Chain rule (derivatives of composite functions)

1. Given $f(u,v) = u^2 \ln v$, compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ when you know that $u(x,y) = \frac{x}{y}$ and $v(x,y) = 3x - 2y$.

2. Given unknown function $z(x,y) = f(u,v) = f(u(x,y), v(x,y))$ and functions $u(x,y) = x^2 - y^2$, $v(x,y) = e^{xy}$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at point $A = [1;2]$ when you know (from physics) that $\frac{\partial f}{\partial u}(A) = 1$ and $\frac{\partial f}{\partial v}(A) = 0$.

Implicitly defined functions

3. Given $F(x,y) = x^3 + y^3 - 6xy + 4$,
verify that by the equation $F(x,y) = 0$ is implicitly defined function $y = f(x)$ near the point $A = [1;1]$. Compute its derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$ at point $x_0 = 1$ and find an equation of tangent to the graph of $f(x)$.

4. Verify that by equation $x^3 y + y^3 x + x^2 y - 3 = 0$ is implicitly defined function $y = f(x)$ near the point $A = [1;1]$.
Compute its derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$ at point $x_0 = 1$ and find an equation of normal to the graph of $f(x)$.

5. Given $F(x,y) = \sin(x+y) - y^2 \cos x$,
verify that by the equation $F(x,y) = 0$ is implicitly defined function $y = f(x)$ in the neighborhood of the point $A = [\pi; 0]$.
Compute its derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$ at point $x_0 = \pi$ and describe the behavior of $f(x)$ near point A (is it increasing or decreasing, how fast?).

6. Given $F(x,y) = x^3 + 2x^2 y + y^4$
verify that by the equation $F(x,y) = 1$ is implicitly defined function $y = f(x)$ near the point $A = [2; -1]$.
Compute the first and the second derivative of f at point $x_0 = 2$.

7. Given $F(x,y) = x^2 + \frac{1}{2} y^2 + xy - 9\ln(x)$
   (a) Find iso-curve $\iota: \quad F(x,y) = 1$.
   (b) Verify that by the iso-curve equation ($F(x,y) = 1$) is implicitly defined function $y = f(x)$ near the point $A = [1;0]$.
   (c) Approximate the iso-curve ι in a point A with the 2^{nd} order Taylor polynomial.

Results

1. $\frac{\partial f}{\partial x} = 2\frac{x}{y^2}\ln(3x-2y) + \frac{3x^2}{y^2(3x-2y)}$
   $\frac{\partial f}{\partial y} = -\frac{2x^2}{y^3}\ln(3x-2y) - \frac{2x^2}{y^2(3x-2y)}$
2. $\frac{\partial z}{\partial x}(A) = -4$, $\frac{\partial z}{\partial y}(A) = 2$.
3. $\frac{\mathrm{d}f}{\mathrm{d}x}(1) = y'(1) = -1$ and tangent line: $y - 1 = -1(x-1)$
4. $\frac{\mathrm{d}f}{\mathrm{d}x}(1) = y'(1) = -\frac{6}{5}$ and normal line: $y - 1 = \frac{5}{6}(x-1)$
5. $\frac{\mathrm{d}f}{\mathrm{d}x}(\pi) = y'(\pi) = -1 < 0$, function is decreasing with the angle $\alpha = -\frac{\pi}{4}$
6. $\frac{\mathrm{d}f}{\mathrm{d}x}(2) = y'(2) = -1$, $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(2) = y''(2) = -1$
7. (a) $x^2 + \frac{1}{2}y^2 + xy - 9\ln(x) = 1$
   (c) $f(x) \approx T_2(x) = 0 + 7(x-1) - 37(x-1)^2$