Matematická propedeutika
totiž na co bychom určitě neměli zapomenout

při studiu MKP

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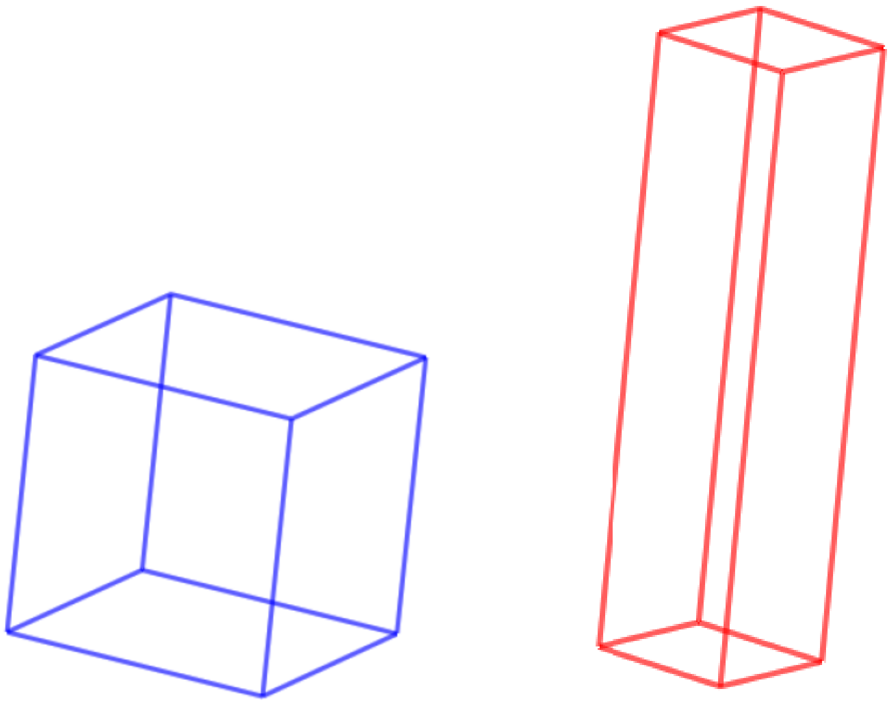
Tělesa

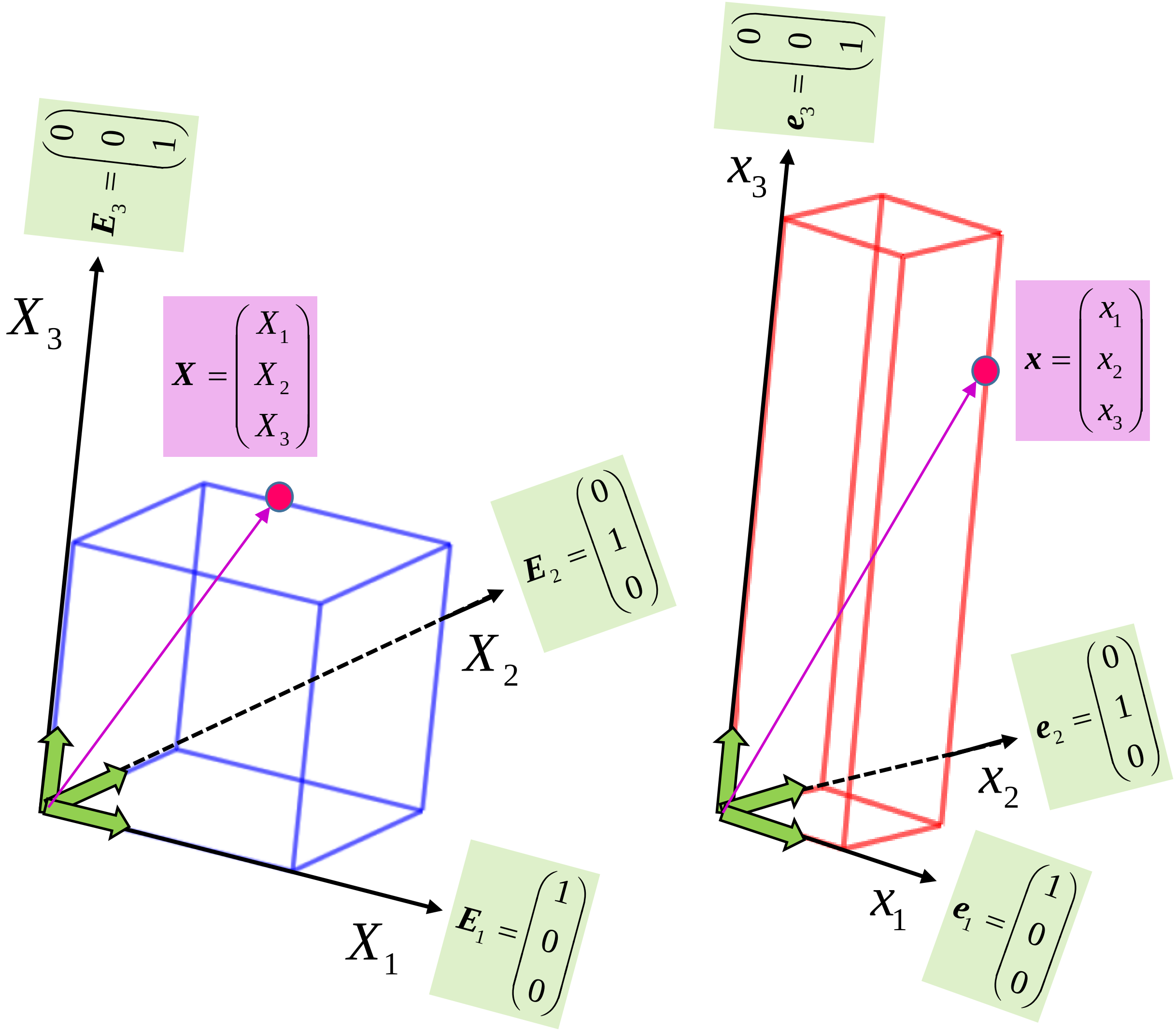
Mějme dvě tělesa…



a představme si je v matematických prostorech, které známe

- $\mathbb{R}^3$
- myslíme na prostor, kde každý bod má tři reálné souřadnice
- kde umíme měřit vzdálenosti
- kde každý bod umíme zaměřit vektorem jdoucím z počátku souřadnic

Vektorová algebra a analytická geometrie



Vzpomeňme si na

- *Báze*
- *Operace (součiny, součty,…)*
- *Souřadnice*
- *Transformace*

Vektorová algebra a analytická geometrie

složky vektoru

v dané bázi

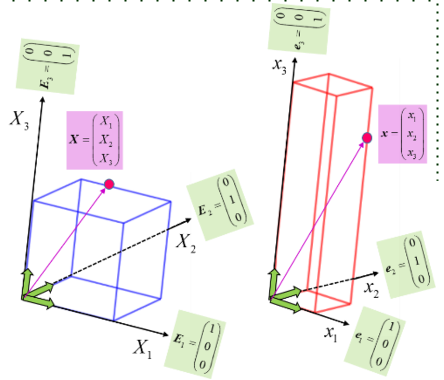
$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3 = u_1\begin{pmatrix}1\\0\\0\end{pmatrix} + u_2\begin{pmatrix}0\\1\\0\end{pmatrix} + u_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$$

standardní báze v $\mathbb{R}^3$

ortonormální, kartézská

$$\boldsymbol{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \boldsymbol{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \boldsymbol{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Vektorová algebra a analytická geometrie



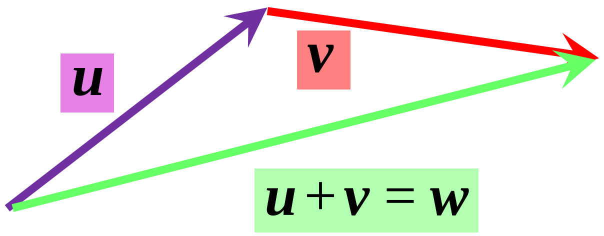
Mějme vektory $\boldsymbol{u}$ a $\boldsymbol{v}$ a $\boldsymbol{w}$ z vektorového prostoru $\mathcal{V}$ nad $\mathbb{R}$

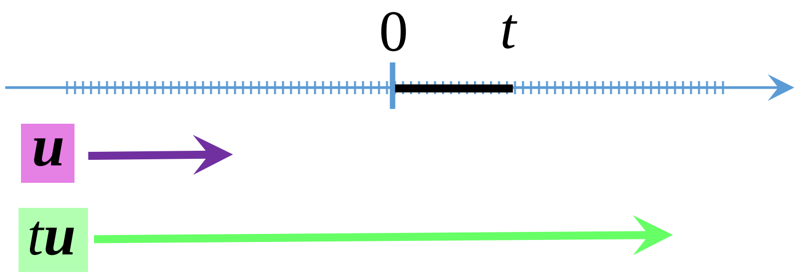
$$\boldsymbol{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} \quad \boldsymbol{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$$ a $t\in\mathbb{R}$

Ve $\mathcal{V}$ platí:

$$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{w}=\begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{pmatrix}$$

$$t\cdot\boldsymbol{u}=\begin{pmatrix}t\cdot u_1\\t\cdot u_2\\t\cdot u_3\end{pmatrix}$$



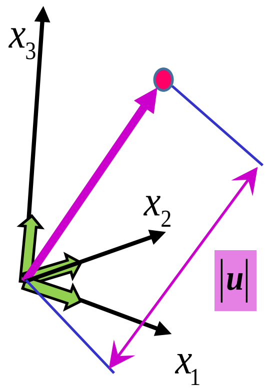


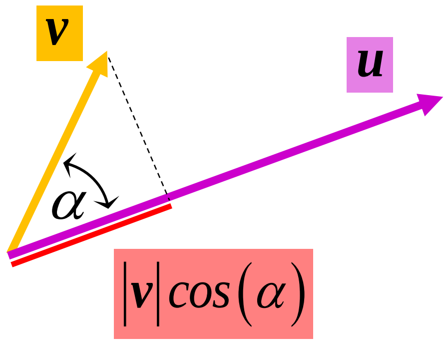
Vektorová algebra a analytická geometrie

norma vektoru

je číslo vyjadřující jeho délku

$$u = |\boldsymbol{u}| = \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}} = \sqrt{[\boldsymbol{u}]^T[\boldsymbol{u}]}$$



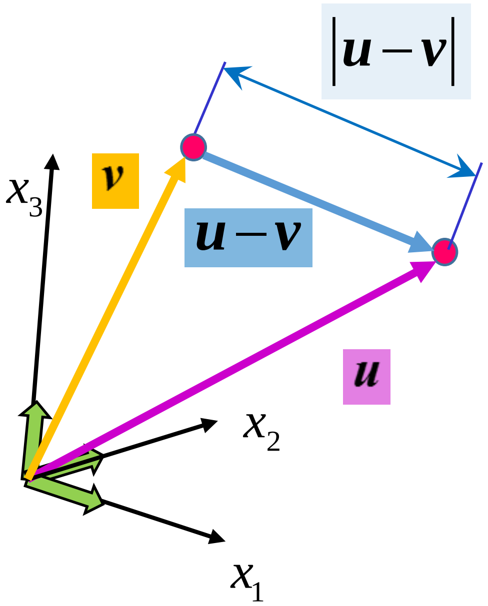


$$cos(\alpha) = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

skalární součin

je číslo

$$\boldsymbol{u}\cdot\boldsymbol{v} = \sum_{i=1}^{3} u_i v_i = u_i v_i = [\boldsymbol{u}]^T[\boldsymbol{v}] = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\boldsymbol{u}||\boldsymbol{v}|cos(\alpha)$$



vzdálenost vektorů

(jimi zaměřených bodů) je číslo

$$\rho(\boldsymbol{u},\boldsymbol{v}) = |\boldsymbol{u}-\boldsymbol{v}| = \sqrt{(\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})}$$

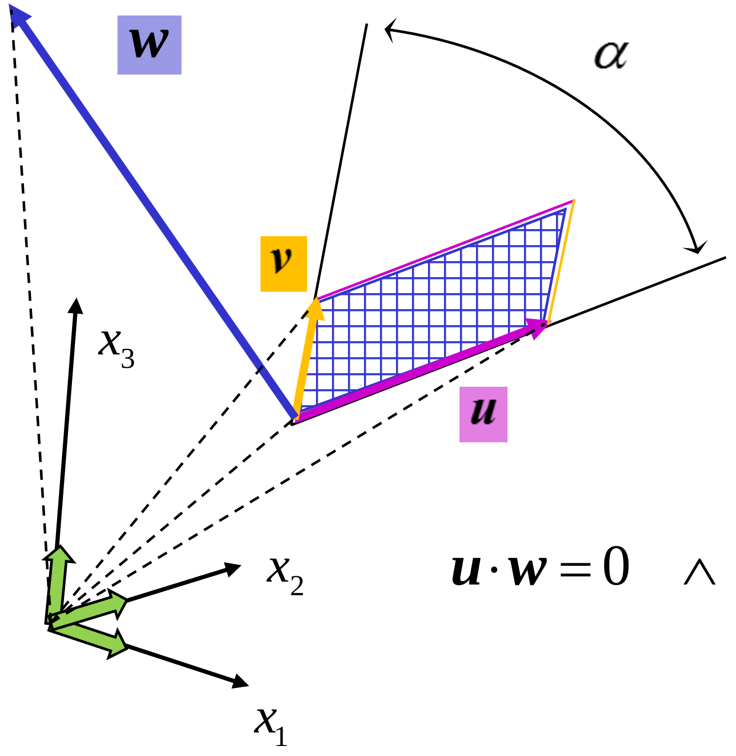
Vektorová algebra a analytická geometrie

vektorový součin

je vektor

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=\begin{pmatrix}u_2v_3-u_3v_2\\ u_3v_1-u_1v_3\\ u_1v_2-u_2v_1\end{pmatrix}=\boldsymbol{w}$$

$$|\boldsymbol{w}|=|\boldsymbol{u}||\boldsymbol{v}|\,sin(\alpha)$$

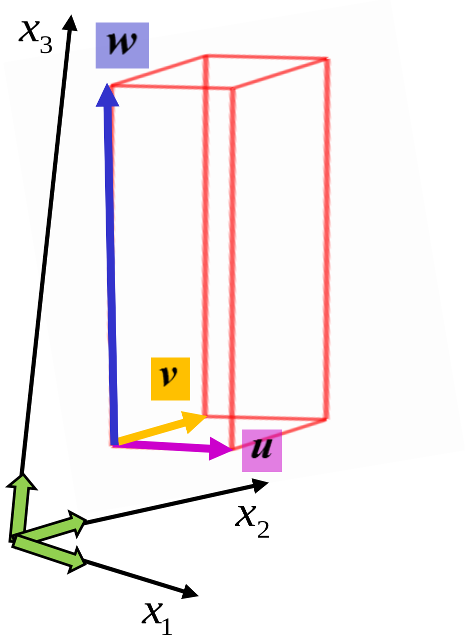


$$\boldsymbol{u}\cdot\boldsymbol{w}=0 \quad\wedge\quad \boldsymbol{v}\cdot\boldsymbol{w}=0$$

smíšený součin

je číslo vyjadřující objem rovnoběžnostěnu daného vektory ***u*, *v*, *w***

$$V=(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}=\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})=\boldsymbol{v}\cdot(\boldsymbol{w}\times\boldsymbol{u})$$



