

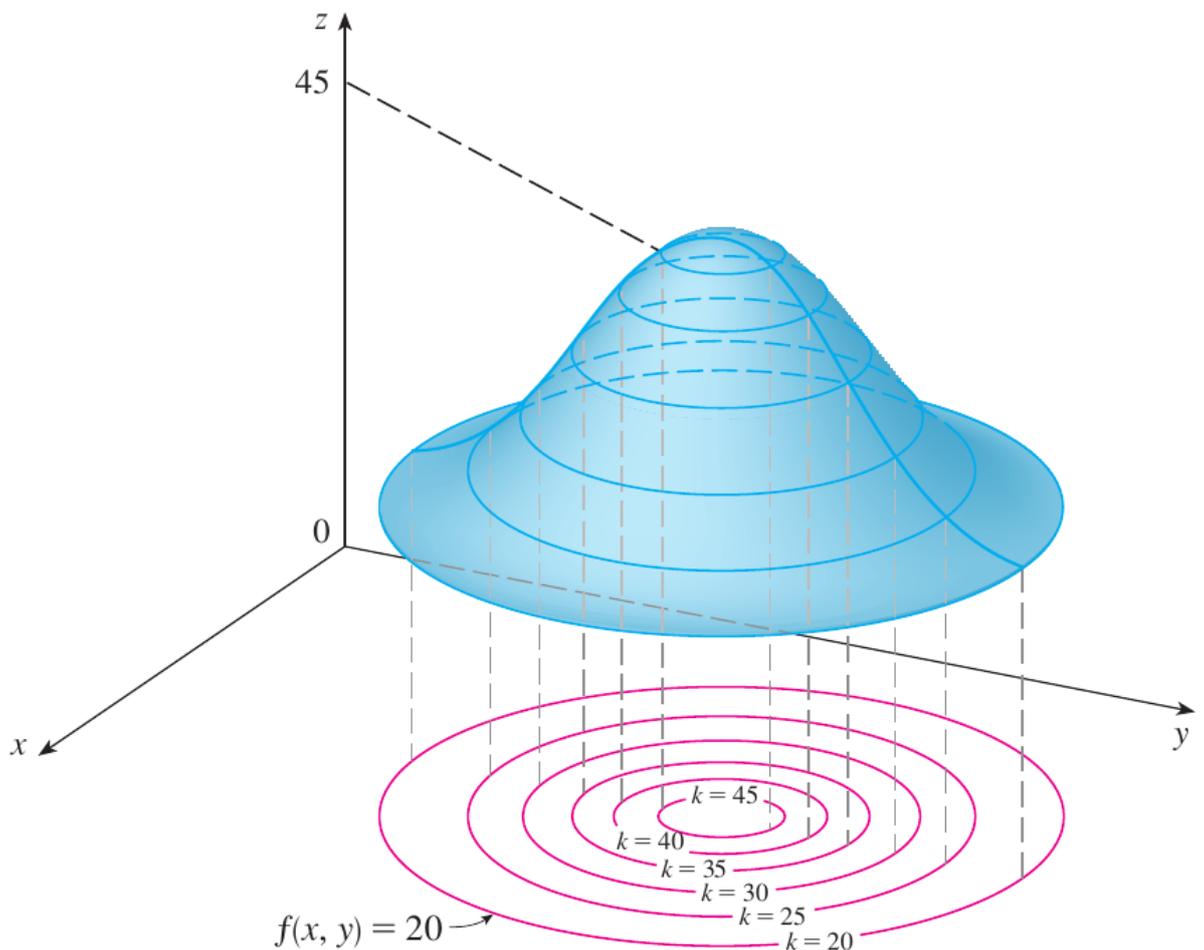
Function of Two Variables

Traces and level curves

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k . In other words, it shows where the graph of f has height k .

