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CONSTRUCTIVE GEOMETRY

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Contents

Introduction

This book has been written as a textbook for the Constructive Geometry course (first semester, Faculty of Mechanical Engineering, Czech Technical University in Prague), but it can be also used for self-study.

The textbook is organized as follows. Chapter 1 Geometrical figures brings the overview of differential geometry of points, curves, surfaces and solids. This chapter is very important for understanding many terms used in the textbook but it is not the matter of the Constructive Geometry course. Chapter 2 Planar kinematic geometry is focused on investigation and constructive solution of trajectories generated by planar motion of points and envelopes generated by planar motion of curves. A brief overview of projection methods enabling to solve three-dimensional problems graphically in two dimensions is given in chapter 3 Methods of projection. Chapter 4 Analytic geometry gives a brief overview of basic vector operations and analytic representations of straight line, conic sections, plane and quadratic surfaces. Chapters 5 to 8 are devoted to geometrical properties of surfaces in engineering practice. In particular, chapter 5 Surfaces of revolution and their intersections is focused on surfaces generated by revolution of a curve about axis and chapter 6 Helicoidal surfaces on surfaces generated by screw motion of a curve. Surfaces generated by a motion of another surface are described in chapter 7 Envelope surfaces. In chapter 8 Developable surfaces, the graphical solution of construction of planar figure into which is possible to unfold or unroll special types of ruled surfaces is presented. A list of basis literature in which more information about the topics described in this textbook can be found in Bibliography.

The theoretical part of each individual chapter is presented together with step-by-step solved example problems so that the construction is easy to follow. To demonstrate important geometrical properties of the studied objects, pictorial drawings are provided whenever possible.

I would like to thank all colleagues who contributed to this textbook by providing useful advice, by reading chapters and suggesting changes, and by finding and correcting errors.

Prague, September 2021 Ivana Linkeová

Chapter 1

Geometrical figures

In this chapter, the most important terms, definitions and fundamental properties of points, curves, surfaces and solids are given. All these figures are represented by n-variate vector function, where n is the dimension of the figure: a point ($n = 0$), a curve ($n = 1$), a surface ($n = 2$) and a solid ($n = 3$).

1.1 Points

In Euclidean three-dimensional space E^3, the Cartesian coordinate system (O, x, y, z) is given. The coordinate axes x, y and z are mutually perpendicular straight lines with common point at the origin O. Length units on all three axes are equal given by the magnitude of coordinate vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. Similarly, Cartesian coordinate system (O, x, y) in Euclidean two-dimensional space E^2 can be defined. Here, the magnitude of coordinate vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ determine length units on two perpendicular axes x and y.

Point $A = [x_A, y_A]$ in E^2 is unambiguously given by a pair of ordered real numbers x_A and y_A (Cartesian coordinates) which determine the oriented distance of point A from coordinate axes y and x in the given order, see fig. 1.1 a). Point $A = [x_A, y_A, z_A]$ in E^3 is unambiguously given by a triplet of ordered real numbers x_A, y_A and z_A (Cartesian coordinates) which determine the oriented distance of point A from coordinate plane (y, z), (x, z) and (x, y) in the given order, see fig. 1.1 b).