Vektorová algebra a analytická geometrie

Lineární transformace $\mathbf{A}$ prostoru $\mathcal{V}$

$$\mathbf{A}:\mathcal{V}\to\mathcal{V}$$

$$\mathbf{A}(\boldsymbol{u}+\boldsymbol{v})=\mathbf{A}(\boldsymbol{u})+\mathbf{A}(\boldsymbol{v})$$ Aditivita

$$\mathbf{A}(t\boldsymbol{u})=t\,\mathbf{A}(\boldsymbol{u})$$ Homogenita

Lineární transformaci $\mathbf{A}$ prostoru $\mathcal{V}$ reprezentujeme maticí $\mathbf{A}$

$$\mathbf{A}(\boldsymbol{u}+\boldsymbol{v})=\mathbf{A}\boldsymbol{u}+\mathbf{A}\boldsymbol{v}$$

$$\mathbf{A}(t\boldsymbol{u})=t\,\mathbf{A}\boldsymbol{u}$$

$$\mathbf{A}\boldsymbol{u}=\boldsymbol{w}$$

$$\mathbf{A}\boldsymbol{u}=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}=\begin{pmatrix}a_{11}u_1+a_{12}u_2+a_{13}u_3\\ a_{21}u_1+a_{22}u_2+a_{23}u_3\\ a_{31}u_1+a_{32}u_2+a_{33}u_3\end{pmatrix}=\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix}$$

Vektorová algebra a analytická geometrie

Násobení matic $\mathbf{B}$ a $\mathbf{A}$ interpretujeme jako *skládání zobrazení*
Vzniká nové zobrazení $\mathbf{C} = \mathbf{BA}$
Celou událost čteme jako $\mathbf{B}$ po $\mathbf{A}$

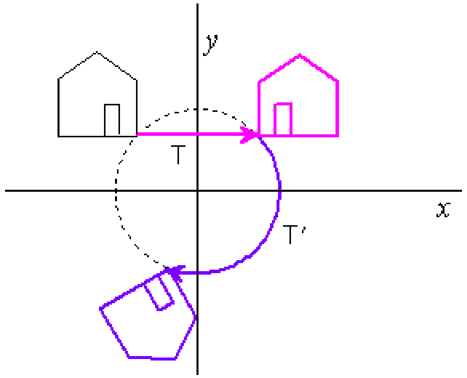
$$\mathbf{C} = \mathbf{B}\,\mathbf{A} \neq \mathbf{AB} = \mathbf{C}'$$

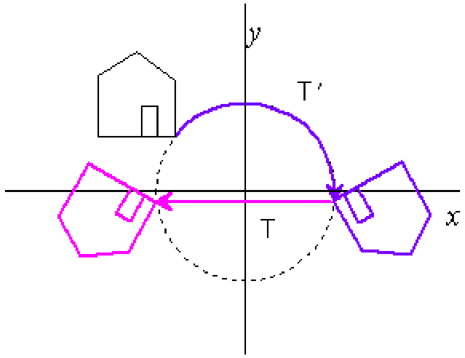
$$\mathbf{C}(\boldsymbol{u}) = \mathbf{B} \circ \mathbf{A}(\boldsymbol{u}) = \mathbf{B}\big(\mathbf{A}(\boldsymbol{u})\big) = \mathbf{B}(\boldsymbol{w}) = \boldsymbol{z}$$

$\mathbf{A}\boldsymbol{u} = \boldsymbol{w}$ $\quad$ $\mathbf{B}\boldsymbol{w} = \boldsymbol{z}$

$$\mathbf{C}'(\boldsymbol{u}) = \mathbf{A} \circ \mathbf{B}(\boldsymbol{u}) = \mathbf{A}\big(\mathbf{B}(\boldsymbol{u})\big) = \mathbf{A}(\boldsymbol{w}') = \boldsymbol{z}'$$

$\mathbf{B}\boldsymbol{u} = \boldsymbol{w}'$ $\quad$ $\mathbf{A}\boldsymbol{w}' = \boldsymbol{z}'$





Vektorová algebra a analytická geometrie

$$\mathbf{C} = \mathbf{B}\,\mathbf{A} \quad \Leftrightarrow \quad C_{ij} = \sum_{k=1}^{3} B_{ik} A_{kj} = B_{ik} A_{kj}$$

$$C_{11} = B_{11}A_{11} + B_{12}A_{21} + B_{13}A_{31} = \sum_{k=1}^{3} B_{1k}A_{k1}$$

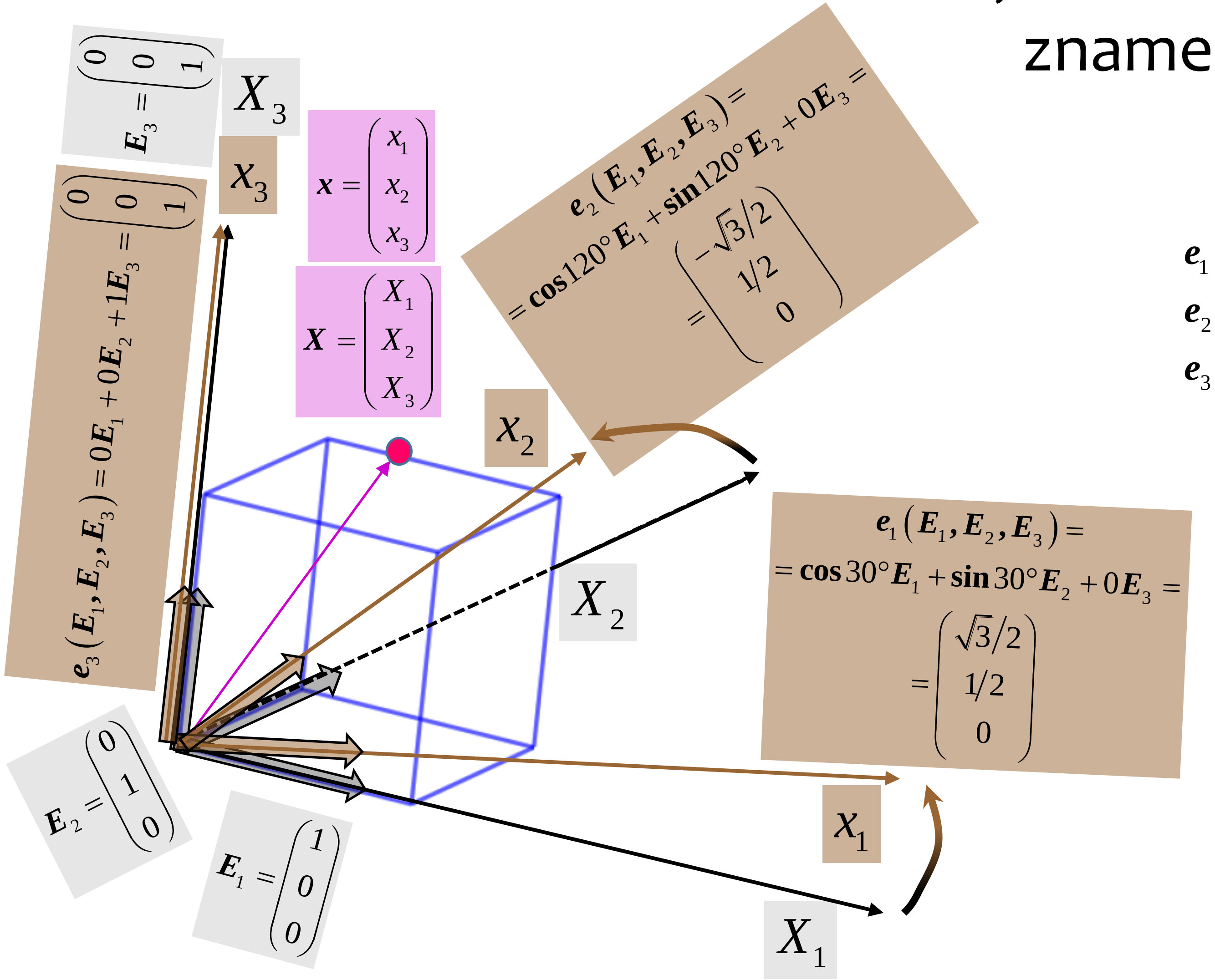
$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$C_{23} = B_{21}A_{13} + B_{22}A_{23} + B_{23}A_{33} = \sum_{k=1}^{3} B_{2k}A_{k3}$$

$$C_{33} = B_{31}A_{13} + B_{32}A_{23} + B_{33}A_{33} = \sum_{k=1}^{3} B_{3k}A_{k3}$$

Vektorová algebra a analytická geometrie

Dvě báze v jednom prostoru: Natočit soustavu souřadnic znamená —> přejít od jedné vektorové báze k druhé



$$
\begin{aligned}
e_1 &= \cos 30° E_1 + \sin 30° E_2 + 0 E_3 \\
e_2 &= \cos 120° E_1 + \sin 120° E_2 + 0 E_3 \\
e_3 &= 0 E_1 + 0 E_2 + 1 E_3
\end{aligned}
\qquad
Mat_E^e = \begin{pmatrix} \sqrt{3}/2 & -\sqrt{3}/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} \sqrt{3}/2 & -\sqrt{3}/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}
$$

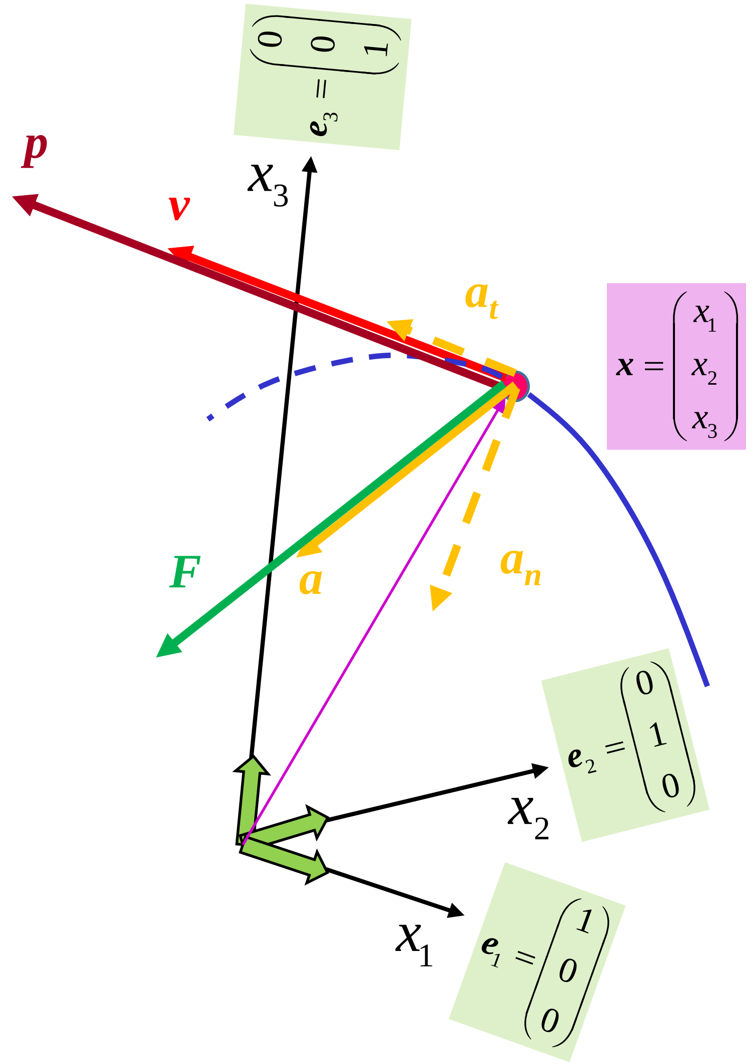
$$
Mat_E^e = \mathbf{R}^{-1}
$$

$$
\mathbf{R}^{-1} x = X \qquad \mathbf{R} X = x
$$

Vektorová algebra a analytická geometrie

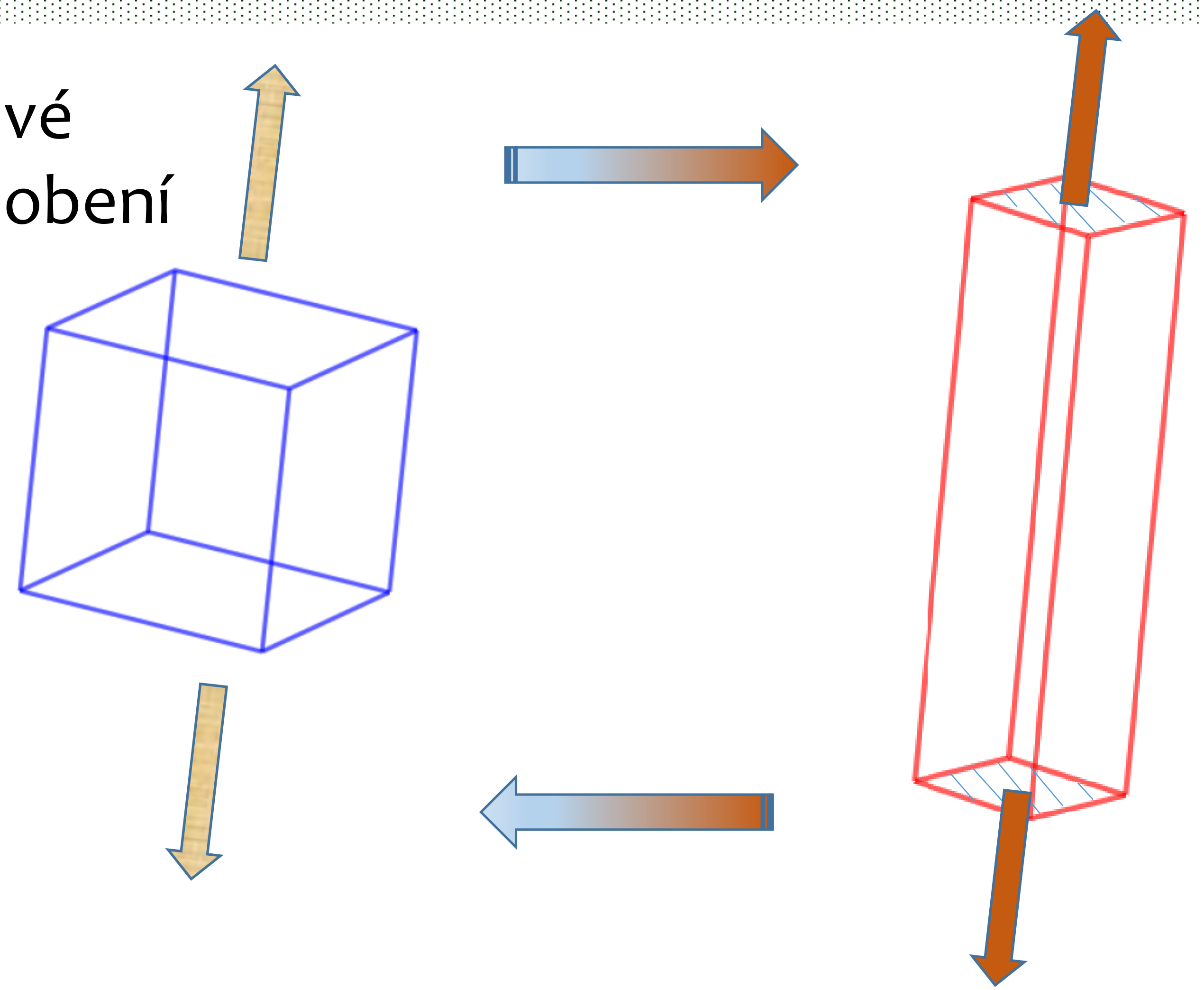
Připomeňme si některé, v mechanice běžné, vektorové veličiny

