}F_{33}\right)\boldsymbol{e}_3\otimes\boldsymbol{e}_1=\mathbf{b}$$

Eigenvalues and eigenvectors

- Tensors $\mathbf{U}$ and $\mathbf{v}$ (and $\mathbf{C}$ and $\mathbf{b}$) have the same eigenvalues λ_i (and λ_i^2) i =1,2,3
- λ_i is referred to as principal stretch
- Tensors $\mathbf{U}$ and $\mathbf{C}$ (and $\mathbf{V}$ and $\mathbf{b}$) have the same eigenvectors $\boldsymbol{N}_I$ (and $\boldsymbol{n}_i$) i,I =1,2,3
- It results in so-called spectral decomposition

Eigenvalue problem

$$\mathbf{A}\boldsymbol{u} = \lambda \boldsymbol{u}$$

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{u} = \boldsymbol{0}$$

$$det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$$\mathbf{C} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} = \lambda_1^2 \boldsymbol{N}_1 \otimes \boldsymbol{N}_1 + \lambda_2^2 \boldsymbol{N}_2 \otimes \boldsymbol{N}_2 + \lambda_3^2 \boldsymbol{N}_3 \otimes \boldsymbol{N}_3 \neq \lambda_1^2 \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \lambda_2^2 \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + \lambda_3^2 \boldsymbol{n}_3 \otimes \boldsymbol{n}_3 = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} = \mathbf{b}$$

Eigenvalue problem and invariants

$$\mathbf{A}\boldsymbol{u} = \lambda \boldsymbol{u} \qquad \left\{ P + t\boldsymbol{u}, t \in \mathbb{R} \right\}$$

$$det\left(\mathbf{A} - \lambda \mathbf{I}\right) = 0$$

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

$$I_1 = tr\left(\mathbf{A}\right) = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \frac{1}{2}\left(\left(tr\left(\mathbf{A}\right)\right)^2 - tr\left(\mathbf{A}^2\right)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$I_3 = det\left(\mathbf{A}\right) = \lambda_1\lambda_2\lambda_3$$

$$I_1^{\mathbf{C}} = tr\left(\mathbf{C}\right) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2^{\mathbf{C}} = \frac{1}{2}\left(\left(tr\left(\mathbf{C}\right)\right)^2 - tr\left(\mathbf{C}^2\right)\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$

$$I_3^{\mathbf{C}} = det\left(\mathbf{C}\right) = \lambda_1^2\lambda_2^2\lambda_3^2$$

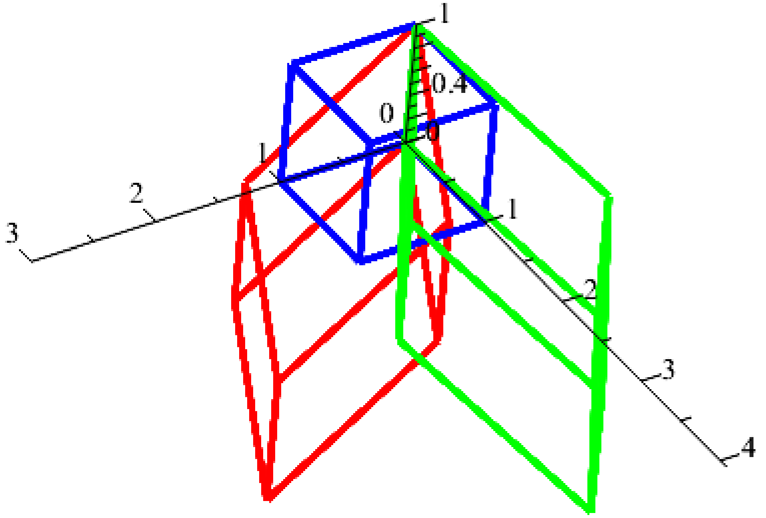
Eigenvalues and eigenvectors

$$x_1 = \left(\frac{\sqrt{3}}{2} - 1\right) X_1 + \left(\sqrt{3} - \frac{1}{2}\right) X_2$$

$$x_2 = \left(\sqrt{3} + \frac{1}{2}\right) X_1 + \left(\frac{\sqrt{3}}{2} + 1\right) X_2$$

$$x_3 = \qquad X_3$$

$$\mathbf{F} = \begin{pmatrix} \frac{\sqrt{3}}{2} - 1 & \sqrt{3} - \frac{1}{2} & 0 \\ \sqrt{3} + \frac{1}{2} & \frac{\sqrt{3}}{2} + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

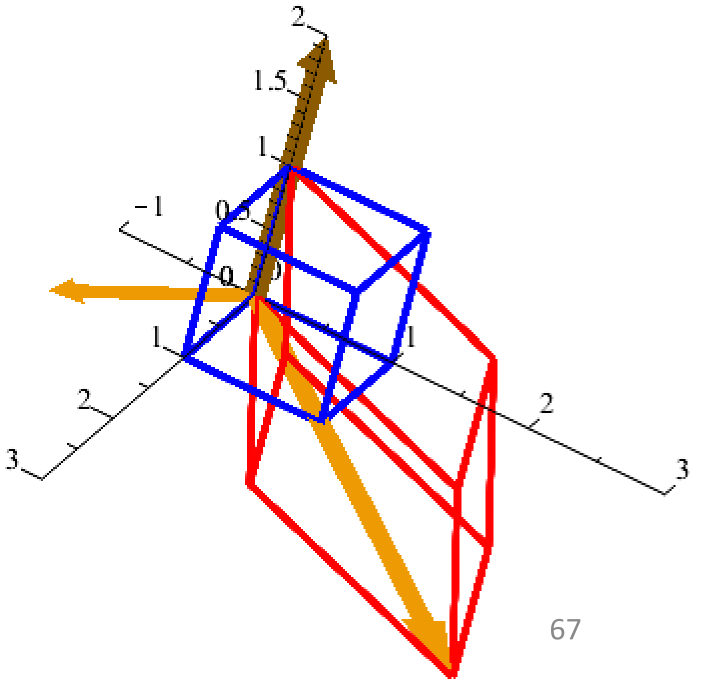


$$\mathbf{F} = \mathbf{R}\mathbf{U} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(30°) & -\sin(30°) & 0 \\ \sin(30°) & \cos(30°) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_1^{\mathbf{U}} = 3 \qquad \boldsymbol{N}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2^{\mathbf{U}} = 1 \qquad \boldsymbol{N}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3^{\mathbf{U}} = -1 \qquad \boldsymbol{N}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$



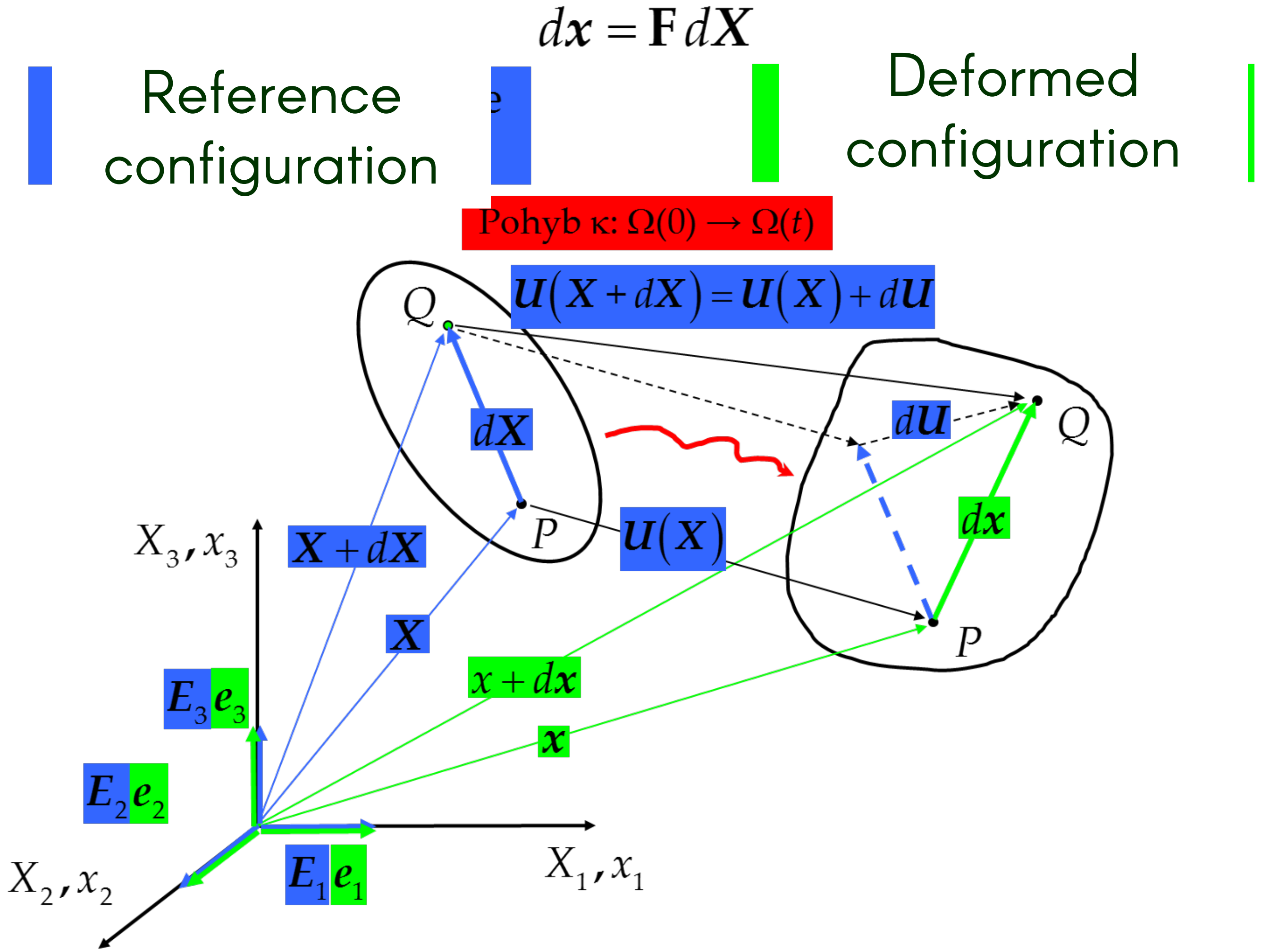
Linearization ≡ connection classical elasticity

- Displacement vector $\boldsymbol{U}$

$$\boldsymbol{U} = \boldsymbol{x} - \boldsymbol{X}$$

$$\nabla_X \boldsymbol{U} = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} - \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{X}} = \mathbf{F} - \mathbf{I}$$

$$\mathbf{F} = \nabla_X \boldsymbol{U} + \mathbf{I}$$



Linearization ≡ connection classical elasticity

- As a consequence of $\boldsymbol{U} = \boldsymbol{x} - \boldsymbol{X} \quad \Rightarrow \quad \mathbf{F} = \mathbf{I} + \nabla_X \boldsymbol{U}$

  we have

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\left(\mathbf{I} + \nabla_X \boldsymbol{U}\right)^T \left(\mathbf{I} + \nabla_X \boldsymbol{U}\right) - \mathbf{I}\right) = \frac{1}{2}\left(\left(\mathbf{I}^T + \left(\nabla_X \boldsymbol{U}\right)^T\right)\left(\mathbf{I} + \nabla_X \boldsymbol{U}\right) - \mathbf{I}\right) =$$

$$\frac{1}{2}\left(\mathbf{I}^T\mathbf{I} + \nabla_X \boldsymbol{U} + \left(\nabla_X \boldsymbol{U}\right)^T + \left(\nabla_X \boldsymbol{U}\right)^T \nabla_X \boldsymbol{U} - \mathbf{I}\right) = \frac{1}{2}\left(\nabla_X \boldsymbol{U} + \left(\nabla_X \boldsymbol{U}\right)^T + \left(\nabla_X \boldsymbol{U}\right)^T \nabla_X \boldsymbol{U}\right)$$

$$E_{IK} = \frac{1}{2}\left(\frac{\partial U_I}{\partial X_K} + \frac{\partial U_K}{\partial X_I} + \frac{\partial U_J}{\partial X_I}\frac{\partial U_J}{\partial X_K}\right)$$

Linearized Green tensor = Engineering strain

$$E_{IK} = \frac{1}{2}\left(\frac{\partial U_I}{\partial X_K} + \frac{\partial U_K}{\partial X_I} + \boxed{\frac{\partial U_J}{\partial X_I}\frac{\partial U_J}{\partial X_K}}\right) \doteq \varepsilon_{IK} = \frac{1}{2}\left(\frac{\partial U_I}{\partial X_K} + \frac{\partial U_K}{\partial X_I}\right)$$

$$E_{11} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_1} + \frac{\partial U_1}{\partial X_1} + \frac{\partial U_1}{\partial X_1}\frac{\partial U_1}{\partial X_1} + \frac{\partial U_2}{\partial X_1}\frac{\partial U_2}{\partial X_1} + \frac{\partial U_3}{\partial X_1}\frac{\partial U_3}{\partial X_1}\right)$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1} + \frac{\partial U_1}{\partial X_1}\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1}\frac{\partial U_2}{\partial X_2} + \frac{\partial U_3}{\partial X_1}\frac{\partial U_3}{\partial X_2}\right)$$

$$E_{13} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_3} + \frac{\partial U_3}{\partial X_1} + \frac{\partial U_1}{\partial X_1}\frac{\partial U_1}{\partial X_3} + \frac{\partial U_2}{\partial X_1}\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_1}\frac{\partial U_3}{\partial X_3}\right)$$

$$E_{22} = \frac{1}{2}\left(\frac{\partial U_2}{\partial X_2} + \frac{\partial U_2}{\partial X_2} + \frac{\partial U_1}{\partial X_2}\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_2}\frac{\partial U_2}{\partial X_2} + \frac{\partial U_3}{\partial X_2}\frac{\partial U_3}{\partial X_2}\right)$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_2} + \frac{\partial U_1}{\partial X_2}\frac{\partial U_1}{\partial X_3} + \frac{\partial U_2}{\partial X_2}\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_2}\frac{\partial U_3}{\partial X_3}\right)$$

$$E_{33} = \frac{1}{2}\left(\frac{\partial U_3}{\partial X_3} + \frac{\partial U_3}{\partial X_3} + \frac{\partial U_1}{\partial X_3}\frac{\partial U_1}{\partial X_3} + \frac{\partial U_2}{\partial X_3}\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_3}\frac{\partial U_3}{\partial X_3}\right)$$

$$\doteq$$

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_1} + \frac{\partial U_1}{\partial X_1}\right)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1}\right)$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_3} + \frac{\partial U_3}{\partial X_1}\right)$$

$$\varepsilon_{22} = \frac{1}{2}\left(\frac{\partial U_2}{\partial X_2} + \frac{\partial U_2}{\partial X_2}\right)$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_2}\right)$$

$$\varepsilon_{33} = \frac{1}{2}\left(\frac{\partial U_3}{\partial X_3} + \frac{\partial U_3}{\partial X_3}\right)$$

Linearized Euler tensor = Engineering strain

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}(\boldsymbol{x}) \qquad \nabla_x \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}} - \frac{\partial \boldsymbol{X}}{\partial \mathrm{x}} = \mathbf{I} - \mathbf{F}^{-1} \quad \Rightarrow \quad \mathbf{F}^{-1} = \mathbf{I} - \nabla_x \boldsymbol{u}$$

- For Almansi (Euler) strain tensor we can write

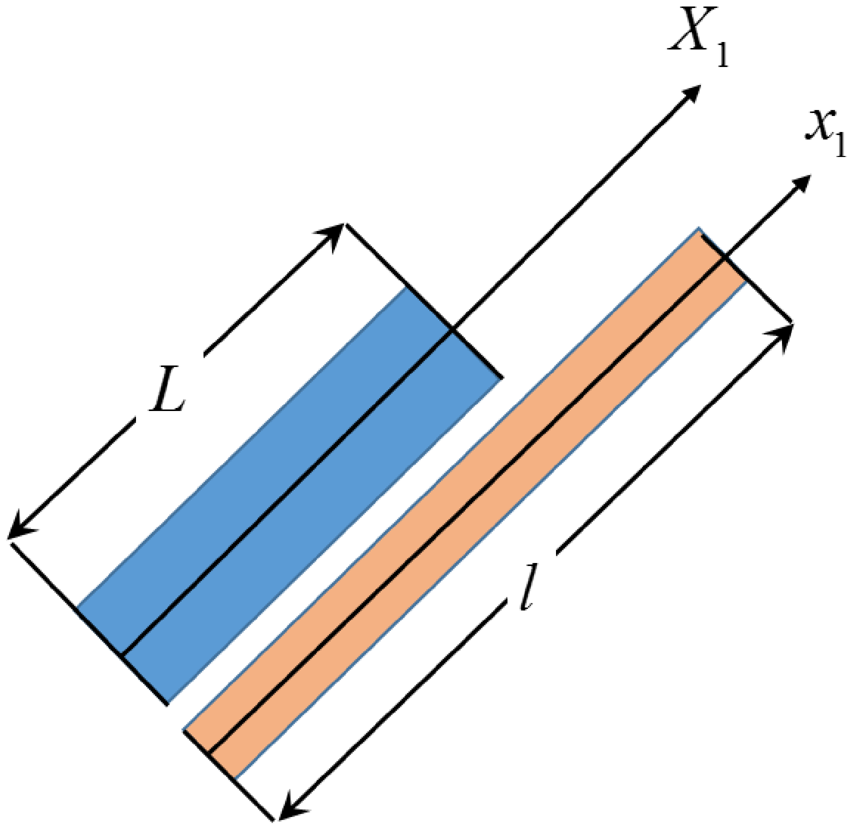
$$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - (\mathbf{I} - \nabla_x \boldsymbol{u})^T (\mathbf{I} - \nabla_x \boldsymbol{u})\right) = \frac{1}{2}\left(\mathbf{I} - \left(\mathbf{I}^T - (\nabla_x \boldsymbol{u})^T\right)(\mathbf{I} - \nabla_x \boldsymbol{u})\right) =$$

$$\frac{1}{2}\left(\mathbf{I} - \mathbf{I}^T\mathbf{I} + \nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T - (\nabla_x \boldsymbol{u})^T \nabla_x \boldsymbol{u}\right) = \frac{1}{2}\left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T - (\nabla_x \boldsymbol{u})^T \nabla_x \boldsymbol{u}\right)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \boxed{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}\right) \doteq \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Too many deformation measures

- Compare them in one component expressing simple tension

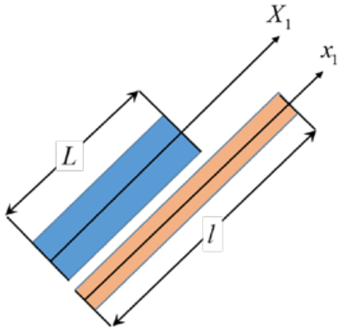


$$x_1 = \lambda_1 X_1$$

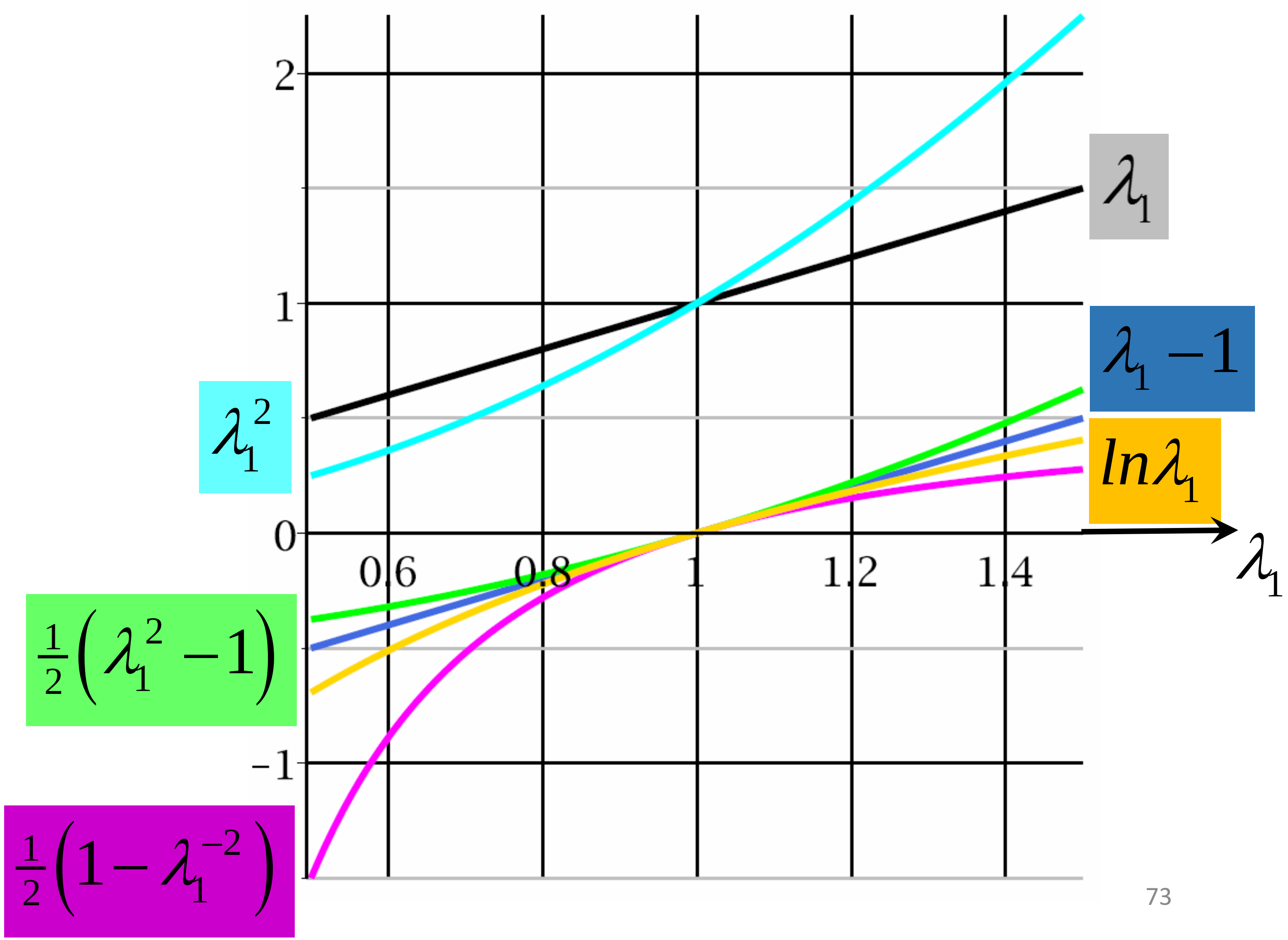
$$F_{11} = \lambda_1 = \frac{\partial x_1}{\partial X_1} = \frac{l}{L}$$

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Comparison of numerical values



$\mathbf{F}$	$F_{11} = \lambda_1$
$\mathbf{C} = \mathbf{F}^T\,\mathbf{F}$	$C_{11} = \lambda_1^2$
$\mathbf{U} = \sqrt{\mathbf{C}}$	$U_{11} = \lambda_1$
$\mathbf{b} = \mathbf{F}\,\mathbf{F}^T$	$b_{11} = \lambda_1^2$
$\mathbf{v} = \sqrt{\mathbf{b}}$	$v_{11} = \lambda_1$
$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right)$	$E_{11} = \frac{1}{2}\left(\lambda_1^2 - 1\right)$
$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{b}^{-1}\right)$	$e_{11} = \frac{1}{2}\left(1 - \lambda_1^{-2}\right)$
$ln\,\mathbf{U}$	$lnU_{11} = ln\lambda_1$
$\boldsymbol{\varepsilon}$	$\varepsilon_{11} = \lambda_1 - 1$

