Transport equations, Navier Stokes equations

Remark: foils with „black background“ could be skipped, they are aimed to the more advanced courses
Navier Stokes Equations

MOMENTUM transport

Newton’s law (mass times acceleration=force)

\[ \rho \frac{D\vec{u}}{Dt} = pressure\ force + viscous\ stress + gravity + centrifugal\ forces \]

mass \hspace{2cm} acceleration \hspace{2cm} Sum of forces on fluid particle
Pressure forces on fluid element surface

Resulting pressure force acting on sides W and E in the x-direction

\[ \delta x \delta y \delta z \frac{\partial p}{\partial x} \]

\[ \delta y \delta z (p - \frac{\partial p}{\partial x} \frac{\delta x}{2}) \]

\( n_x = -1 \)

\[ \delta y \delta z (p + \frac{\partial p}{\partial x} \frac{\delta x}{2}) \]

\( n_x = 1 \)
Viscous forces on fluid element surface

Resulting viscous force acting on all sides (W,E,N,S,T,B) in the x-direction

\[ \vec{f} = \bar{n} \cdot \vec{\tau} \]

\[ f_x = \sum_i n_i \tau_{ix} = n_i \tau_{ix} \]

\[ \delta x \delta y \delta z \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \]
**Balance of forces**

\[
\rho \frac{D\mathbf{u}}{Dt} \delta x\delta y\delta z = \text{pressure\_force} + \text{viscous\_stress} + \text{gravity}
\]

\[
\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_x
\]

\[
\rho \frac{Du_y}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_y
\]

\[
\rho \frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + S_z
\]

Cauchy’s equation of momentum balances (in fact 3 equations)

\[
\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \mathbf{\tau} + \mathbf{S}_M [\text{N/m}^3]
\]
**Balance of ENERGY**

**TOTAL ENERGY transport**

\[
\rho \frac{DE}{Dt} = \text{heat} + \text{mechanical work}
\]

\[
E = i + \frac{1}{2} (u^2 + v^2 + w^2)
\]

- **Total energy [J/kg]**
- **Kinetic energy [J/kg]**
- **Internal energy** all form of energies (chemical, intermolecular, thermal) independent of coordinate system

- Heat added to FE by diffusion only (convective transport is included in DE/Dt)
- Power of mechanical forces is velocity times force. Internal forces are pressure and viscous forces.
Heat conduction

Heat transfer by conduction is described by Fourier’s law

\[ \vec{q} = -k \nabla T \]

\[ q_x = -k \frac{\partial T}{\partial x} \]
\[ q_y = -k \frac{\partial T}{\partial y} \]
\[ q_z = -k \frac{\partial T}{\partial z} \]

\[ \rho \frac{DE}{Dt} = -\nabla \cdot \vec{q} = \nabla \cdot (k \nabla T) = \frac{\partial}{\partial x} (k \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) \]
Mechanical work - pressure

\[ \delta y \delta z (pu - \frac{\partial pu}{\partial x} \frac{\delta x}{2}) \]

\[ n_x = -1 \]

\[ \delta y \delta z (pu + \frac{\partial pu}{\partial x} \frac{\delta x}{2}) \]

\[ n_x = 1 \]

\[ \rho \frac{DE}{Dt} = \nabla \cdot (k \nabla T) - \nabla \cdot (pu) = \]

\[ = \frac{\partial}{\partial x} (k \frac{\partial T}{\partial x} - pu) + \frac{\partial}{\partial y} (k \frac{\partial T}{\partial y} - pv) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z} - pw) \]
The situation is more complicated because not only the work of normal but also shear stresses must be included.
This is scalar equation for total energy, comprising internal energy (temperature) and also kinetic energy.

\[
\rho \frac{D\mathcal{E}}{Dt} = \nabla \cdot (k \nabla T) - \nabla \cdot (p \vec{u}) + \nabla \cdot (\vec{\tau} \cdot \vec{u}) + S_E
\]

\[
\rho \frac{D\mathcal{E}}{Dt} = \nabla \cdot (k \nabla T - p\vec{u} + \vec{\tau} \cdot \vec{u}) + S_E
\]
Fourier Kirchhoff equation