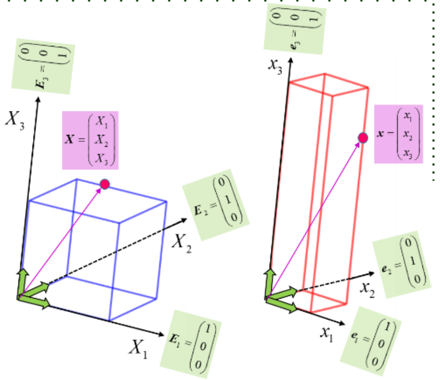
- Polohový vektor $\boldsymbol{x}$
- Rychlost a zrychlení $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \qquad \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$
- Síla $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$
- Hybnost $\boldsymbol{p} = m\boldsymbol{v}$
- Moment síly vzhledem k počátku $\boldsymbol{M} = \boldsymbol{x} \times \boldsymbol{F}$
- Moment hybnosti vzhledem k počátku $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$



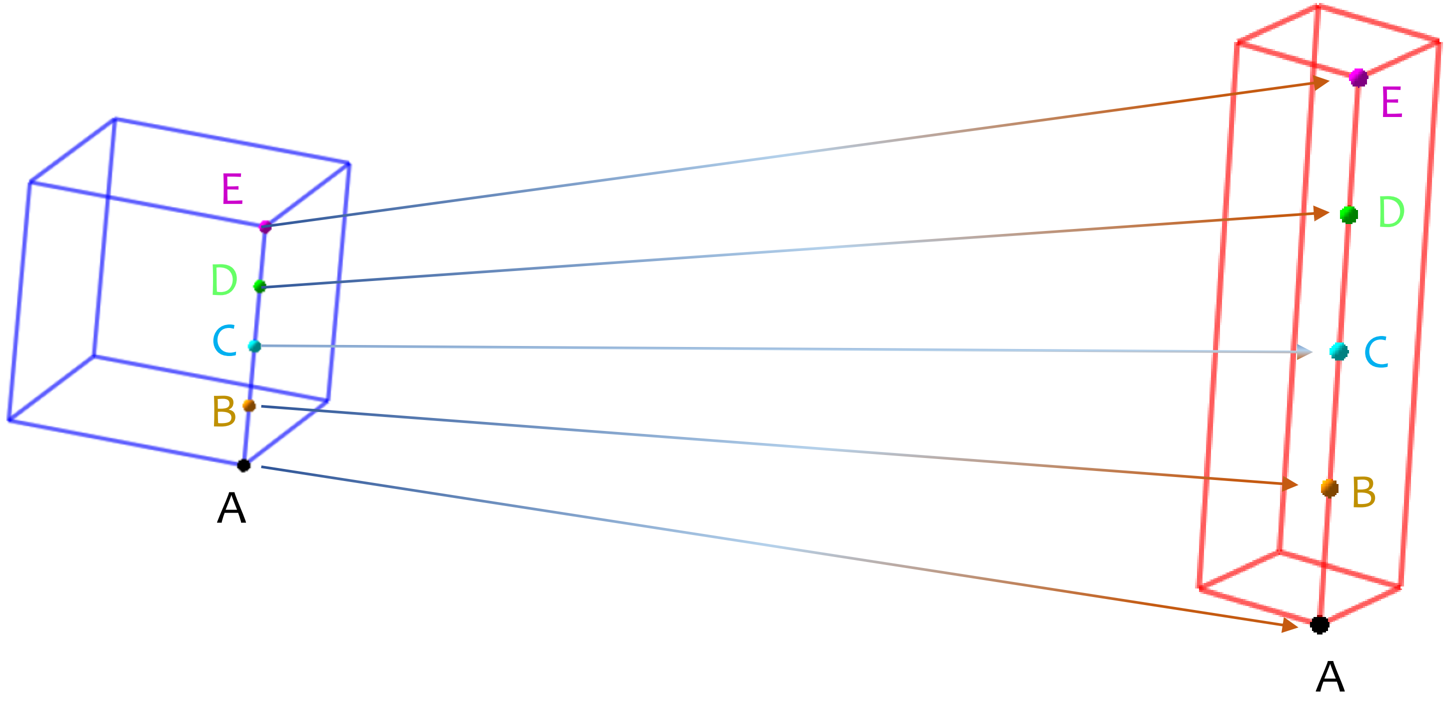
Deformace

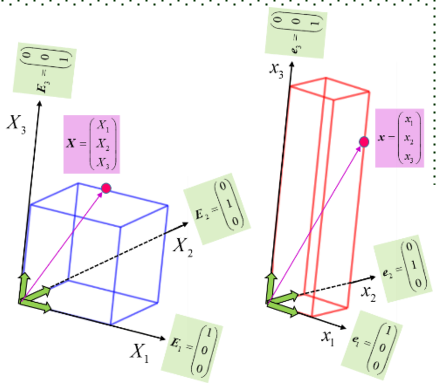
Silové
působení



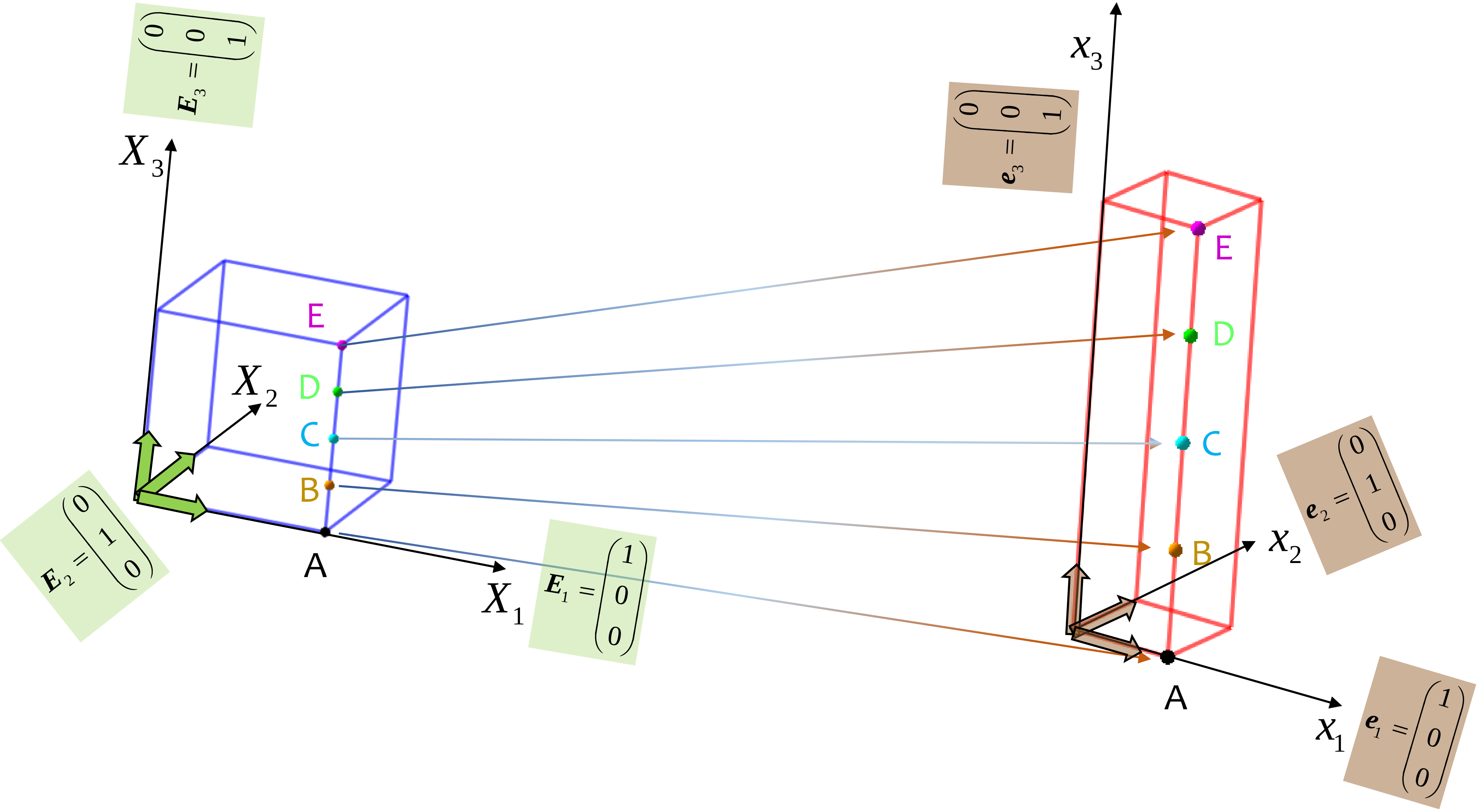


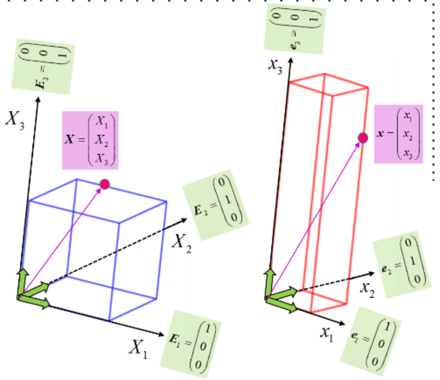
Deformace = Zobrazení



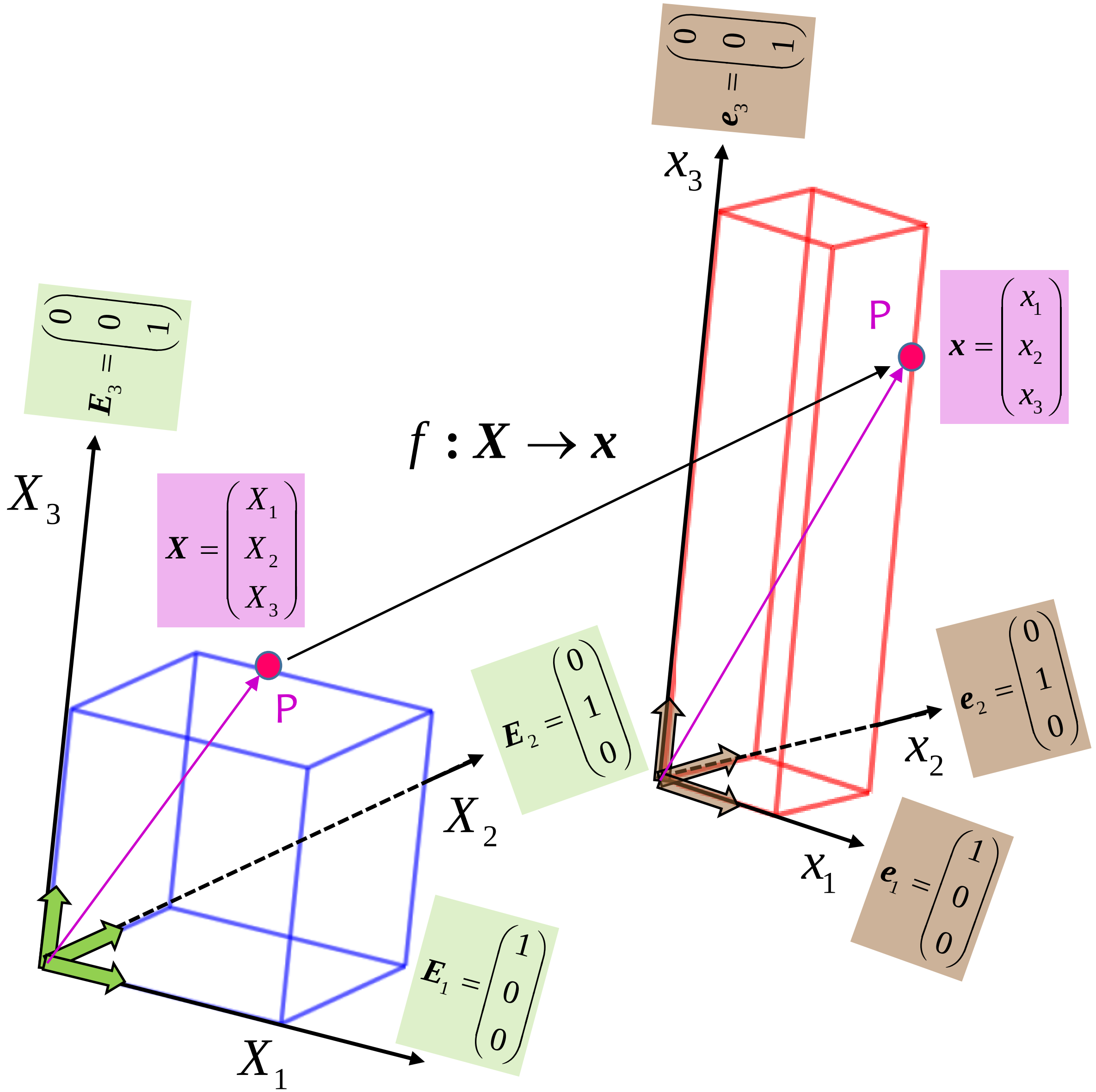


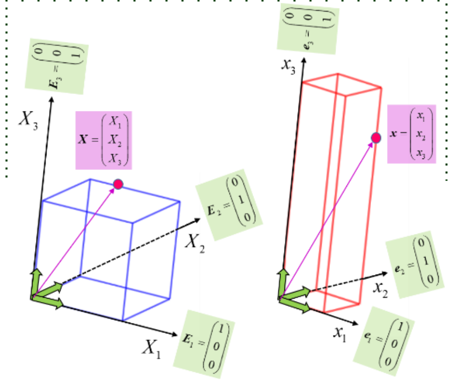
Deformace = Zobrazení





Zobrazení $\quad f : \boldsymbol{X} \to \boldsymbol{x}$





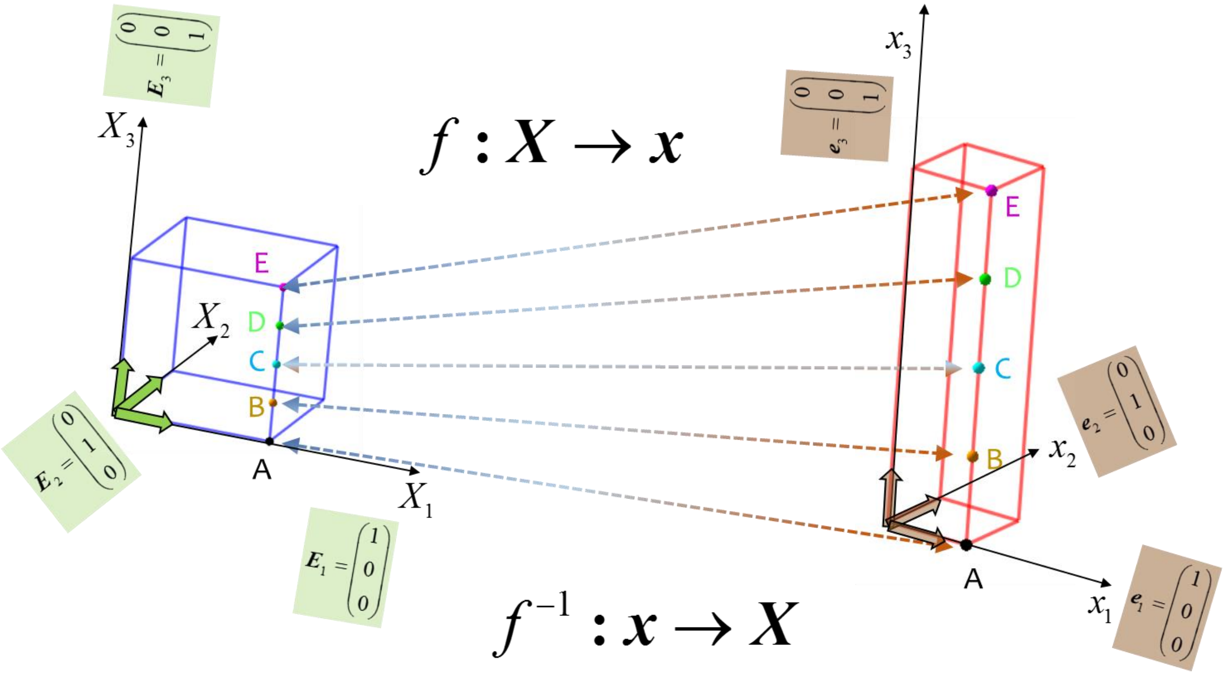
$$f : \boldsymbol{X} \to \boldsymbol{x}$$
$$\boldsymbol{x} = f(\boldsymbol{X})$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1(X_1, X_2, X_3) \\ f_2(X_1, X_2, X_3) \\ f_3(X_1, X_2, X_3) \end{pmatrix} =$$
$$= f_1(X_1, X_2, X_3)\boldsymbol{e}_1 + f_2(X_1, X_2, X_3)\boldsymbol{e}_2 + f_3(X_1, X_2, X_3)\boldsymbol{e}_3$$