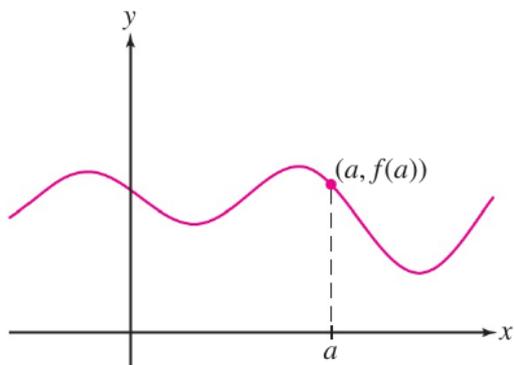
- **Horizontal trace at height c :** Intersection of the graph with the horizontal plane $z = k$, consisting of the points $[x; y; f(x, y)]$ such that $f(x, y) = k$.
- **Level curve:** The curve $f(x, y) = k$ in the xy -plane.
- **Contour map** is a plot in xy -plane that shows the level curves $f(x, y) = k$ for equally spaced values of k .

You can see from Figure the relation between **level curves and horizontal traces**. The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy -plane.

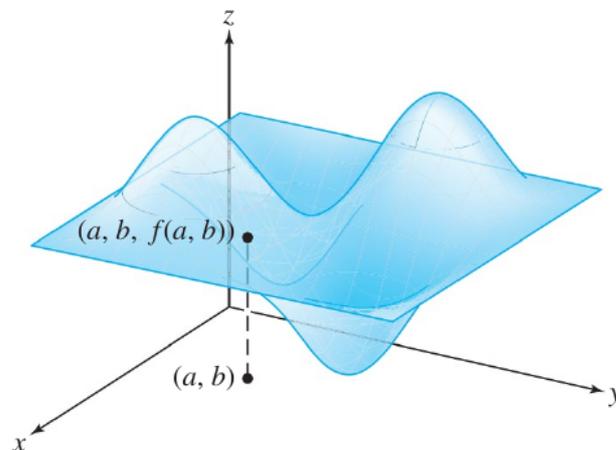


So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph. The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

One way of analyzing the graph of $f(x, y)$ is to freeze the x -coordinate by setting $x = a$ and examine the resulting curve given by $z = f(a, y)$. Similarly, we may set $y = b$ and consider the curve $z = f(x, b)$. Curves of this type are called **vertical traces**. They are obtained by intersecting the graph with planes parallel to a vertical coordinate plane.



(A) Graph of $y = f(x)$



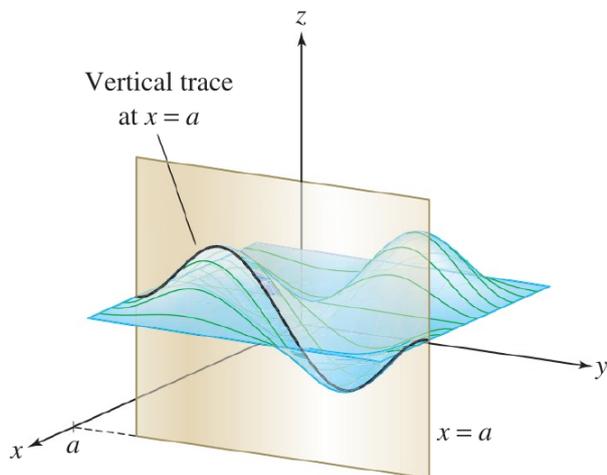
(B) Graph of $z = f(x, y)$

- **Vertical trace in the plane $x = a$:**

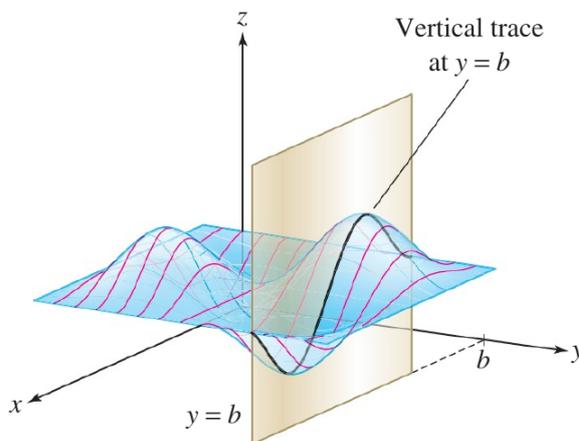
Intersection of the graph with the vertical plane $x = a$, consisting of all points $[a; y; f(a, y)]$.

