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Strain energy function for arterial walls based on limiting fiber extensibility

L. Horny[1], R. Zitny[2] and H. Chlup[1]

[1] Department of Mechanics, Biomechanics and Mechatronics, Faculty of Mechanical Engineering Czech Technical University in Prague, Prague, Czech Republic

[2] Department of Process Engineering, Faculty of Mechanical Engineering Czech Technical University in Prague, Prague, Czech Republic

Abstract— **An anisotropic strain energy density function based on limiting fiber extensibility assumption was suggested. The function was deduced directly from isotropic Gent model. A material was modeled as a composite reinforced with two families of helical fibers. The anisotropy of the strain energy function was incorporated via pseudo-invariants I_4 and I_6. Mathematical expression includes three material parameters. Suitability of the model for a description of arterial mechanical response was verified by regression analysis of experimental data. Computational model based on a cylindrical thick–walled tube with residual strains was used to estimate material parameters. Identified material model fits pressure–radius data of an aortic inflation test successfully. Further upgrades of the model are discussed.**

Keywords— **aorta, constitutive model, limiting fiber extensibility, orthotropy, strain energy function**

I. INTRODUCTION

Arterial walls exhibit anisotropic, nonlinear and inelastic response to external loads. This response does not only occur as passive deformation but active contraction and dilation of smooth muscle cells can cause changes in their mechanical behavior. Moreover arterial wall is non–homogenous material with complicated internal structure. All these facts make the question about the best material model for arterial wall still unanswered. There are two basic approaches to material modeling. First of them is phenomenological where mechanical qualities are modeled with no information about internal structure and its interactions. Second approach is characterized by incorporating structural information when considering e.g. layers, fibers, fiber orientation or waviness. Typical representative of phenomenological approach is exponential strain energy density function suggested by Fung et al. [1]. This function or its modifications have been successfully used by many authors. In two–dimensional formulation (thin tube) it can be written in the form below

$$\psi = \frac{c}{2}\left(e^{b_1 E_{tt}^2 + b_2 E_{zz}^2 + 2 b_3 E_{tt} E_{zz}} - 1\right). \tag{1}$$

Here ψ means strain energy density function. E_{tt} and E_{zz} denote Green strains in circumferential and axial direction, respectively. Material parameters c, b_1, b_2 and b_3 enable description of the anisotropic behavior of wall. Isotropic form of the function (1) was suggested by Fung in 1967.

Structural approach asserts at present. The most frequent method how to incorporate structural information is to regard arterial wall as a fiber reinforced composite. Probably the first who presented this idea was Lanir [2] in 1983. Nowadays models can be divided into two groups according to number of reinforcement directions. One can considered reinforcement in a finite number of directions; e.g. two like in the case of Holzapfel et al. [3]; or infinite number where a probability density of fiber orientation must be considered.

These considerations about preferred directions are subsequently implemented into the framework of continuum mechanics. The leading approach to building constitutive models is the theory of hyperelastic materials. Thus mechanical response of an arterial wall is supposed to be governed by a strain energy (or free energy) density function like in (1). The theory of hyperelastic materials is widely applied and studied in details in polymer science. Due to some phenomenological and structural similarities between rubber–like materials and biological tissues, methods of polymer physics are frequently applied in biomechanics, see Holzapfel [4]. Gent [5] suggested the new isotropic model for strain energy density function which was based on an assumption of limiting chain extensibility in polymer materials. This model has become quickly popular and now is implemented in usual FEA packages like e.g. ANSYS or ABAQUS. The Gent model expresses strain energy as a function of first invariant I_1 of the right Cauchy-Green strain tensor

$$\psi = -\frac{1}{2}\mu J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right). \tag{2}$$

In equation (2) μ denotes infinitesimal shear modulus and J_m denotes limiting value of I_1 - 3. The domain of logarithm requires $I_1 - 3 < J_m$ and J_m can be interpreted as limiting value for macromolecular chains stretch.

The main goal of our study is to show anisotropic upgrade of the model (2) and verifying its suitability for arterial walls based on experimental data.

II. Constitutive model

It was proved many times that arteries exhibit anisotropic behavior. The model proposed by Gent (2) is isotropic. Horgan and Saccomandi in [6] suggested its anisotropic extension. They recently published its modification based on usual concept of anisotropic materials where anisotropy arises from fiber reinforcement, see paper [7]. Horgan and Saccomandi consider transversely isotropic material where anisotropy is induced by reinforcement with one family of fibers (one preferred direction). They use rational approximations to relate a strain energy expression to Cauchy stress representation formula. Final form of the strain energy density function for transversely isotropic material with limiting fibers extensibility follows

$$\psi = \frac{\mu_m}{2}(I_1 - 3) - \frac{1}{2}\mu_f J_m \ln\left(1 - \frac{(I_4 - 1)^2}{J_m}\right). \qquad (3)$$