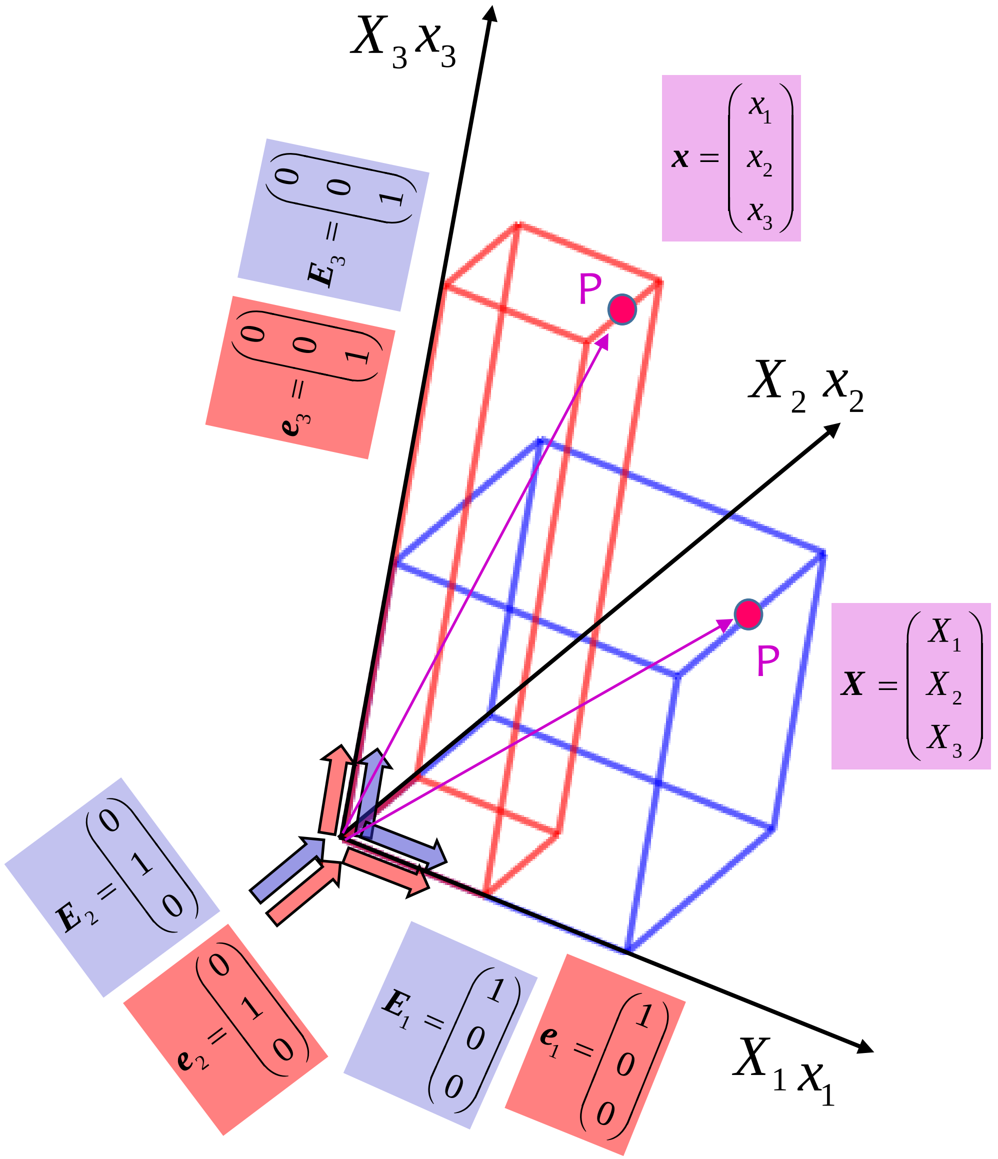
Zobrazení $f^{-1}: \boldsymbol{x} \to \boldsymbol{X}$

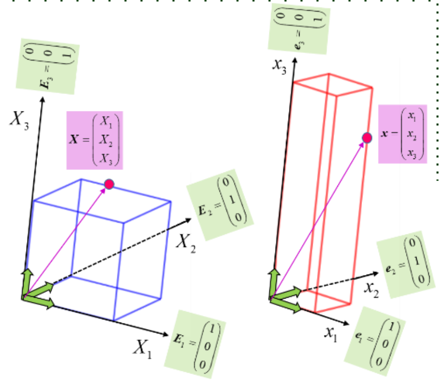
Po zobrazení $f: \boldsymbol{X} \to x$ požadujeme, aby bylo *bijektivní* (vzájemně jednoznačné) a *spojité*

Požadujeme též existenci *spojitého* f^{-1}



Zobrazení $f: \boldsymbol{X} \rightarrow \boldsymbol{x}$

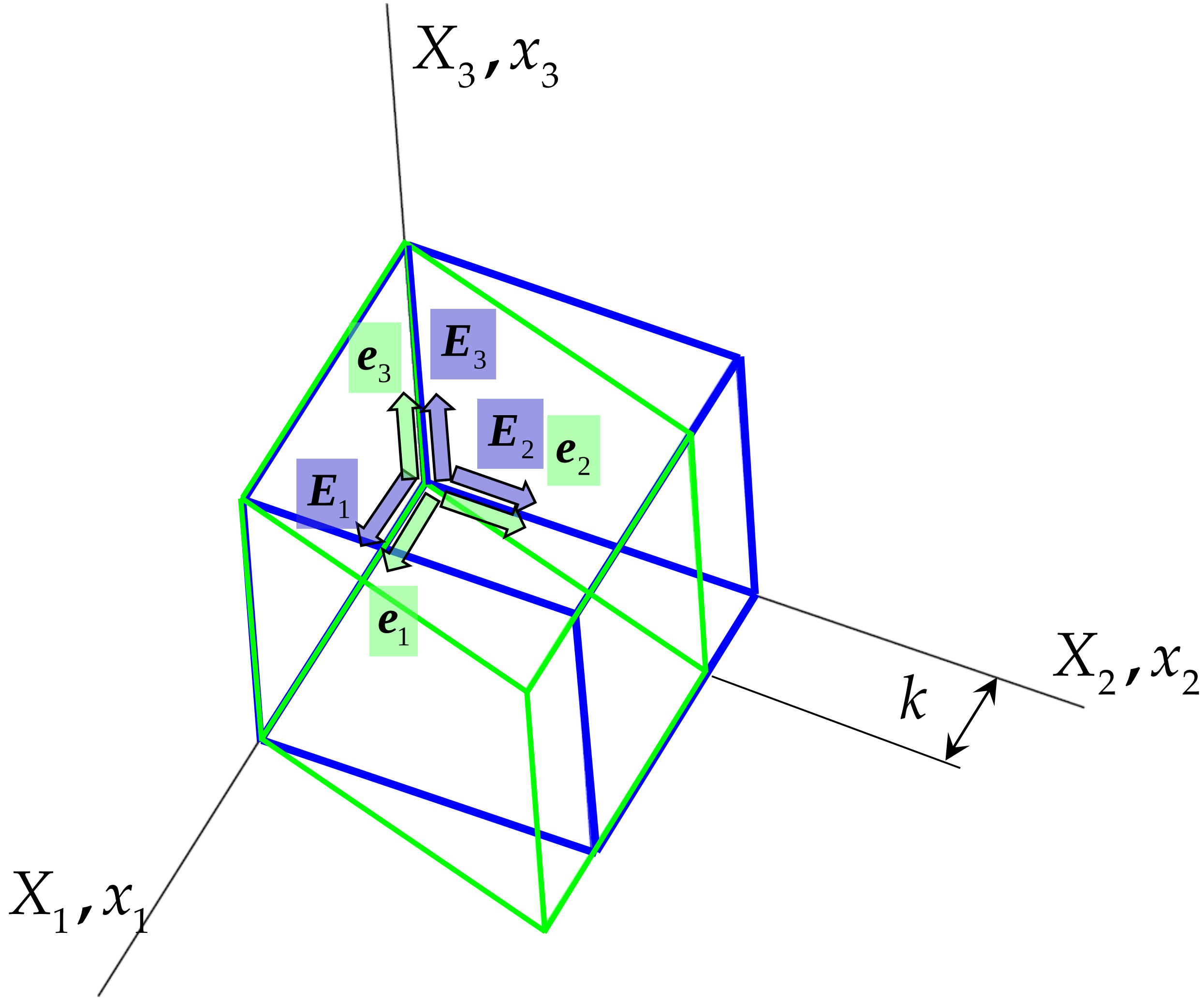


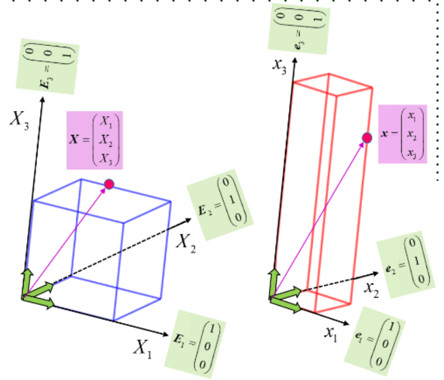


$$f: \boldsymbol{X} \rightarrow \boldsymbol{x}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.5X_1 \\ 0.5X_2 \\ 2X_3 \end{pmatrix} = 0.5X_1\boldsymbol{e}_1 + 0.5X_2\boldsymbol{e}_2 + 2X_3\boldsymbol{e}_3$$

Zobrazení $\quad f : \boldsymbol{X} \to \boldsymbol{x}$

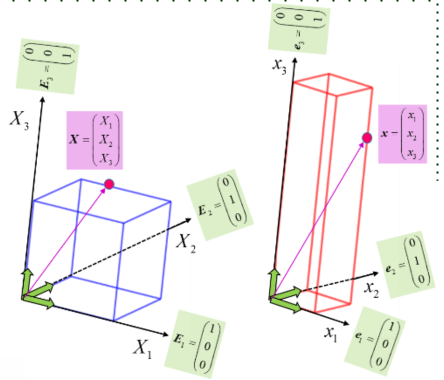




$$f : \boldsymbol{X} \to \boldsymbol{x}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 + kX_2 \\ X_2 \\ X_3 \end{pmatrix} = \left(X_1 + kX_2\right)\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3$$

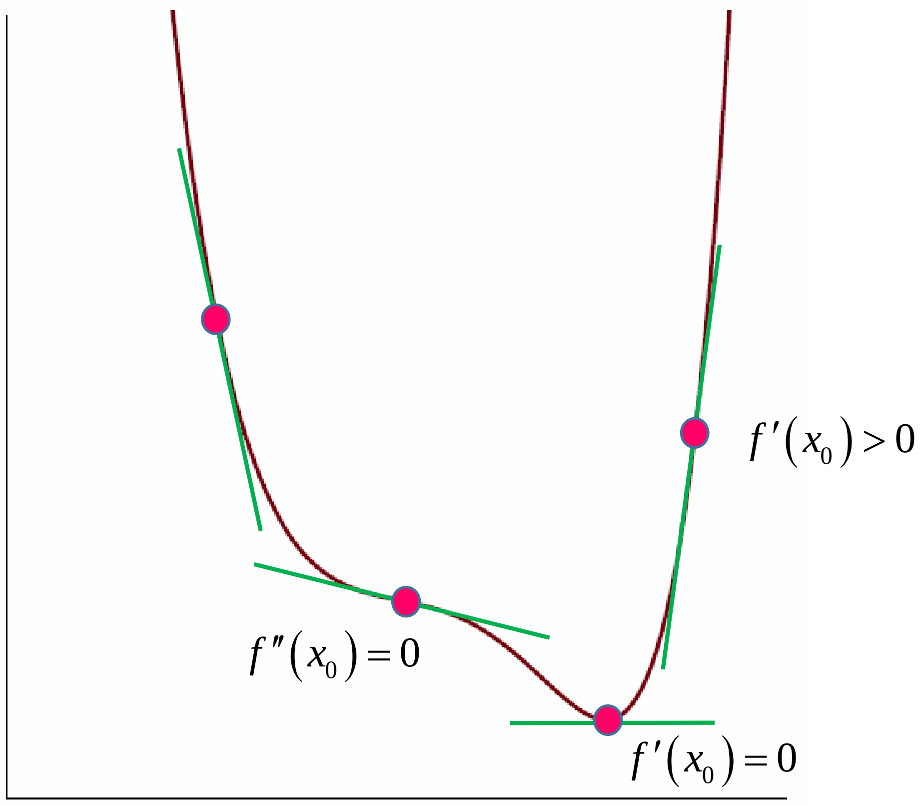
Zjišťování lokálních vlastností zobrazení



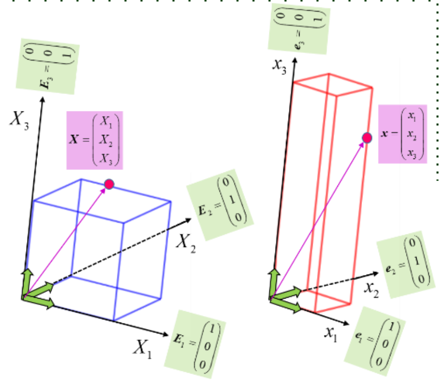
Derivace

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \to f(x)$$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Zjišťování lokálních vlastností zobrazení



Směrová derivace a gradient

$$f:\mathbb{R}^3 \to \mathbb{R} \qquad \boldsymbol{x} \to f(\boldsymbol{x})$$

$$\nabla_{\boldsymbol{v}} f(\boldsymbol{x}_0) = \lim_{h\to 0} \frac{f(\boldsymbol{x}_0 + h\boldsymbol{v}) - f(\boldsymbol{x}_0)}{h} = \nabla f(\boldsymbol{x}_0)\cdot \boldsymbol{v}$$

Směrová derivace

$$\nabla_{(1,0,0)} f(\boldsymbol{x}_0) = \lim_{h\to 0} \frac{f(\boldsymbol{x}_0 + h(1,0,0)) - f(\boldsymbol{x}_0)}{h} = \frac{\partial f(\boldsymbol{x}_0)}{\partial x_1}$$

Parciální derivace

$$\nabla f(\boldsymbol{x}_0) = \begin{pmatrix} \dfrac{\partial f(\boldsymbol{x}_0)}{\partial x_1} \\ \dfrac{\partial f(\boldsymbol{x}_0)}{\partial x_2} \\ \dfrac{\partial f(\boldsymbol{x}_0)}{\partial x_3} \end{pmatrix} = \frac{\partial f(\boldsymbol{x}_0)}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial f(\boldsymbol{x}_0)}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial f(\boldsymbol{x}_0)}{\partial x_3}\boldsymbol{e}_3$$