- **Vertical trace in the plane $y = b$:**

Intersection of the graph with the vertical plane $y = b$, consisting of all points $[x; b; f(x, b)]$.

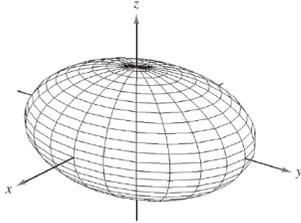
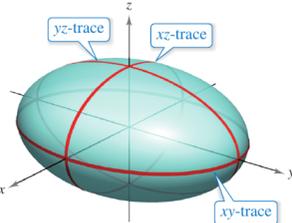
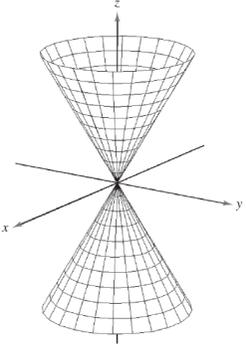
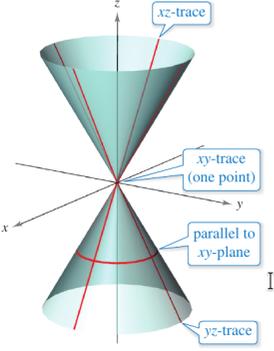
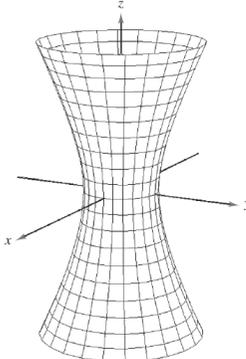
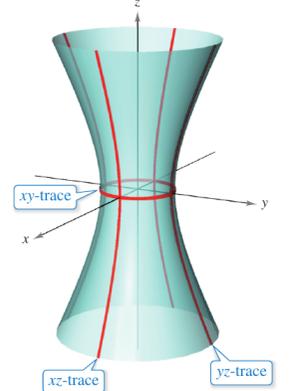
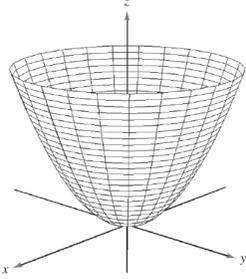
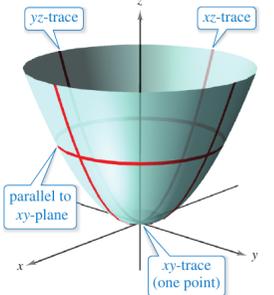
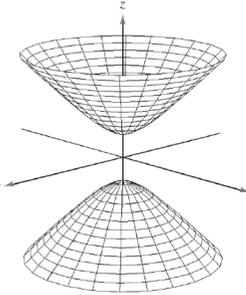
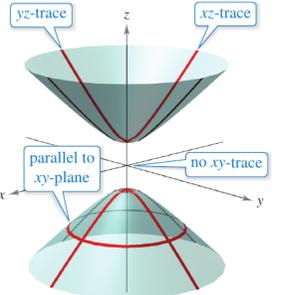
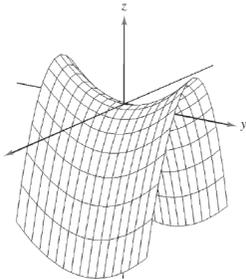
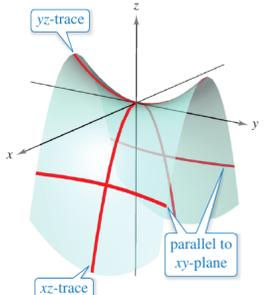