The first term (Neo–Hook) in equation (3) is related to energy stored in isotropic matrix as usual. Second part of (3) is related to energy stored in fibers. In (3) μ_m and μ_f denote shear modulus for matrix and fibers, respectively. J_m is the material parameter related to limiting extensibility of fibers. The similar definitional inequality like in (2) must be hold for logarithm in (3). Thus I_4 must satisfy $(I_4 - 1)^2 < J_m$. I_4 denotes so called fourth pseudo–invariant of the strain tensor which arises from the existence of one preferred direction in continuum. It is worth to note that total number of invariants of the strain tensor is five in the case of transversely isotropic material and nine in the case of orthotropy. Details can be found in e.g. Holzapfel [8]. Introducing of I_4 lies in its clear physical interpretation. The value of I_4 is equal to square of the stretch in the fiber direction. Thus it can be written in the following form

$$I_4 = \lambda_f^2 = \lambda_1^2 \cos^2\beta + \lambda_2^2 \sin^2\beta\,. \qquad (4)$$

In equation (4) λ_1, λ_2 denotes stretches in (x_1,x_2) plane. Parameter β is related to the internal structure of a material and characterizes the direction of fibers in (x_1,x_2).

The model (3) can be modified to a form suitable for locally orthotropic material in a similar way like e.g. Holzapfel, Gasser and Ogden in [9]. Assuming that an artery is a composite material reinforced by two families of mechanically equivalent fibers, the corresponding general form of the anisotropic strain energy density function is:

$$\psi = \frac{\mu_m}{2}(I_1 - 3) + \\ -\frac{1}{2}\mu_{f1} J_{m1} \ln\left(1 - \frac{(I_4 - 1)^2}{J_{m1}^2}\right) - \frac{1}{2}\mu_{f2} J_{m2} \ln\left(1 - \frac{(I_6 - 1)^2}{J_{m2}^2}\right). \qquad (5)$$

The first term in (5) is the same as in (3). Remaining terms reflect the energy stored in two families of fibers (two preferred directions). Due to symmetry of fiber coils and mechanical equivalence of fibers $\mu_{f1} = \mu_{f2} = \mu_f$, $J_{m1} = J_{m2}$, $I_4 = I_6$ ($\Rightarrow \beta_1 = -\beta_2$), the model (5) reduces to the form seemingly similar to Eq.(3),

$$\psi = \frac{\mu_m}{2}(I_1 - 3) - \mu_f J_m \ln\left(1 - \frac{(I_4 - 1)^2}{J_m^2}\right). \qquad (6)$$

It is necessary to note the difference in the argument of logarithm in (6), where J_m^2 is used instead of simple J_m in (3). This modification allows splitting the logarithm according to logarithmic rules

$$\psi = \frac{\mu_m}{2}(I_1 - 3) - \mu_f J_m \left(\ln\left(1 - \frac{I_4 - 1}{J_m}\right) + \ln\left(1 + \frac{I_4 - 1}{J_m}\right)\right). \qquad (7)$$

This is the form, which is advantageous for solutions of several boundary value problems.

In the following section the model (7) will be used for regression analysis of data collected during an inflation test of an artery. The model is reduced only the logarithmic term

$$\psi = -\mu_f J_m \left(\ln\left(1 - \frac{I_4 - 1}{J_m}\right) + \ln\left(1 + \frac{I_4 - 1}{J_m}\right)\right), \qquad (8)$$

neglecting the effect of isotropic matrix and taking into account only the energy stored in the clockwise and anticlockwise coils of collagen fibers in the arterial wall.

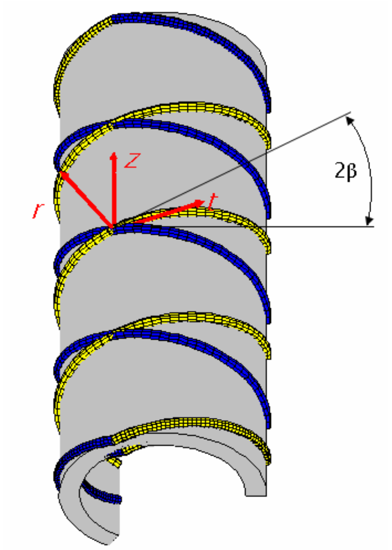

Fig. 1 Fiber reinforced vessel in the reference (open) configuration: blue and yellow – two equivalent families of fibers; red – local coordinate system (cylindrical); gray – matrix.

III. EXPERIMENT AND REGRESSION ANALYSIS

In order to verify capability of (8) to govern multi–axial mechanical response of an artery, previously published experimental data will be adopted, see Horny et al. [9] and [10].