Gradient skalární funkce

Zjišťování lokálních vlastností zobrazení

Gradient $f:\mathbb{R}^3 \to \mathbb{R}^3$ $\boldsymbol{x} = f\left(\boldsymbol{X}\right)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1\left(X_1, X_2, X_3\right) \\ f_2\left(X_1, X_2, X_3\right) \\ f_3\left(X_1, X_2, X_3\right) \end{pmatrix}$$

$$\nabla f\left(\boldsymbol{x}_0\right) = \begin{pmatrix} \frac{\partial f_1\left(\boldsymbol{x}_0\right)}{\partial x_1} & \frac{\partial f_1\left(\boldsymbol{x}_0\right)}{\partial x_2} & \frac{\partial f_1\left(\boldsymbol{x}_0\right)}{\partial x_3} \\ \frac{\partial f_2\left(\boldsymbol{x}_0\right)}{\partial x_1} & \frac{\partial f_2\left(\boldsymbol{x}_0\right)}{\partial x_2} & \frac{\partial f_2\left(\boldsymbol{x}_0\right)}{\partial x_3} \\ \frac{\partial f_3\left(\boldsymbol{x}_0\right)}{\partial x_1} & \frac{\partial f_3\left(\boldsymbol{x}_0\right)}{\partial x_2} & \frac{\partial f_3\left(\boldsymbol{x}_0\right)}{\partial x_3} \end{pmatrix}$$

Zjišťování lokálních vlastností zobrazení

Gradient $\quad \boldsymbol{x} = f\left(\boldsymbol{X}\right)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1\left(X_1, X_2, X_3\right) \\ f_2\left(X_1, X_2, X_3\right) \\ f_3\left(X_1, X_2, X_3\right) \end{pmatrix} = f_1\left(X_1, X_2, X_3\right)\boldsymbol{e}_1 + f_2\left(X_1, X_2, X_3\right)\boldsymbol{e}_2 + f_3\left(X_1, X_2, X_3\right)\boldsymbol{e}_3$$

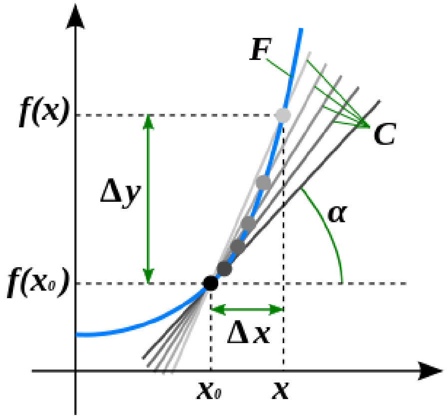
$$\nabla f\left(\boldsymbol{x}_0\right) = \begin{pmatrix} \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_1} & \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_2} & \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_3} \\ \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_1} & \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_2} & \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_3} \\ \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_1} & \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_2} & \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_3} \end{pmatrix} = \begin{array}{cc} & \begin{array}{ccc} \quad\boldsymbol{e}_1 & \qquad\boldsymbol{e}_2 & \qquad\boldsymbol{e}_3 \end{array} \\ \begin{array}{c} \boldsymbol{e}_1 \\[2ex] \boldsymbol{e}_2 \\[2ex] \boldsymbol{e}_3 \end{array} & \begin{pmatrix} \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_1} & \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_2} & \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_3} \\ \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_1} & \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_2} & \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_3} \\ \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_1} & \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_2} & \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_3} \end{pmatrix} \end{array}$$

$$\frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_1}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_2}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \frac{\partial f_1(\boldsymbol{x}_0)}{\partial x_3}\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_1}\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_2}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \frac{\partial f_2(\boldsymbol{x}_0)}{\partial x_3}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_1}\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_2}\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \frac{\partial f_3(\boldsymbol{x}_0)}{\partial x_3}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

Zjišťování lokálních vlastností zobrazení

Podstatou stále zůstává derivování…
měření velikosti změny $\Delta f(x)$ vzhledem ke změně Δx

$$\frac{df(x)}{dx}$$



Gradient $\boldsymbol{x} = f(\boldsymbol{X})$

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

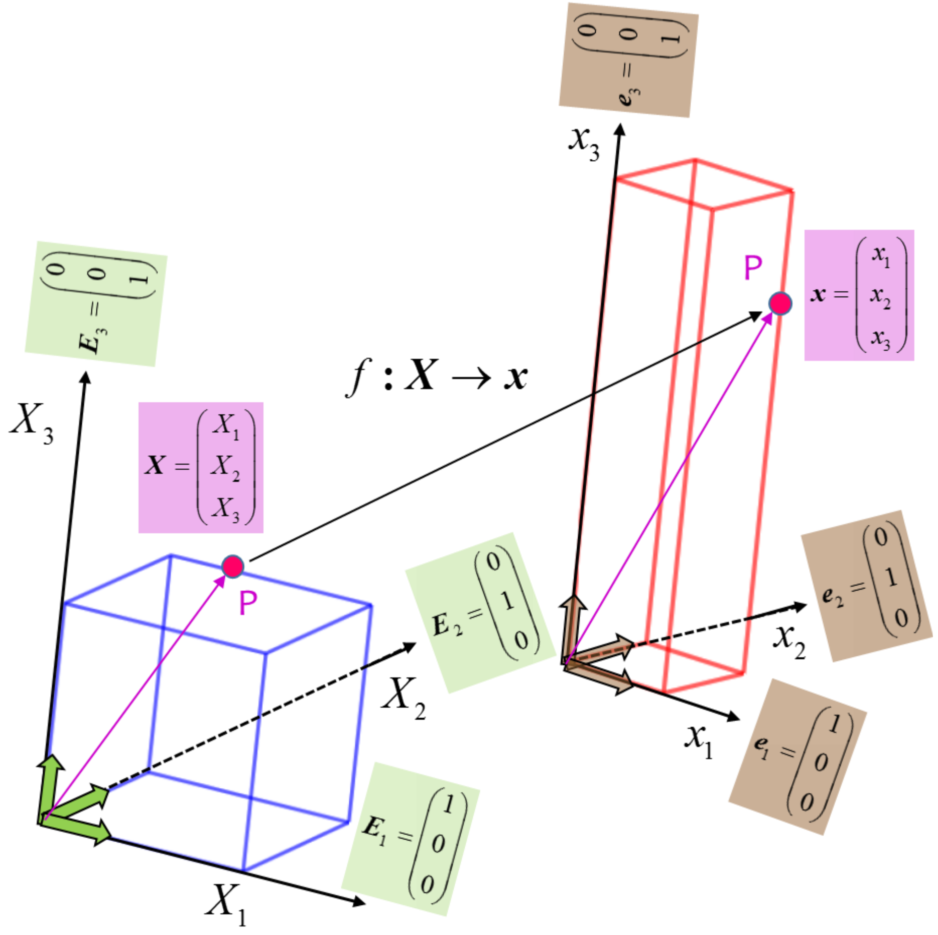
$$\frac{\partial f_i(X_1, X_2, X_3)}{\partial x_j} \qquad i, j = 1, 2, 3$$

Deformace

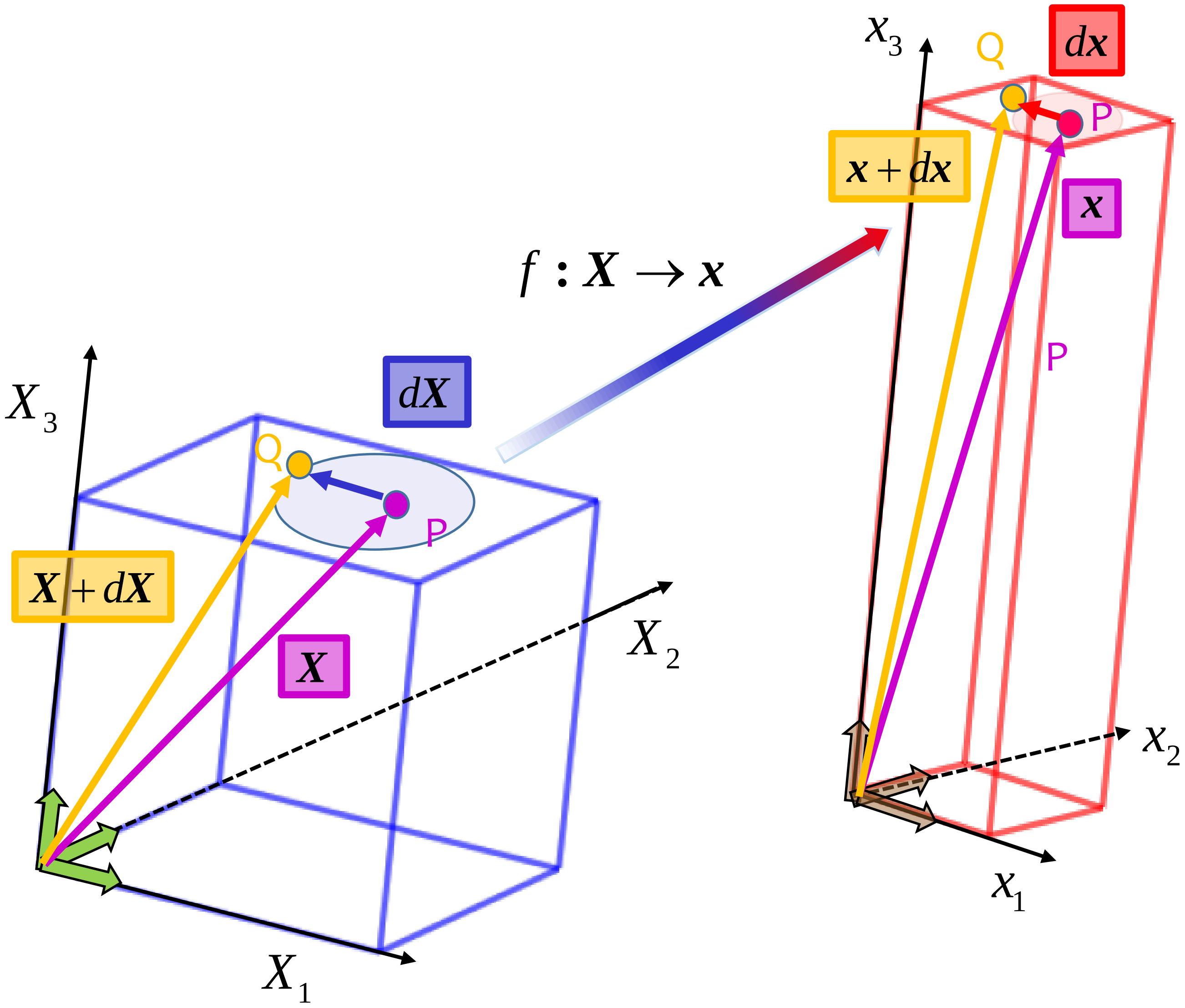
Takže odpověď na otázku, co se děje s tělesem při deformování, budeme opět hledat pomocí derivací…

Pomocí tzv.

deformačního gradientu



Deformace



Referenční konfigurace:

bod P zaměřen $\boldsymbol{X}$

Bod v elementárním okolí Q

zaměřen $\boldsymbol{X} + d\boldsymbol{X}$

Deformovaná konfigurace:

bod P zaměřen $\boldsymbol{x}$

Bod v elementárním okolí Q

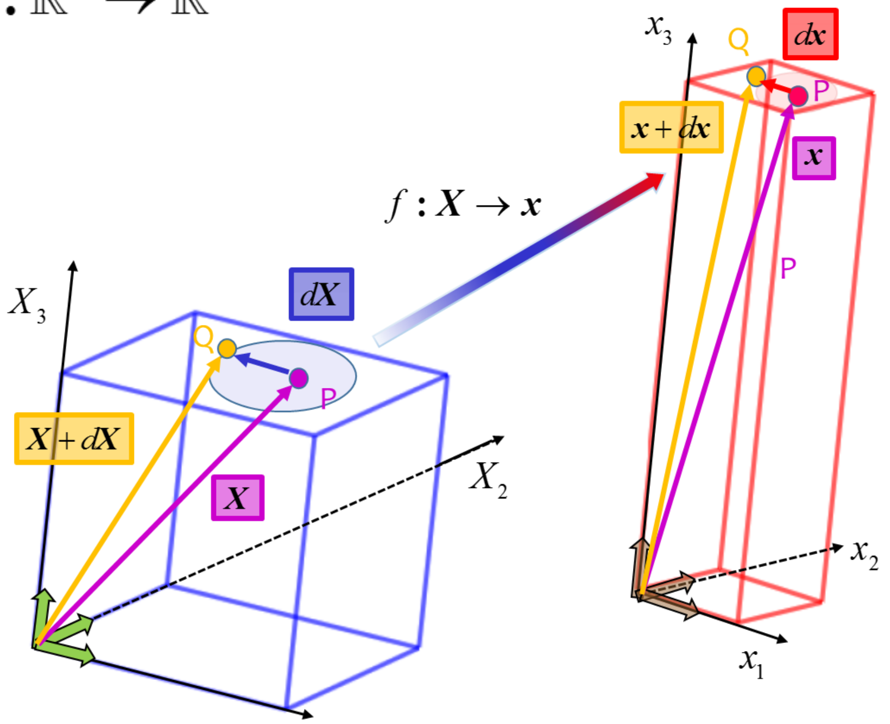
zaměřen $\boldsymbol{x} + d\boldsymbol{x}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1(X_1, X_2, X_3) \\ f_2(X_1, X_2, X_3) \\ f_3(X_1, X_2, X_3) \end{pmatrix}$$

Deformační gradient $\mathbf{F}$

$$f : \boldsymbol{X} \to \boldsymbol{x}$$

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$



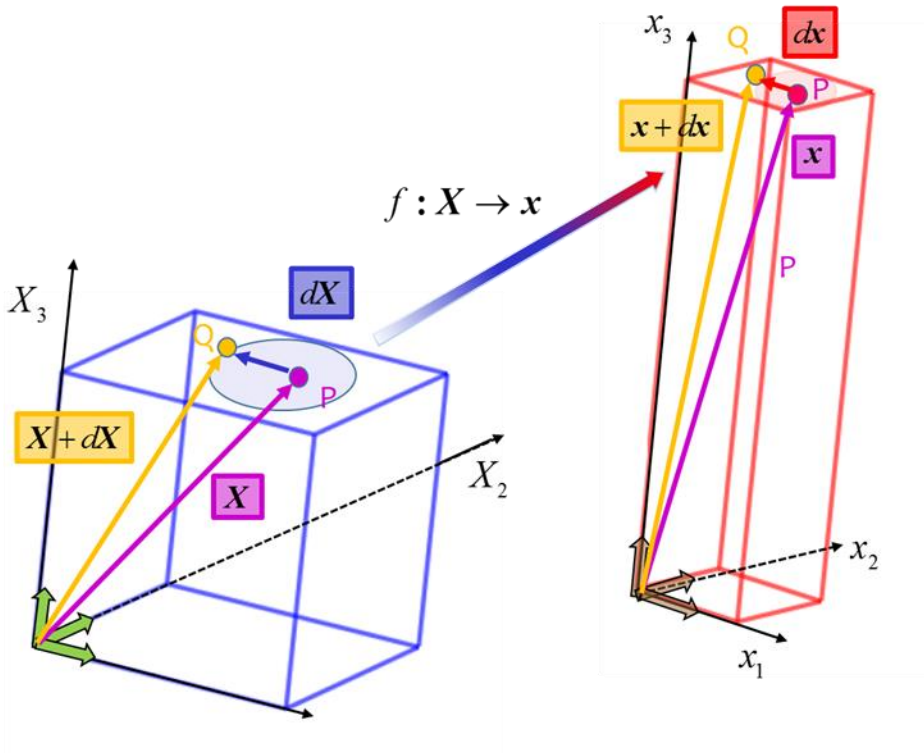
Infinitesimální změna („diferenciál“) zobrazení f