Kinetic energy can be eliminated from total energy equation

\[ D(i + \frac{1}{2} \ddot{u} \cdot \ddot{u}) \]
\[ \rho \frac{2}{Dt} = \nabla \cdot (k \nabla T) - \nabla \cdot (p \ddot{u}) + \nabla \cdot (\bar{\tau} \cdot \ddot{u}) + S_E \] (1)

using Cauchy’s equation multiplied by velocity vector (scalar product, this is the way how to obtain scalar equation from the vector equation)

\[ \rho \ddot{u} \cdot \frac{D \ddot{u}}{Dt} = \rho \frac{1}{2} \ddot{u} \cdot \ddot{u} \]
\[ \rho \frac{D \ddot{u}}{Dt} = \rho \frac{D \ddot{u}}{Dt} = -\ddot{u} \cdot \nabla p + \dddot{u} \cdot \nabla \bar{\tau} + \ddot{u} \cdot \dot{S}_M \] (2)

Subtracting Eq.(2) from Eq.(1) we obtain transport equation for internal energy

\[ \rho \frac{Di}{Dt} = \nabla \cdot (k \nabla T) - p \nabla \cdot \ddot{u} + \bar{\tau} : \nabla \ddot{u} + S_E - \ddot{u} \cdot \dot{S}_M \] [W/m³]
Fourier Kirchhoff equation

Interpretation using First law of thermodynamics

\[ di = dq - p \, dv \]

\[ \rho \frac{Di}{Dt} = \nabla \cdot (k \nabla T) - p \nabla \cdot \vec{u} + \vec{t} : \nabla \vec{u} + S_E - \vec{u} \cdot \vec{S}_M \]

- Heat transferred by conduction into FE
- Expansion cools down working fluid
- This term is zero for incompressible fluid
- Dissipation of mechanical energy to heat by viscous friction
Dissipation term

\[\tau : \nabla \ddot{u} = \sum_i \sum_j \tau_{ij} \frac{\partial u_i}{\partial x_j} = \]

\[\tau_{xx} \frac{\partial u_x}{\partial x} + \tau_{xy} \frac{\partial u_x}{\partial y} + \tau_{xz} \frac{\partial u_x}{\partial z} + \]

\[\tau_{yx} \frac{\partial u_y}{\partial x} + \tau_{yy} \frac{\partial u_y}{\partial y} + \tau_{yz} \frac{\partial u_y}{\partial z} + \]

\[\tau_{zx} \frac{\partial u_z}{\partial x} + \tau_{zy} \frac{\partial u_z}{\partial y} + \tau_{zz} \frac{\partial u_z}{\partial z} \]

Heat dissipated in unit volume [W/m³] by viscous forces
Dissipation term

\[ \bar{\tau} \cdot \nabla \bar{u} = \frac{1}{2} \bar{\tau} : (\nabla \bar{u} + (\nabla \bar{u})^T) = \bar{\tau} : \bar{e} \]

This identity follows from the stress tensor symmetry

\[ \bar{\varepsilon} = \frac{1}{2} (\nabla \bar{u} + (\nabla \bar{u})^T) \]

Rate of deformation tensor

Example: Simple shear flow (flow in a gap between two plates, lubrication)

\[ e_{xy} = e_{yx} = \frac{1}{2} (\nabla_x u_y + \nabla_y u_x) = \frac{1}{2} \frac{\partial u_x}{\partial y} = \frac{1}{2} \dot{\gamma} \]

\[ \bar{\tau} : \nabla \bar{u} = \bar{\tau} : \bar{e} = \tau_{xy} e_{yx} + \tau_{yx} e_{xy} = \tau_{xy} \dot{\gamma} \]
Example tutorial

Gap width $H=0.1\text{mm}$, $U=10\text{ m/s}$, oil M9ADS-II at $0^\circ\text{C}$

$\mu=3.4\text{ Pa.s, } \gamma=10^5\text{ 1/s, } \tau=3.4\cdot10^5\text{ Pa, } \tau\gamma=3.4\cdot10^{10}\text{ W/m}^3$

At contact surface $S=0.0079\text{ m}^2$ the dissipated heat is $26.7\text{ kW}$ !!!!
Internal energy can be expressed in terms of temperature as \( \text{d}i = c_p \text{d}T \) or \( \text{d}i = c_v \text{d}T \). Especially simple form of this equation holds for liquids when \( c_p = c_v \) and divergence of velocity is zero (incompressibility constraint):

\[
\rho c_p \frac{\text{D}T}{\text{D}t} = \nabla \cdot (k \nabla T) + \vec{\tau} : \nabla \vec{u} + S_i
\]