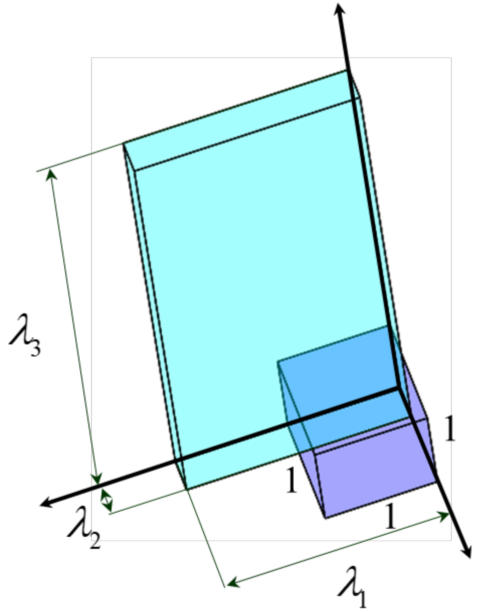


Deformation of line, area, volume element

- Line element is $d\boldsymbol{X}$, thus its change is given directly by $\mathbf{F}$

$$d\boldsymbol{x} = \mathbf{F}\, d\boldsymbol{X}$$

- A change of the volume is also clear and left without exact proof



$$J = \frac{dv}{dV} = \lambda_1 \lambda_2 \lambda_3 = det(\mathbf{F}) = I_3^{\mathbf{U}} = \sqrt{I_3^{\mathbf{C}}}$$

If $J = 1$, the material and the deformation is said to be incompressible (isochoric, isovolumic)

Deformation of line, area, volume element

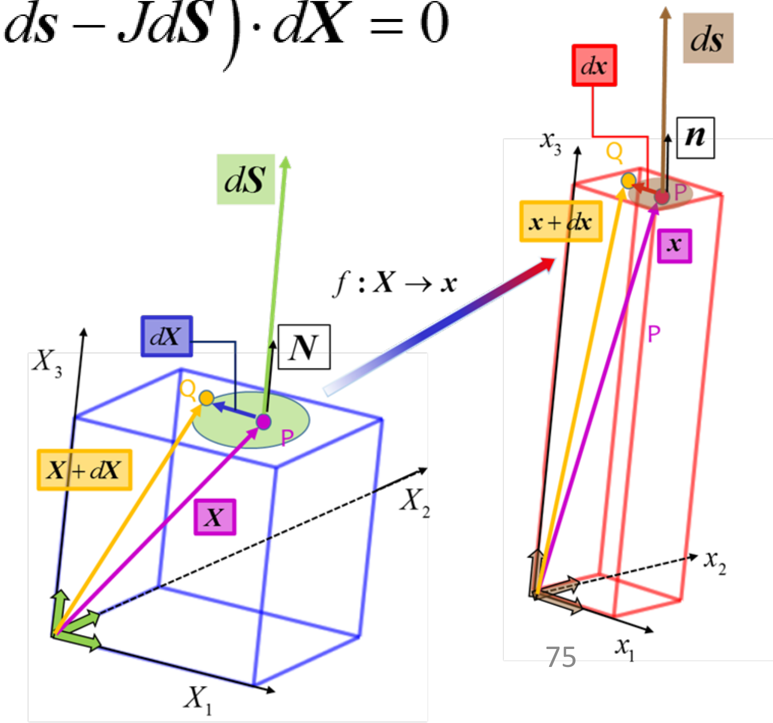
- A little math is necessary to derive the rule for are, so-called Nanson formula

$$d\boldsymbol{s} = J\,\mathbf{F}^{-T}\,d\boldsymbol{S}$$

$$dv = JdV \quad \Rightarrow \quad d\boldsymbol{s}\cdot d\boldsymbol{x} = Jd\boldsymbol{S}\cdot d\boldsymbol{X} \quad \Rightarrow \quad d\boldsymbol{s}\cdot\mathbf{F}\,d\boldsymbol{X} = Jd\boldsymbol{S}\cdot d\boldsymbol{X} \quad \Rightarrow$$

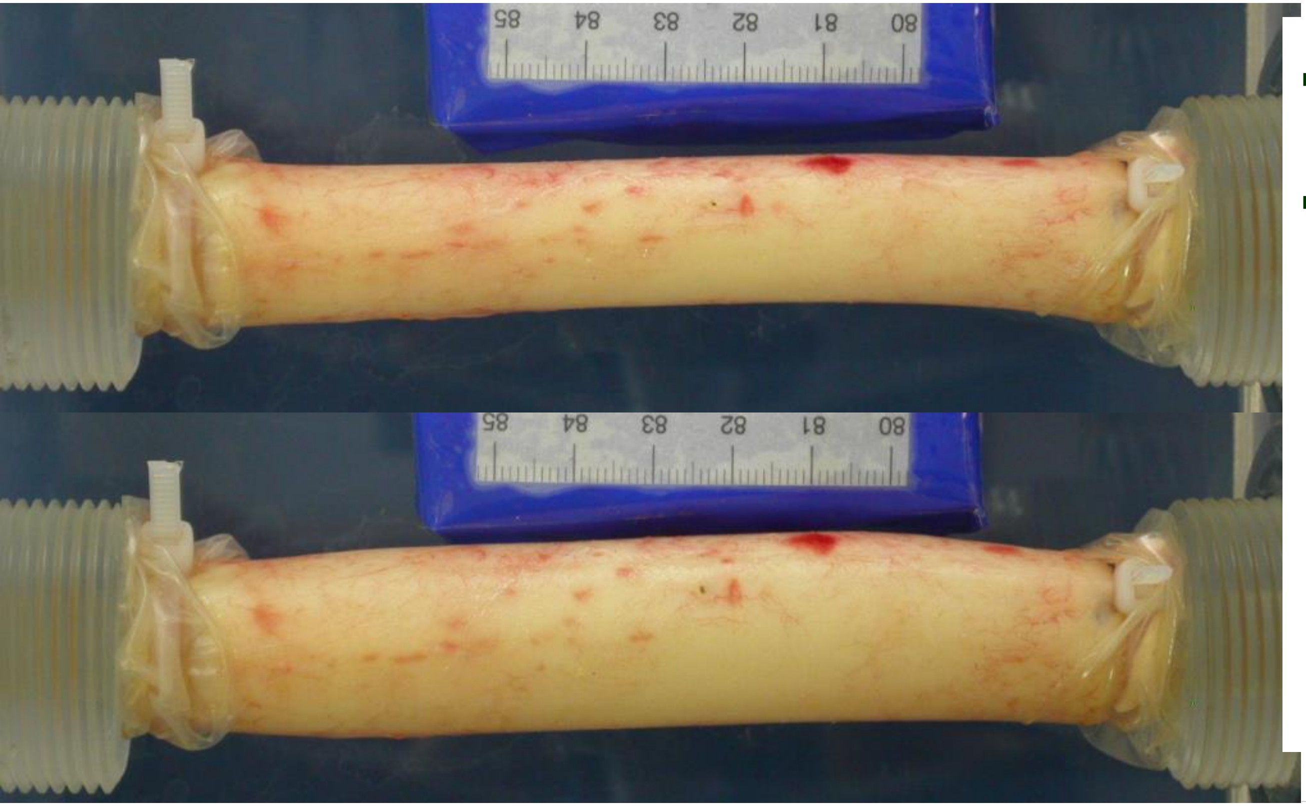
$$d\boldsymbol{s}\cdot\mathbf{F}\,d\boldsymbol{X} - Jd\boldsymbol{S}\cdot d\boldsymbol{X} = 0 \quad \Rightarrow \quad \mathbf{F}^{T}\,d\boldsymbol{s}\cdot d\boldsymbol{X} - Jd\boldsymbol{S}\cdot d\boldsymbol{X} = 0 \quad \Rightarrow \quad \left(\mathbf{F}^{T}\,d\boldsymbol{s} - Jd\boldsymbol{S}\right)\cdot d\boldsymbol{X} = 0$$

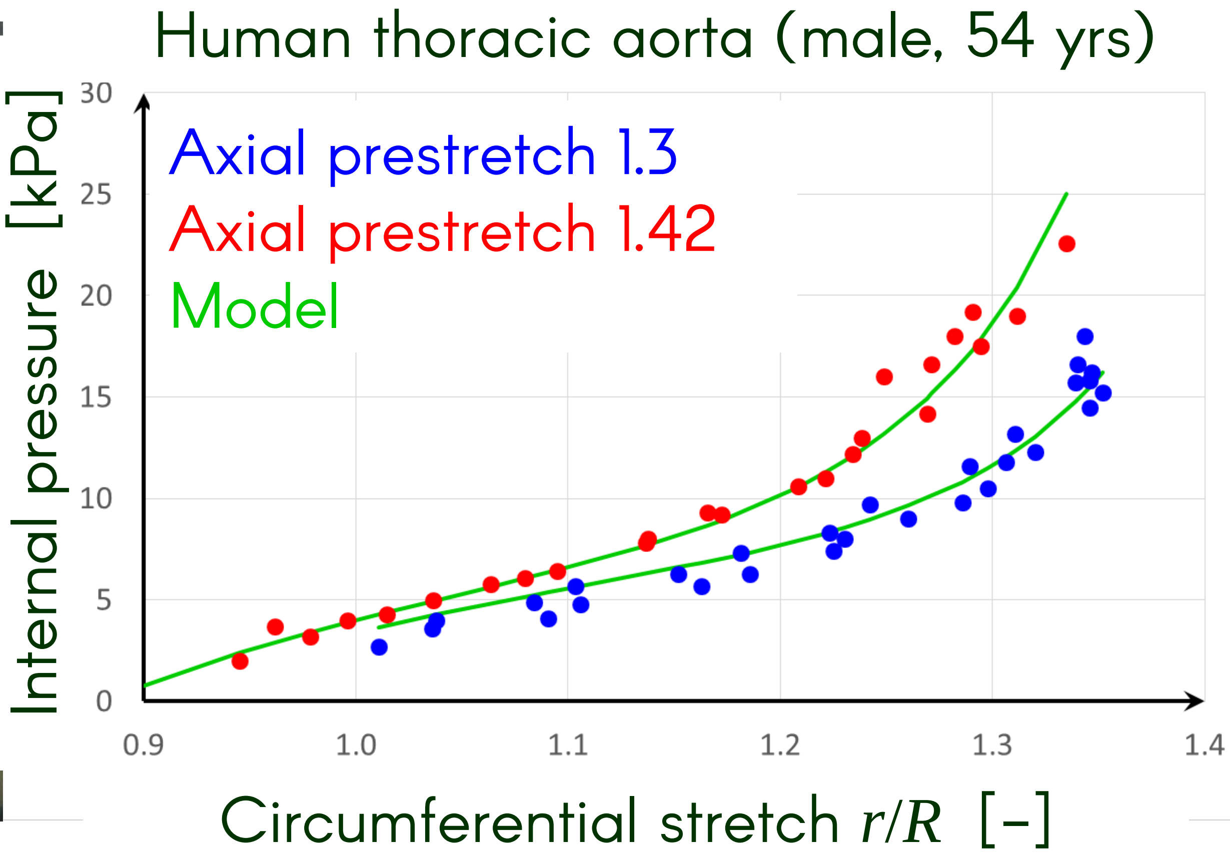
$$\Rightarrow \quad \mathbf{F}^{T}\,d\boldsymbol{s} - Jd\boldsymbol{S} = 0 \quad \Rightarrow \quad d\boldsymbol{s} = J\,\mathbf{F}^{-T}\,d\boldsymbol{S}$$



Where we can see large deformations?