$$df : d\boldsymbol{X} \to d\boldsymbol{x}$$

$$\mathbf{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}}$$

$$\mathbf{F} = Grad\left(\boldsymbol{x}\left(\boldsymbol{X}\right)\right)$$

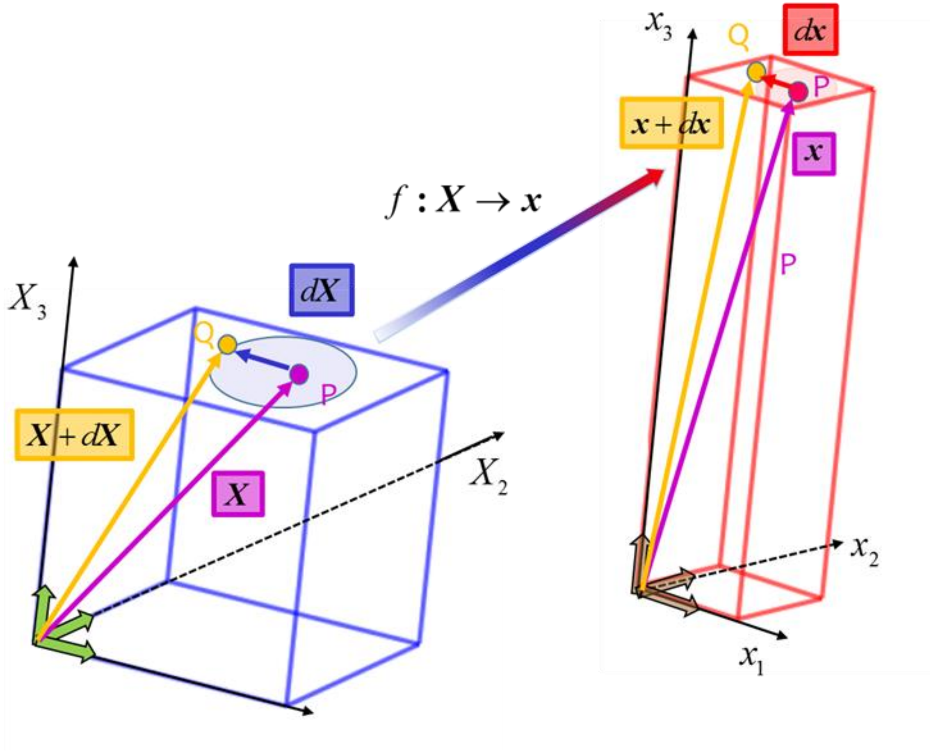
Deformační gradient $\mathbf{F}$



$$\mathbf{F}=\frac{d\boldsymbol{x}}{d\boldsymbol{X}}$$

$$\mathbf{F}=\begin{pmatrix}\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3}\\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3}\\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}\end{pmatrix}=\begin{pmatrix}\frac{\partial x_1(X_1,X_2,X_3)}{\partial X_1} & \frac{\partial x_1(X_1,X_2,X_3)}{\partial X_2} & \frac{\partial x_1(X_1,X_2,X_3)}{\partial X_3}\\ \frac{\partial x_2(X_1,X_2,X_3)}{\partial X_1} & \frac{\partial x_2(X_1,X_2,X_3)}{\partial X_2} & \frac{\partial x_2(X_1,X_2,X_3)}{\partial X_3}\\ \frac{\partial x_3(X_1,X_2,X_3)}{\partial X_1} & \frac{\partial x_3(X_1,X_2,X_3)}{\partial X_2} & \frac{\partial x_3(X_1,X_2,X_3)}{\partial X_3}\end{pmatrix}$$

Deformační gradient $\mathbf{F}$



$$\mathbf{F}=\frac{d\boldsymbol{x}}{d\boldsymbol{X}} \qquad d\boldsymbol{x}=\mathbf{F}\,d\boldsymbol{X}$$

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}=\begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix}\begin{pmatrix} dX_1 \\ dX_2 \\ dX_3 \end{pmatrix}$$

Deformační gradient $\mathbf{F}$

$\mathbf{F}$ představuje lineární transformaci mezi dvěma vektorovými prostory (prostor referenčních a deformovaných elementárních vektorů)

$$d\boldsymbol{x} = \mathbf{F}\, d\boldsymbol{X} \qquad \mathbf{F} : d\boldsymbol{X} \rightarrow d\boldsymbol{x}$$

$$dx_1\boldsymbol{e}_1 + dx_2\boldsymbol{e}_2 + dx_3\boldsymbol{e}_3 = \mathbf{F}\left\{dX_1\boldsymbol{E}_1 + dX_2\boldsymbol{E}_2 + dX_3\boldsymbol{E}_3\right\}$$

Takovou veličinu nazýváme tenzor

$\mathbf{F}$ je konkrétně tenzor druhého řádu (tzv. smíšený, též dvou-bodový)

Deformační gradient $\mathbf{F}$

$$\mathbf{F} : d\boldsymbol{X} \rightarrow d\boldsymbol{x}$$

$$\begin{pmatrix} dx_1\boldsymbol{e}_1 \\ dx_2\boldsymbol{e}_2 \\ dx_3\boldsymbol{e}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1\boldsymbol{e}_1}{\partial X_1\boldsymbol{E}_1} & \frac{\partial x_1\boldsymbol{e}_1}{\partial X_2\boldsymbol{E}_2} & \frac{\partial x_1\boldsymbol{e}_1}{\partial X_3\boldsymbol{E}_3} \\ \frac{\partial x_2\boldsymbol{e}_2}{\partial X_1\boldsymbol{E}_1} & \frac{\partial x_2\boldsymbol{e}_2}{\partial X_2\boldsymbol{E}_2} & \frac{\partial x_2\boldsymbol{e}_2}{\partial X_3\boldsymbol{E}_3} \\ \frac{\partial x_3\boldsymbol{e}_3}{\partial X_1\boldsymbol{E}_1} & \frac{\partial x_3\boldsymbol{e}_3}{\partial X_2\boldsymbol{E}_2} & \frac{\partial x_3\boldsymbol{e}_3}{\partial X_3\boldsymbol{E}_3} \end{pmatrix} \begin{pmatrix} dX_1\boldsymbol{E}_1 \\ dX_2\boldsymbol{E}_2 \\ dX_3\boldsymbol{E}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1\boldsymbol{e}_1}{\partial X_1\boldsymbol{E}_1} dX_1\boldsymbol{E}_1 + \frac{\partial x_1\boldsymbol{e}_1}{\partial X_2\boldsymbol{E}_2} dX_2\boldsymbol{E}_2 + \frac{\partial x_1\boldsymbol{e}_1}{\partial X_3\boldsymbol{E}_3} dX_3\boldsymbol{E}_3 \\ \frac{\partial x_2\boldsymbol{e}_2}{\partial X_1\boldsymbol{E}_1} dX_1\boldsymbol{E}_1 + \frac{\partial x_2\boldsymbol{e}_2}{\partial X_2\boldsymbol{E}_2} dX_2\boldsymbol{E}_2 + \frac{\partial x_2\boldsymbol{e}_2}{\partial X_3\boldsymbol{E}_3} dX_3\boldsymbol{E}_3 \\ \frac{\partial x_3\boldsymbol{e}_3}{\partial X_1\boldsymbol{E}_1} dX_1\boldsymbol{E}_1 + \frac{\partial x_3\boldsymbol{e}_3}{\partial X_2\boldsymbol{E}_2} dX_2\boldsymbol{E}_2 + \frac{\partial x_3\boldsymbol{e}_3}{\partial X_3\boldsymbol{E}_3} dX_3\boldsymbol{E}_3 \end{pmatrix} =$$

Deformační gradient $\mathbf{F}$

$$
=\begin{pmatrix}
\frac{\partial x_1 \boldsymbol{e}_1}{\partial X_1 \cancel{\boldsymbol{E}_1}} dX_1 \cancel{\boldsymbol{E}_1} + \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_2 \cancel{\boldsymbol{E}_2}} dX_2 \cancel{\boldsymbol{E}_2} + \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_3 \cancel{\boldsymbol{E}_3}} dX_3 \cancel{\boldsymbol{E}_3} \\
\frac{\partial x_2 \boldsymbol{e}_2}{\partial X_1 \cancel{\boldsymbol{E}_1}} dX_1 \cancel{\boldsymbol{E}_1} + \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_2 \cancel{\boldsymbol{E}_2}} dX_2 \cancel{\boldsymbol{E}_2} + \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_3 \cancel{\boldsymbol{E}_3}} dX_3 \cancel{\boldsymbol{E}_3} \\
\frac{\partial x_3 \boldsymbol{e}_3}{\partial X_1 \cancel{\boldsymbol{E}_1}} dX_1 \cancel{\boldsymbol{E}_1} + \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_2 \cancel{\boldsymbol{E}_2}} dX_2 \cancel{\boldsymbol{E}_2} + \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_3 \cancel{\boldsymbol{E}_3}} dX_3 \cancel{\boldsymbol{E}_3}
\end{pmatrix}
=\begin{pmatrix}
\frac{\partial x_1 \boldsymbol{e}_1}{\partial X_1} dX_1 + \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_2} dX_2 + \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_3} dX_3 \\
\frac{\partial x_2 \boldsymbol{e}_2}{\partial X_1} dX_1 + \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_2} dX_2 + \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_3} dX_3 \\
\frac{\partial x_3 \boldsymbol{e}_3}{\partial X_1} dX_1 + \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_2} dX_2 + \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_3} dX_3
\end{pmatrix} =
$$

$$
=\begin{pmatrix}
\frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3 \\
\frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3 \\
\frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3
\end{pmatrix}
=\left(\frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3\right)\boldsymbol{e}_1 + \left(\frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3\right)\boldsymbol{e}_2 + \left(\frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3\right)\boldsymbol{e}_3
$$

Deformační gradient $\mathbf{F}$

$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_1 \boldsymbol{E}_1} & \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_2 \boldsymbol{E}_2} & \frac{\partial x_1 \boldsymbol{e}_1}{\partial X_3 \boldsymbol{E}_3} \\ \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_1 \boldsymbol{E}_1} & \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_2 \boldsymbol{E}_2} & \frac{\partial x_2 \boldsymbol{e}_2}{\partial X_3 \boldsymbol{E}_3} \\ \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_1 \boldsymbol{E}_1} & \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_2 \boldsymbol{E}_2} & \frac{\partial x_3 \boldsymbol{e}_3}{\partial X_3 \boldsymbol{E}_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} \boldsymbol{e}_1 \boldsymbol{E}_1 & \frac{\partial x_1}{\partial X_2} \boldsymbol{e}_1 \boldsymbol{E}_2 & \frac{\partial x_1}{\partial X_3} \boldsymbol{e}_1 \boldsymbol{E}_3 \\ \frac{\partial x_2}{\partial X_1} \boldsymbol{e}_2 \boldsymbol{E}_1 & \frac{\partial x_2}{\partial X_2} \boldsymbol{e}_2 \boldsymbol{E}_2 & \frac{\partial x_2}{\partial X_3} \boldsymbol{e}_2 \boldsymbol{E}_3 \\ \frac{\partial x_3}{\partial X_1} \boldsymbol{e}_3 \boldsymbol{E}_1 & \frac{\partial x_3}{\partial X_2} \boldsymbol{e}_3 \boldsymbol{E}_2 & \frac{\partial x_3}{\partial X_3} \boldsymbol{e}_3 \boldsymbol{E}_3 \end{pmatrix} =$$

$$= \frac{\partial x_1}{\partial X_1}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_1) + \frac{\partial x_1}{\partial X_2}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_2) + \frac{\partial x_1}{\partial X_3}(\boldsymbol{e}_1 \otimes \boldsymbol{E}_3) + \frac{\partial x_2}{\partial X_1}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_1) + \frac{\partial x_2}{\partial X_2}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_2) +$$

$$+ \frac{\partial x_2}{\partial X_3}(\boldsymbol{e}_2 \otimes \boldsymbol{E}_3) + \frac{\partial x_3}{\partial X_1}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_1) + \frac{\partial x_3}{\partial X_2}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_2) + \frac{\partial x_3}{\partial X_3}(\boldsymbol{e}_3 \otimes \boldsymbol{E}_3)$$

Deformační gradient $\mathbf{F}$

$$\mathbf{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} \qquad F_{iK} = \frac{\partial x_i}{\partial X_K}$$

$$\mathbf{F} = F_{11}\left(\boldsymbol{e}_1 \otimes \boldsymbol{E}_1\right) + F_{12}\left(\boldsymbol{e}_1 \otimes \boldsymbol{E}_2\right) + F_{13}\left(\boldsymbol{e}_1 \otimes \boldsymbol{E}_3\right) + F_{21}\left(\boldsymbol{e}_2 \otimes \boldsymbol{E}_1\right) + F_{22}\left(\boldsymbol{e}_2 \otimes \boldsymbol{E}_2\right) +$$

$$+ F_{23}\left(\boldsymbol{e}_2 \otimes \boldsymbol{E}_3\right) + F_{31}\left(\boldsymbol{e}_3 \otimes \boldsymbol{E}_1\right) + F_{32}\left(\boldsymbol{e}_3 \otimes \boldsymbol{E}_2\right) + F_{33}\left(\boldsymbol{e}_3 \otimes \boldsymbol{E}_3\right)$$

Tenzory druhého řádu

Lineární zobrazení (transformace) mezi dvěma vektorovými prostory

Tenzor $\mathbf{A}$ tedy je $\mathbf{A}:\mathcal{V}\to\mathcal{V}$ nebo $\mathbf{A}:\mathcal{V}\to\mathcal{W}$

Jako lineární transformace je tenzor druhého řádu reprezentován maticí A_{ij} nebo A_{iK}

$$\mathbf{A}:\mathcal{V}\to\mathcal{W}$$

$$\boldsymbol{u}\in\mathcal{V}\left(\boldsymbol{E}_1,\boldsymbol{E}_2,\boldsymbol{E}_3\right)$$
$$\boldsymbol{v}\in\mathcal{W}\left(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\right)$$

$$\mathbf{A}=A_{iK}\boldsymbol{e}_i\otimes\boldsymbol{E}_K$$

$$\mathbf{A}:\mathcal{V}\to\mathcal{W}$$

$$\boldsymbol{v}=\mathbf{A}\boldsymbol{u}$$

$$[\boldsymbol{v}]=[\mathbf{A}][\boldsymbol{u}]$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Operace s tenzory druhého řádu

„po složkách“ $\boldsymbol{u},\boldsymbol{v} \in \mathcal{V}$ $t \in \mathbb{R}$

$$(\boldsymbol{u} \otimes \boldsymbol{v})_{ij} = u_i v_j \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix} \qquad A_{ij} = u_i v_j$$

$\mathbf{A} + \mathbf{B} = \mathbf{C}$ tak, že $A_{ij} + B_{ij} = C_{ij}$

$t \cdot \mathbf{A} = \mathbf{C}$ tak, že $t \cdot A_{ij} = C_{ij}$

Operace s tenzory druhého řádu

„po složkách“ $\boldsymbol{u},\boldsymbol{v} \in \mathcal{V}$ $t \in \mathbb{R}$

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = A_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \cdots + A_{23}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + A_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 =$$

$$= A_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A_{12}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + A_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + A_{33}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Operace s tenzory druhého řádu

„po složkách“ $\boldsymbol{u},\boldsymbol{v} \in \mathcal{V}$ $t \in \mathbb{R}$

$$(\boldsymbol{u}\otimes\boldsymbol{v})^T = \boldsymbol{v}\otimes\boldsymbol{u} \quad \text{též} \quad \boldsymbol{v}\cdot(\mathbf{A}\boldsymbol{u}) = \boldsymbol{u}\cdot\left(\mathbf{A}^T\boldsymbol{v}\right) \quad \text{též} \quad \mathbf{A}\boldsymbol{u} = \boldsymbol{u}\mathbf{A}^T$$

$$tr(\mathbf{A}) = \sum_{i=1}^{3} A_{ii} = A_{ii} \quad \text{stopa tenzoru} \quad \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$\mathbf{A}:\mathbf{B} = \sum_{i,j=1}^{3} A_{ij}B_{ij} = A_{ij}B_{ij}$$ *vnitřní součin* (analogon skalárního) „po složkách“