(A) Vertical traces parallel to yz -plane



(B) Vertical traces parallel to xz -plane

Quadric surfaces

	<p>Ellipsoid</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>Trace Ellipse Ellipse Ellipse</p> <p>Plane Parallel to xy-plane Parallel to xz-plane Parallel to yz-plane</p> <p>The surface is a sphere when $a = b = c \neq 0$.</p>			<p>Elliptic Cone</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <p>Trace Ellipse Hyperbola Hyperbola</p> <p>Plane Parallel to xy-plane Parallel to xz-plane Parallel to yz-plane</p> <p>The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.</p>	
	<p>Hyperboloid of One Sheet</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p>Trace Ellipse Hyperbola Hyperbola</p> <p>Plane Parallel to xy-plane Parallel to xz-plane Parallel to yz-plane</p> <p>The axis of the hyperboloid corresponds to the variable whose coefficient is negative.</p>			<p>Elliptic Paraboloid</p> $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Trace Ellipse Parabola Parabola</p> <p>Plane Parallel to xy-plane Parallel to xz-plane Parallel to yz-plane</p> <p>The axis of the paraboloid corresponds to the variable raised to the first power.</p>	
	<p>Hyperboloid of Two Sheets</p> $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <p>Trace Ellipse Hyperbola Hyperbola</p> <p>Plane Parallel to xy-plane Parallel to xz-plane Parallel to yz-plane</p> <p>The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.</p>			<p>Hyperbolic Paraboloid</p> $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ <p>Trace Hyperbola Parabola Parabola</p> <p>Plane Parallel to xy-plane Parallel to xz-plane Parallel to yz-plane</p> <p>The axis of the paraboloid corresponds to the variable raised to the first power.</p>	

Example 1

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

[Solution]

Identify the type of a Quadric and sketch for $a = 3$, $b = 4$ and $c = 5$.

[Process]

Example 2

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

[Solution]

Identify the type of a Quadric and sketch for $a = 4$ and $b = 2$.

[Process]

Example 3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

[Solution]

Identify the type of a Quadric and sketch for $a = 2$, $b = 3$ and $c = 1$.

[Process]

Example 4

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

[Solution]

Identify the type of a Quadric and sketch for $a = 1, b = 2$.

[Process]

Example 5

Sketch the surface

$$z = y^2 - x^2$$

[Solution]