Male 54–year–old sample of thoracic aorta underwent inflation test in order to find suitable material model and estimate its material parameters. The sample was obtained during autopsy at the Institute of Forensic Medicine of the University Hospital Na Kralovskych Vinohradech in Prague. No significant atherosclerotic changes were found. The time between the presumptive death and the inflation test was approximately 66 hours. Before experiments the specimen was stored at temperature approximately of 4°C; inflation test was performed under room temperature. The inflation experiments were performed under the following conditions. A tubular sample was 6 times pressurized in the range 0 kPa – 18 kPa – 0 kPa under axial pre–stretch λ_z = 1.3 and 3 times in the pressure range 0 kPa – 20 kPa – 0 kPa under λ_z = 1.42, respectively. The opening angle was measured after a radial cut of specially prepared ring of the artery before pressurization to account residual strains and to find a reference configuration. Geometrical characteristic of sample were as follows: thickness in reference state H = 2.04 mm; opening angle α = 83°, reference outer radius R_o = 19.33 mm; reference inner radius R_i = 17.29 mm, outer radius of the closed but not pressurized artery r_o = 10.88 mm; inner radius of the closed but not pressurized artery r_i = 8.84 mm.

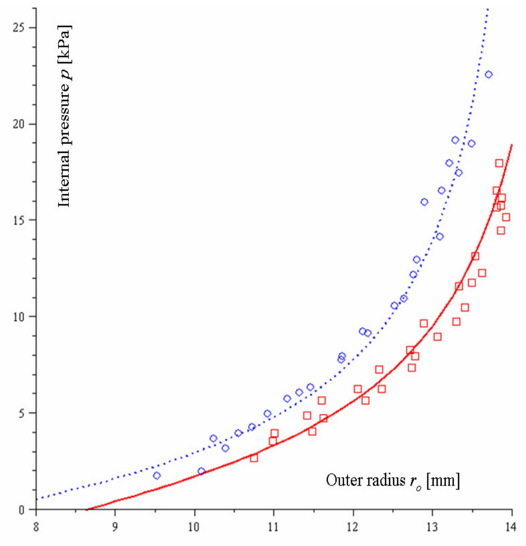

Fig. 2 Inflation test and model prediction: red/square – λ_z=1.3; blue/circle – λ_z=1.42.

Measured values of the internal pressure and the outer radius with predictions by the model (8) are shown at Fig. 1. Boxes (red color) and circles (blue color) at Fig. 1 display observation points for axial pre–stretch λ_z = 1.3 and λ_z = 1.42, respectively. Predictions of internal pressure based on the model (8) at the given axial pre–stretch λ_z = 1.3 and λ_z = 1.42 are displayed by solid (red color) and doted (blue color) curves, respectively.

Regression analysis based on least square method gave the estimations for material parameters μ, J_m and β in the model (8). A system of nonlinear equations was solved by Levenberg – Marquardt algorithm using in-house software package FEMINA.

Least square optimization was based on a comparison of measured and predicted values of internal pressure during inflation of cylindrical vessel. Computational model was based on radial equilibrium in axially pre–stretched thick–walled tube with residual strains, assuming incompressible material. Shear strains were not included into the model.

All assumptions mentioned above lead to the solution of boundary value problem for thick–walled tube with internal pressure given by following equation

$$p(r_i)=\int_{r_i}^{r_o}\lambda_t\frac{\partial\psi}{\partial\lambda_t}\frac{dr}{r}\,. \tag{9}$$

In the equation (9) p denotes internal pressure. λ_t means stretch in the circumferential direction and r denotes radius. The derivative in (9) for (8) has the form as below

$$\frac{\partial\psi}{\partial\lambda_t}=\frac{4\mu J_m\lambda_t\cos^2\beta\left(\lambda_t^2\cos^2\beta+\lambda_z^2\sin^2\beta-1\right)}{J_m^2-\left(\lambda_t^2\cos^2\beta+\lambda_z^2\sin^2\beta-1\right)^2}\,. \tag{10}$$

If one wants to integrate (9), circumferential stretch λ_t must be expressed as a function of the radius r. If residual strains are included it can be done in the form

$$\lambda_t(r)=\frac{\pi r}{(\pi-\alpha)\sqrt{R_o^2-\dfrac{\pi\lambda_z\left(r_o^2-r^2\right)}{\pi-\alpha}}}\,. \tag{11}$$

The denotation used in (11) is following: α – opening angle; r – variable radius; R_o – outer radius in the reference configuration (opened up circular sector of the artery); λ_z – axial stretch. It is obvious that inserting (10) and (11) into (9) causes (9) to be rather complicated. However, in contrast to Fung–type material models now the antiderivative for (9) in a closed form of elementary function exists. The antiderivative was found and all algebraic manipulations were performed using MAPLE 11 (Maplesoft, Waterloo, Canada).

IV. Results and conclusions

Nonlinear regression analysis described in the above section gives estimations for material parameters of the model (8) summarized in the Table 1. Results are also displayed graphically in the Fig. 1. We can conclude that proposed material model fits experimental data successfully. Thus strain energy density function given in (8) is suitable to govern arterial response during its inflation and extension. We can expect that incorporating Neo–Hookean term, like in (7), will improve model predictions under low pressures. However, fitting of data from the inflation test is only one task which must good material model carry out. Final decision about the appropriateness of the model should be made after successful application in all types of mechanical test usual in arterial mechanics. Especially biaxial extension tests are necessary and this analysis must be performed in future.

Table 1 Material parameters (8)

Material parameter	μ	J_m	β
[dimension]	[kPa]	[1]	[°]
value	26	1.044	37.2

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