Operace s tenzory druhého řádu

„po složkách“

$\mathbf{AB} = \mathbf{C}$ skládání tenzorů (čili skládání zobrazení, neboli násobení matic)

$$C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} = A_{ik} B_{kj}$$

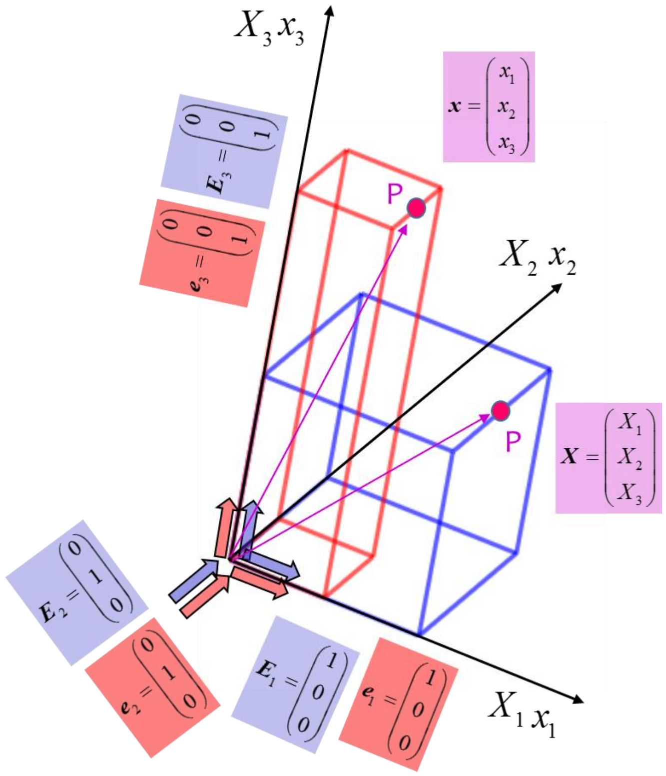
$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{x} \otimes \boldsymbol{y}) = (\boldsymbol{v} \cdot \boldsymbol{x})(\boldsymbol{u} \otimes \boldsymbol{y}) = (\boldsymbol{u} \otimes \boldsymbol{y})(\boldsymbol{v} \cdot \boldsymbol{x}) \qquad \boldsymbol{v} \cdot \boldsymbol{x} = \sum_{k=1}^{3} v_k x_k = v_k x_k$$

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \qquad I_{ij} = \delta_{ij}, \quad \text{kde} \quad \delta_{ij} = 1 \quad \text{pro } i = j, \quad \text{jinak} \quad \delta_{ij} = 0$$

Inverzní tenzor jako inverzní matice; *jednotkový tenzor* (matice)

Příklady pro deformační gradient $\mathbf{F}$

$$f : \boldsymbol{X} \to \boldsymbol{x} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.5X_1 \\ 0.5X_2 \\ 2X_3 \end{pmatrix} = 0.5X_1\boldsymbol{e}_1 + 0.5X_2\boldsymbol{e}_2 + 2X_3\boldsymbol{e}_3 \qquad \mathbf{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} \quad F_{iK} = \frac{\partial x_i}{\partial X_K}$$



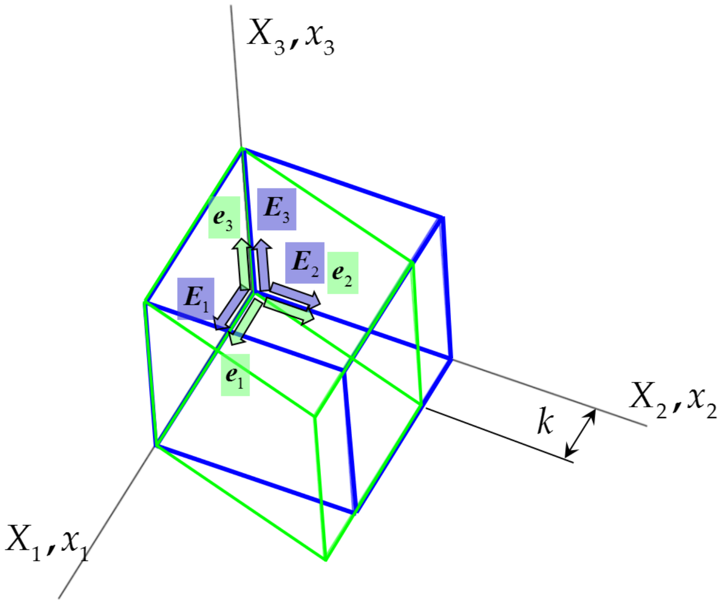
$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Příklady pro deformační gradient $\mathbf{F}$

$$f : \boldsymbol{X} \rightarrow \boldsymbol{x}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 + kX_2 \\ X_2 \\ X_3 \end{pmatrix} = (X_1 + kX_2)\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3$$

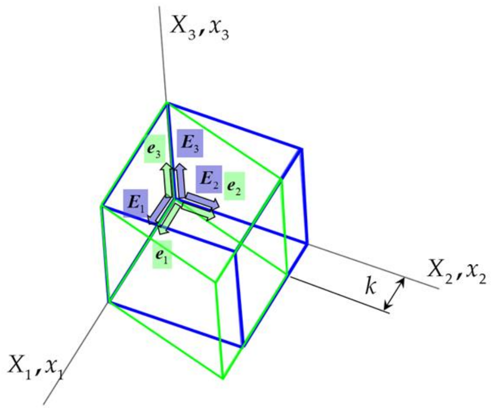
$$\mathbf{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} \qquad F_{iK} = \frac{\partial x_i}{\partial X_K}$$



$$\mathbf{F} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Nesymetrie $\mathbf{F}$

$\mathbf{F}$ má devět nezávislých složek $F_{iK} \quad i,K = 1,2,3$



$$\mathbf{F} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To je důsledek přítomnosti rotací $\mathbf{R}$ při zobrazení (pohybu) $f : \boldsymbol{X} \rightarrow \boldsymbol{x}$ mezi konfiguracemi. f obsahuje informaci o *translaci*, *rotaci* i *deformaci*. Od popisu deformace samozřejmě očekáváme, že bude založen pouze na změně relativní vzdálenosti bodů tělesa vůči sobě. Dodejme, že translace je konstanta, a tak derivována na 0, tudíž není v $\mathbf{F}$.

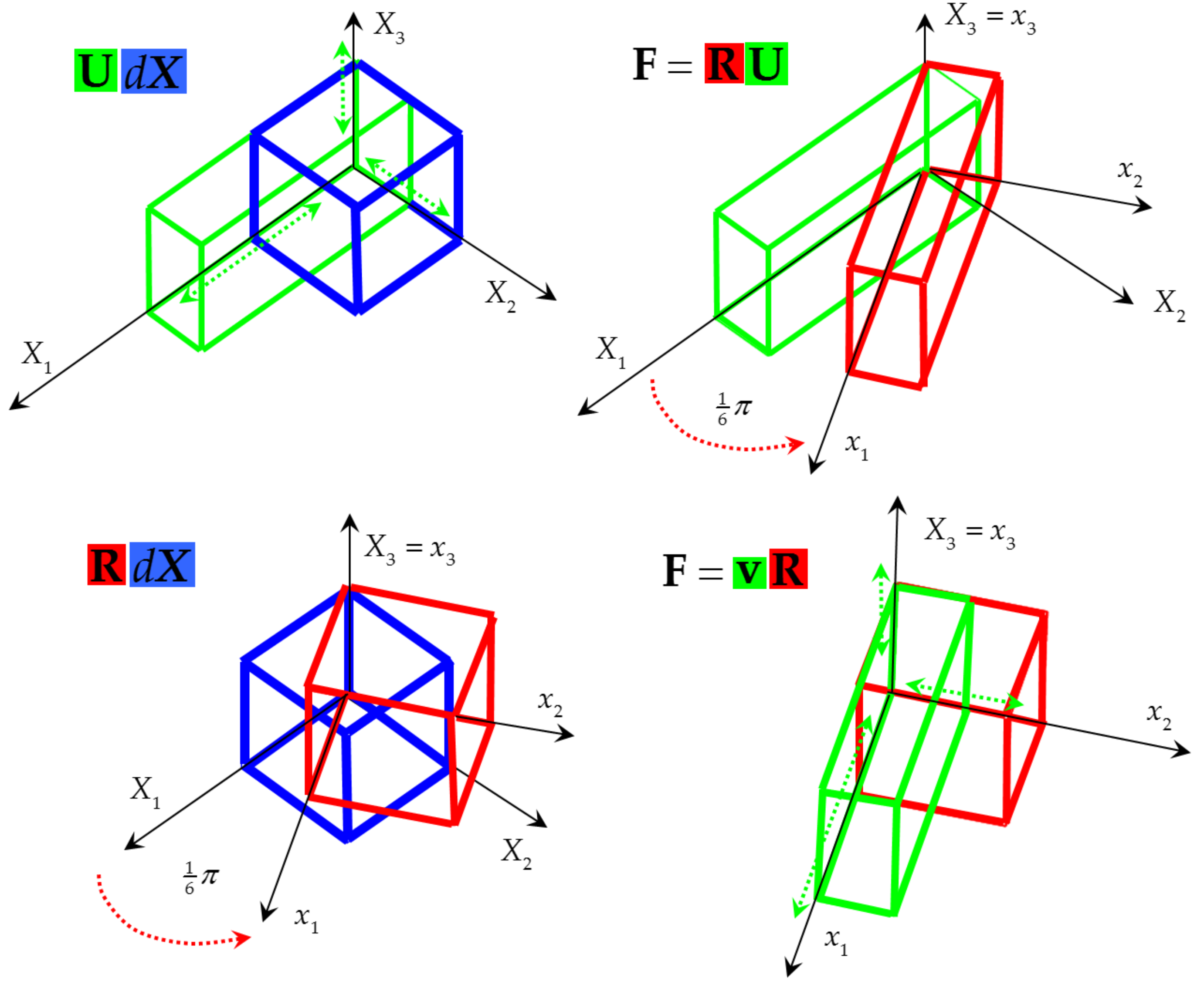
$$\mathbf{F} = \frac{df\left(\boldsymbol{X}\right)}{d\boldsymbol{X}} = \mathbf{RU} = \mathbf{vR}$$

Polární rozklad $\mathbf{F}$

$$\mathbf{F} = \mathbf{RU} = \mathbf{vR}$$

$$\begin{array}{ccccc} \mathbf{F} & & \mathbf{R} & & \mathbf{U} \\ \begin{pmatrix} \sqrt{3} & -\frac{1}{4} & 0 \\ 1 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} & = & \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & & \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{ccccc} \mathbf{F} & & \mathbf{v} & & \mathbf{R} \\ \begin{pmatrix} \sqrt{3} & -\frac{1}{4} & 0 \\ 1 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} & = & \begin{pmatrix} \frac{13}{8} & \sqrt{3}\,\frac{3}{8} & 0 \\ \sqrt{3}\,\frac{3}{8} & \frac{7}{8} & 0 \\ 0 & 0 & 1 \end{pmatrix} & & \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$



Tenzory deformace

$\mathbf{C} = \mathbf{F}^T\,\mathbf{F} = (\mathbf{RU})^T\,\mathbf{RU} = \mathbf{U}^T\,\mathbf{R}^T\,\mathbf{RU} = \mathbf{U}\,\mathbf{R}^{-1}\,\mathbf{RU} = \mathbf{UIU} = \mathbf{U}^2$ pravý Cauchyův-Greenův

$\mathbf{b} = \mathbf{F}\,\mathbf{F}^T = \mathbf{v}^2$ levý Cauchyův-Greenův

$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ Greenův (též Lagrangeův)

$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1})$ Eulerův (též Almansiho)

$ln(\mathbf{U})$ logaritmický

$$\mathbf{C}=\mathbf{F}^T\,\mathbf{F}=\begin{pmatrix} F_{11}\boldsymbol{E}_1\otimes\boldsymbol{e}_1 & F_{21}\boldsymbol{E}_1\otimes\boldsymbol{e}_2 & F_{31}\boldsymbol{E}_1\otimes\boldsymbol{e}_3\\ F_{12}\boldsymbol{E}_2\otimes\boldsymbol{e}_1 & F_{22}\boldsymbol{E}_2\otimes\boldsymbol{e}_2 & F_{32}\boldsymbol{E}_2\otimes\boldsymbol{e}_3\\ F_{13}\boldsymbol{E}_3\otimes\boldsymbol{e}_1 & F_{23}\boldsymbol{E}_3\otimes\boldsymbol{e}_2 & F_{33}\boldsymbol{E}_3\otimes\boldsymbol{e}_3 \end{pmatrix}\begin{pmatrix} F_{11}\boldsymbol{e}_1\otimes\boldsymbol{E}_1 & F_{12}\boldsymbol{e}_1\otimes\boldsymbol{E}_2 & F_{13}\boldsymbol{e}_1\otimes\boldsymbol{E}_3\\ F_{21}\boldsymbol{e}_2\otimes\boldsymbol{E}_1 & F_{22}\boldsymbol{e}_2\otimes\boldsymbol{E}_2 & F_{23}\boldsymbol{e}_2\otimes\boldsymbol{E}_3\\ F_{31}\boldsymbol{e}_3\otimes\boldsymbol{E}_1 & F_{32}\boldsymbol{e}_3\otimes\boldsymbol{E}_2 & F_{33}\boldsymbol{e}_3\otimes\boldsymbol{E}_3 \end{pmatrix}=$$

$$C_{11}\sim\begin{array}{l} F_{11}F_{11}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_1\right)\cdot\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_1\right)+F_{21}F_{21}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_2\right)\cdot\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_1\right)+F_{31}F_{31}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_3\right)\cdot\left(\boldsymbol{e}_3\otimes\boldsymbol{E}_1\right)=\\ =F_{11}F_{11}\boldsymbol{E}_1\otimes\boldsymbol{E}_1+F_{21}F_{21}\boldsymbol{E}_1\otimes\boldsymbol{E}_1+F_{31}F_{31}\boldsymbol{E}_1\otimes\boldsymbol{E}_1=\left(F_{11}F_{11}+F_{21}F_{21}+F_{31}F_{31}\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_1 \end{array}$$

$$C_{12}\sim\begin{array}{l} F_{11}F_{12}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_1\right)\cdot\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_2\right)+F_{21}F_{22}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_2\right)\cdot\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_2\right)+F_{31}F_{32}\left(\boldsymbol{E}_1\otimes\boldsymbol{e}_3\right)\cdot\left(\boldsymbol{e}_3\otimes\boldsymbol{E}_2\right)=\\ =F_{11}F_{12}\boldsymbol{E}_1\otimes\boldsymbol{E}_2+F_{21}F_{22}\boldsymbol{E}_1\otimes\boldsymbol{E}_2+F_{31}F_{32}\boldsymbol{E}_1\otimes\boldsymbol{E}_2=\left(F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32}\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_2 \end{array}$$

$$C_{21}\sim\begin{array}{l} F_{12}F_{11}\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_1\right)\cdot\left(\boldsymbol{e}_1\otimes\boldsymbol{E}_1\right)+F_{22}F_{21}\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_2\right)\cdot\left(\boldsymbol{e}_2\otimes\boldsymbol{E}_1\right)+F_{32}F_{31}\left(\boldsymbol{E}_2\otimes\boldsymbol{e}_3\right)\cdot\left(\boldsymbol{e}_3\otimes\boldsymbol{E}_1\right)=\\ =F_{12}F_{11}\boldsymbol{E}_2\otimes\boldsymbol{E}_1+F_{22}F_{21}\boldsymbol{E}_2\otimes\boldsymbol{E}_1+F_{32}F_{31}\boldsymbol{E}_2\otimes\boldsymbol{E}_1=\left(F_{12}F_{11}+F_{22}F_{21}+F_{32}F_{31}\right)\boldsymbol{E}_2\otimes\boldsymbol{E}_1 \end{array}$$

$$=\begin{pmatrix} F_{11}^2+F_{21}^2+F_{31}^2 & F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32} & F_{11}F_{13}+F_{21}F_{23}+F_{31}F_{33}\\ F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32} & F_{12}^2+F_{22}^2+F_{32}^2 & F_{12}F_{13}+F_{22}F_{23}+F_{32}F_{33}\\ F_{11}F_{13}+F_{21}F_{23}+F_{31}F_{33} & F_{12}F_{13}+F_{22}F_{23}+F_{32}F_{33} & F_{13}^2+F_{23}^2+F_{33}^2 \end{pmatrix}=$$

$$=\left(F_{11}^2+F_{21}^2+F_{31}^2\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_1+\left(F_{12}^2+F_{22}^2+F_{32}^2\right)\boldsymbol{E}_2\otimes\boldsymbol{E}_2+$$

$$+\left(F_{13}^2+F_{23}^2+F_{33}^2\right)\boldsymbol{E}_3\otimes\boldsymbol{E}_3+2\left(F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32}\right)\boldsymbol{E}_1\otimes\boldsymbol{E}_2+$$

$$+2\left(F_{12}F_{13}+F_{22}F_{23}+F_{32}F_{33}\right)\boldsymbol{E}_2\otimes\boldsymbol{E}_3+2\left(F_{11}F_{13}+F_{21}F_{23}+F_{31}F_{33}\right)\boldsymbol{E}_3\otimes\boldsymbol{E}_1=\mathbf{C}$$