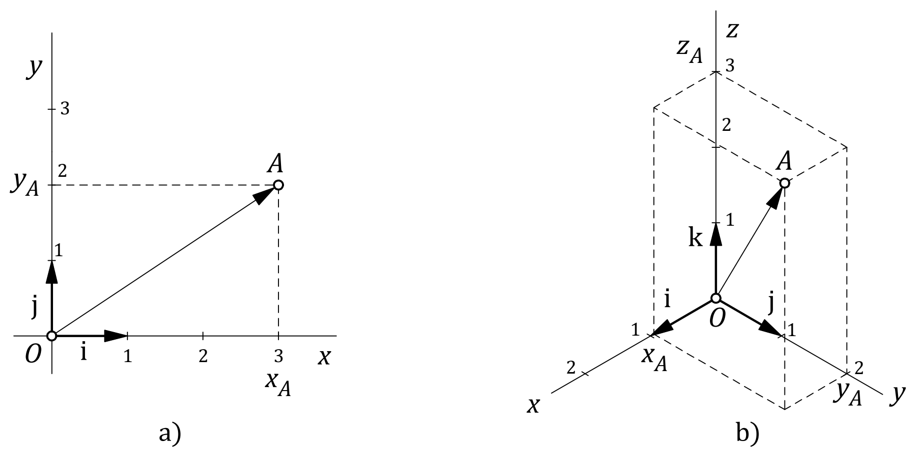


Figure 1.1: Point A in Euclidean two- and three-dimensional space

Considering a point as a function value of a vector function, it can be represented by radius vector with tail at the origin O of coordinate system and head at the point A, fig. 1.1. In the following analytic representations in this textbook, all points are considered in vector form and denoted as vectors, e.g. $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A}) = (1, 2)$ or $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A}, z_\mathbf{A}) = (1, 2, 3)$. In all graphical representations in this textbook, points are drawn by symbol $\circ$ and denoted by capital letters, e.g. $A, B, C \ldots$.

1.2 Curves

From physical point of view, a *curve* is considered to be a trajectory of a point moving in dependence on time. From geometrical point of view, a curve is characterized by one-parametric sequence of points (i.e. their radius vectors in vector representation) in two- or three-dimensional space.

■ **Definition 1.1 – Curve.** A *curve* is any connected non-empty subset p in R^2 or R^3, which is a continuous mapping of a real interval $I \subset R$. If $n = 2$, the curve is called *planar*. If $n = 3$, the curve is called *spatial*.
Analytic representation of planar or spatial curve is given by *vector equation*

$$\mathbf{P}(t) = (x(t), y(t)),\ t \in I \tag{1.1}$$

or

$$\mathbf{P}(t) = (x(t), y(t), z(t)),\ t \in I, \tag{1.2}$$

where $\mathbf{P}(t)$ is vector function of one real variable. This function is defined, continuous and at least once differentiable on the interval I. *Parametric equations* of a planar or a spatial curve are obtained by itemizing the coordinate functions of the curve given by vector equation (1.1) or (1.2)

$$\begin{aligned} x &= x(t), \\ y &= y(t),\ t \in I \end{aligned} \tag{1.3}$$

or

$$\begin{aligned} x &= x(t), \\ y &= y(t), \\ z &= z(t),\ t \in I. \end{aligned} \tag{1.4}$$

A curve defined by eq. (1.3) or eq. (1.4) is referred to as a *curve defined parametrically.* Equations (1.3) or (1.4) are referred to as a *parametric expression* or *parametrization* of a curve. □

In the following definitions, spatial curve is considered only. The modifications necessary to obtain the definitions related to planar curve are obvious.

■ **Definition 1.2 – Curve point.** A *curve point* is the function value of vector function eq. (1.2) for $t = \alpha$, $\alpha \in [a, b]$

$$\mathbf{P}(\alpha) = (x(\alpha), y(\alpha), z(\alpha))\,.$$

The parameter value $t = \alpha$ that unambiguously determines the position of the point on the curve is called *parametric (curvilinear) coordinate* of a curve point. □

The orientation of the curve is defined by orientation of its vector function. The curvilinear coordinate of the start curve point is equal to a, the curvilinear coordinate of the terminal curve point is equal to b. The start and terminal curve points are called the *endpoints* of the curve.

■ **Definition 1.3 – Regular and singular curve point.** Curve point $\mathbf{P}(\alpha)$, $\alpha \in [a, b]$ is called a *regular curve point* if the vector $\mathbf{P}'(\alpha) = (x'(\alpha), y'(\alpha), z'(\alpha))$ is a non-zero vector and only one value of parameter $t = \alpha$, $\alpha \in (a, b)$ corresponds to this point. Every other curve point is called a *singular curve point*. The coinciding endpoints of the curve are not considered singular points. □

■ **Definition 1.4 – Tangent vector, binormal vector and principal normal vector at curve point.** The first derivative of vector function (1.2)

$$\mathbf{P}'(t) = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}, \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \frac{\mathrm{d}z(t)}{\mathrm{d}t} \right) = \left(x'(t),\ y'(t),\ z'(t) \right),\ t \in [a, b]$$

is a vector function that expresses for $\alpha \in [a, b]$ a *tangent vector of a curve* $\mathbf{P}(t)$ at its regular point $\mathbf{P}(\alpha)$

$$\mathbf{P}'(\alpha) = \left(x'(\alpha),\ y'(\alpha),\ z'(\alpha) \right).$$

Orientation of tangent vector $\mathbf{P}'(t)$ is identical to the orientation of curve $\mathbf{P}(t)$. The unit tangent vector $\mathbf{t}(\alpha)$ at regular point $\mathbf{P}(\alpha)$ of curve $\mathbf{P}(t)$ is given by

$$\mathbf{t}(\alpha) = \frac{\mathbf{P}'(\alpha)}{||\mathbf{P}'(\alpha)||}. \tag{1.5}$$