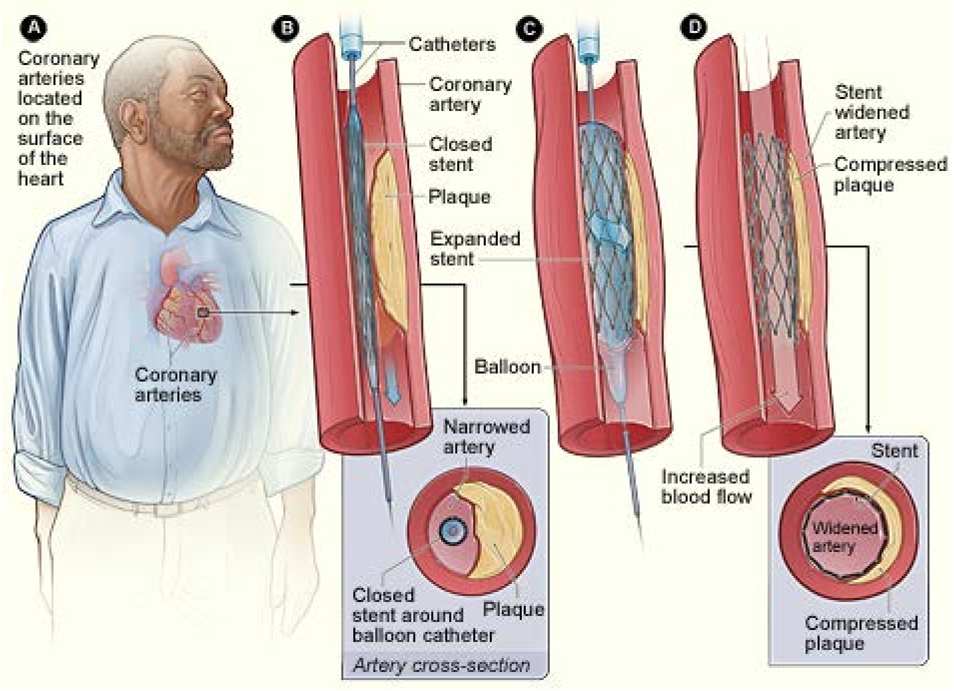
- Ex vivo inflation of the human aorta

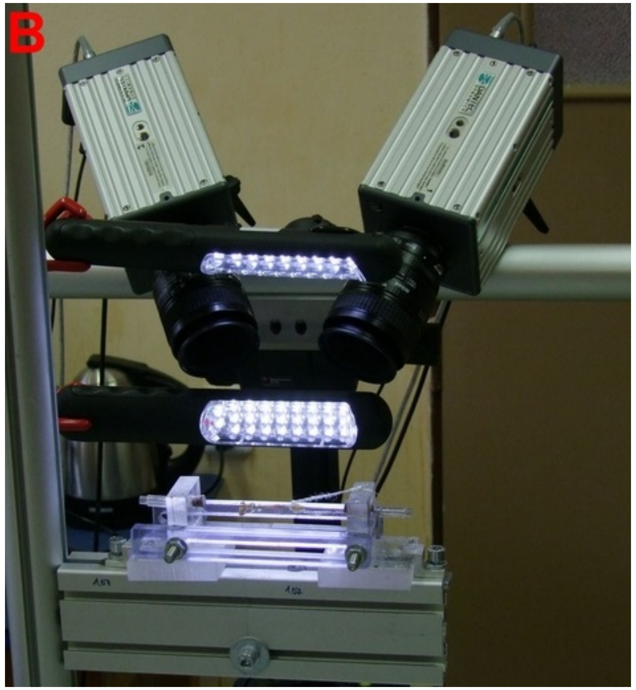


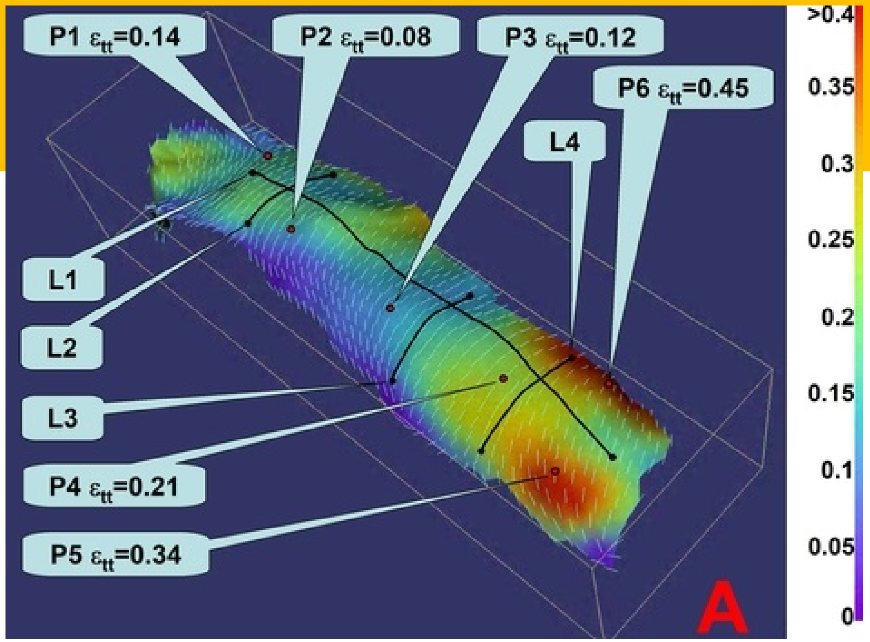


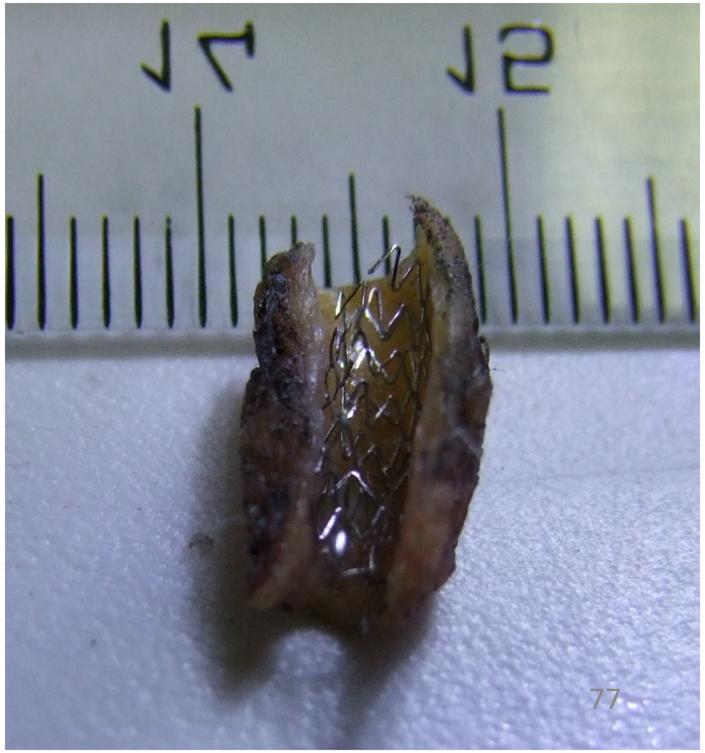
Finite deformations

- Stent implantation



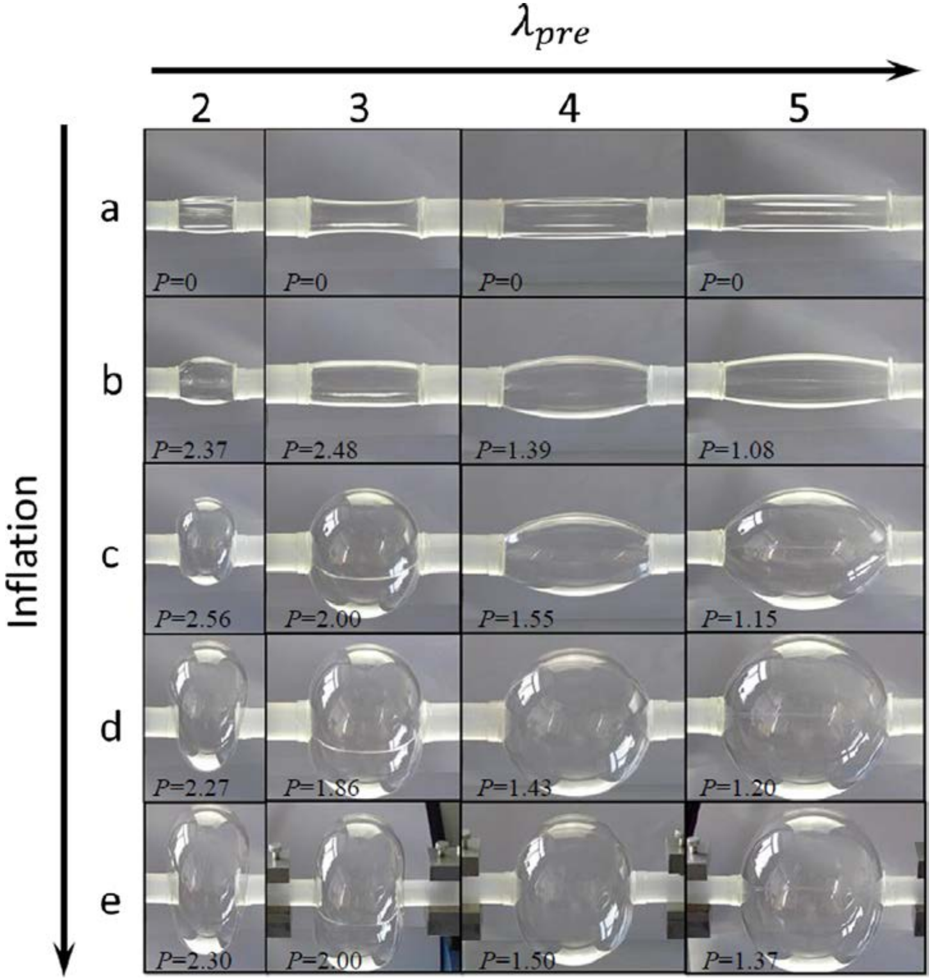


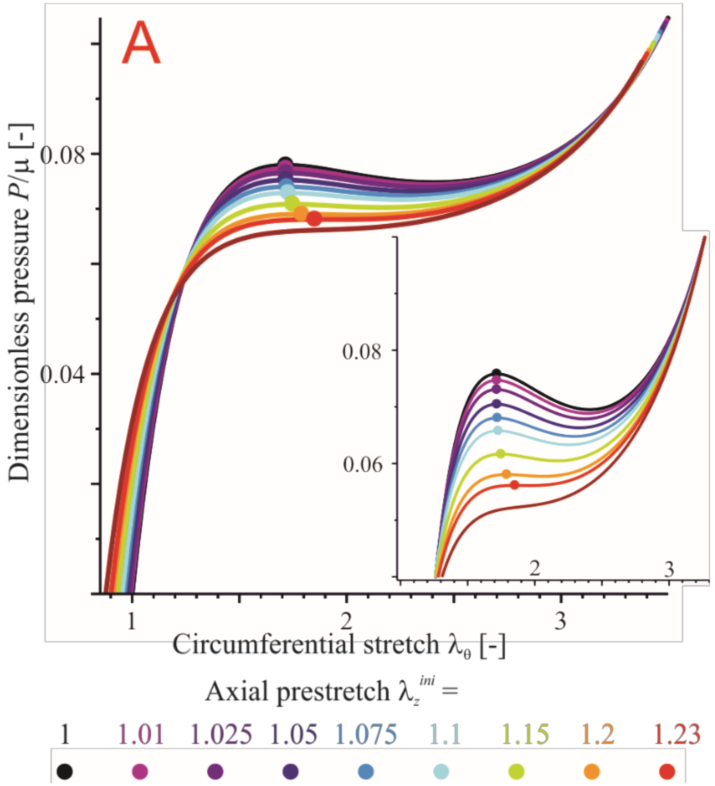




Finite deformations

- Nonlinearity allows an instability to occur
  Inflation of thin elastomer tube

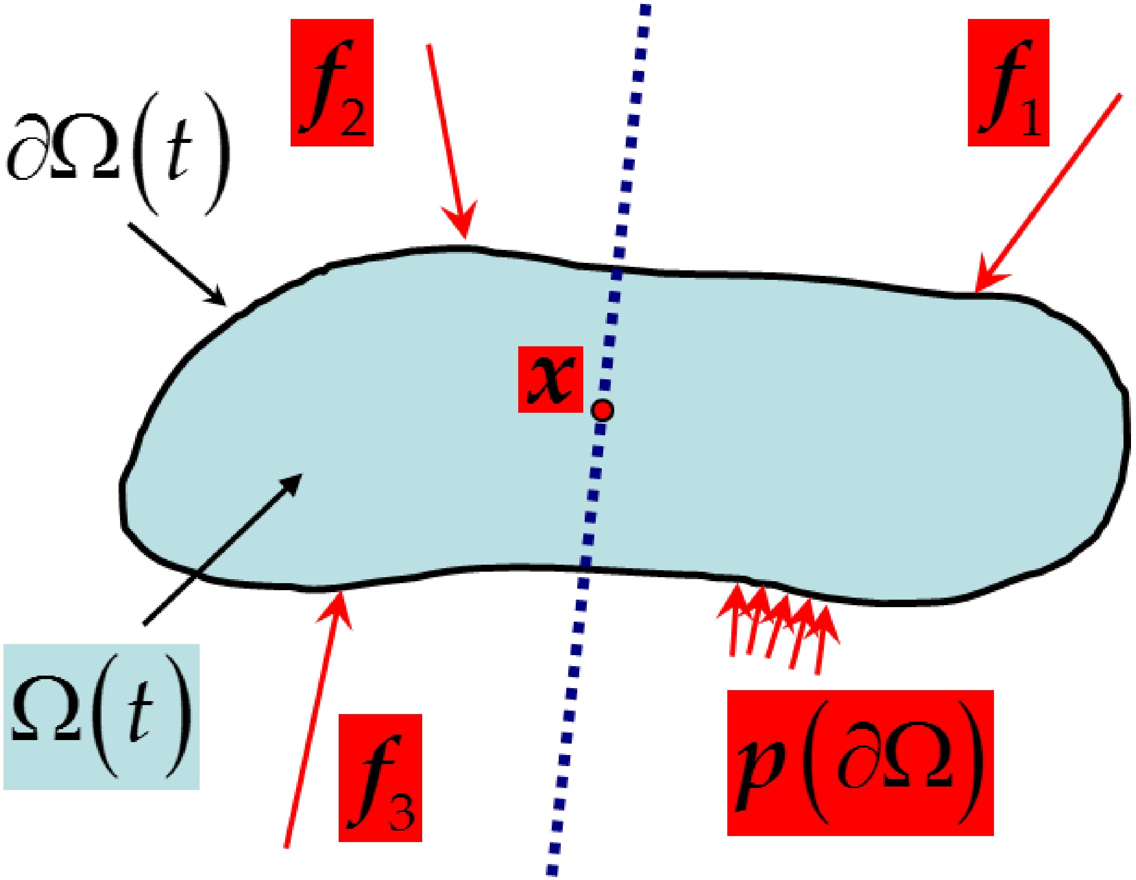




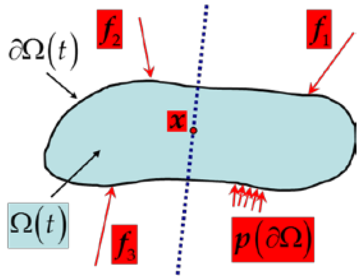
- Recall
  party
  balloon

State of stress and its mathematical expression

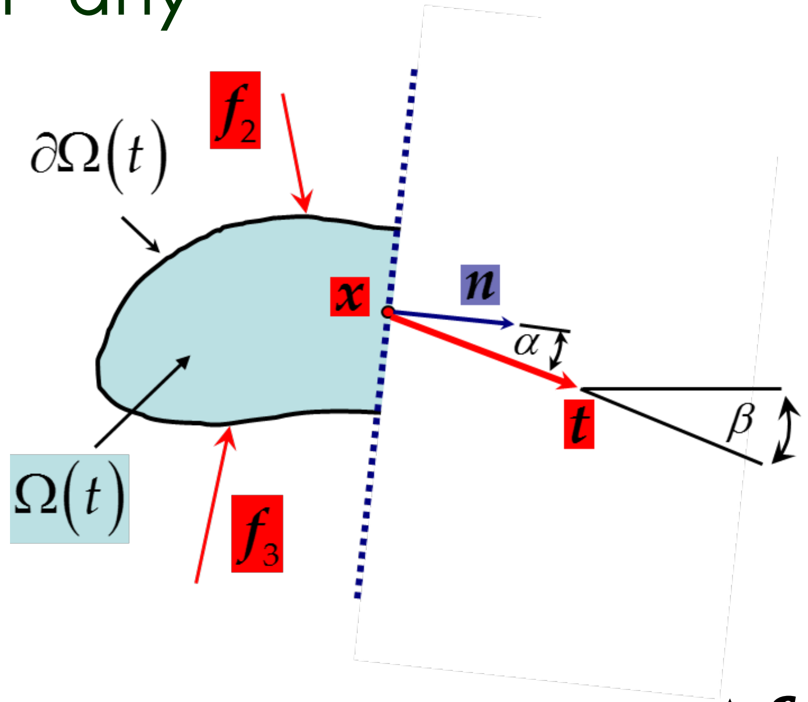
Body and forces acting on it



Body in equilibrium



- Let the body be cut by some plane passing through any given point $\boldsymbol{x}$



- To satisfy equilibrium, resultant internal force $\boldsymbol{f}$ has to act in $\boldsymbol{x}$ to compensate interaction with cut part of the body

- We introduce traction vector $\boldsymbol{t}$ as

$$d\boldsymbol{f} = \boldsymbol{t}ds \qquad \boldsymbol{t} = \lim_{\Delta s \to 0} \frac{\Delta \boldsymbol{f}}{\Delta s}$$

Stress tensor and traction vector

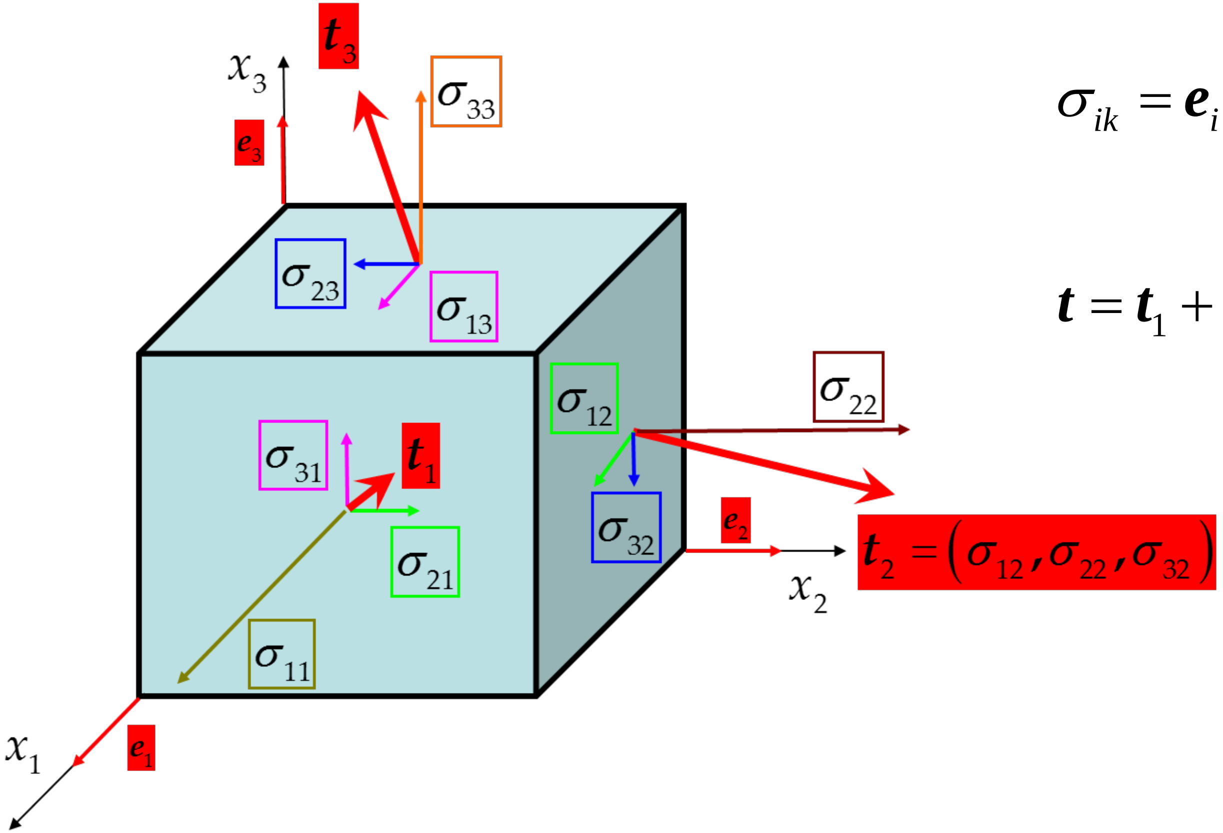
- Cauchy's stress theorem: There is unique second-order tensor $\boldsymbol{\sigma}$ such that $\boldsymbol{t} = \boldsymbol{\sigma n}$ holds.

$$\boldsymbol{t} = \boldsymbol{\sigma n} \qquad t_i = \sigma_{ij} n_j$$

- Cauchy stress $\boldsymbol{\sigma}$ is a linear transformation of normal vector $\boldsymbol{n}$

$$\boldsymbol{\sigma} : \boldsymbol{n} \rightarrow \boldsymbol{t} \qquad \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

Cauchy stress tensor



$$\sigma_{ik} = \boldsymbol{e}_i \cdot \boldsymbol{\sigma} \boldsymbol{e}_k = \boldsymbol{e}_i \cdot \boldsymbol{t}_k$$

$$\boldsymbol{t} = \boldsymbol{t}_1 + \boldsymbol{t}_2 + \boldsymbol{t}_3 = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix} + \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix} + \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} + \sigma_{12} + \sigma_{13} \\ \sigma_{21} + \sigma_{22} + \sigma_{23} \\ \sigma_{31} + \sigma_{32} + \sigma_{33} \end{pmatrix}$$

$$\boldsymbol{t}_1 = \boldsymbol{\sigma} \boldsymbol{e}_1 = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}$$

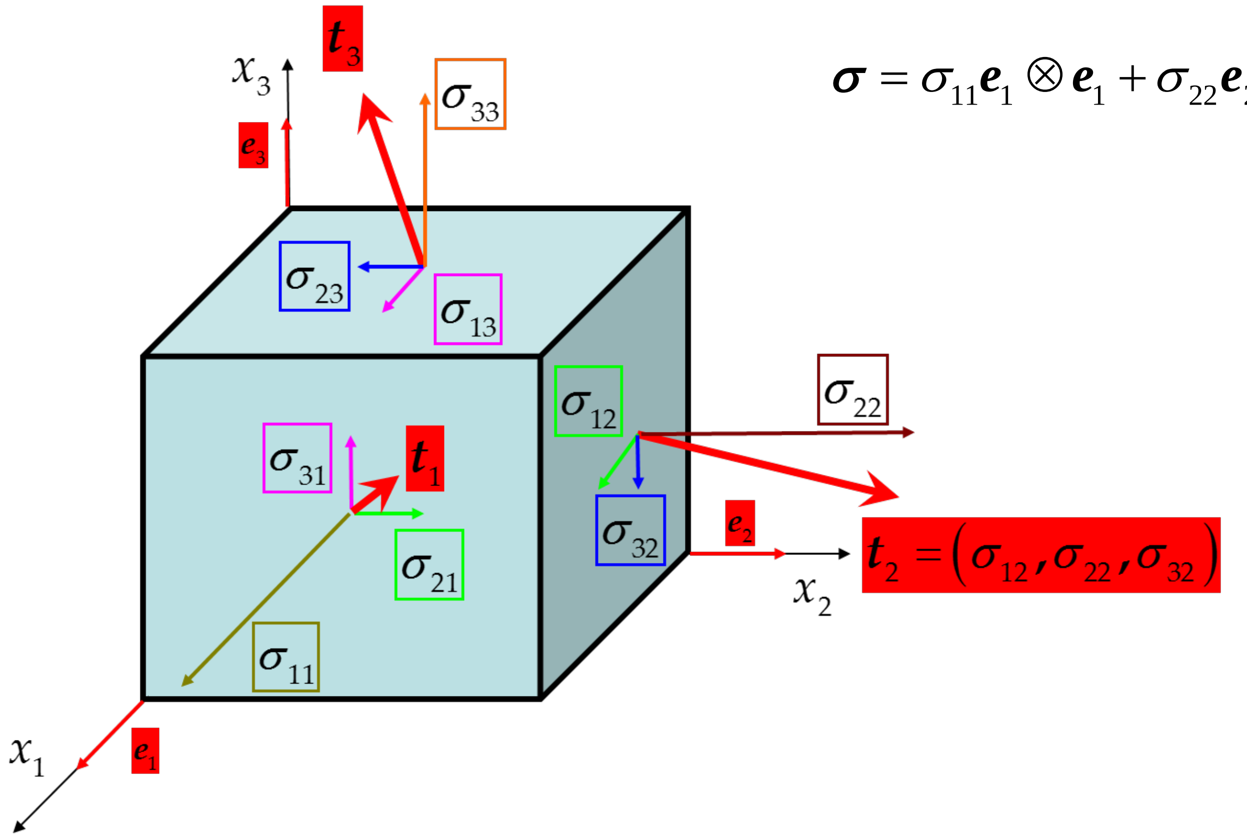
$$\boldsymbol{t}_2 = \boldsymbol{\sigma} \boldsymbol{e}_2 = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix}$$

$$\boldsymbol{t}_3 = \boldsymbol{\sigma} \boldsymbol{e}_3 = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix}$$

- Be careful when interpreting *i,j* in σ_{ij}. Two possible conventions exist.

Cauchy stress tensor is symmetric

- Cauchy stress symmetry is a consequence of a balance of angular momentum



$$\boldsymbol{\sigma} = \sigma_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + 2\sigma_{12}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + 2\sigma_{23}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + 2\sigma_{31}\boldsymbol{e}_3 \otimes \boldsymbol{e}_1$$

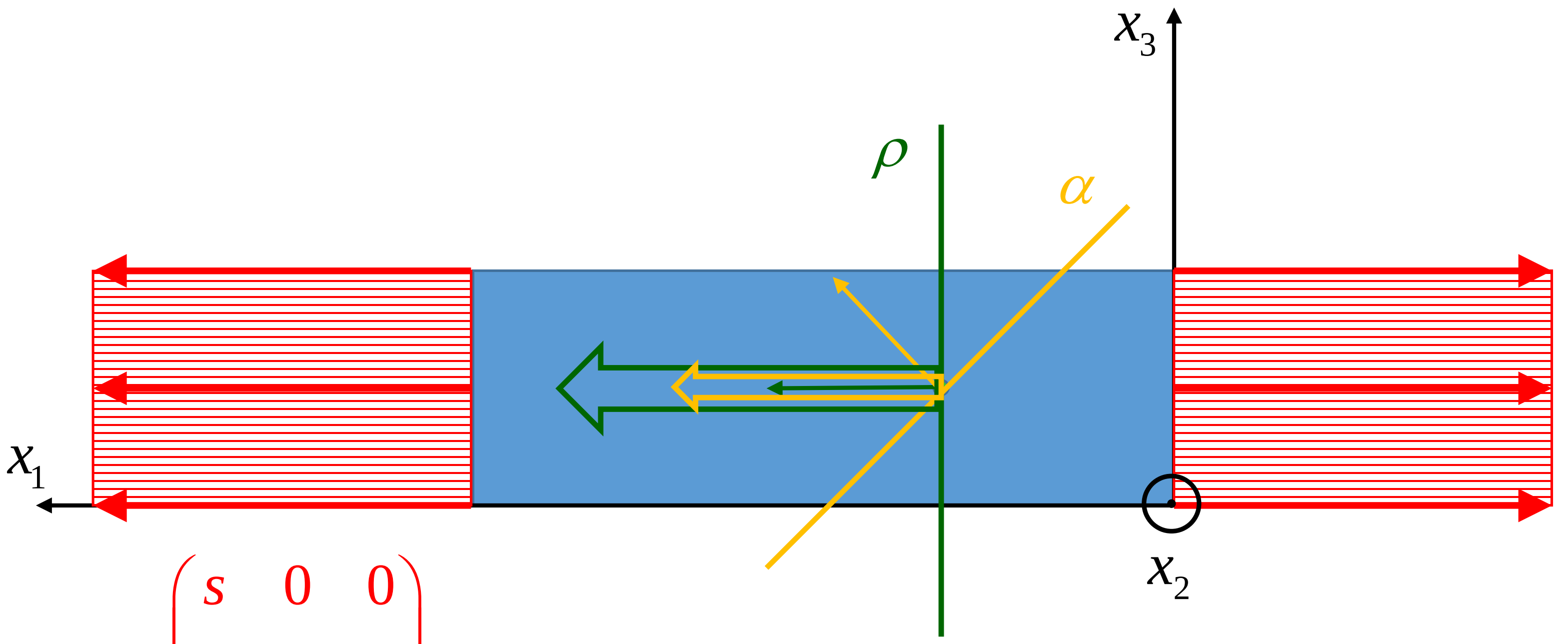
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

$$\sigma_{ik} = \sigma_{ki}$$

Two planes, two tractions, but unique tensor $\boldsymbol{\sigma}$

$$\boldsymbol{n}_\rho = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{t}_\rho = \boldsymbol{\sigma}\boldsymbol{n}_\rho = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{n}_\alpha = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix} \quad \boldsymbol{t}_\alpha = \boldsymbol{\sigma}\boldsymbol{n}_\alpha = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} s\frac{1}{2}\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$



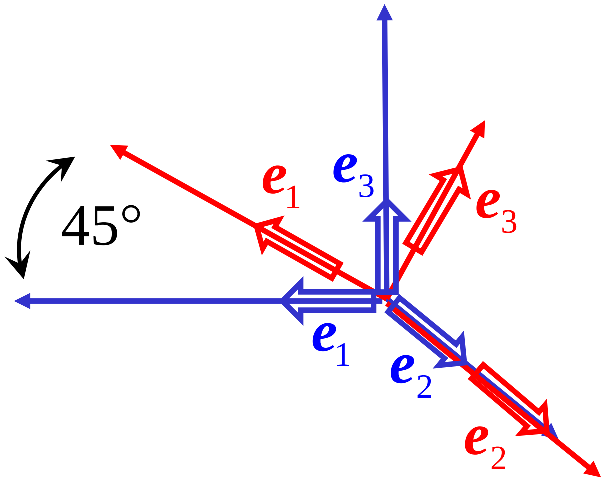
$$\boldsymbol{\sigma} = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Once more. At a point $\boldsymbol{x}$, we have infinite number of planes and their normal and traction vectors, but only one stress tensor which relates each normal and each traction vector together.

Once again about tensors: Transformations

$$\mathbf{Q}=\begin{pmatrix} \boldsymbol{e}_1\cdot\boldsymbol{e}_1 & \boldsymbol{e}_1\cdot\boldsymbol{e}_2 & \boldsymbol{e}_1\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}_2\cdot\boldsymbol{e}_1 & \boldsymbol{e}_2\cdot\boldsymbol{e}_2 & \boldsymbol{e}_2\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}_3\cdot\boldsymbol{e}_1 & \boldsymbol{e}_3\cdot\boldsymbol{e}_2 & \boldsymbol{e}_3\cdot\boldsymbol{e}_3 \end{pmatrix}=\begin{pmatrix} cos(\boldsymbol{e}_1,\boldsymbol{e}_1) & cos(\boldsymbol{e}_1,\boldsymbol{e}_2) & cos(\boldsymbol{e}_1,\boldsymbol{e}_3) \\ cos(\boldsymbol{e}_2,\boldsymbol{e}_1) & cos(\boldsymbol{e}_2,\boldsymbol{e}_2) & cos(\boldsymbol{e}_2,\boldsymbol{e}_3) \\ cos(\boldsymbol{e}_3,\boldsymbol{e}_1) & cos(\boldsymbol{e}_3,\boldsymbol{e}_2) & cos(\boldsymbol{e}_3,\boldsymbol{e}_3) \end{pmatrix}$$

$$\boldsymbol{e}_i=\mathbf{Q}\boldsymbol{e}_i$$



$$\mathbf{Q}=\begin{pmatrix} cos(45°) & cos(90°) & cos(135°) \\ cos(90°) & cos(0°) & cos(90°) \\ cos(45°) & cos(90°) & cos(45°) \end{pmatrix}=\begin{pmatrix} \tfrac{1}{2}\sqrt{2} & 0 & -\tfrac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \tfrac{1}{2}\sqrt{2} & 0 & \tfrac{1}{2}\sqrt{2} \end{pmatrix}$$

$\boldsymbol{e}_1=\mathbf{Q}\boldsymbol{e}_1$

$$\boldsymbol{e}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\longrightarrow\boldsymbol{e}_1=\begin{pmatrix}\tfrac{1}{2}\sqrt{2}\\0\\\tfrac{1}{2}\sqrt{2}\end{pmatrix}$$

$\boldsymbol{e}_2=\mathbf{Q}\boldsymbol{e}_2$

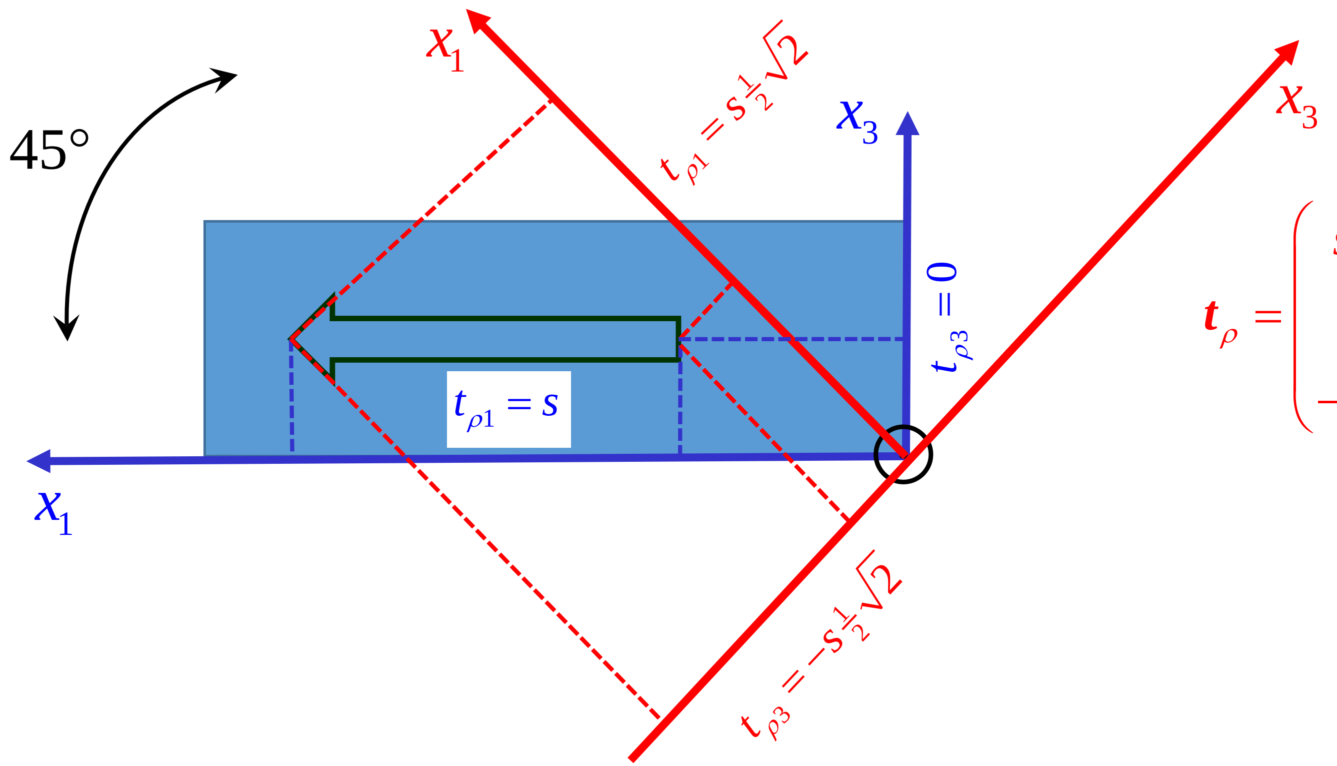
$$\boldsymbol{e}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}\longrightarrow\boldsymbol{e}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\boldsymbol{e}_3=\mathbf{Q}\boldsymbol{e}_3$

$$\boldsymbol{e}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}\longrightarrow\boldsymbol{e}_3=\begin{pmatrix}-\tfrac{1}{2}\sqrt{2}\\0\\\tfrac{1}{2}\sqrt{2}\end{pmatrix}$$