Linearizace tenzorů deformace

V lineární pružnosti, kde pokládáme posuvy za „malé“ (a tak i deformace, někdy hovoříme o infinitesimálních deformacích), přejdou tenzory $\mathbf{E}$, $\mathbf{e}$ a $\ln\mathbf{U}$ v $\boldsymbol{\varepsilon}$

Vektor posuvů $\quad \boldsymbol{U} = \boldsymbol{x}(\boldsymbol{X}) - \boldsymbol{X} \qquad \boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}(\boldsymbol{x})$

$$\mathbf{E} = \frac{1}{2}\left(\nabla_X \boldsymbol{U} + (\nabla_X \boldsymbol{U})^T + \underset{\overset{\doteq}{0}}{(\nabla_X \boldsymbol{U})^T \nabla_X \boldsymbol{U}}\right) = \frac{1}{2}\left(\nabla_X \boldsymbol{U} + (\nabla_X \boldsymbol{U})^T\right) = \frac{1}{2}\left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T\right) = \frac{1}{2}\left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T - \underset{\overset{\doteq}{0}}{\nabla_x \boldsymbol{u}(\nabla_x \boldsymbol{u})^T}\right) = \mathbf{e}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\begin{pmatrix} 2\dfrac{\partial u_1}{\partial x_1} & \left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & 2\dfrac{\partial u_2}{\partial x_2} & \left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\ \left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) & \left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) & 2\dfrac{\partial u_3}{\partial x_3} \end{pmatrix}$$

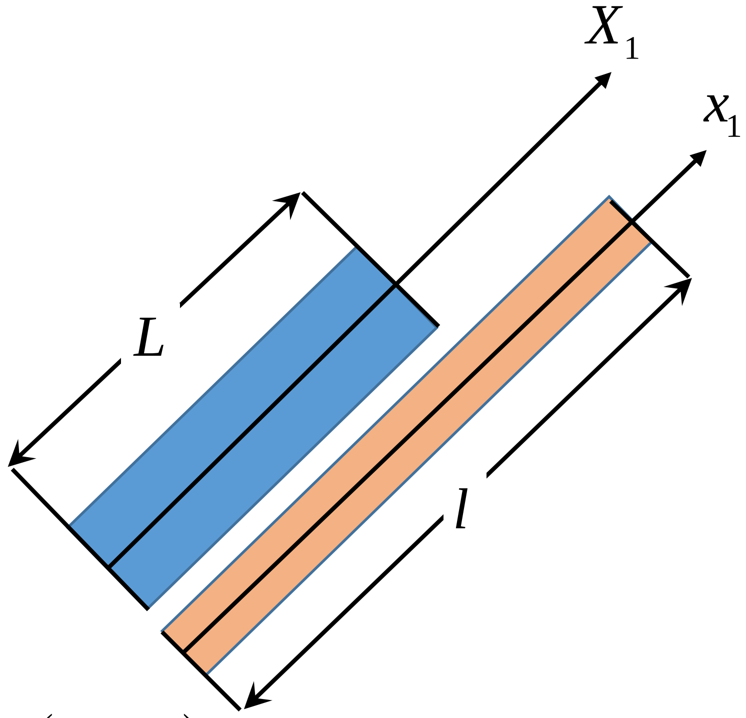
Tenzory deformace v MKP systémech

Vždy je třeba zkontrolovat manuál!

ANSYS, ABAQUS:

NLGEOM: OFF (default)

$$\boldsymbol{\varepsilon} = \frac{1}{2}\begin{pmatrix} 2\frac{\partial u_1}{\partial x_1} & \left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & 2\frac{\partial u_2}{\partial x_2} & \left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & 2\frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

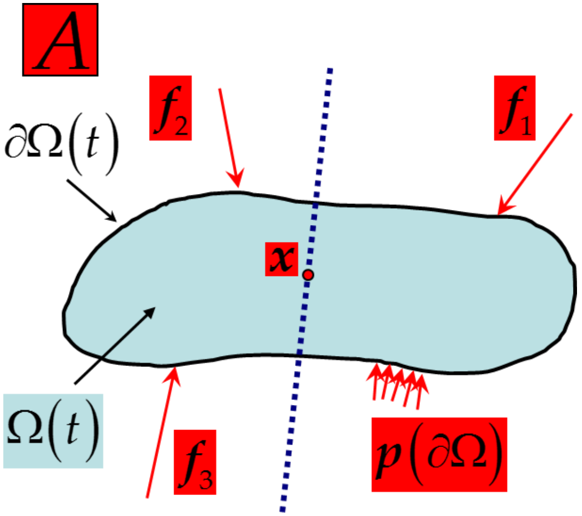


NLGEOM: ON (option)

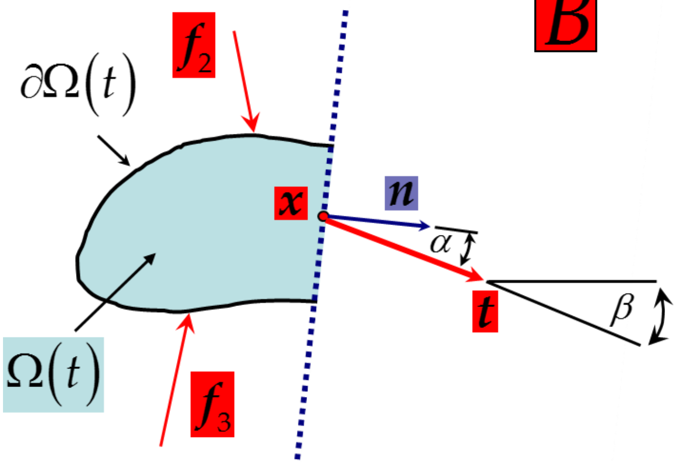
$$ln(\mathbf{U})$$

$$ln(\mathbf{U})_{11} = ln\frac{l}{L} = ln\frac{L+\Delta l}{L} = ln(1+\varepsilon_{11})$$

Tenzor napětí $\boldsymbol{\sigma}$



$\boxed{SR}$



Zavádíme vektor (plošné) intenzity vnitřních sil $\boldsymbol{t}$ (tzv. trakční nebo též napěťový vektor), tak, že platí

$$d\boldsymbol{f} = \boldsymbol{t}ds$$

zde ds je velikost plochy elementárního okolí bodu $\boldsymbol{x}$ v rovině řezu a $d\boldsymbol{f}$ je infinitesimální vektor vnitřní síly uvádějící řez do rovnováhy

Tenzor napětí $\boldsymbol{\sigma}$

Tenzor druhého řádu, čili zobrazení $\boldsymbol{\sigma}$, které promítá $\boldsymbol{n}$, což je vektor vnější normály roviny řezu v bodě $\boldsymbol{x}$, na napěťový vektor $\boldsymbol{t}$

$$\boldsymbol{\sigma} : \boldsymbol{n} \to \boldsymbol{t} \qquad \boldsymbol{t} = \boldsymbol{\sigma} \boldsymbol{n}$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

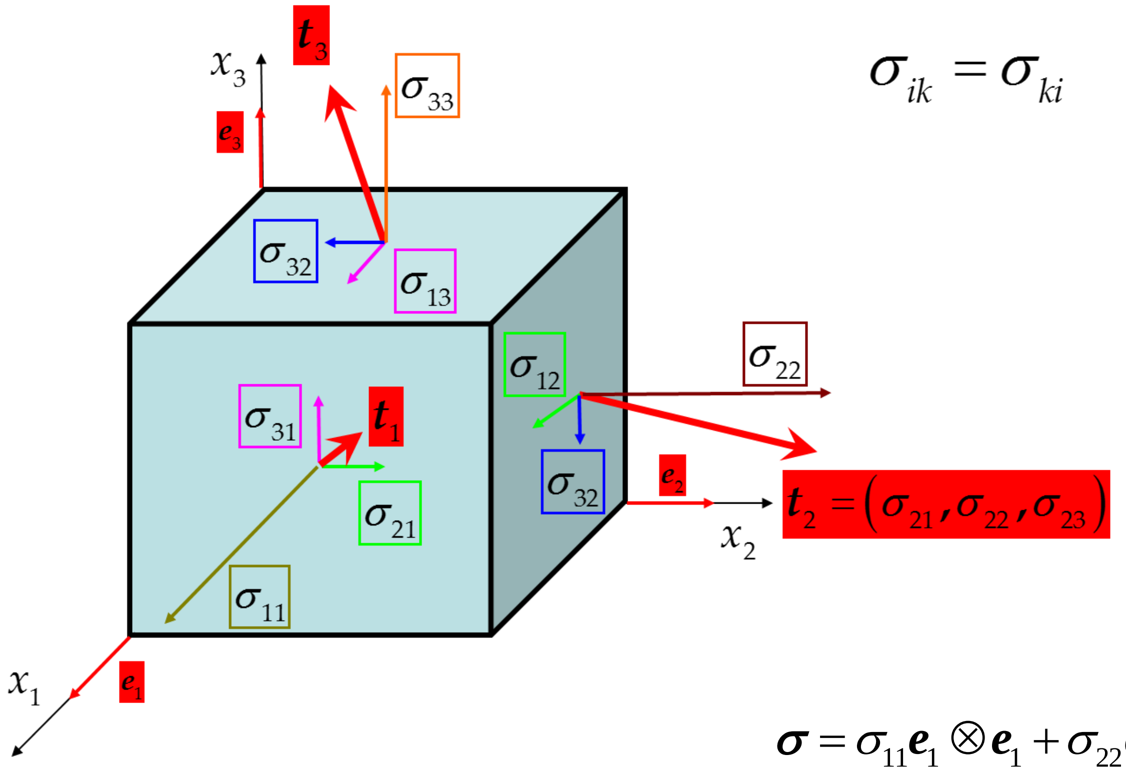
Takže tenzor napětí $\boldsymbol{\sigma}$ umožňuje určit napěťový vektor $\boldsymbol{t}$ v bodě $\boldsymbol{x}$ v libovolném řezu, čili pro libovolný vektor $\boldsymbol{n}$. To je úplná informace o stavu napjatosti v bodě.

Tenzor napětí $\boldsymbol{\sigma}$

$$\boldsymbol{t} = (t_1, t_2, t_3) = \boldsymbol{t}_1 + \boldsymbol{t}_2 + \boldsymbol{t}_3 = (\sigma_{11} + \sigma_{21} + \sigma_{31}, \sigma_{12} + \sigma_{22} + \sigma_{32}, \sigma_{13} + \sigma_{23} + \sigma_{33})$$

Pozor na rozdíl v označování složek tenzoru napětí σ_{ij}, zde i je směr průmětu a j směr normály stěny krychle

$$\sigma_{ik} = \sigma_{ki}$$

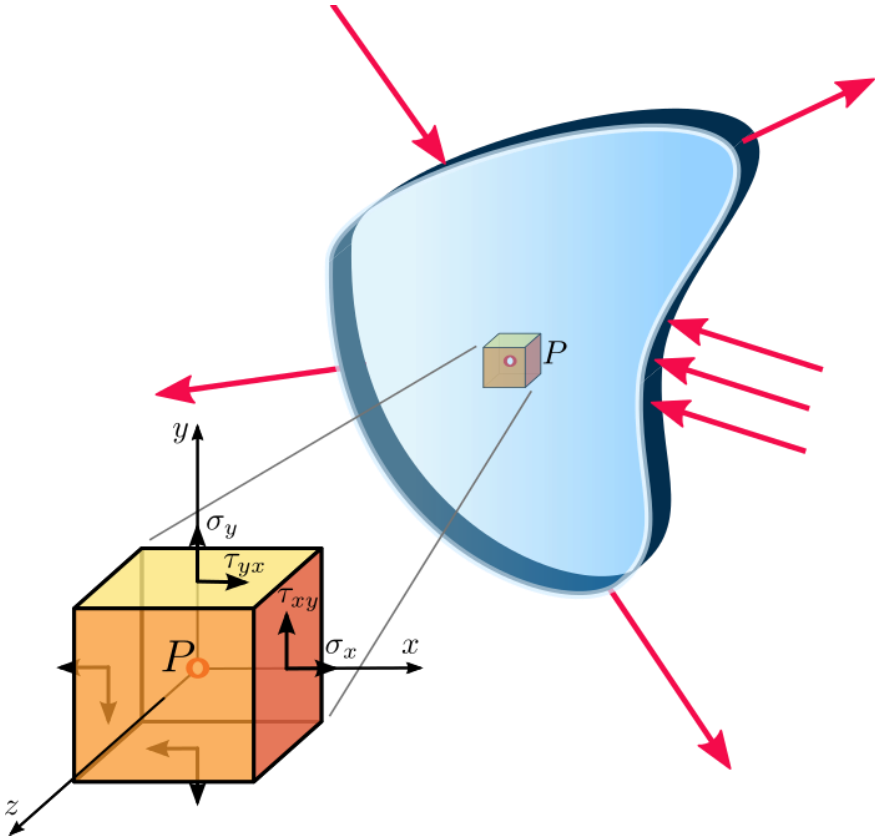


$$\boldsymbol{\sigma} = \sigma_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + 2\sigma_{12}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + 2\sigma_{23}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + 2\sigma_{31}\boldsymbol{e}_3 \otimes \boldsymbol{e}_1$$

Rovinná napjatost

Některé stavy napjatosti a deformace umožňují zjednodušení formulace úloh pružnosti, což v MKP výpočtech může vést významnému *snížení výpočetní náročnosti*

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}$$



Typickým příkladem jsou tenkostěnná tělesa zatížená ve své střední ploše – stěny. Za přibližně rovinnou napjatost můžeme považovat i napjatost v tenkých deskách a trubkách. Stav rovinné napjatosti je též ve všech nezatížených bodech povrchu těles.

Rovinná deformace

Některé stavy napjatosti a deformace umožňují zjednodušení formulace úloh pružnosti, což v MKP výpočtech může vést významnému *snížení výpočetní náročnosti*

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Za přibližně rovinný stav deformace můžeme považovat deformaci prutových těles (lépe řečeno všech velmi dlouhých prizmatických) zatížených kolmo ke střednici