The straight line given by point $\mathbf{P}(\alpha)$ and direction vector $\mathbf{t}(\alpha)$ is the *tangent line* t_α of curve $\mathbf{P}(t)$ at its point $\mathbf{P}(\alpha)$.
The *binormal vector* is obtained as a cross product (see chapter 4) of the first and second derivatives of vector function (1.2) at its regular and non-inflection (see def. 1.14) point $\mathbf{P}(\alpha)$, $\alpha \in [a, b]$. The unit binormal vector $\mathbf{b}(\alpha)$ is given by

$$\mathbf{b}(\alpha) = \frac{\mathbf{P}'(\alpha) \times \mathbf{P}''(\alpha)}{||\mathbf{P}'(\alpha) \times \mathbf{P}''(\alpha)||}.$$

The straight line given by point $\mathbf{P}(\alpha)$ and direction vector $\mathbf{b}(\alpha)$ is the *binormal line* b_α of curve $\mathbf{P}(t)$ at its non-inflection point $\mathbf{P}(\alpha)$.
Cross product of binormal vector and tangent vector of curve $\mathbf{P}(t)$, $t \in [a, b]$ at point $\mathbf{P}(\alpha)$, $\alpha \in [a, b]$ is called the *principal normal vector*. The unit principal normal vector $\mathbf{n}(\alpha)$ is given by

$$\mathbf{n}(\alpha) = \mathbf{b}(\alpha) \times \mathbf{t}(\alpha).$$

The straight line given by point $\mathbf{P}(\alpha)$ and direction vector $\mathbf{n}(\alpha)$ is the *principal normal line* n_α of curve $\mathbf{P}(t)$ at its point $\mathbf{P}(\alpha)$. □

A planar curve has one tangent line and one normal line at its regular point, see fig. 1.2 a).

■ **Definition 1.5 – Double (node) and multiple curve point.** If there exist real numbers

$$\alpha_1,\ \alpha_2 \in (a, b),\ \alpha_1 \neq \alpha_2,$$

for which $\mathbf{P}(\alpha_1) = \mathbf{P}(\alpha_2)$, i.e. the curve point given by curvilinear coordinate α_1 and curve point given by curvilinear coordinate α_2 coincide, the point $\mathbf{P}(\alpha_1) = \mathbf{P}(\alpha_2)$ is called the *double curve point* or *curve node*. If there exist $k > 2$ such numbers from the interval I, the point is called the *k-multiple curve point*. □

Double (multiple) curve point is a point where a curve intersects itself. The curve has two (k) distinct tangent lines at its double (multiple) point, see fig. 1.2 b).

■ **Definition 1.6 – Cuspidal curve point.** The curve point, in which the tangent vector is zero vector and at least one of coordinate functions $x'(t)$, $y'(t)$ or $z'(t)$ of tangent vector changes its sign is called a *cuspidal curve point* or a *cusp.* □

A curve has "sharp corner" at its cuspidal point, see fig. 1.2 c).

■ **Definition 1.7 – Vertex of the curve.** The curve point, in which the normal line is identical to the axis of symmetry of a curve is called the *vertex* of the curve. □

An example of vertex of a curve is shown in fig. 1.2 d).