Rotation of the coordinate system

$$\boldsymbol{u} = \mathbf{Q}^T \boldsymbol{u} \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \cos(\boldsymbol{e}_1,\boldsymbol{e}_1) & \cos(\boldsymbol{e}_2,\boldsymbol{e}_1) & \cos(\boldsymbol{e}_3,\boldsymbol{e}_1) \\ \cos(\boldsymbol{e}_1,\boldsymbol{e}_2) & \cos(\boldsymbol{e}_2,\boldsymbol{e}_2) & \cos(\boldsymbol{e}_3,\boldsymbol{e}_2) \\ \cos(\boldsymbol{e}_1,\boldsymbol{e}_3) & \cos(\boldsymbol{e}_2,\boldsymbol{e}_3) & \cos(\boldsymbol{e}_3,\boldsymbol{e}_3) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad u_i = Q_{ji} u_j$$



$$\boldsymbol{t}_\rho = \begin{pmatrix} s\frac{1}{2}\sqrt{2} \\ 0 \\ -s\frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} = \mathbf{Q}^T \boldsymbol{t}_\rho$$

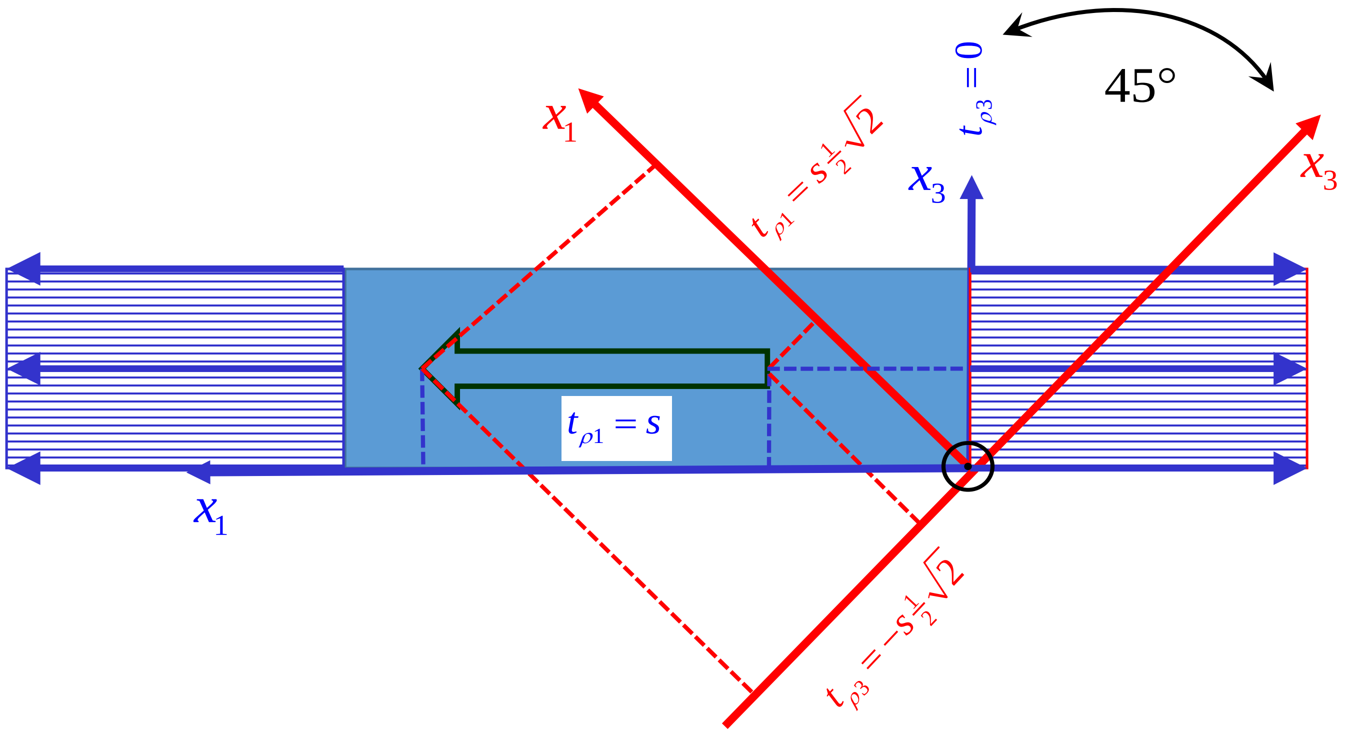
Rotation of the coordinate system – Tensors

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \left(Q_{ki}\mathbf{e}_k\right)\cdot\mathbf{T}\left(Q_{mj}\mathbf{e}_m\right) = Q_{ki}Q_{mj}\left(\mathbf{e}_k\cdot\mathbf{T}\mathbf{e}_m\right) = Q_{ki}Q_{mj}T_{km} = Q_{ki}T_{km}Q_{mj}$$

$$\mathbf{T} = \mathbf{Q}^T\mathbf{T}\mathbf{Q}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{2}s & 0 & \frac{1}{2}s \\ 0 & 0 & 0 \\ -\frac{1}{2}s & 0 & \frac{1}{2}s \end{pmatrix} = \mathbf{Q}^T\boldsymbol{\sigma}\mathbf{Q} =$$

$$= \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{pmatrix}$$



$$\mathbf{t}_\rho = \boldsymbol{\sigma}\mathbf{n}_\rho = \mathbf{Q}^T\mathbf{t}_\rho = \left(\mathbf{Q}^T\boldsymbol{\sigma}\mathbf{Q}\right)\left(\mathbf{Q}^T\mathbf{n}_\rho\right) = \begin{pmatrix} s\frac{1}{2}\sqrt{2} \\ 0 \\ -s\frac{1}{2}\sqrt{2} \end{pmatrix}$$

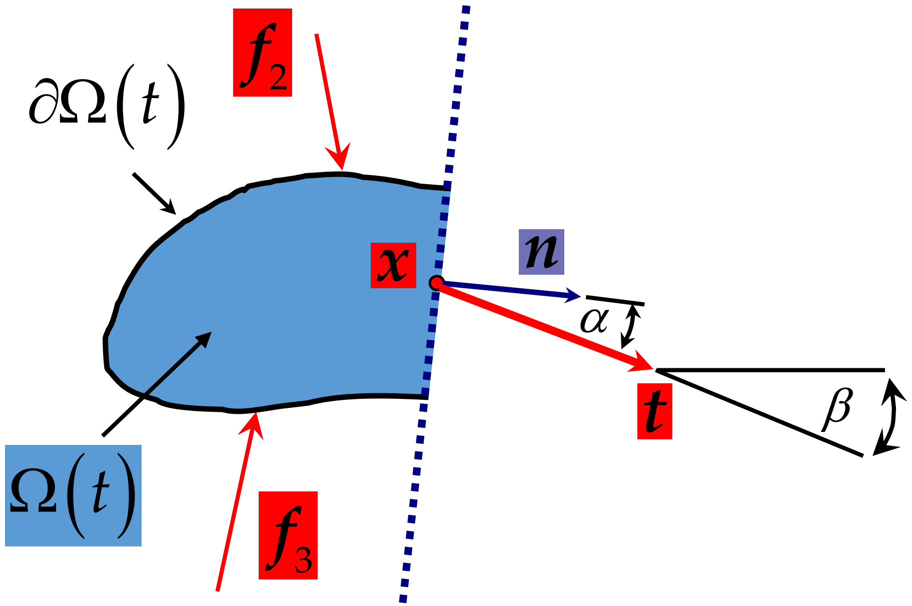
Rotation of the coordinate system – Tensors

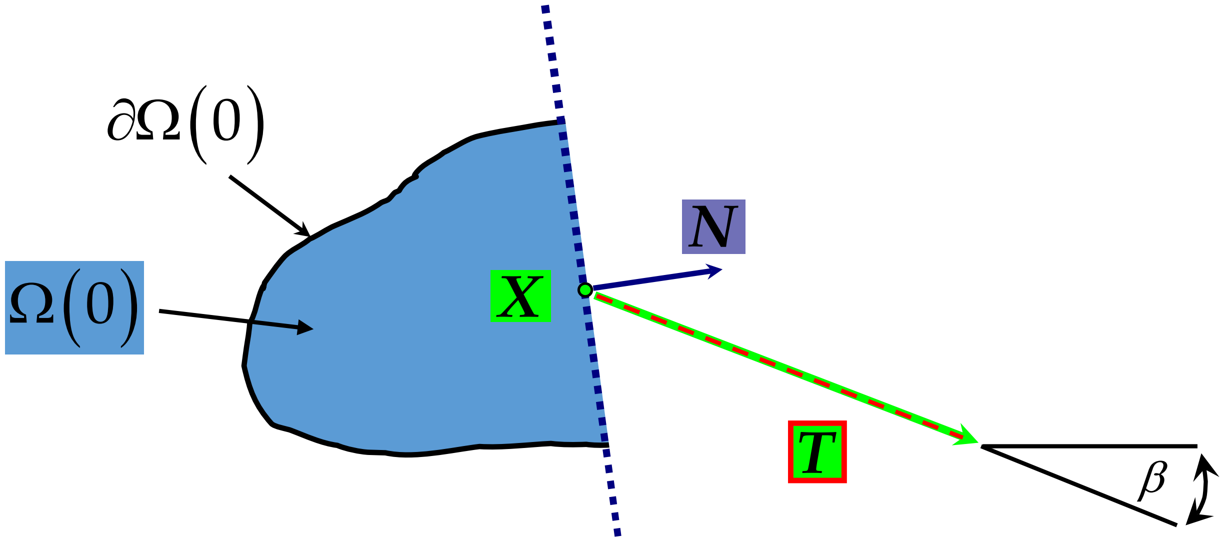
- 0-roder tensor: scalar $t = t$
- 1st-order tensor: vector $t_i = Q_{ji} t_j$
- 2nd-order tensor (ε_{ij}, E_{IJ}, F_{iK}, σ_{ij},…) $t_{ij} = Q_{ki} Q_{mj} t_{km}$
- 3rd-order tensor (ε_{ijk}) $t_{ijl} = Q_{ki} Q_{mj} Q_{nl} t_{kmn}$
- 4th-order tensor $E_{ijkl} = \partial\sigma_{ij}/\partial\varepsilon_{kl}$ $t_{ijlp} = Q_{ki} Q_{mj} Q_{nl} Q_{qp} t_{kmnq}$
- … …

Components of vectors and tensors depend on the basis! In this text, we always use orthonormal basis. Such components are referred to as physical components of a tensor.

Stress seen from the reference configuration

- Cut by a plane in the deformed state



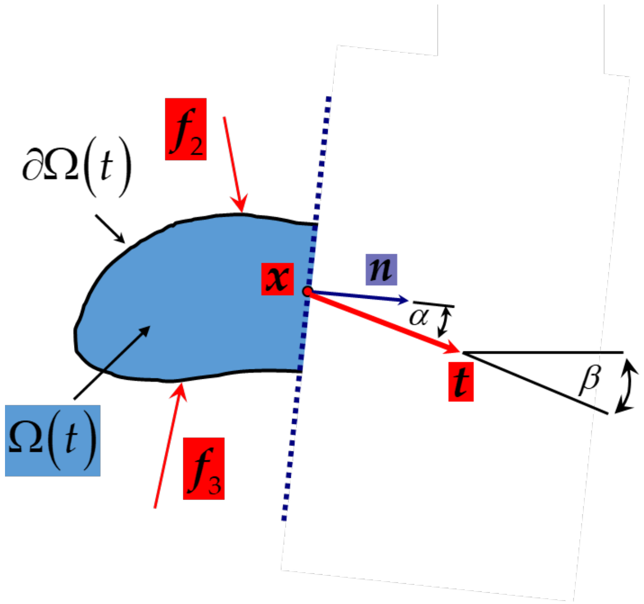


- Cut by a plane in the undeformed state

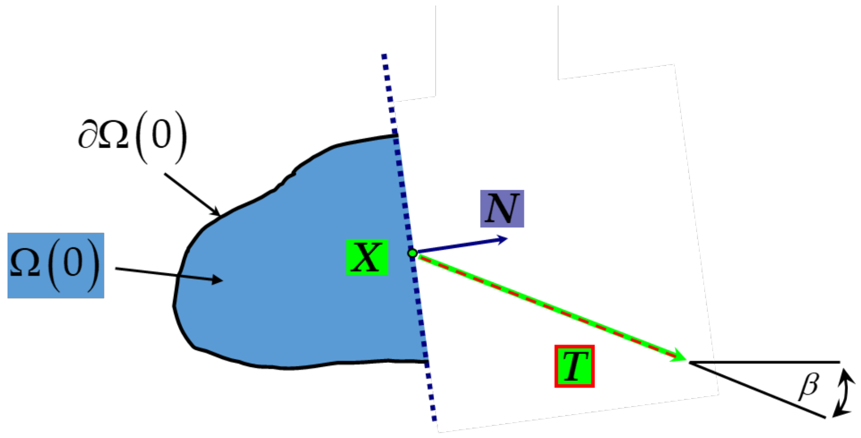
$$d\boldsymbol{f} = \boldsymbol{t}ds = \boldsymbol{T}dS$$

True (Cauchy) stress vector $\boldsymbol{t}$ and 1st Piola-Kirchhoff stress vector $\boldsymbol{T}$ form equivalent force systems with respect to infinitesimal force resultant $d\boldsymbol{f}$

Cauchy $\boldsymbol{\sigma}$ vs 1st Piola-Kirchhoff $\mathbf{P}$ stress tensor



$$d\boldsymbol{f} = \boldsymbol{t}ds = \boldsymbol{T}dS$$



- Cauchy (true) stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$$

$$t_i = \sigma_{ij} n_j$$

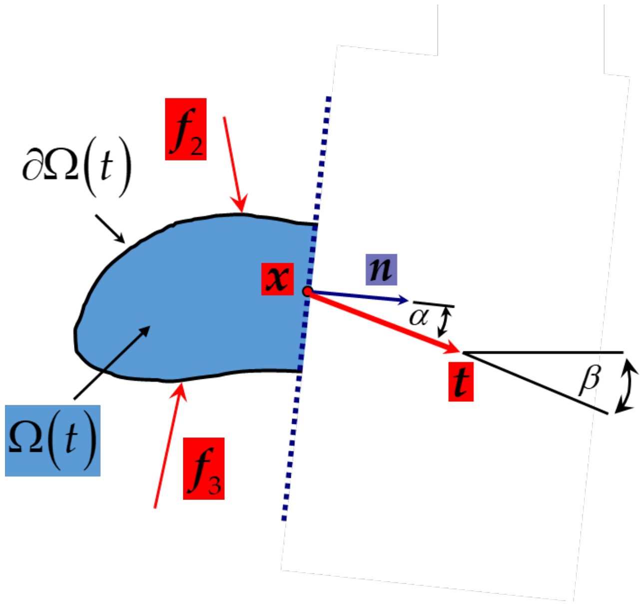
- 1st Piola-Kirchhoff stress tensor $\mathbf{P}$

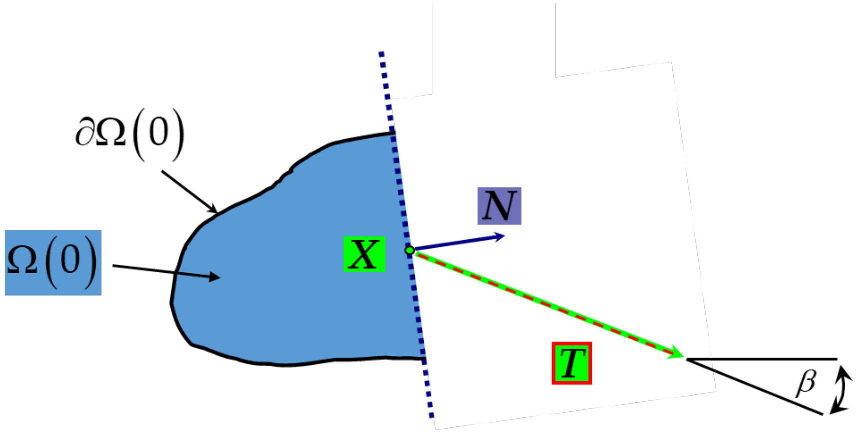
$$\boldsymbol{T} = \mathbf{P}\,\boldsymbol{N}$$

$$T_i = P_{iK} N_K$$

- Transpose of $\mathbf{P}$ stress tensor is so-called nominal stress tensor

Cauchy $\boldsymbol{\sigma}$ vs 1st Piola-Kirchhoff $\mathbf{P}$ stress tensor





$$d\boldsymbol{s} = J\,\mathbf{F}^{-T}\,d\boldsymbol{S}$$

$$d\boldsymbol{f} = d\boldsymbol{f} \quad \Leftrightarrow \quad \boldsymbol{t}ds = \boldsymbol{T}dS \quad \Leftrightarrow$$

$$\boldsymbol{\sigma n}ds = \mathbf{P}\,\boldsymbol{N}dS \quad \Leftrightarrow \quad \boldsymbol{\sigma}d\boldsymbol{s} = \mathbf{P}\,d\boldsymbol{S} \quad \Leftrightarrow$$

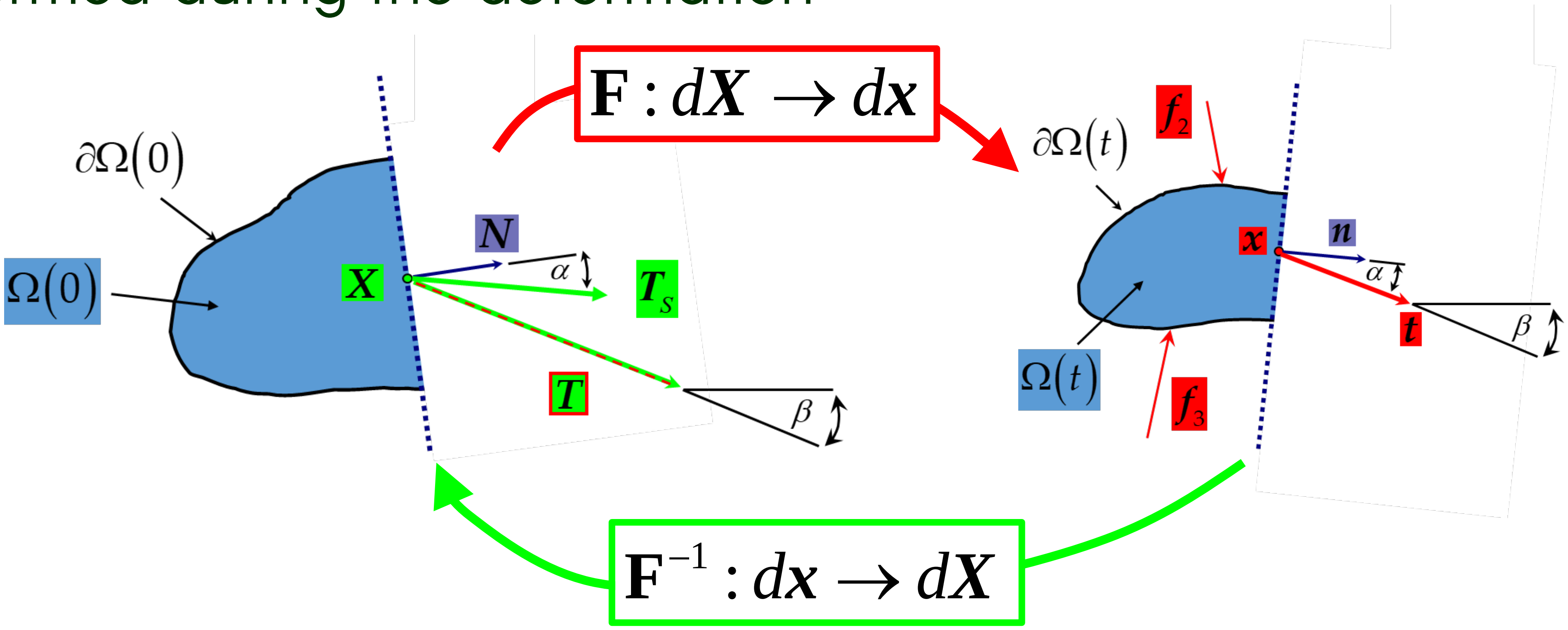
$$J\boldsymbol{\sigma}\,\mathbf{F}^{-T}\,d\boldsymbol{S} = \mathbf{P}\,d\boldsymbol{S} \quad \Leftrightarrow$$

$$\mathbf{P} = J\boldsymbol{\sigma}\,\mathbf{F}^{-T}$$

$$P_{iK} = J\sigma_{ij}F_{Kj}^{-1}$$

2nd Piola-Kirchhoff stress vector $\boldsymbol{T}_S$

- Let be the force transformed in the same way as is geometry transformed during the deformation



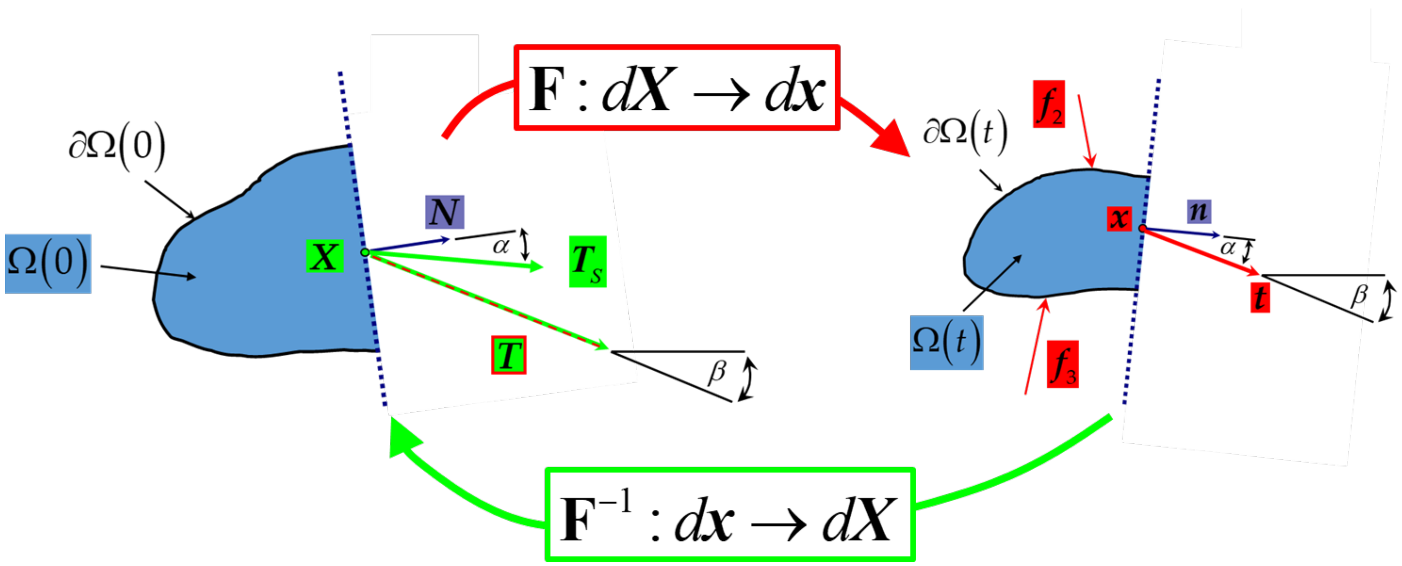
- By analogy $d\boldsymbol{X} = \mathbf{F}^{-1} d\boldsymbol{x}$

$$\mathbf{F}^{-1} d\boldsymbol{f} = \mathbf{F}^{-1} \boldsymbol{T} dS = \mathbf{F}^{-1} \boldsymbol{t} ds = \boldsymbol{T}_S dS$$