[\text{W/m}^3]

An alternative form of energy equation substitute internal energy by enthalpy

\[
\rho \frac{\text{D}H}{\text{D}t} = \nabla \cdot (k \nabla T) - p \nabla \cdot \vec{u} + \frac{\partial p}{\partial t} + \nabla \cdot (\vec{\tau} \cdot \vec{u}) + S_i
\]

where total enthalpy is defined as

\[
H = i + \frac{p}{\rho} + \frac{1}{2} \vec{u} \cdot \vec{u}
\]
Calculate evolution of temperature in a gap assuming the same parameters as previously (H=0.1 mm, U=10 m/s, oil M9ADS-II). Assume constant value of heat production term $3.4 \times 10^{10}$ W/m$^3$, uniform inlet temperature $T_0=0^\circ$C and thermally insulated walls, or constant wall temperature, respectively.

Parameters: density $= 800$ kg/m$^3$, $c_p=1.9$ kJ/(kg.K), $k=0.14$ W/(m.K).

Approximate FK equation in 2D by finite differences. Use upwind differences in convection terms:

$$\rho c \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \vec{\tau} : \nabla \vec{u}$$
Mass conservation (continuity equation)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \]

Momentum balance (3 equations)

\[ \rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \cdot \vec{\tau} + \vec{S}_M \]

Energy balance

\[ \rho c \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \vec{\tau} : \nabla \vec{u} + S_i \]

State equation \( F(p,T,\rho)=0, \) e.g.
\[ \rho = \rho_0 + \beta T \]

Thermodynamic equation
\[ di = c_p \cdot dT \]
Constitutive equations represent description of material properties

**Kinematics** (rate of deformation) – **stress** (dynamic response to deformation)

\[
\ddot{\varepsilon} = \frac{1}{2} (\nabla \ddot{u} + (\nabla \ddot{u})^T)
\]

\[
\ddot{\sigma} = -p \ddot{\delta} + \ddot{\tau}
\]

- Rate of deformation is symmetric part of gradient of velocity
- Gradient of velocity is tensor with components
  \[
  \nabla_i u_j = \frac{\partial u_j}{\partial x_i}
  \]

- Viscous stresses affected by fluid flow. Stress is in fact momentum flux due to molecular diffusion

\[
\ddot{\tau} = \lambda \ddot{\delta} \nabla \cdot \ddot{u} + 2 \mu (\nabla \cdot \ddot{u}) \ddot{\varepsilon}
\]

- Second viscosity [Pa.s]
- Dynamic viscosity [Pa.s]
Constitutive equations

\[ \vec{\tau} = \lambda \delta \vec{\nabla} \cdot \vec{u} + 2 \mu (II) \vec{\dot{e}} \]

Rheological behaviour is quite generally expressed by viscosity function

\[ \mu(II), \text{ where } II = \vec{\dot{e}} : \vec{\dot{e}} = \sum_{i=1}^{3} \sum_{j=1}^{3} e_{ij} e_{ji} \]

and by the coefficient of second viscosity, that represents resistance of fluid to volumetric expansion or compression. According to Lamb’s hypothesis the second (volumetric) viscosity can be expressed in terms of dynamic viscosity \( \mu \)

\[ \lambda = -\frac{2}{3} \mu \]

This follows from the requirement that the mean normal stresses are zero (this mean value is absorbed in the pressure term)

\[ \text{trace } \vec{\tau} = \tau_{xx} + \tau_{yy} + \tau_{zz} = 3 \lambda \vec{\nabla} \cdot \vec{u} + 2 \mu \vec{\nabla} \cdot \vec{u} = 0 \]
Constitutive equations

\[ \vec{\tau} = 2\mu (II)(\vec{\varepsilon} - \frac{\nabla \cdot \vec{u}}{3}\vec{\delta}) \]

The simplest form of rheological model is NEWTONIAN fluid, characterized by viscosity independent of rate of deformation. Example is water, oils and air.