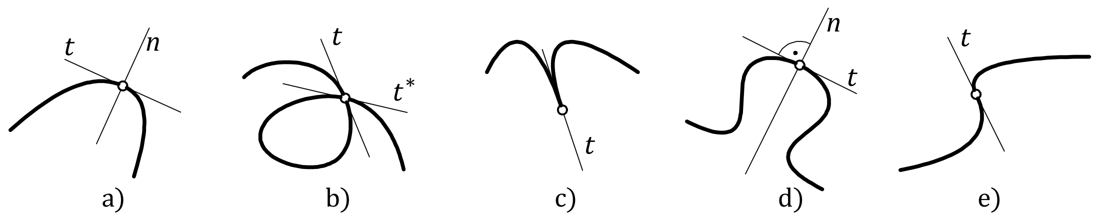


Figure 1.2: Classification of curve points

The derivatives of vector function of a curve and their function values are very important for shape modelling and curves joining. In the following in this textbook, the term *k-th curve derivative* means k-th derivative of vector function analytically representing the curve $\mathbf{P}(t)$, $t \in [a,b]$. The first three curve derivatives are denoted with $\mathbf{P}'(t)$, $\mathbf{P}''(t)$ and $\mathbf{P}'''(t)$. The function value of k-th curve derivative for parameter value $t = \alpha$, $\alpha \in [a,b]$ is referred to as *vector of k-th curve derivative.*

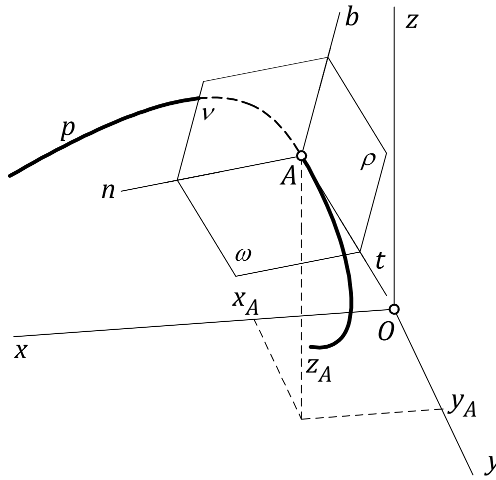
■ **Definition 1.8 – Frenet moving trihedron of a curve.** The normalized orthogonal right-handed (with positive orientation) trihedron created at a regular point $\mathbf{P}(\alpha)$, $\alpha \in [a,b]$ of curve $\mathbf{P}(t)$, $t \in [a,b]$ by unit tangent vector $\mathbf{t}(\alpha)$, unit principal normal vector $\mathbf{n}(\alpha)$ and unit binormal vector $\mathbf{b}(\alpha)$ is called *Frenet moving trihedron of a curve.* □

Frenet moving trihedron of a curve is used to describe intrinsic (geometric) properties of the curve in the neighbourhood of its regular point.

■ **Definition 1.9 – Normal, rectification and osculation plane.** The plane given at a regular point $\mathbf{P}(\alpha)$, $\alpha \in [a,b]$ of curve $\mathbf{P}(t)$, $t \in [a,b]$ by principal normal line and binormal line is called *normal plane* ν_α. The plane given at a regular point $\mathbf{P}(\alpha)$, $\alpha \in [a,b]$ of curve $\mathbf{P}(t)$, $t \in [a,b]$ by binormal line and tangent line is called *rectification plane* ρ_α. The plane given at a regular point $\mathbf{P}(\alpha)$, $\alpha \in [a,b]$ of curve $\mathbf{P}(t)$, $t \in [a,b]$ by principal normal line and tangent line is called *osculation plane* ω_α. □

Normal plane is perpendicular to the tangent line, rectification plane is perpendicular to the principal normal line and osculation plane is perpendicular to the binormal line.

Example of Frenet moving trihedron and normal, rectification and osculation plane at a point on a spatial curve $p:\ \mathbf{P}(t)$ is given in fig. 1.3. Note that in graphical representations in this textbook, a slightly simplified denotation of drawn geometrical figures is used.



Denotation of geometrical figures
in graphical representation

$p \ldots$ curve $\mathbf{P}(t)$
$A \ldots$ point on curve $\mathbf{P}(\alpha)$
$x_A, y_A, z_A \ldots$ Cartesian coordinates $x(\alpha), y(\alpha), y(\alpha)$
$t \ldots$ tangent line t_α
$n \ldots$ normal line n_α
$b \ldots$ binormal line b_α
$\nu \ldots$ normal plane ν_α
$\rho \ldots$ rectification plane ρ_α
$\omega \ldots$ osculation plane ω_α

Figure 1.3: Frenet moving trihedron
and normal, rectification and osculation plane of a curve