2nd Piola-Kirchhoff stress vector T_S

$$d\boldsymbol{x} = \mathbf{F}\, d\boldsymbol{X} \quad \Rightarrow \quad d\boldsymbol{X} = \mathbf{F}^{-1}\, d\boldsymbol{x}$$

$$d\boldsymbol{f} = \mathbf{F}\, d\boldsymbol{F} \quad \Rightarrow \quad d\boldsymbol{F} = \mathbf{F}^{-1}\, d\boldsymbol{f}$$



$$\Rightarrow \quad d\boldsymbol{F} = \mathbf{F}^{-1}\, d\boldsymbol{f} = \mathbf{F}^{-1}\boldsymbol{T} dS = \boldsymbol{T}_S\, dS$$

$$\boldsymbol{T}_S = \mathbf{F}^{-1}\boldsymbol{T}$$

2nd Piola-Kirchhoff stress tensor $\mathbf{S}$

- Second Piola-Kirchhoff stress tensor is defined with respect to undeformed configuration („undeformed“ force per undeformed area)

- $\mathbf{S}$ is again a linear transformation of $\boldsymbol{N}$ to $\boldsymbol{T}_S$

$$\boldsymbol{T}_S = \mathbf{S}\,\boldsymbol{N}$$

$$T_{S_I} = S_{IK} N_K$$

- Similarly to $\boldsymbol{\sigma}$, $\mathbf{S}$ is symmetric. It is in contrast to $\mathbf{P}$ (mixed tensor P_{iK}) which is not symmetric (situation is the same as with the deformation gradient F_{iK})

3 stress measures

- Cauchy stress tesnor

$$\boldsymbol{t} = \boldsymbol{\sigma n}$$

$$t_i = \sigma_{ij} n_j$$

- 1st Piola-Kirchhoff stress tensor

$$\boldsymbol{T} = \mathbf{P}\,\boldsymbol{N}$$

$$T_i = P_{iK} N_K$$

- 2nd Piola-Kirchhoff stress tensor

$$\boldsymbol{T}_S = \mathbf{S}\,\boldsymbol{N}$$

$$T_{S_I} = S_{IK} N_K$$

3 stress measures

- Consequences of equivalent force systems for *df* expression

$$d\boldsymbol{f} = \boldsymbol{t}ds = \boldsymbol{T}dS = \mathbf{F}\,\boldsymbol{T}_S\,dS$$

$$d\boldsymbol{f} = \boldsymbol{\sigma}\boldsymbol{n}ds = \mathbf{P}\,\boldsymbol{N}dS = \mathbf{F}\mathbf{S}\,\boldsymbol{N}dS$$

$$d\boldsymbol{f} = \boldsymbol{\sigma}d\boldsymbol{s} = \mathbf{P}\,d\boldsymbol{S} = \mathbf{F}\mathbf{S}\,d\boldsymbol{S}$$

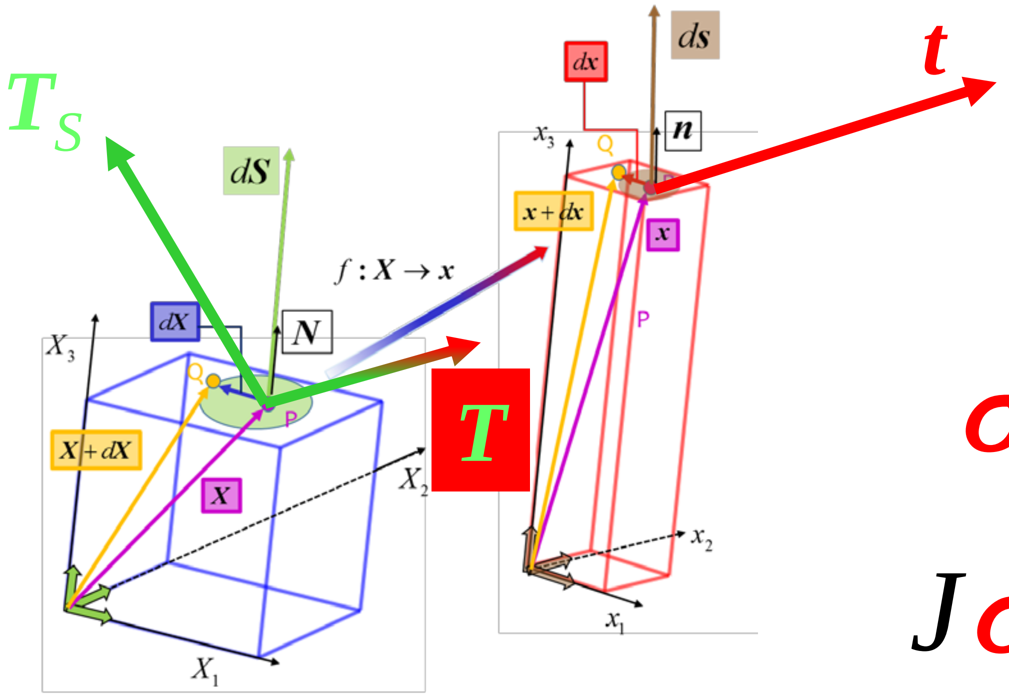
$$d\boldsymbol{s} = J\,\mathbf{F}^{-T}\,d\boldsymbol{S}$$

$$d\boldsymbol{S} = J^{-1}\,\mathbf{F}^{T}\,d\boldsymbol{s}$$

$$d\boldsymbol{f} = J\boldsymbol{\sigma}\mathbf{F}^{-T}\,d\boldsymbol{S} = \mathbf{P}\,d\boldsymbol{S} = \mathbf{F}\mathbf{S}\,d\boldsymbol{S}$$

$$d\boldsymbol{f} = \boldsymbol{\sigma}d\boldsymbol{s} = J^{-1}\,\mathbf{P}\mathbf{F}^{T}\,d\boldsymbol{s} = J^{-1}\,\mathbf{F}\mathbf{S}\mathbf{F}^{T}\,d\boldsymbol{s}$$

Transformation rules



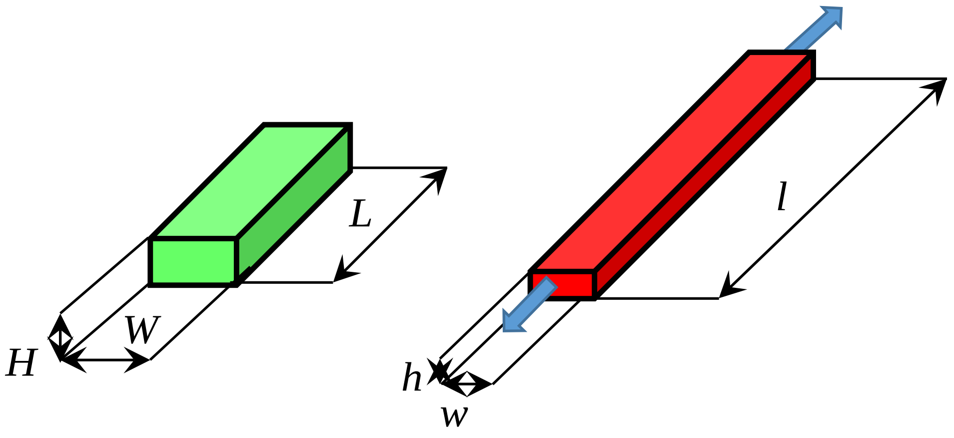
$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{T} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^{T}$$

$$J\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{P} = \mathbf{F}\mathbf{S}$$

$$J\,\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{P} = \mathbf{S}$$

Example: Uniaxial tension

- Consider homogenous deformation with no shear of an isotropic bar made from linearly elastic and incompressible material characterized elastic constant E



- 1st Piola–Kirchhoff stress $\mathbf{P}$

$$P_{11} = E\varepsilon_{11}$$

State of uniaxial tension

$$\mathbf{P} = \begin{pmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Homogenous deformation with no shear

$$\mathbf{F} = \begin{pmatrix} F_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{pmatrix} = \begin{pmatrix} l/L & 0 & 0 \\ 0 & w/W & 0 \\ 0 & 0 & h/H \end{pmatrix}$$

Example: Uniaxial tension

- Cauchy stress $\boldsymbol{\sigma}$ $\quad \boldsymbol{\sigma} \quad = J^{-1}\mathbf{P}\mathbf{F}^{T}$
- Incompressibility dictates $J = 1$

$$\boldsymbol{\sigma} = \mathbf{P}\mathbf{F}^{T} = \begin{pmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} F_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{pmatrix}^{T} = \begin{pmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} F_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} P_{11} \cdot F_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} P_{11} \cdot (1+\varepsilon_{11}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{11} = P_{11} \cdot (1+\varepsilon_{11}) = E\varepsilon_{11} \cdot (1+\varepsilon_{11})$$

Example: Uniaxial tension

- 2nd Piola-Kirchhoff stress **S** $\quad \mathbf{F}^{-1}\,\mathbf{P} \quad = \quad \mathbf{S}$

$$\mathbf{S}=\mathbf{F}^{-1}\mathbf{P}=\begin{pmatrix} F_{11} & 0 & 0 \\ 0 & F_{22} & 0 \\ 0 & 0 & F_{33} \end{pmatrix}^{-1}\begin{pmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1/F_{11} & 0 & 0 \\ 0 & 1/F_{22} & 0 \\ 0 & 0 & 1/F_{33} \end{pmatrix}\begin{pmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} P_{11}/F_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} P_{11}/\left(1+\varepsilon_{11}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{11}=P_{11}/\left(1+\varepsilon_{11}\right)=E\varepsilon_{11}/\left(1+\varepsilon_{11}\right)$$

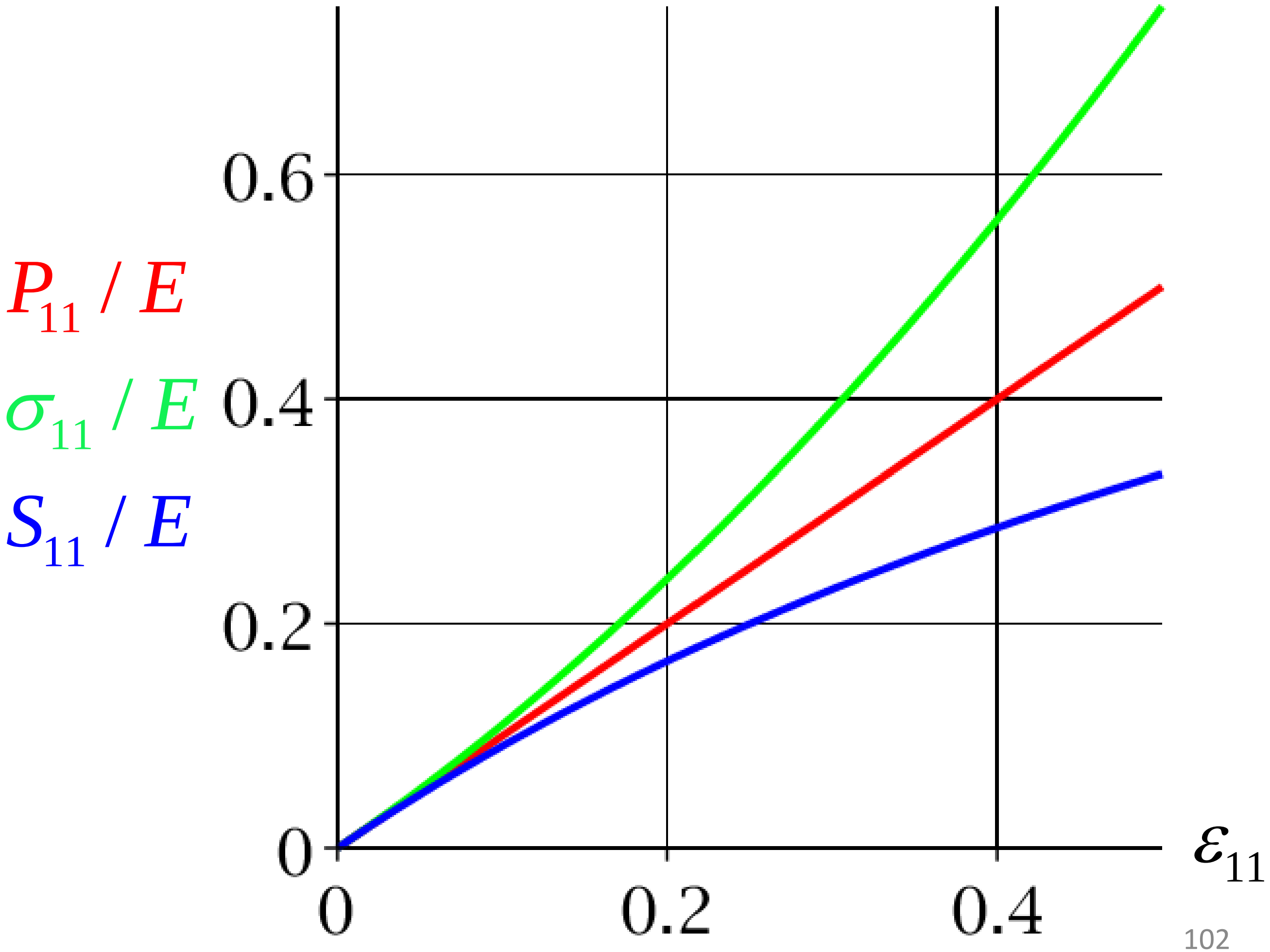
Example: Uniaxial tension

- One state of stress but 3 possible expression of the same physical reality

$$P_{11} = E\varepsilon_{11}$$

$$\sigma_{11} = E\varepsilon_{11} \cdot (1 + \varepsilon_{11})$$

$$S_{11} = E\varepsilon_{11} / (1 + \varepsilon_{11})$$



Constitutitve modeling of non-linearly elastic materials

Constitutive equation

- …is mathematical form of the relation between state variables, in case of deformable solid bodies, the state variables are stress and strain
- Constitutive means state. There is analogy to the equation of state in thermodynamics.

$$pV = nRT$$

Constitutive equation

- Classical approach, so-called Cuachy's elasticity, is to directly find system of equations relating 6 components of the stress tensor with 6 components of the strain tensor

$$\boldsymbol{\sigma} = f\left(\boldsymbol{\varepsilon}\right)$$

- In uniaxial tension it is $\sigma_{11} = E\varepsilon_{11}$

Linear isotropic material

- Generalized Hooke's law represents classical (Cauchy) approach

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix} = \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}$$

E is elastic modulus

ν is Poisson ratio

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0\\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0\\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

Linear isotropic material

- Generalized Hooke's law in tensor notation

$$\boldsymbol{\sigma} = \lambda tr(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} tr(\boldsymbol{\sigma})\mathbf{I}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{kk}\delta_{ij}$$

$$\mu = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

λ is 1st parameter of Lamé

μ is shear modulus

Hyperelasticity

- G. Green approach. In contrast to classical approach, elastic potential that is strain energy density function W, is in the heart of the method.
- Density means related to undeformed volume.

$$\boldsymbol{\sigma} = \frac{\partial W\left(\boldsymbol{\varepsilon}\right)}{\partial \boldsymbol{\varepsilon}} \qquad \sigma_{ij} = \frac{\partial W\left(\varepsilon_{ij}\right)}{\partial \varepsilon_{ij}}$$

Work conjugate pairs

- Without proof we state that stress power (time rate of change of the density of mechanical work produced by internal forces) is given by

$$w_{int} = J\boldsymbol{\sigma} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}} = \frac{1}{2}\mathbf{S} : \dot{\mathbf{C}}$$

- Dot denotes (material) time derivative

Hyperelastic means elastic

- Density of dissipated mechanical power D_{int} is zero. Dissipated power is the difference between stress power w_{int} and time rate of change of the strain energy density $\dot{W}$.

$$D_{int}=w_{int}-\dot{W}(\mathbf{F})=\mathbf{P}:\dot{\mathbf{F}}-\dot{W}(\mathbf{F})=\mathbf{P}:\dot{\mathbf{F}}-\frac{\partial W(\mathbf{F})}{\partial\mathbf{F}}:\dot{\mathbf{F}}=\left(\mathbf{P}-\frac{\partial W(\mathbf{F})}{\partial\mathbf{F}}\right):\dot{\mathbf{F}}=0$$

$$\mathbf{P}:\dot{\mathbf{F}}=P_{iK}\dot{F}_{iK}=P_{11}\dot{F}_{11}+P_{12}\dot{F}_{12}+P_{13}\dot{F}_{13}+P_{21}\dot{F}_{21}+P_{22}\dot{F}_{22}+P_{23}\dot{F}_{23}+P_{31}\dot{F}_{31}+P_{23}\dot{F}_{23}+P_{33}\dot{F}_{33}$$

$$\dot{W}(\mathbf{F})=\frac{d}{dt}W(\mathbf{F})=\frac{\partial W(\mathbf{F})}{\partial\mathbf{F}}:\frac{d\,\mathbf{F}}{dt}=\frac{\partial W(\mathbf{F})}{\partial\mathbf{F}}:\frac{d\,\mathbf{F}}{dt}=\frac{\partial W}{\partial F_{iK}}\frac{dF_{iK}}{dt}=$$

$$=\frac{\partial W}{\partial F_{11}}\dot{F}_{11}+\frac{\partial W}{\partial F_{12}}\dot{F}_{12}+\frac{\partial W}{\partial F_{13}}\dot{F}_{13}+\frac{\partial W}{\partial F_{21}}\dot{F}_{21}+\frac{\partial W}{\partial F_{22}}\dot{F}_{22}+\frac{\partial W}{\partial F_{32}}\dot{F}_{32}+\frac{\partial W}{\partial F_{31}}\dot{F}_{31}+\frac{\partial W}{\partial F_{32}}\dot{F}_{32}+\frac{\partial W}{\partial F_{33}}\dot{F}_{33}$$

Hyperelastic means elastic

- Density of dissipated mechanical power D_{int} is zero. Dissipated power is the difference between stress power w_{int} and time rate of change of the strain energy density $\dot{W}$.

$$D_{int} = \left(\mathbf{P} - \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\right) : \dot{\mathbf{F}} = 0 \Leftrightarrow \mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}$$

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \qquad P_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}$$

Hyperelasticity

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \qquad \boldsymbol{\sigma} = 2J^{-1}\frac{\partial W(\boldsymbol{b})}{\partial \boldsymbol{b}}\boldsymbol{b} \qquad \mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}$$

$$P_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \qquad \sigma_{ij} = 2J^{-1}\frac{\partial W(\boldsymbol{b})}{\partial b_{ik}}b_{kj} \qquad S_{IK} = \frac{\partial W(\mathbf{E})}{\partial E_{IK}} = 2\frac{\partial W(\mathbf{C})}{\partial C_{IK}}$$

Hooke's law in hyperelastic notation

$$W = \frac{1}{2}\lambda I_1^2 + 2\mu I_2 = \frac{E\nu}{2(1+\nu)(1-2\nu)} I_1^2 + \frac{E}{1+\nu} I_2$$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = \frac{E\nu}{(1+\nu)(1-2\nu)} \varepsilon_{kk}\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij}$$

$$I_1 = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \qquad I_2 = \frac{1}{2}\varepsilon_{ij}\varepsilon_{ji} = \frac{1}{2}\varepsilon_{11}^2 + \frac{1}{2}\varepsilon_{22}^2 + \frac{1}{2}\varepsilon_{33}^2 + \varepsilon_{12}\varepsilon_{21} + \varepsilon_{23}\varepsilon_{32} + \varepsilon_{31}\varepsilon_{13}$$

Hyperelasticity

- Simple substitution of the large strain tensor E instead of e into the Generalized Hooke‘s is known as Saint-Venant-Kirchhoff material model. However, is gives unrealistic predictions in uniaxial compression. This simple way fails.

$$W = \frac{\lambda}{2} tr^2(\mathbf{E}) + \mu tr(\mathbf{E}^2)$$

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \lambda tr(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$$

Saint-Venant-Kirchhoff model fails

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \lambda tr(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$$

$$S_{11} = \frac{E\nu}{(1+\nu)(1-2\nu)}(E_{11}+E_{22}+E_{33}) + \frac{E}{1+\nu}E_{11} = \frac{E\nu}{(1+\nu)(1-2\nu)}(E_{11}-2\nu E_{11}) + \frac{E}{1+\nu}E_{11} =$$

$$= \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{1}{2}(\lambda_{11}^2-1) - 2\nu\frac{1}{2}(\lambda_{11}^2-1)\right) + \frac{E}{1+\nu}\frac{1}{2}(\lambda_{11}^2-1) = \frac{E\nu(\lambda_{11}^2-1)}{(1+\nu)(1-2\nu)}\left(\frac{1}{2}-\nu\right) + \frac{E}{2(1+\nu)}(\lambda_{11}^2-1) =$$

$$= (\lambda_{11}^2-1)\left(\frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{1}{2}-\nu\right) + \frac{E}{2(1+\nu)}\right) = (\lambda_{11}^2-1)\frac{2E\nu\left(\frac{1}{2}-\nu\right)+E(1-2\nu)}{2(1+\nu)(1-2\nu)} =$$

$$= (\lambda_{11}^2-1)\frac{E\nu(1-2\nu)+E(1-2\nu)}{2(1+\nu)(1-2\nu)} = E(\lambda_{11}^2-1)\frac{1-\nu-2\nu^2}{2(1+\nu)(1-2\nu)}$$

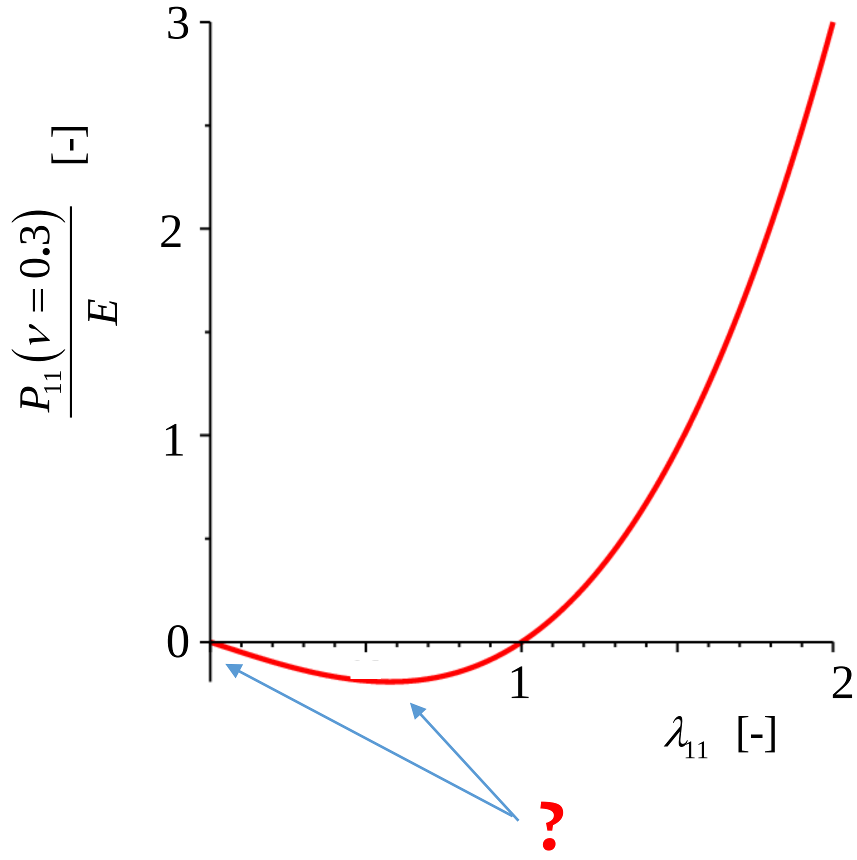
Saint-Venant-Kirchhoff model fails

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \lambda tr(\mathbf{E})\mathbf{I} + 2\mu \mathbf{E}$$

$$S_{11} = E\left(\lambda_{11}^2 - 1\right)\frac{1-\nu-2\nu^2}{2(1+\nu)(1-2\nu)}$$

$$\mathbf{P} = \mathbf{FS}$$

$$P_{11} = E\lambda_{11}\left(\lambda_{11}^2 - 1\right)\frac{1-\nu-2\nu^2}{2(1+\nu)(1-2\nu)}$$



Conditions to be met by the model

- Realistic and meaningful models of W cannot be arbitrary. There are some physical and mathematical conditions that has to be satisfied.