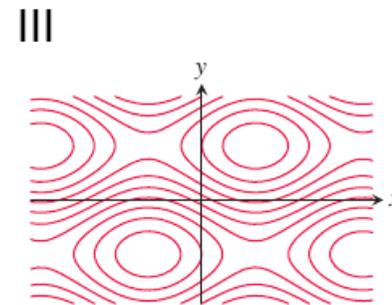
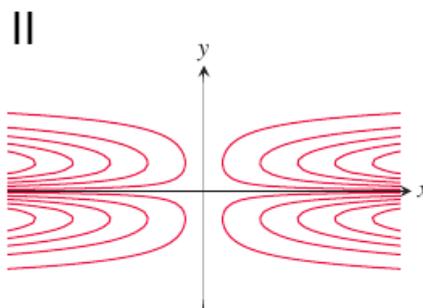
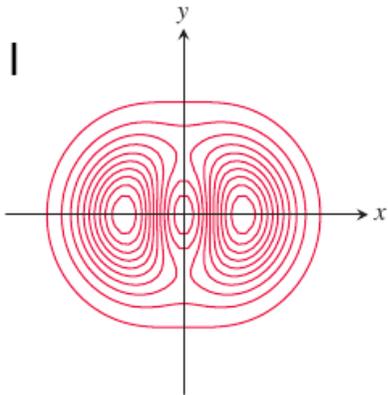
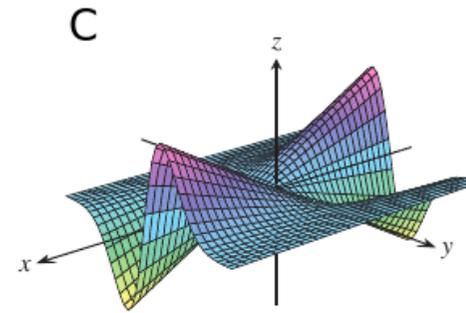
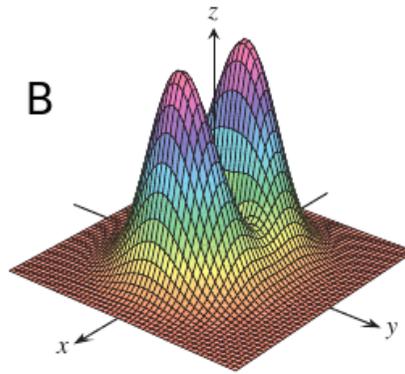
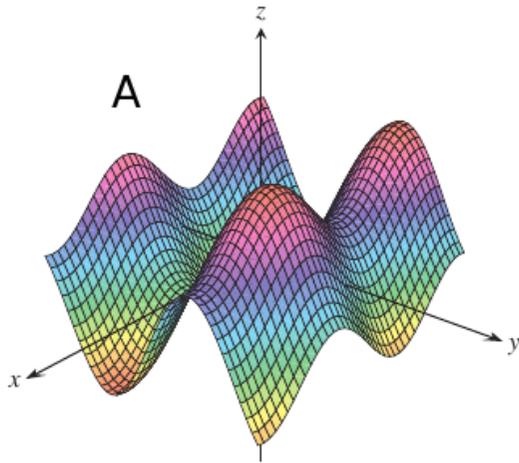
[Process]

Example 6

Match the **function** with its **graph** (labeled A–C) and with its **contour map** (labeled I–III).
Give reasons for your choices.

[Solution]

1. $z = (4x^2 + y^2)e^{-x^2-y^2}$ 2. $z = \sin x + 2 \sin y$ 3. $z = xye^{-y^2}$



Example 7

Match the function with its graph (labeled A–F) and with its contour map (labeled I–VI). Give reasons for your choices.

[Solution]