■ **Definition 1.10 – The first curvature – flection.** *The first curvature – flection* ${}^1k(\alpha)$ at a regular point $\mathbf{P}(\alpha)$, $\alpha \in [a, b]$ of curve $\mathbf{P}(t)$, $t \in [a, b]$ is a non-negative number expressed by

$$ {}^1k(\alpha) = \frac{||\mathbf{P}'(\alpha) \times \mathbf{P}''(\alpha)||}{||\mathbf{P}'(\alpha)||^3}, \tag{1.6} $$

where $||\mathbf{P}'(\alpha) \times \mathbf{P}''(\alpha)||$ is the magnitude of cross product of the first and second derivatives of the curve and $||\mathbf{P}'(\alpha)||$ is the magnitude of tangent vector of the curve at point $\mathbf{P}(\alpha)$. □

The first curvature of the curve corresponds to elevation of the curve from its tangent line. If ${}^1k(t) = 0$, $t \in [a, b]$, the curve $\mathbf{P}(t)$ is a straight line.

■ **Definition 1.11 – Radius of the first curvature.** The number

$$ r(\alpha) = \frac{1}{{}^1k(\alpha)} $$

is called *radius of the first curvature* at point $\mathbf{P}(\alpha)$. □

■ **Definition 1.12 – Centre of the first curvature.** The point

$$ \mathbf{S}(\alpha) = \mathbf{P}(\alpha) + r(\alpha)\mathbf{n}(\alpha) $$

lying in osculation plane ω_α on the halfline given by point $\mathbf{P}(\alpha)$ and direction vector $\mathbf{n}(\alpha)$ at the distance $r(\alpha)$ from $\mathbf{P}(\alpha)$ is called *centre of the first curvature* at point $\mathbf{P}(\alpha)$. □

■ **Definition 1.13 – Osculation circle.** The circle with the centre at point $\mathbf{S}(\alpha)$ and radius $r(\alpha)$ is called *osculation circle* at point $\mathbf{P}(\alpha)$. □

■ **Definition 1.14 – Inflection curve point.** Curve point $\mathbf{P}(\alpha)$, $\alpha \in [a, b]$ is called *inflection curve point* or *point of inflection* if the curvature ${}^1k(\alpha)$ at this point is equal to zero and unit tangent vector is continuous. □

The tangent line crosses the curve at its inflection point, see in fig. 1.2 e).

■ **Definition 1.15 – The second curvature – torsion.** *The second curvature – torsion* $^2k(\alpha)$ at a non-inflection point $\mathbf{P}(\alpha)$, $\alpha \in [a, b]$ of curve $\mathbf{P}(t)$, $t \in [a, b]$ is a real number expressed by

$$^2k(\alpha) = \frac{[\mathbf{P}'(\alpha)\mathbf{P}''(\alpha)\mathbf{P}'''(\alpha)]}{||\mathbf{P}'(\alpha) \times \mathbf{P}''(\alpha)||^2}, \tag{1.7}$$

where $[\mathbf{P}'(\alpha)\mathbf{P}''(\alpha)\mathbf{P}'''(\alpha)]$ is scalar triple product (see chapter 4) of vectors of the first, second and third curve derivatives at point $\mathbf{P}(\alpha)$. □

The second curvature corresponds to elevation of the curve from its osculation plane. If $^2k(t) = 0$, $t \in [a, b]$, the curve $\mathbf{P}(t)$ is a planar curve.

■ **Definition 1.16 – Contact of two curves.** Two curves have a contact at a common point, when they have a common tangent line at this point. □

A contact of a curve and its tangent line is drawn in fig. 1.4 a), contact of a curve and its osculation circle is drawn in fig. 1.4 b) and contact of two general curves is shown in fig. 1.4 c).