$$W(\mathbf{I})=0 \qquad W(\mathbf{F})\geq 0 \qquad \mathbf{P}(\mathbf{I})=\frac{\partial W(\mathbf{I})}{\partial \mathbf{F}}=\mathbf{0} \qquad \frac{\partial^2 W(\mathbf{I})}{\partial \mathbf{F}\,\partial \mathbf{F}}>\mathbf{O}$$

$$J=\det(\mathbf{F})\to\infty \;\Rightarrow\; W(\mathbf{F})\to\infty \qquad J=\det(\mathbf{F})\to 0^+ \;\Rightarrow\; W(\mathbf{F})\to\infty$$

$$W(\mathbf{Q}\mathbf{F})=W(\mathbf{F}) \qquad W(\mathbf{F})=W(\mathbf{U})=W(\mathbf{C})=W(\mathbf{E})$$

Tensors of elasticity

$$\mathbf{A} = \frac{\partial \mathbf{P}(\mathbf{F})}{\partial \mathbf{F}} = \frac{\partial^2 W(\mathbf{F})}{\partial \mathbf{F} \partial \mathbf{F}}$$

$$\mathbf{C} = \frac{\partial \mathbf{S}(\mathbf{E})}{\partial \mathbf{E}} = 2\frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial^2 W(\mathbf{E})}{\partial \mathbf{E} \partial \mathbf{E}} = 4\frac{\partial^2 W(\mathbf{C})}{\partial \mathbf{C} \partial \mathbf{C}}$$

$$\mathbf{c} = 2\mathbf{b}\frac{\partial \boldsymbol{\sigma}(\mathbf{b})}{\partial \mathbf{b}} = 4J^{-1}\mathbf{b}\frac{\partial^2 W(\mathbf{b})}{\partial \mathbf{b} \partial \mathbf{b}}\mathbf{b}$$

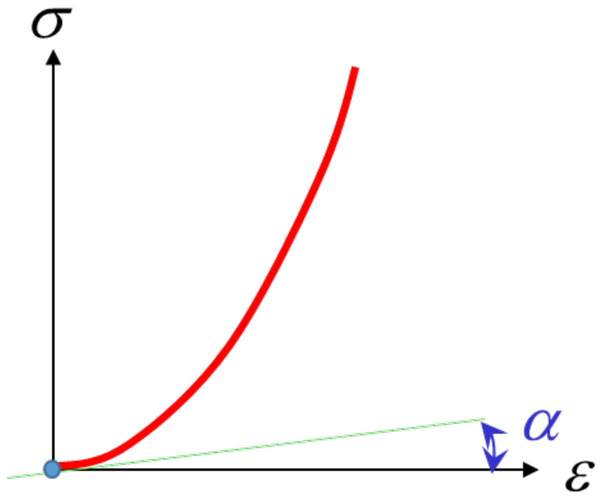
$$c_{ijkl} = J^{-1} F_{iK} F_{jL} F_{kM} F_{lN} C_{KLMN}$$

Restriction imposed on elasticity tensor

- Major symmetry $C_{IJKL} = C_{KLIJ}$

- Minor symmetry (in isotropic materials only) $C_{IJKL} = C_{JIKL} = C_{IJLK}$

- Solid body has to have positive elastic modulus

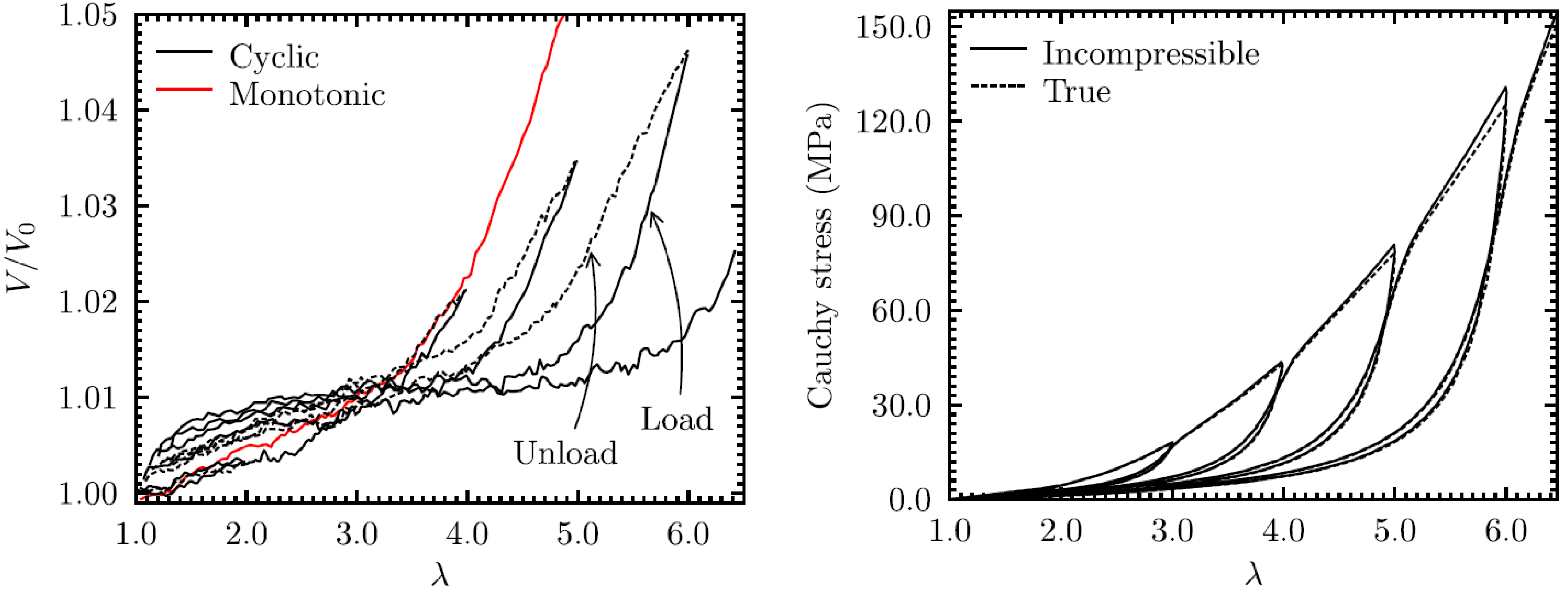


$$\frac{\partial^2 W(\mathbf{I})}{\partial \mathbf{F} \partial \mathbf{F}} > \mathbf{O}$$

$$E(\varepsilon = 0) = \frac{d\sigma(\varepsilon = 0)}{d\varepsilon} = tg(\alpha) > 0$$

Hyperelasticity of incompressible materials

- Elastomers, as well as soft tissues, are known to exhibit negligible volumetric changes during their deformations



Merckel, Y., Diani, J., Brieu, M., & Caillard, J. (2013). Constitutive modeling of the anisotropic behavior of mullins softened filled rubbers. *Mechanics of Materials, 57*, 30-41.
http://www.sciencedirect.com/science/article/pii/S0167663612001834#

Hyperelasticity of incompressible materials

- Soft tissues and elastomers are modeled as incompressible (no change of a volume)
- However, if they do not change volume, hydrostatic part of stress tensor do no work because this work is given as hydrostatic stress times volumetric deformation which is zero…
- By means of $\partial W/\partial \mathbf{F}$, we cannot obtain hydrostatic part of the stress tensor… because it does not contributed to W
- No deformation means no work thus no energy

$$\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

Hyperelasticity of incompressible materials

- Constitutive equation for a material with a constraint
- Lagrange multiplier p, to be determined from force boundary condition, is employed

$$W = W(\mathbf{F}) - p(J-1)$$

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + \mathbf{F}\frac{\partial W}{\partial \mathbf{F}}$$

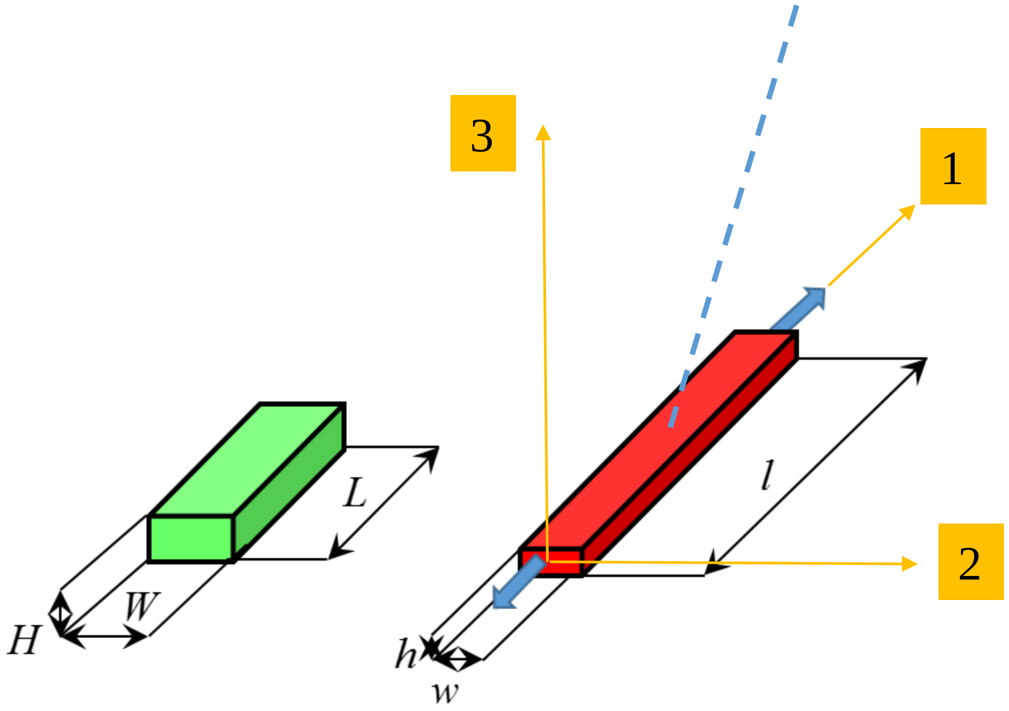
$$J = 1$$

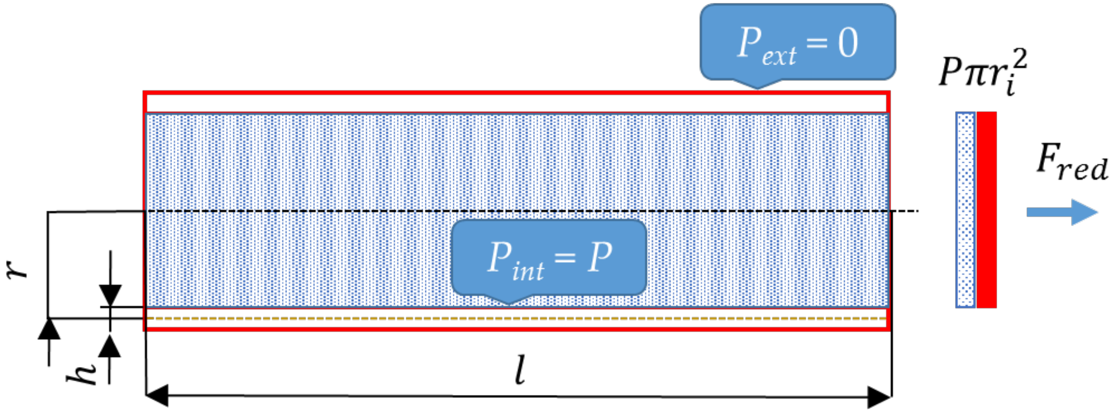
$$\mathbf{P} = -p\,\mathbf{F}^{-T} + \frac{\partial W}{\partial \mathbf{F}}$$

$$\mathbf{S} = -p\,\mathbf{C}^{-1} + 2\frac{\partial W}{\partial \mathbf{C}} = -p(2\mathbf{E}+\mathbf{I})^{-1} + \frac{\partial W}{\partial \mathbf{E}}$$

How to handle with p

$$\sigma_{33} = 0 \quad \Rightarrow \quad \lambda_3 \frac{\partial W}{\partial \lambda_3} - p = 0$$



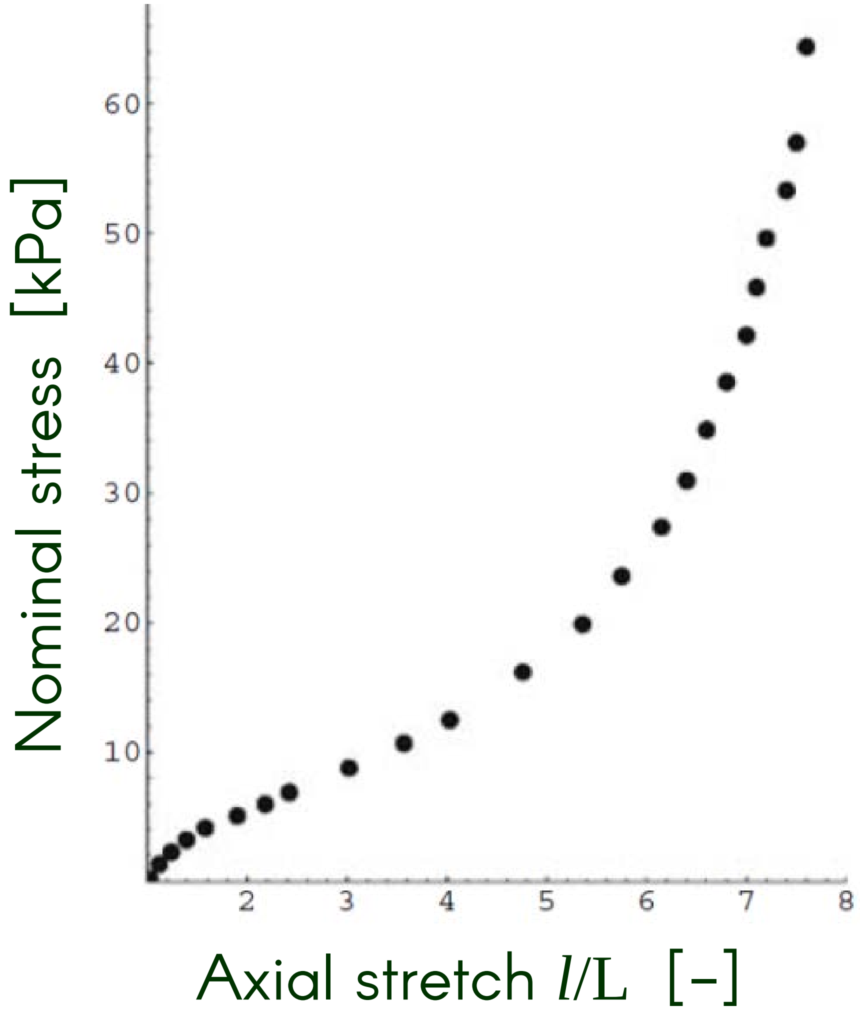


$$\sigma_{rr}\left(r = r_i\right) = -P \quad \Rightarrow \quad \lambda_{rR} \frac{\partial W}{\partial \lambda_{rR}} - p = -P \Bigg|_{r = r_i}$$

Models of W

Observation

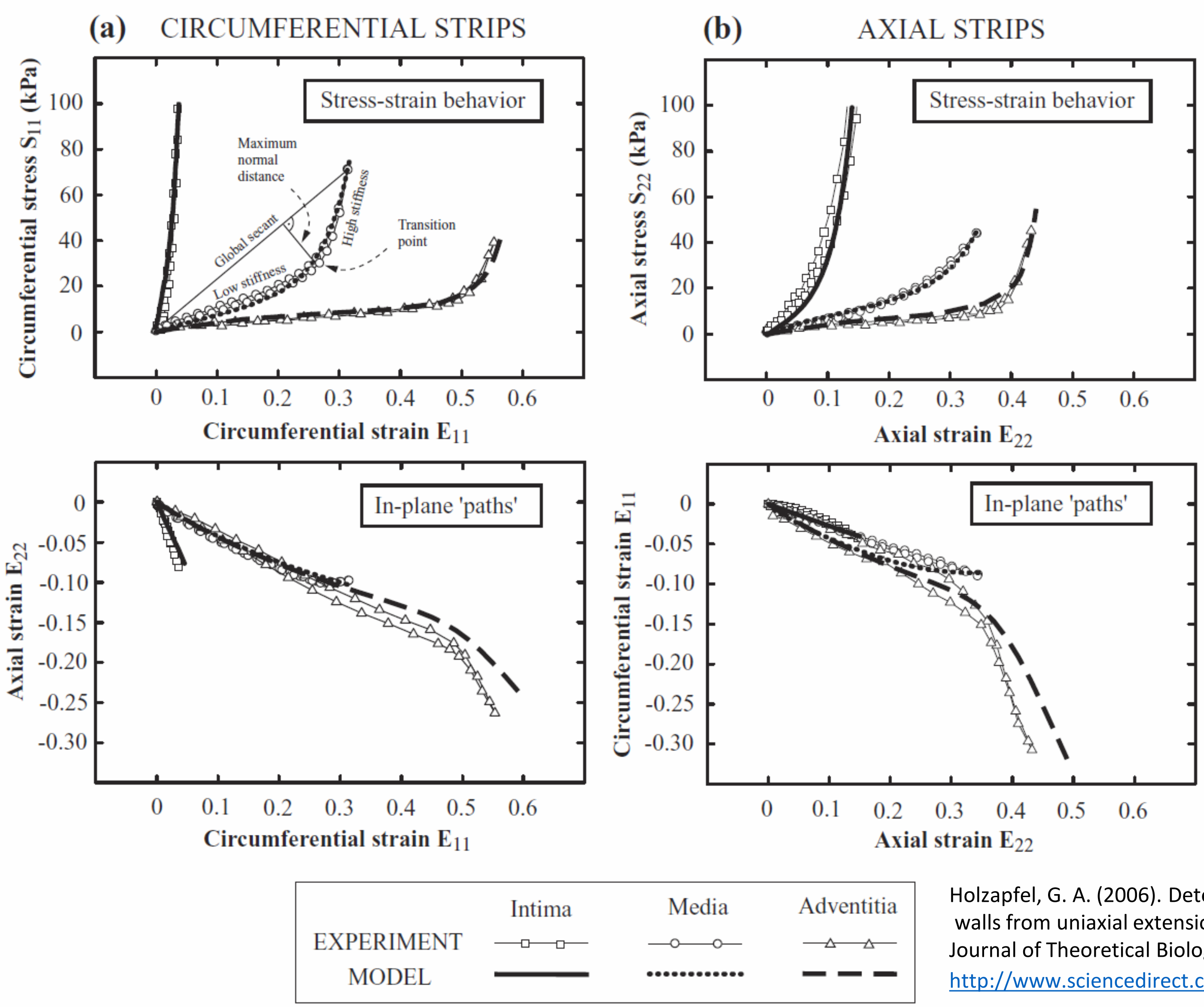
- Uniaxial tensile test with a rubber strip
- L.R.G. Treolar (1944)



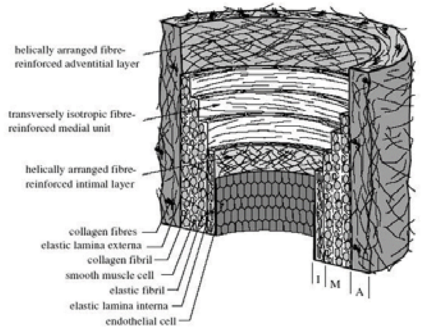
http://en.wikipedia.org/wiki/L._R._G._Treloar

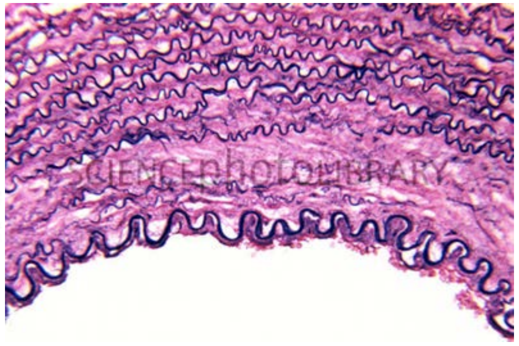
http://books.google.cz/books/about/The_Physics_of_Rubber_Elasticity.html?id=-iyDehYpoAQC&redir_esc=y

Observation



- Uniaxial tensile tests with strips cut from human abdominal aorta separated to individual layers (female 80 yrs)





Gasser, T. C., Ogden, R. W., & Holzapfel, G. A. (2006). Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. *Journal of the Royal Society Interface, 3*(6), 15-35. http://rsif.royalsocietypublishing.org/content/3/6/15

http://www.sciencephoto.com/image/115279/530wm/C0051117-Human_artery_wall_cross-section._LM-SPL.jpg

Holzapfel, G. A. (2006). Determination of material models for arterial walls from uniaxial extension tests and histological structure. Journal of Theoretical Biology, 238(2), 290-302. http://www.sciencedirect.com/science/article/pii/S0022519305002080

The simplest model

- Neo-Hookean model

$$W = \frac{\mu}{2}\left(I_1 - 3\right)$$

$$\mu = \frac{NkT}{V} = nkT = \frac{\rho RT}{M} > 0$$

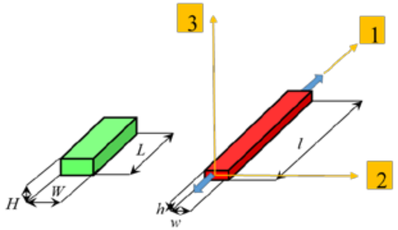
- The model can be deduced from entropic principle applied to freely jointed chain with Gaussian probability density for end points locations

I_1 1st invariant of $\mathbf{C}$
μ initial shear modulus
k Boltzmann constant
T thermodynamic temperature
N number of chains in a volume
n chain density
R universal gas constant
ρ mass density
M molecular mass

Neo-Hookean model

- Uniaxial tension for incompressible material

$$W = \frac{\mu}{2}(I_1 - 3)$$



$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{F}\frac{\partial W}{\partial \mathbf{F}}$$

$$\sigma_3 = 0 \quad \Rightarrow \quad p = \lambda_3^2 \mu$$

$$\sigma_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - p \qquad \sigma_1 = \lambda_1 \frac{\partial}{\partial \lambda_1}\left[\frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)\right] - p = \lambda_1^2 \mu - p$$

$$\sigma_2 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - p \qquad \sigma_2 = \lambda_2 \frac{\partial}{\partial \lambda_2}\left[\frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)\right] - p = \lambda_2^2 \mu - p$$

$$\sigma_3 = \lambda_3 \frac{\partial W}{\partial \lambda_3} - p \qquad \sigma_3 = \lambda_3 \frac{\partial}{\partial \lambda_3}\left[\frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)\right] - p = \lambda_3^2 \mu - p$$

$$J = \sqrt{det(\mathbf{C})} = \lambda_1 \lambda_2 \lambda_3 = 1 \qquad \lambda_2 = \lambda_3$$

$$\lambda_2 = \lambda_3 = \frac{1}{\sqrt{\lambda_1}}$$

$$\sigma_1 = \mu\lambda_1^2 - \frac{\mu}{\lambda_1}$$

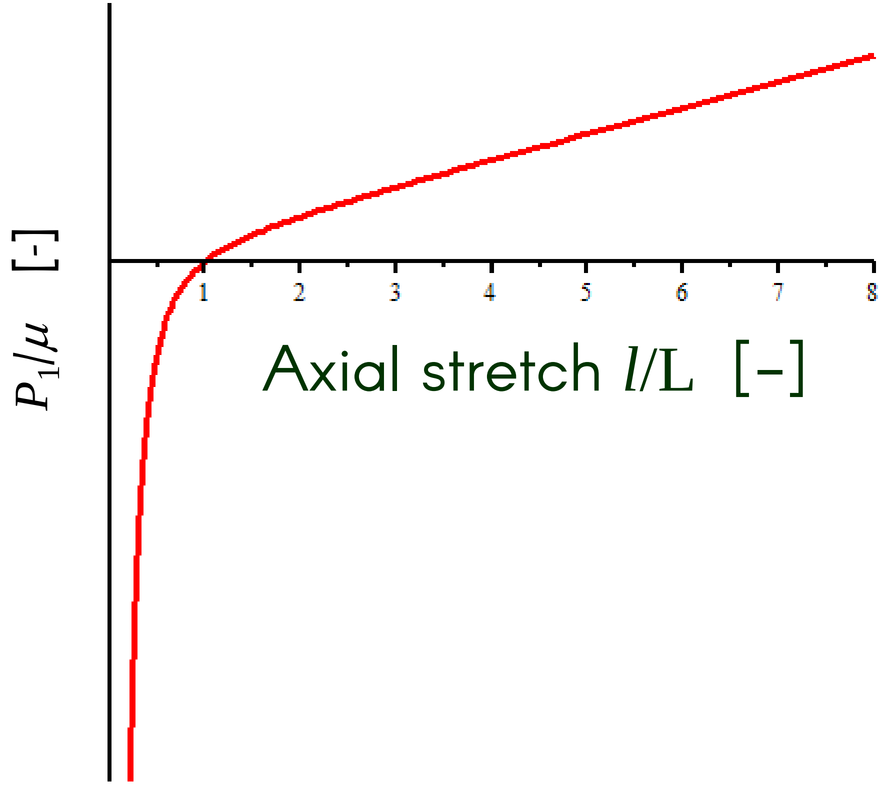
The simplest and meaningful but inadequate