1. $z = \sin(xy)$

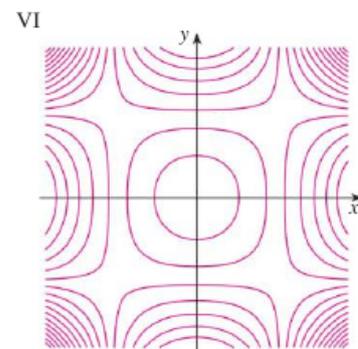
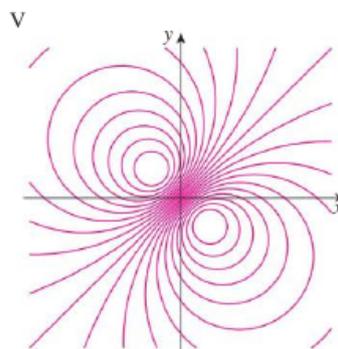
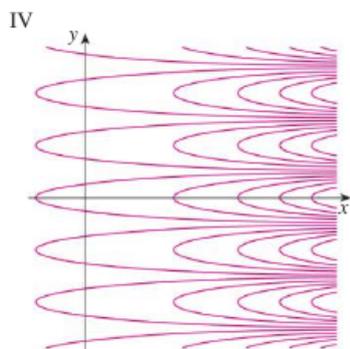
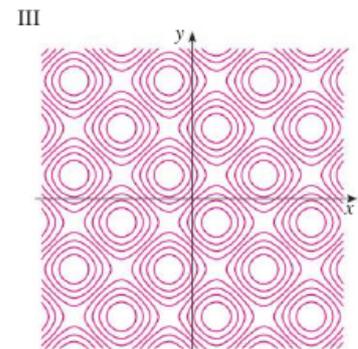
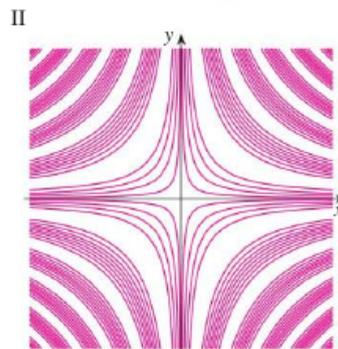
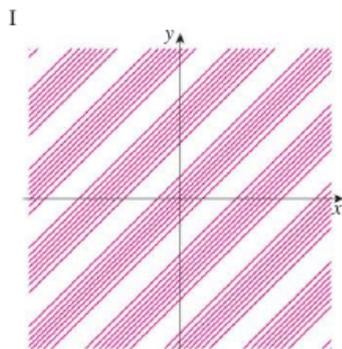
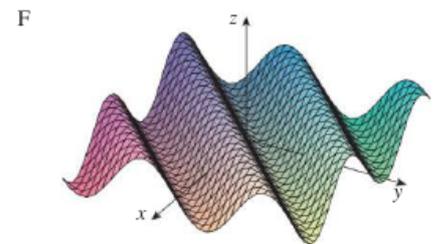
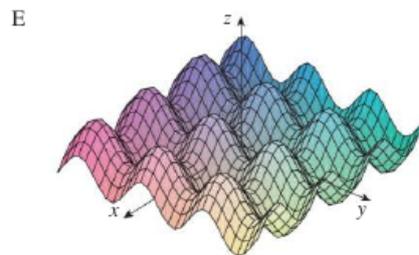
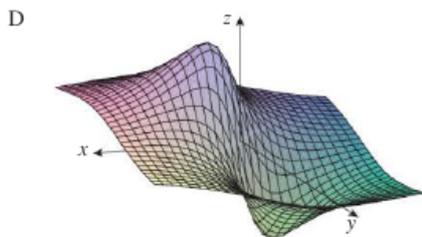
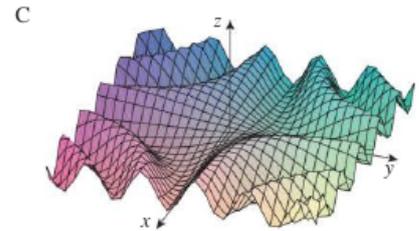
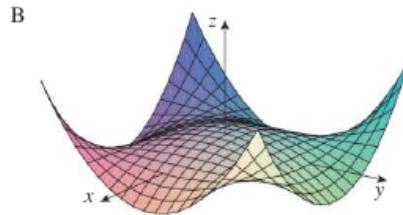
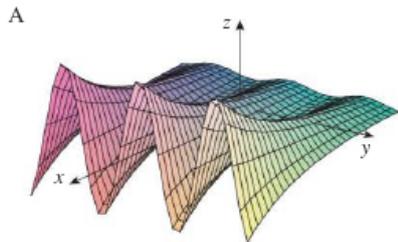
4. $z = \sin x - \sin y$

2. $z = e^x \cos y$

5. $z = (1 - x^2)(1 - y^2)$

3. $z = \sin(x - y)$

6. $z = \frac{x-y}{1+x^2+y^2}$



Example 8

Classify and sketch the surface $x - y^2 - 4z^2 = 0$

[Solution]

[Process]

Example 9

Classify and sketch the surface $4x^2 - 3y^2 + 12z^2 + 12 = 0$

[Solution]

[Process]

Example 10

Classify and sketch the surface $x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0$

[Solution]

[Process]

Example 11

Match the equation with its graphs.

[Solution]

$$1. \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

$$2. 15x^2 - 4y^2 + 15z^2 = -4$$

$$3. 4x^2 - y^2 + 4z^2 = 4$$

$$4. y^2 = 4x^2 + 9z^2$$

$$5. 4x^2 - 4y + z^2 = 0$$

$$6. 4x^2 - y^2 + 4z = 0$$

[Process]