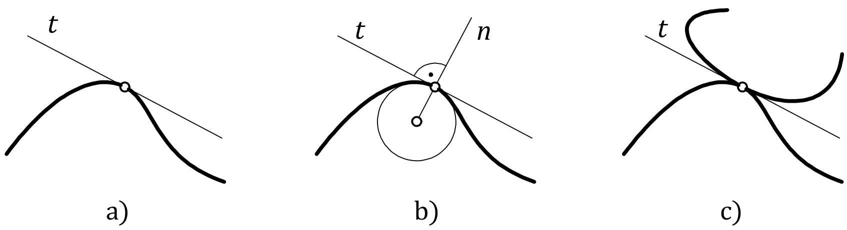


Figure 1.4: Contact of two curves

■ **Definition 1.17 – Angle of two curves.** Angle of two curves is formed by tangent lines to these curves at their common point. □

1.3 Surfaces

■ **Definition 1.18 – Surface.** A *surface* is any connected non-empty subset σ in R^3, which is a continuous mapping of a real region $I \subset R^2$. Analytic representation of a surface is given by vector equation

$$\mathbf{P}(u, v) = (x(u, v),\ y(u, v),\ z(u, v)),\ (u, v) \in I, \tag{1.8}$$

where $\mathbf{P}(u, v)$ is vector function of two real variables. This function is defined, continuous and at least once differentiable on the region I. Parametric equations of the surface are obtained by itemizing the coordinate functions of the surface given by vector equation (1.8)

$$\begin{aligned} x &= x(u, v), \\ y &= y(u, v), \\ z &= z(u, v),\ (u, v) \in I. \end{aligned} \tag{1.9}$$

A surface defined by (1.9) is referred to as a *surface defined parametrically.* Equations (1.9) are referred to as a parametric expression or parametrization of the surface.

■ **Definition 1.19 – Surface point.** A *surface point* is the function value of vector function (1.8) for $(\alpha, \beta) \in I$

$$\mathbf{P}(\alpha, \beta) = (x(\alpha, \beta),\ y(\alpha, \beta),\ z(\alpha, \beta)).$$

The parameter values $u = \alpha$ and $v = \beta$ that unambiguously define the position of a point on the surface are called parametric (curvilinear) coordinates of surface point. □

For a;constant value of one variable in a vector function of two variables, we obtain vector function of one variable representing a curve located on the surface. This curve is called *parametric curve of the surface.*

■ **Definition 1.20 – Parametric curves of a surface.** Let $\mathbf{P}(u, v)$, $(u, v) \in I$ be a vector equation of a surface and α and β parameter values from I. Then the curve

$$\mathbf{P}(u, \beta) = (x(u, \beta),\ y(u, \beta),\ z(u, \beta))$$

is called *parametric u-curve of the surface* and the curve

$$\mathbf{P}(\alpha, v) = (x(\alpha, v),\ y(\alpha, v),\ z(\alpha, v))$$

is called *parametric v-curve of the surface.* □

On a surface, parametric curves form two systems of curves, where each curve from one system intersects all curves from the other system. Two parametric curves each from different systems intersect at a common point located on the surface. Curvilinear coordinates of this point correspond to constant values of parameters u and v, see fig. 1.5 a).

■ **Definition 1.21 – Tangent vectors of parametric curves.** The first partial derivative

$$\mathbf{P}^u(u, v) = \frac{\partial \mathbf{P}(u, v)}{\partial u} = (x^u(u, v),\ y^u(u, v),\ z^u(u, v)),\ (u, v) \in I$$

is a vector function which determines for $(\alpha, \beta) \in I$ *tangent vector of parametric u-curve* $\mathbf{P}^u(u, v)$ at point $\mathbf{P}(\alpha, \beta)$

$$\mathbf{P}^u(\alpha, \beta) = \left.\frac{\partial \mathbf{P}(u, v)}{\partial u}\right|_{u=\alpha, v=\beta} = (x^u(\alpha, \beta),\ y^u(\alpha, \beta),\ z^u(\alpha, \beta)).$$

The first partial derivative

$$\mathbf{P}^v(u, v) = \frac{\partial \mathbf{P}(u, v)}{\partial v} = (x^v(u, v),\ y^v(u, v),\ z^v(u, v)),\ (u, v) \in I$$

is a vector function which determines for $(\alpha, \beta) \in I$ *tangent vector of parametric v-curve* $\mathbf{P}^v(u, v)$ at point $\mathbf{P}(\alpha, \beta)$

$$\mathbf{P}^v(\alpha, \beta) = \left.\frac{\partial \mathbf{P}(u, v)}{\partial v}\right|_{u=\alpha, v=\beta} = (x^v(\alpha, \beta),\ y^v(\alpha, \beta),\ z^v(\alpha, \beta)).$$