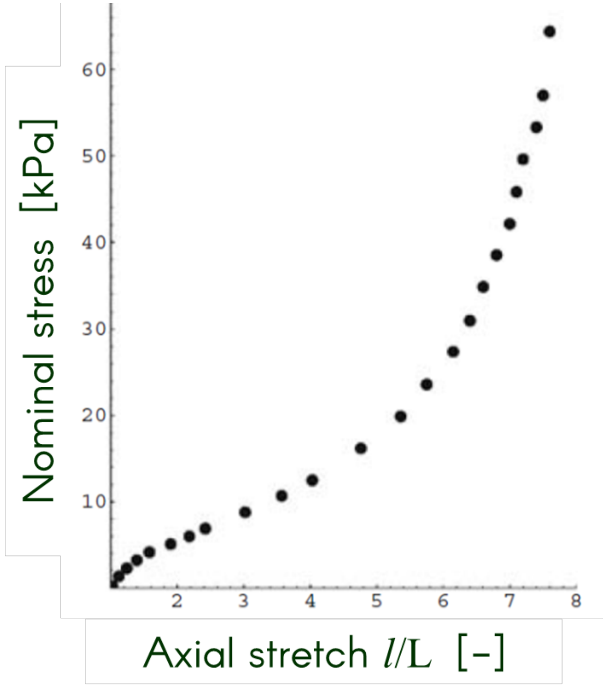
- Inadequate at large strains (say higher than 0.2)

$$\sigma_1 = \mu\lambda_1^2 - \frac{\mu}{\lambda_1}$$

$$P_1 = \mu\lambda_1 - \frac{\mu}{\lambda_1^2}$$

$$S_1 = \mu - \frac{\mu}{\lambda_1^3}$$





Models for rubber

- Mooney-Rivlin model

$$W = \frac{\mu}{2}\left[\alpha\left(I_1 - 3\right) + \left(1-\alpha\right)\left(I_2 - 3\right)\right]$$

$$\mu > 0 \quad \wedge \quad 0 < \alpha \leq 1$$

$$W = c_1\left(I_1 - 3\right) + c_2\left(I_2 - 3\right)$$

http://en.wikipedia.org/wiki/Melvin_Mooney

Melvin Mooney (1893-1968) American physicist who achieved results in the rheology of polymers

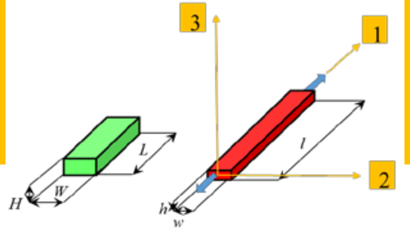
$$\mathbf{C} = \mathbf{F}^T\mathbf{F}$$

$$I_1 = tr\left(\mathbf{C}\right) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \frac{1}{2}\left[tr^2\left(\mathbf{C}\right) - tr\left(\mathbf{C}^2\right)\right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$

$$I_3 = det\left(\mathbf{C}\right) = \lambda_1^2\lambda_2^2\lambda_3^2$$

Models for rubber



- Mooney-Rivlin model $W = \frac{\mu}{2}\left[\alpha\left(I_1 - 3\right) + \left(1-\alpha\right)\left(I_2 - 3\right)\right]$

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{F}\frac{\partial W}{\partial \mathbf{F}}$$

$$\sigma_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - p \qquad \sigma_1 = \lambda_1 \frac{\partial}{\partial \lambda_1}\left\{\frac{\mu}{2}\left[\alpha\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + \left(1-\alpha\right)\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3\right)\right]\right\} - p = \mu\lambda_1\left[\alpha\lambda_1 + \left(1-\alpha\right)\left(\lambda_1\lambda_2^2 + \lambda_1\lambda_3^2\right)\right] - p$$

$$\sigma_2 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - p \qquad \sigma_2 = \lambda_2 \frac{\partial}{\partial \lambda_2}\left\{\frac{\mu}{2}\left[\alpha\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + \left(1-\alpha\right)\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3\right)\right]\right\} - p = \mu\lambda_2\left[\alpha\lambda_2 + \left(1-\alpha\right)\left(\lambda_2\lambda_1^2 + \lambda_2\lambda_3^2\right)\right] - p$$

$$\sigma_3 = \lambda_3 \frac{\partial W}{\partial \lambda_3} - p \qquad \sigma_3 = \lambda_3 \frac{\partial}{\partial \lambda_3}\left\{\frac{\mu}{2}\left[\alpha\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + \left(1-\alpha\right)\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3\right)\right]\right\} - p = \mu\lambda_3\left[\alpha\lambda_3 + \left(1-\alpha\right)\left(\lambda_3\lambda_1^2 + \lambda_3\lambda_2^2\right)\right] - p$$

$$J = \sqrt{det\left(\mathbf{C}\right)} = \lambda_1\lambda_2\lambda_3 = 1 \quad \wedge \quad \lambda_2 = \lambda_3 \quad \Rightarrow \quad \lambda_2 = \lambda_3 = \frac{1}{\sqrt{\lambda_1}} \qquad \sigma_3 = 0 \quad \Rightarrow \quad p = \mu\lambda_3\left[\alpha\lambda_3 + \left(1-\alpha\right)\left(\lambda_3\lambda_1^2 + \lambda_3\lambda_2^2\right)\right]$$

$$\sigma_1 = \alpha\mu\lambda_1^2 + \mu\left(1-\alpha\right)\lambda_1 - \alpha\mu\lambda_1^{-1} + \mu\left(\alpha - 1\right)\lambda_1^{-2}$$

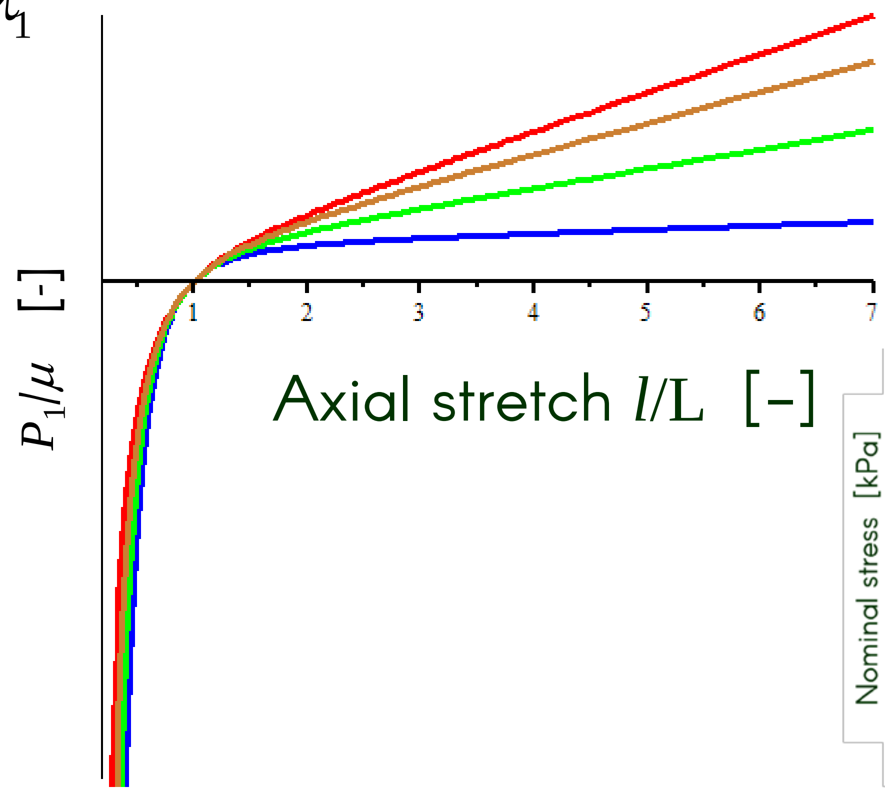
Models for rubber

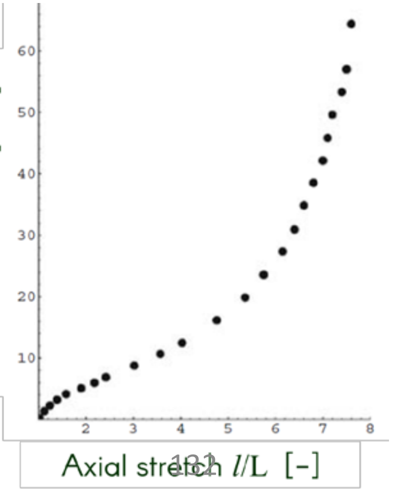
- Mooney-Rivlin model $W = \frac{\mu}{2}\left[\alpha\left(I_1 - 3\right) + \left(1-\alpha\right)\left(I_2 - 3\right)\right]$

$$\sigma_1 = \alpha\mu\lambda_1^2 + \mu\left(1-\alpha\right)\lambda_1 - \alpha\mu\lambda_1^{-1} + \mu\left(\alpha - 1\right)\lambda_1^{-2}$$

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$$

$$P_1 = \alpha\mu\lambda_1 + \mu\left(1-\alpha\right) - \alpha\mu\lambda_1^{-2} + \mu\left(\alpha - 1\right)\lambda_1^{-3}$$



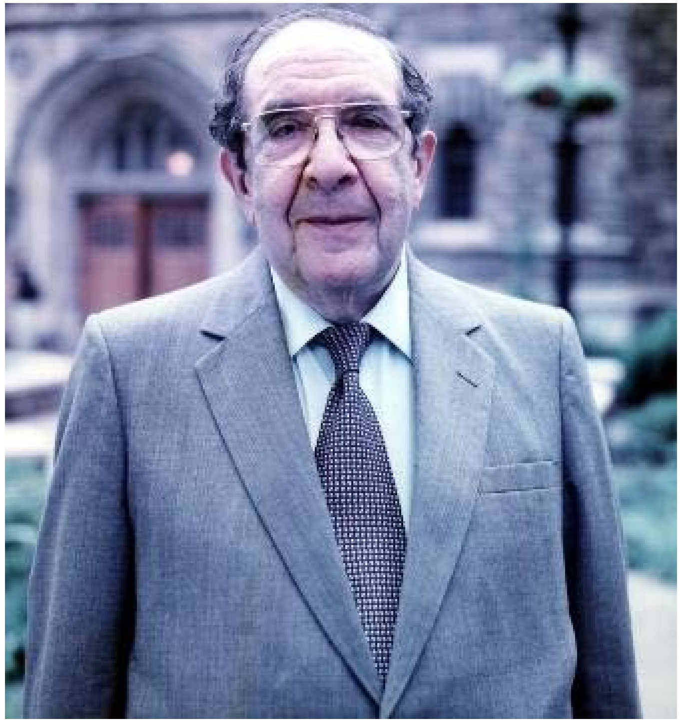


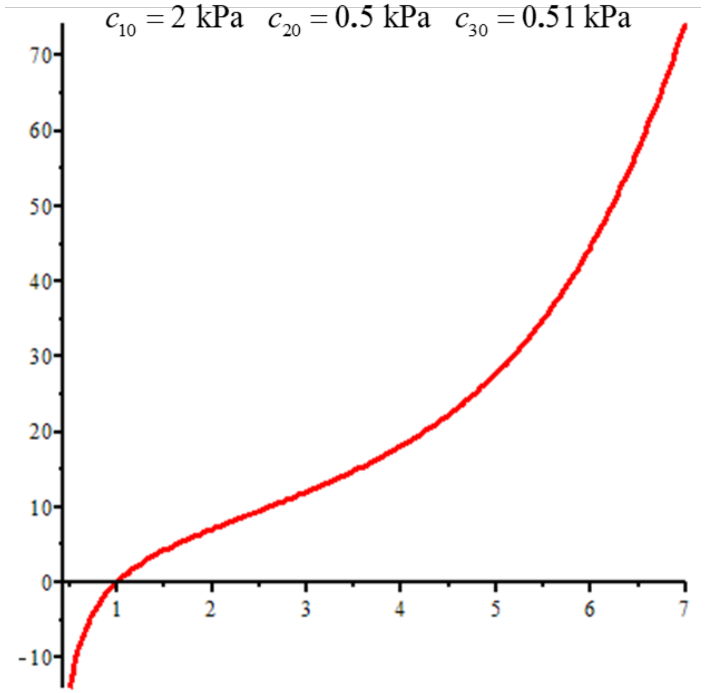
Family of Ronald Rivlin models

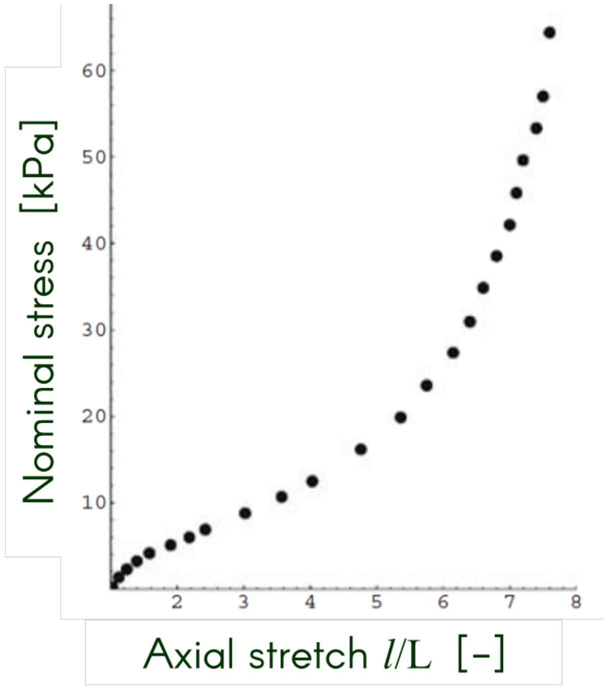
http://www.lehigh.edu/engineering/about/alumni/bio_rivlin_r.html

http://en.wikipedia.org/wiki/Ronald_Rivlin

$$W = \sum_{i,j=0}^{n} c_{ij} \left(I_1 - 3\right)^i \left(I_2 - 3\right)^j \qquad c_{00} = 0$$







Ronald Rivlin (1915–2005) Famous British-American physicist and mathematician, significant progress in nonlinear continuum mechanics and description of nonlinear materials. 3000 pages of scientific papers, unfortunately no monograph.

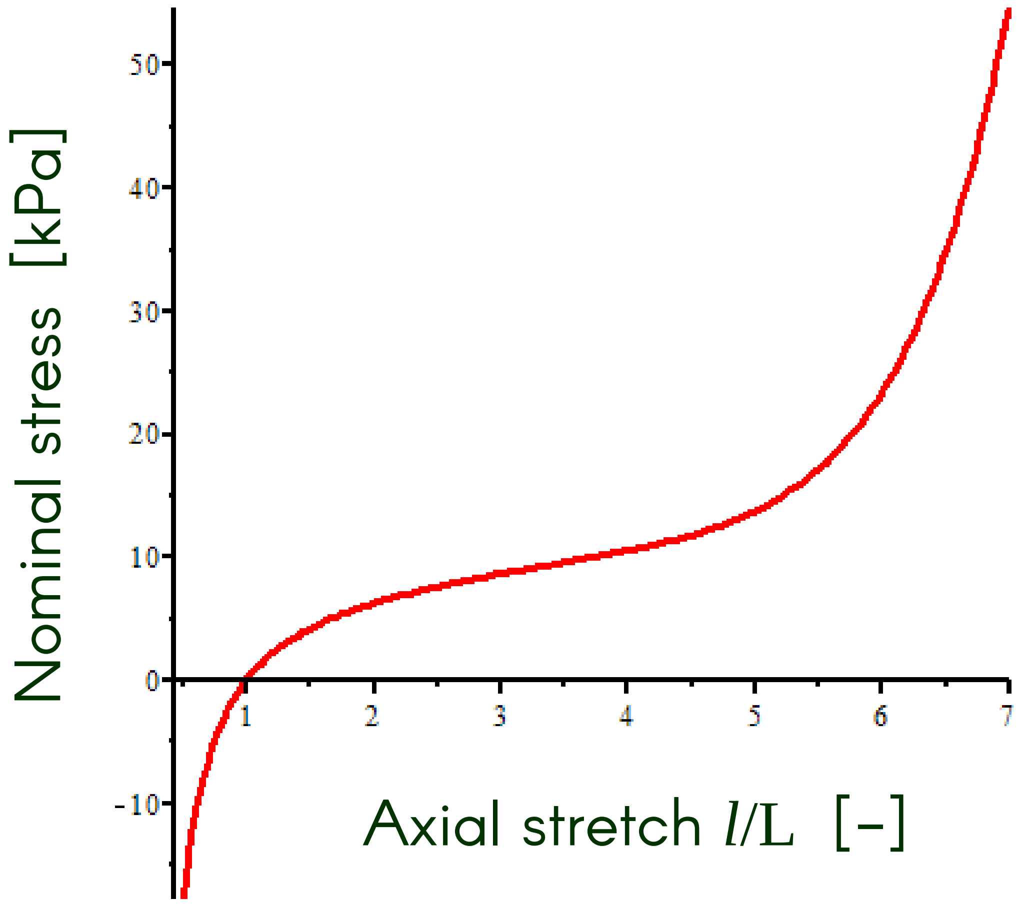
Model by R.W. Ogden

http://www.gla.ac.uk/schools/mathematicsstatistics/staff/raymondogden/

$\mu_1 = 5.39\ \text{kPa} \quad \mu_2 = -0.531\ \text{kPa} \quad \mu_3 = 0.19\cdot 10^{-5}\ \text{kPa} \alpha_1 = 1.46 \quad \alpha_2 = -2.03 \quad \alpha_3 = 9.68$

$$W = \sum_{k=1}^{n} \frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3\right)$$

$$\sum_{k=1}^{n} \alpha_k \mu_k = 2\mu > 0$$



$$P_1 = \mu_1 \lambda_1^{\alpha_1 - 1} + \mu_2 \lambda_1^{\alpha_2 - 1} + \mu_3 \lambda_1^{\alpha_3 - 1} - \mu_1 \lambda_1^{-\frac{\alpha_1}{2} - 1} - \mu_2 \lambda_1^{-\frac{\alpha_2}{2} - 1} + \mu_3 \lambda_1^{-\frac{\alpha_3}{2} - 1}$$

Models of W suitable to describe soft tissues

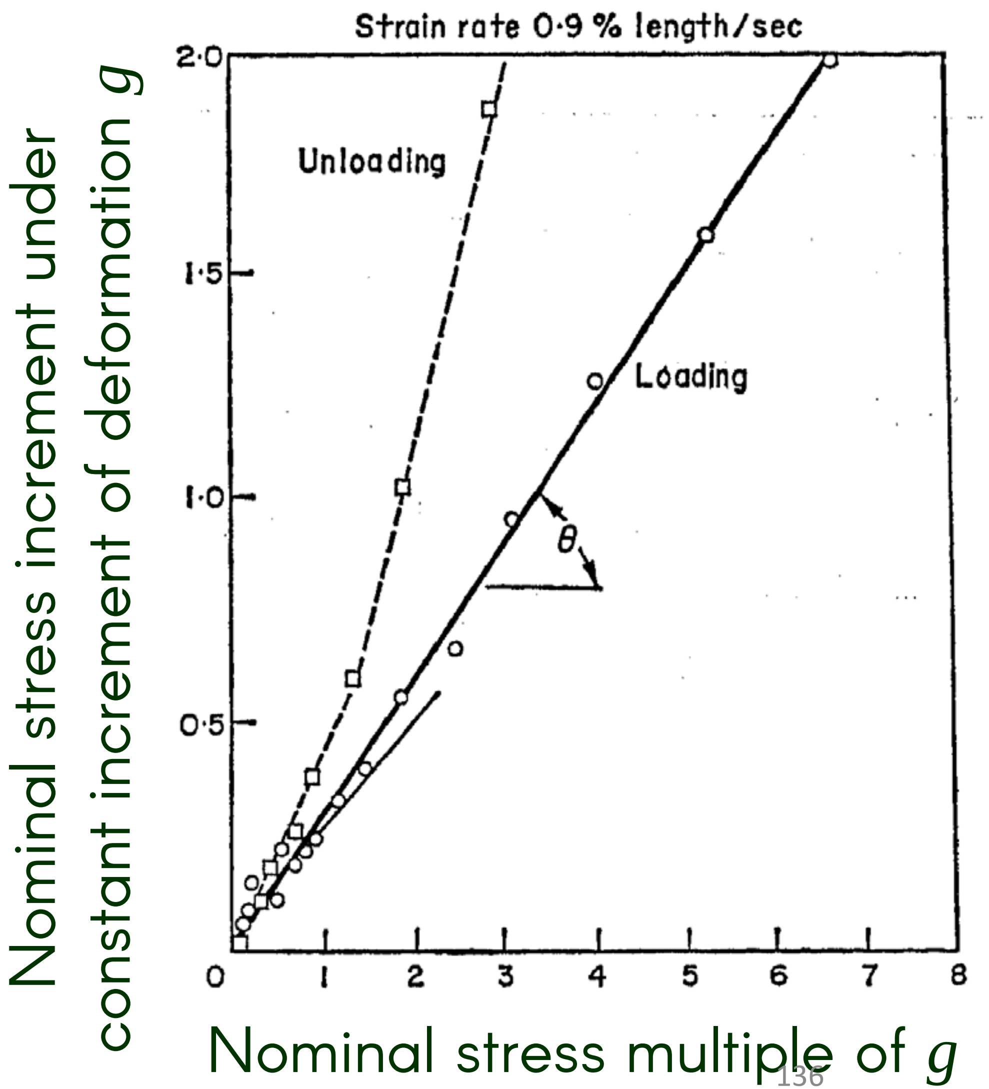
Exponential model by Fung

- Y.C. Fung (1919) UCSD

$$E = \frac{d\sigma}{d\lambda} = a + b\sigma \quad \Rightarrow \quad \sigma = \frac{a}{b}\left(e^{b(\lambda-1)} - 1\right)$$

$$y' = a + by$$

$$y(0) = 0 \quad \vee \quad y'(0) = E_{ini}$$



http://en.wikipedia.org/wiki/Yuan-Cheng_Fung

http://www.jacobsschool.ucsd.edu/faculty/faculty_bios/index.sfe?fmp_recid=21

Exponential model by Fung

$$W = \frac{c}{2}\left(e^{Q} - 1\right)$$

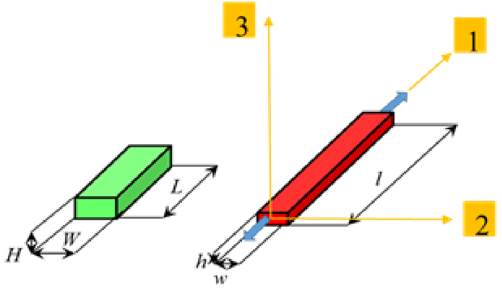
$$Q = b_{11}E_{11}^2 + b_{22}E_{22}^2 + b_{33}E_{33}^2 + 2b_{12}E_{12}^2 + 2b_{13}E_{13}^2 + 2b_{23}E_{23}^2 + 2b_{21}E_{11}E_{22} + 2b_{32}E_{22}E_{33} + 2b_{31}E_{11}E_{33}$$

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{T}\,\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right)$$