More complicated constitutive equations exist for fluids exhibiting

- **yield stress** (fluid flows only if stress exceeds a threshold, e.g. ketchup, tooth paste, many food products),
- **generalized newtonian fluids** (viscosity depends upon the actual state of deformation rate, example are power law fluids \( \mu = K (\sqrt{2II})^{n-1} \))
- **thixotropic** fluids (viscosity depends upon the whole deformation history, examples thixotropic paints, plasters, yoghurt)
- **viscoelastic** fluids (exhibiting recovery of strains and relaxation of stresses). Examples are polymers.
There are 13 unknowns:
\[ u, v, w, \text{ (3 velocities)}, \ p, T, \ \rho, \ i, \ \tau_{xx}, \ \tau_{xy}, \ldots \text{(6 components of symmetric stress tensor)} \]

And the same number of equations

**Continuity equation**
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

**3 Cauchy’s equations**
\[
\rho \frac{D \mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \mathbf{\tau} + \mathbf{S}_M
\]

**Energy equation**
\[
\rho \frac{Di}{Dt} = \nabla \cdot (k \nabla T) - p \nabla \cdot \mathbf{u} + \mathbf{\tau} : \nabla \mathbf{u} + S_E - \mathbf{u} \cdot \mathbf{S}_M
\]

**State equation**
\[
p/\rho = RT
\]

**Thermodynamic equation**
\[
di = c_p \, d\mathbf{T}
\]

**6 Constitutive equations**
\[
\mathbf{\tau} = 2\mu(II)(\dot{\mathbf{e}} - \frac{\nabla \cdot \mathbf{u}}{3} \mathbf{\delta})
\]
Navier Stokes equations

Using constitutive equation the divergence of viscous stresses can be expressed

\[ \nabla \cdot \tau = 2 \nabla \cdot (\mu(II)(\tilde{\varepsilon} - \frac{\nabla \cdot \vec{u}}{3} \delta)) = \nabla \cdot (\mu(II)(\nabla \vec{u} + (\nabla \vec{u})^T)) - \frac{2}{3} \nabla \cdot (\mu(II)(\nabla \cdot \vec{u}) \delta) \]

This is the same, but written in the index notation (you cannot make mistakes when calculating derivatives)

\[ \frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i} (\mu \frac{\partial u_j}{\partial x_i}) + \frac{\partial}{\partial x_i} (\mu \frac{\partial u_i}{\partial x_j}) - \frac{2}{3} \frac{\partial}{\partial x_i} (\mu \frac{\partial u_k}{\partial x_i} \delta_{ij}) = \]

\[ = \frac{\partial}{\partial x_i} (\mu \frac{\partial u_j}{\partial x_i}) + \frac{\partial}{\partial x_i} (\mu \frac{\partial u_i}{\partial x_j}) - \frac{2}{3} \frac{\partial}{\partial x_j} (\mu \frac{\partial u_i}{\partial x_i}) = \]

\[ = \frac{\partial}{\partial x_i} (\mu \frac{\partial u_j}{\partial x_i}) + \frac{\partial}{\partial x_i} (\mu \frac{\partial u_i}{\partial x_j}) + \mu \frac{\partial^2 u_i}{\partial x_i \partial x_i} - \frac{2}{3} (\frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_i}) = \]

\[ = \frac{\partial}{\partial x_i} (\mu \frac{\partial u_j}{\partial x_i}) + \frac{\partial}{\partial x_i} (\mu \frac{\partial u_i}{\partial x_j}) + \mu \frac{\partial^2 u_i}{\partial x_i \partial x_i} - \frac{2}{3} \frac{\partial}{\partial x_i} (\frac{\partial u_j}{\partial x_i}) \]

\[ \nabla \cdot \tau = \nabla \cdot (\mu(II)(\nabla u))^T + \nabla \mu(II) \nabla (\nabla \cdot u) - 2 \nabla \cdot (\nabla \cdot u) \nabla \mu(II) \]

These terms are small and will be replaced by a parameter \( s_m \)

These terms are ZERO for incompressible fluids
Navier Stokes equations

General form of Navier Stokes equations valid for compressible/incompressible
Non-Newtonian (with the exception of viscoelastic or thixotropic) fluids

\[ \rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \cdot (\mu(II)\nabla \vec{u}) + \vec{s}_m + \vec{S}_M \]

Special case – Newtonian liquids with constant viscosity

\[ \rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \vec{S}_M \]

Written in the cartesian coordinate system

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x \]

\[ \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y \]

\[ \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z \]