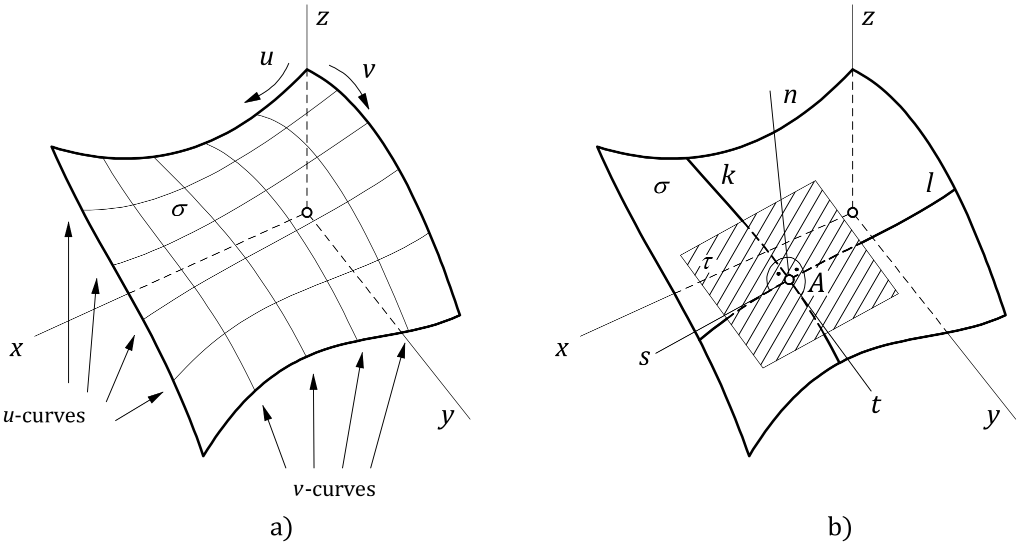
Orientation of tangent vectors of parametric curves is identical to orientation of the corresponding parametric curve. The straight line given by point $\mathbf{P}(\alpha, \beta)$ and tangent vector $\mathbf{P}^u(\alpha, \beta)$ is *tangent line of parametric u-curve* at point $\mathbf{P}(\alpha, \beta)$. The straight line given by point $\mathbf{P}(\alpha, \beta)$ and tangent vector $\mathbf{P}^v(\alpha, \beta)$ is *tangent line of parametric v-curve* at point $\mathbf{P}(\alpha, \beta)$. □

■ **Definition 1.22 – Regular and singular surface point.** Surface point $\mathbf{P}(\alpha,\beta)$, $(\alpha,\beta) \in I$ is called *regular surface point*, if the vectors $\mathbf{P}^u(\alpha,\beta)$ and $\mathbf{P}^v(\alpha,\beta)$ are non-zero, not parallel, and only one pair of parameter values $(u,v) = (\alpha,\beta)$, $(\alpha,\beta) \in I$ corresponds to this point. Every other surface point is called a *singular surface point*.

■ **Definition 1.23 – Tangent plane at surface point.** Plane $\tau(\alpha,\beta)$ given by regular point $\mathbf{P}(\alpha,\beta)$, $(\alpha,\beta) \in I$ on the surface and tangent vectors of parametric curves $\mathbf{P}^u(\alpha,\beta)$ and $\mathbf{P}^v(\alpha,\beta)$ is called the *tangent plane* of surface $\mathbf{P}(u,v)$ at point $\mathbf{P}(\alpha,\beta)$. The vector

$$\mathbf{n}(\alpha,\beta) = \mathbf{P}^u(\alpha,\beta) \times \mathbf{P}^v(\alpha,\beta) \tag{1.10}$$

at regular point $\mathbf{P}(\alpha,\beta)$ on the surface is called *normal vector* of surface $\mathbf{P}(u,v)$ at point $\mathbf{P}(\alpha,\beta)$. The straight line given by point $\mathbf{P}(\alpha,\beta)$ and normal vector $\mathbf{n}(\alpha,\beta)$ is called *normal line* of the surface at point $\mathbf{P}(\alpha,\beta)$, see fig. 1.5 b). □



Denotation of geometrical figures in graphical representation

$\sigma\ldots$ surface $\mathbf{S}(u,v)$	$k,l\ldots$ curves on surface
$A\ldots$ point on surface $\mathbf{P}(\alpha)$	$t,s\ldots$ tangent lines to curves on surface
$n\ldots$ normal line $n_{\alpha,\beta}$	$\tau\ldots$ tangent plane $\tau_{\alpha,\beta}$