Exponential model by Fung

- Isotropic form of Fung model in uniaxial tension

$$W = \frac{\mu}{2\alpha}\left(e^{\alpha(I_1-3)} - 1\right)$$



$$\sigma_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - p \qquad \sigma_1 = \frac{\mu}{2\alpha}\lambda_1 \frac{\partial}{\partial \lambda_1}\left(e^{\alpha\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)} - 1\right) - p = \mu\lambda_1^2 e^{\alpha\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)} - p$$

$$\sigma_2 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - p \qquad \sigma_2 = \frac{\mu}{2\alpha}\lambda_2 \frac{\partial}{\partial \lambda_2}\left(e^{\alpha\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)} - 1\right) - p = \mu\lambda_2^2 e^{\alpha\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)} - p$$

$$\sigma_3 = \lambda_3 \frac{\partial W}{\partial \lambda_3} - p \qquad \sigma_3 = \frac{\mu}{2\alpha}\lambda_3 \frac{\partial}{\partial \lambda_3}\left(e^{\alpha\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)} - 1\right) - p = \mu\lambda_3^2 e^{\alpha\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)} - p$$

$$\sigma_3 = 0 \quad \Rightarrow \quad p = \mu\lambda_3^2 e^{\alpha\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)}$$

$$J = \sqrt{det(\mathbf{C})} = \lambda_1 \lambda_2 \lambda_3 = 1 \quad \wedge \quad \lambda_2 = \lambda_3$$

$$\Rightarrow \quad \lambda_2 = \lambda_3 = \frac{1}{\sqrt{\lambda_1}}$$

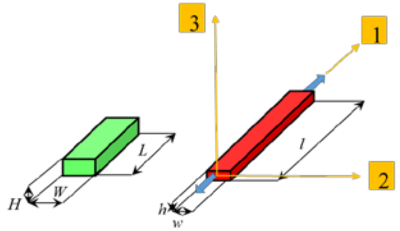
$$P_1 = \mu e^{\alpha\left(\lambda_1^2 + \frac{2}{\lambda_1} - 3\right)}\left(\lambda_1 - \frac{1}{\lambda_1^2}\right)$$

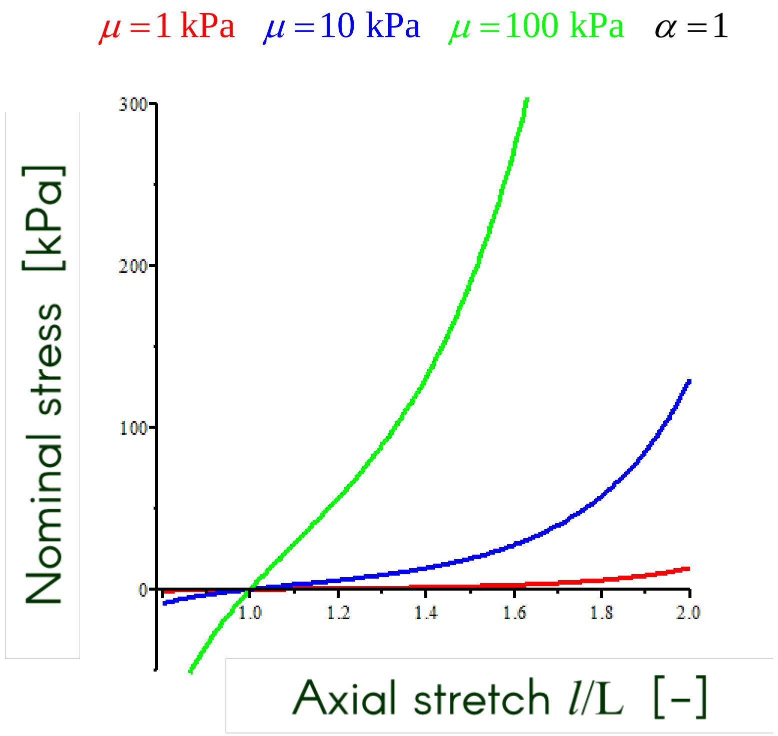
Exponential model by Fung

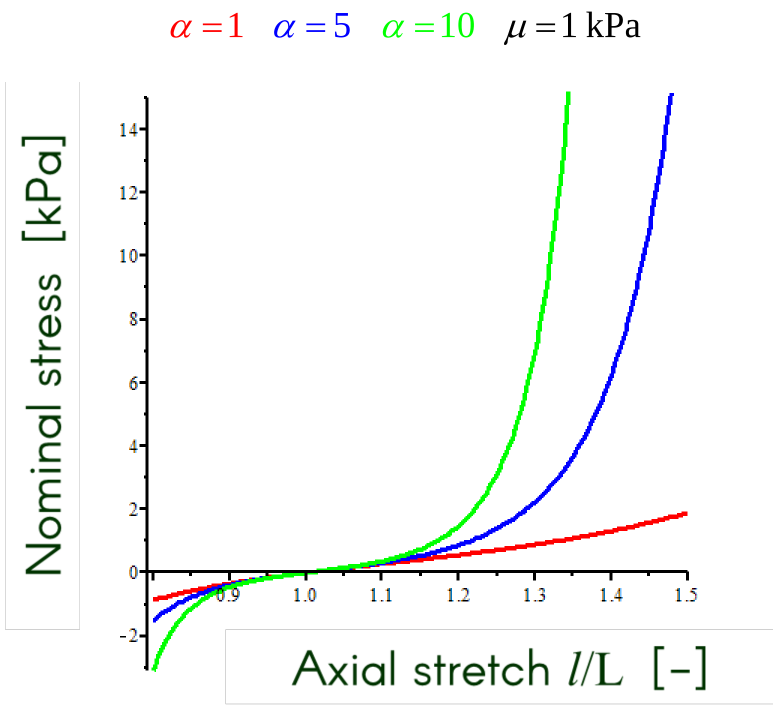
- Isotropic form of Fung model in uniaxial tension

$$W = \frac{\mu}{2\alpha}\left(e^{\alpha(I_1-3)} - 1\right)$$

$$P_1 = \mu e^{\alpha\left(\lambda_1^2+\frac{2}{\lambda_1}-3\right)}\left(\lambda_1 - \frac{1}{\lambda_1^2}\right)$$



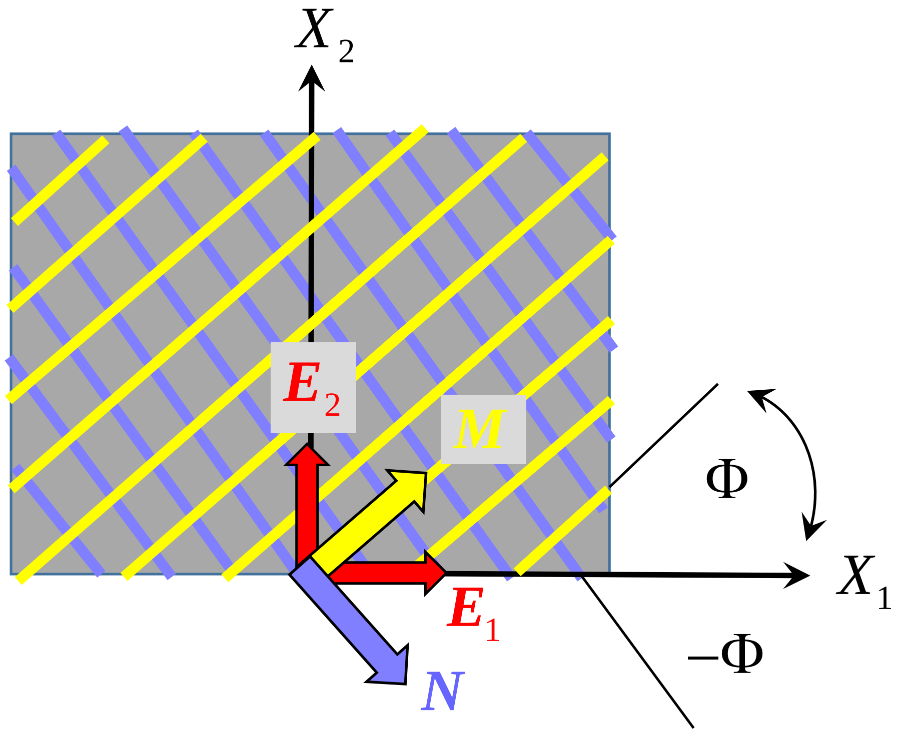




The most frequent model nowadays

- The model introduced by G.A. Holzapfel, T.C. Gasser nad R.W. Ogden in 2000 https://www.biomech.tugraz.at/images/pdf/Holzapfel_et_al-JElasticity-2000.pdf

$$W = \frac{\mu}{2}\left(I_1 - 3\right) + \frac{k_1}{2k_2}\left(e^{k_2\left(I_4-1\right)^2} - 1\right) + \frac{k_1}{2k_2}\left(e^{k_2\left(I_6-1\right)^2} - 1\right)$$

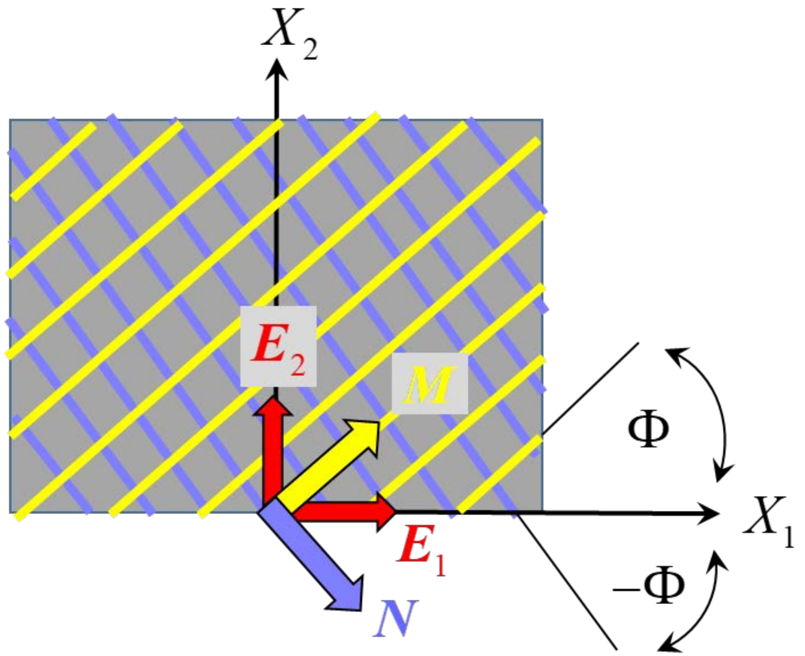


- Material is considered to be homogenized composite consisting of isotropic matrix (cells and compliant extracellular matrix) and two families of collagen fibers

The most frequent model nowadays

- HGO model

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2k_2}\left(e^{k_2(I_4-1)^2} - 1\right) + \frac{k_1}{2k_2}\left(e^{k_2(I_6-1)^2} - 1\right)$$

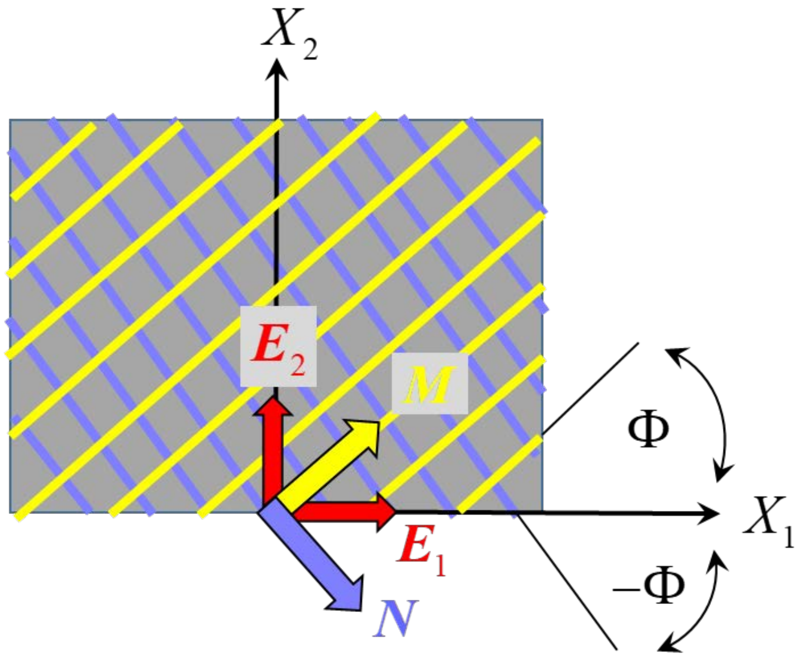


- Collagen fibers are nonlinearly stiff and reinforce the material. They make it anisotropic.
- Collagen fibers form preferred directions, here denoted by unit vectors $\boldsymbol{M}$ and N, in a continuum = anisotropy
- We want to find some invariant capturing preferred directions and the deformation to be incorporated into W

The most frequent model nowadays

- HGO model $W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2k_2}\left(e^{k_2(I_4-1)^2} - 1\right) + \frac{k_1}{2k_2}\left(e^{k_2(I_6-1)^2} - 1\right)$

$$W = W\left(I_1, I_4\left(\boldsymbol{M}\right), I_6\left(\boldsymbol{N}\right)\right)$$

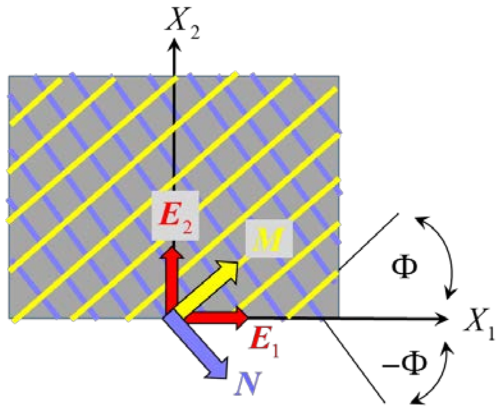


$$\boldsymbol{M} = M_1\boldsymbol{E}_1 + M_2\boldsymbol{E}_2 = \begin{pmatrix} M_1 \\ M_2 \\ 0 \end{pmatrix} = cos\left(\Phi\right)\boldsymbol{E}_1 + sin\left(\Phi\right)\boldsymbol{E}_2 = \begin{pmatrix} cos\left(\Phi\right) \\ sin\left(\Phi\right) \\ 0 \end{pmatrix}$$

$$\boldsymbol{N} = N_1\boldsymbol{E}_1 + N_2\boldsymbol{E}_2 = \begin{pmatrix} N_1 \\ N_2 \\ 0 \end{pmatrix} = cos\left(-\Phi\right)\boldsymbol{E}_1 + sin\left(-\Phi\right)\boldsymbol{E}_2 = \begin{pmatrix} cos\left(-\Phi\right) \\ sin\left(-\Phi\right) \\ 0 \end{pmatrix}$$

The most frequent model nowadays

Reference configuration



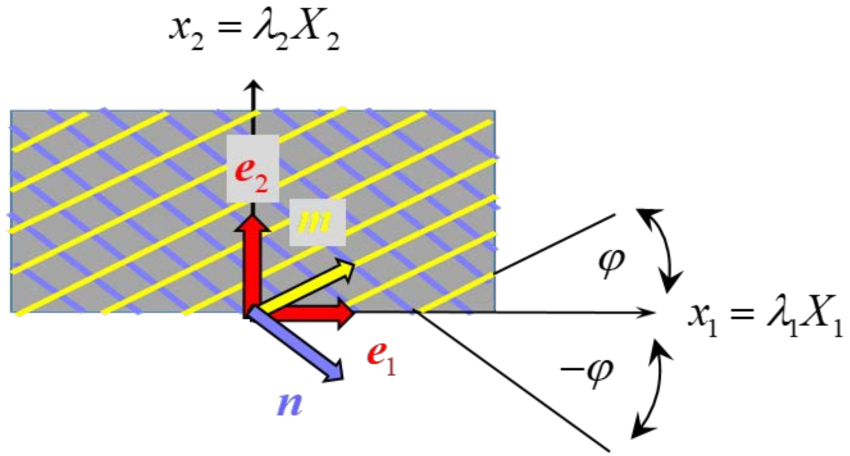
$$W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2k_2}\left(e^{k_2(I_4-1)^2} - 1\right) + \frac{k_1}{2k_2}\left(e^{k_2(I_6-1)^2} - 1\right)$$

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\boldsymbol{m} = \mathbf{F}\boldsymbol{M} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} cos(\Phi) \\ sin(\Phi) \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 cos(\Phi) \\ \lambda_2 sin(\Phi) \\ 0 \end{pmatrix}$$

$$\boldsymbol{n} = \mathbf{F}\boldsymbol{N} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} cos(-\Phi) \\ sin(-\Phi) \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 cos(-\Phi) \\ \lambda_2 sin(-\Phi) \\ 0 \end{pmatrix}$$

Deformed configuration



The most frequent model nowadays

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \boldsymbol{m} = \mathbf{F}\boldsymbol{M}$$

$$\boldsymbol{M} = cos(\Phi)\boldsymbol{E}_1 + sin(\Phi)\boldsymbol{E}_2$$

$$\boldsymbol{m} = cos(\varphi)\boldsymbol{e}_1 + sin(\varphi)\boldsymbol{e}_2 = \lambda_1 cos(\Phi)\boldsymbol{e}_1 + \lambda_2 sin(\Phi)\boldsymbol{e}_2$$

- Scalar product of the deformed preferred vector gives new invariant

$$\boldsymbol{m}\cdot\boldsymbol{m} = \begin{pmatrix} \lambda_1 cos(\Phi) & \lambda_2 sin(\Phi) & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 cos(\Phi) \\ \lambda_2 sin(\Phi) \\ 0 \end{pmatrix} = \lambda_1^2 cos^2(\Phi) + \lambda_2^2 sin^2(\Phi) = \lambda_M^2 = I_4$$

$$\boldsymbol{m}\cdot\boldsymbol{m} = (\mathbf{F}\boldsymbol{M})\cdot(\mathbf{F}\boldsymbol{M}) = \boldsymbol{M}\mathbf{F}^T\mathbf{F}\boldsymbol{M} = \boldsymbol{M}\cdot(\mathbf{C}\boldsymbol{M}) = I_4$$

The most frequent model nowadays

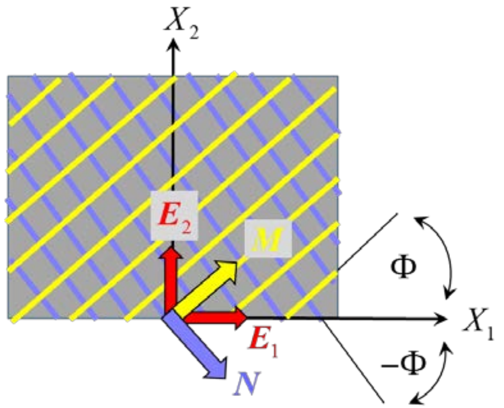
- Holzapfel, Gasser, Ogden

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$I_4 = \boldsymbol{m} \cdot \boldsymbol{m} = \lambda_M^2 = \lambda_1^2 \cos^2(\Phi) + \lambda_2^2 \sin^2(\Phi)$$

$$I_6 = \boldsymbol{n} \cdot \boldsymbol{n} = \lambda_N^2 = \lambda_1^2 \cos^2(-\Phi) + \lambda_2^2 \sin^2(-\Phi)$$

$$I_4 = \lambda_1^2 \cos^2(\Phi) + \lambda_2^2 \sin^2(\Phi) = I_6$$

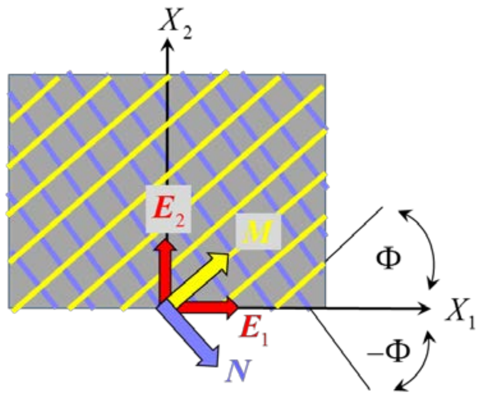


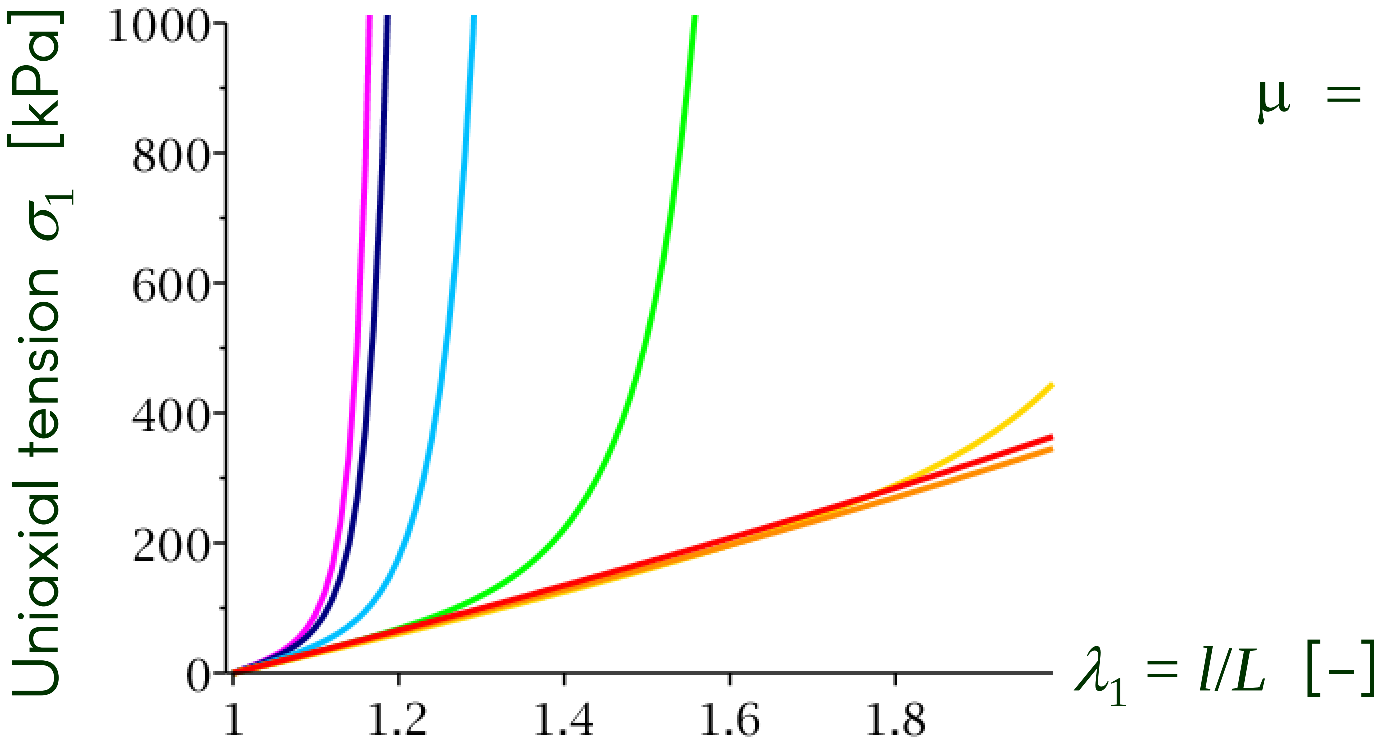
$$W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2k_2}\left(e^{k_2(I_4-1)^2} - 1\right) + \frac{k_1}{2k_2}\left(e^{k_2(I_6-1)^2} - 1\right)$$

The most frequent model nowadays

- Holzapfel, Gasser, Ogden

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{k_1}{2k_2}\left(e^{k_2(I_4-1)^2} - 1\right) + \frac{k_1}{2k_2}\left(e^{k_2(I_6-1)^2} - 1\right)$$





$\mu = 100$ kPa, $k_1 = 20$ kPa, $k_2 = 25$

$\Phi = 0°$
$\Phi = 15°$
$\Phi = 30°$
$\Phi = 45°$
$\Phi = 60°$
$\Phi = 75°$
$\Phi = 90°$