Mathematics II A – May 19, 2022

1. Given real function of two variables: $f(x,y) = x^2 + y^2 - 2x - 4y$
and domain $D = \{[x;y] \in \mathbb{E}_2 : x \geq 0, y \geq x, y \leq 3\}$.

a) *Justify the existence* of absolute (global) extrema of the given function f on the set D.
(Use the sufficient conditions theorem on the existence of global extrema.)

b) *Find* the absolute (global) extrema of the given function f on the set D.

2. Given function $F(x,y) = y + \ln y + x^3$

a) Verify, that by $F(x,y) = 0$ is in the neighborhood of the point $T = [-1;1]$ implicitly defined function $y = f(x)$ which has continuous first and second derivative.
(i.e. write down and verify all the assumptions of the corresponding theorem)

b) Write the equation of the tangent to the graph of the function $y = f(x)$ at the point T.

c) Compute $f''(-1)$ and decide if the function $f(x)$ is convex or concave (concave up or down) at $x = -1$.

d) Write down the Taylor's polynomial $T_2(x)$ of the function $f(x)$ centered at $x = -1$.

3. Given solid body $M \subset \mathbb{E}_3$ bounded by surfaces

$$z = 0,\ z = 2 + \frac{1}{y},\ y = 1,\ y = x \text{ and } y = 8 - x.$$

a) Sketch the projection of the body M into the xy plane, i.e. sketch the region

$$M_{xy} = \{[x;y] \in \mathbb{E}_2 : y \geq 1; y \leq x; y \leq 8 - x\}.$$

Write this domain M_{xy} as an elementary domain of integration with respect to y axis.

b) Compute the volume of the given solid body M.

4. The solid body $B \subset \mathbb{E}_3$ (elliptic cylinder) is defined as

$$B = \{[x;y;z] \in \mathbb{E}_3 : 16x^2 + 9y^2 \leq 144;\ 1 \leq z \leq 3\}.$$

a) Transform the solid B into *generalized cylindrical coordinates*.
(i.e. write down the transformation formulas and determine the bounds for the transformed variables)

b) Use definition to compute the Jacobi matrix and Jacobian of the transformation from a).

c) Compute the integral $\displaystyle\iiint_B z^2 \sqrt{16x^2 + 9y^2}\ dxdydz$.

5. Given vector field $\vec{f}(x,y) = (x+y; x^2-y)$ and domain $\Omega = \{[x;y] \in \mathbb{E}_2 : x^2 \leq y \leq \sqrt{x}\}$.

a) Compute the integral $\oint_C \vec{f} \cdot ds$ where the closed curve $C = +\partial\Omega$ is positively oriented boundary of Ω.

b) Write down the Green's theorem (including all assumptions, notation and the statement).

c) Is the given vector field $\vec{f}(x,y)$ potential (conservative)? Why?

6. A three-dimensional thin plate has a shape of the surface

$$Q = \{[x;y,z] \in \mathbb{E}_3 : x^2 + y^2 = 4; 0 \leq z \leq 4\}$$

and (surface) density prescribed by $\rho(x,y,z) = \mathrm{e}^{-z}$.

a) Sketch the surface Q and suggest a suitable parametrization $P(u,v)$ of Q including the domain for parameters u and v.

b) Compute the mass m of the above described thin plate.