(implicitly defined functions)

1. Prove that the equation $\ln(x+2y)+x-1=0$ imlicitly defines (in some neighborhood) a function $y=f(x)$ which satisfies $f(1)=0$. Aproximate the value $f(1.1)$ by the second order Taylor's polynomial.

2. Given $F(x,y)=x^3+y^3-2x^2-xy+1$,
   verify that by the equation $F(x,y)=0$ is implicitly defined function $y=f(x)$ near the point $A=[1;0]$.
   Compute the first and the second derivative of $y=f(x)$ at point $x_0=1$ and describe the behavior of $y=f(x)$ near point A (is it increasing or decreasing, convex or concave?).

3. a) Find equation of an iso-curve for $F(x,y)=xye^{x-y}$ at point $P=[1;2]$.
   b) Find a tangent line to this iso-curve at point P.

4. Given $F(x,y)=\ln(xy+4)-2\ln 2$ and a point $A=[0;2]$.
   Can the equation $F(x,y)=0$ defined correctly the implicitly defined function $y=f(x)$ near the point A?
   If not, suggest how to compute tangent to the iso-curve $F(x,y)=0$. (hint: switch the variables)

5. Given $F(x,y,z)=x^3+y^3+z^3+xyz-6$,
   a) verify that by the equation $F(x,y,z)=0$ is implicitly defined function $z=f(x,y)$ near the point $A=[1;2;-1]$.
   b) Compute all the partial derivatives of $z=f(x,y)$ at point $T=[1;2]$.
   c) Find an equation of the tangent plain which is tangent to the graph of $z=f(x,y)$ at tangent point A.

6. Verify that by the equation $xz^2-x^2y+y^2z+2x-y=0$ is implicitly defined function $z=f(x,y)$ near the point $A=[0;1;1]$.
   Find a direction in which is the function $z=f(x,y)$ increasing the most at point $[0;1]$.
   Compute a directional derivative at the point $[0;1]$ in direction $\vec{s}=(3,-4)$.

7. Given $F(x,y,z)=z^3+3x^2z-2xy=0$
   (a) $\exists ?\, z=f(x,y)$ near the point $A=[-1;-2;1]$ defined implicitly?
   (b) Find its gradient ($\nabla f=?$) at point $A_0=[-1;-2]$.
   (c) Compute a tangent plane (τ) to $z=f(x,y)$ at point A.

Results

1. $F(x, y) = \ln(x + 2y) + x - 1 = 0,\ A = [1; 0]$
   $f(1.1) \approx T_2(1.1) = -0.0975$

2. $\frac{\mathrm{d}f}{\mathrm{d}x}(1) = y'(1) = -1 < 0$, decreasing
   $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(1) = y''(1) = 4 > 0$, convex

3. a) $F(x, y) = xye^{x-y} = 2/e$
   b) $y - 2 = 4(x - 1)$.

4. NO, but $F(x^*, y^*) = \ln(y^* x^* + 4) - 2\ln 2 = 0$ near point $A^* = [2, 0]$ can.
   tangent line : $y^* - 0 = 0(x^* - 2)$, i.e. $y^* = 0$, in original variables $x = 0$

5. b) $\frac{\partial f}{\partial x}(T) = z_x(T) = -1/5$, $\frac{\partial f}{\partial y}(T) = z_y(T) = -11/5$
   c) tangent plane: $z + 1 = -\frac{1}{5}(x - 1) - -\frac{11}{5}(y - 2)$

6. $\nabla f([0; 1]) = (-3; -1)$
   $\frac{\partial f}{\partial \vec{s}}([0; 1]) = -1$

7. $\nabla f([-1; -2]) = (1/3; -1/3)$
   τ: $z - 1 = \frac{1}{3}(x + 1) - \frac{1}{3}(y + 2)$