Figure 1.5: Parametric curves of a surface,
tangent plane and normal line at point A on a surface

■ **Definition 1.24 – Twist vector.** The second mixed partial derivative of vector function

$$\begin{aligned}\mathbf{P}^{uv}(u,v) &= \frac{\partial^2\mathbf{P}(u,v)}{\partial u\partial v} = \frac{\partial^2\mathbf{P}(u,v)}{\partial v\partial u} = \\ &= (x^{uv}(u,v),\ y^{uv}(u,v),\ z^{uv}(u,v)) = (x^{vu}(u,v),\ y^{vu}(u,v),\ z^{vu}(u,v)), \\ &\quad (u,v) \in I\end{aligned}$$

is a vector function which determines for $(\alpha,\beta) \in I$ *twist vector* of the surface at point $\mathbf{P}(\alpha,\beta)$. □

Twist vector corresponds to elevation of the surface from its tangent plane.

To describe intrinsic properties of a surface, it is necessary to investigate *principal curvature*, *Gaussian curvature* and *mean curvature* of the surface.

To understand the term principal curvature of a surface, consider a regular surface point $\mathbf{P}(\alpha, \beta)$, tangent plane $\tau(\alpha, \beta)$ and normal line $n(\alpha, \beta)$ at this point. Infinitely many curves located on the surface pass through point $\mathbf{P}(\alpha, \beta)$. Thus, infinitely many tangent lines of these curves are lying in tangent plane $\tau(\alpha, \beta)$. Each tangent line together with normal line $n(\alpha, \beta)$ form a *plane of normal section of the surface*. Intersection of this plane of normal section and the surface is called a *curve of normal section*. At point $\mathbf{P}(\alpha, \beta)$, each curve of normal section has the first curvature equal to a certain real number, see eq. (1.6), called a *normal curvature*. Directions in which the normal curvature reaches its minimum or maximum are called *principal directions of the surface*. The corresponding values of the normal curvature are called *principal curvatures of the surface*. We denote the principal curvatures at point $\mathbf{P}(\alpha, \beta)$ with $k_{min}(\alpha, \beta)$ and $k_{max}(\alpha, \beta)$.

We will use the following convention for the sign of the principal curvature: the principal curvature is positive if vector $\overrightarrow{\mathbf{P}(\alpha, \beta)\mathbf{S}(\alpha, \beta)}$ and normal vector $n(\alpha, \beta)$ are identically oriented; the principal curvature is negative if vector $\overrightarrow{\mathbf{P}(\alpha, \beta)\mathbf{S}(\alpha, \beta)}$ and normal vector $n(\alpha, \beta)$ have opposite orientation. $\mathbf{S}(\alpha, \beta)$ is the centre of osculation circle.

■ **Definition 1.25 – Gaussian curvature.** *Gaussian curvature* at regular surface point $\mathbf{P}(\alpha, \beta)$ is given by

$$K(\alpha, \beta) = k_{min}(\alpha, \beta) \cdot k_{max}(\alpha, \beta).$$

□

According to the Gaussian curvature, a regular surface point is referred to as an *elliptic*, *parabolic* or *hyperbolic* point. At elliptic point, the Gaussian curvature is positive and the tangent plane does not contain any other point of the surface in the neighbourhood of this point. At parabolic point, the Gaussian curvature is equal to zero and the tangent plane contacts the surface along a curve. At hyperbolic point, the Gaussian surface is negative and the tangent plane intersects the surface in the intersection curve.

Gaussian surface of a plane is equal to zero. Surfaces with zero Gaussian curvature are developable into a plane.

■ **Definition 1.26 – Mean curvature.** *Mean curvature* at regular point $\mathbf{P}(\alpha, \beta)$ of surface is given by

$$H(\alpha, \beta) = \frac{k_{\min}(\alpha, \beta) + k_{\max}(\alpha, \beta)}{2}.$$

□

Surfaces with zero mean curvature are called *minimal surfaces*.

1.4 Solids

■ **Definition 1.27 – Solid.** A *solid* is any connected non-empty subset κ in R^3, which is a continuous mapping of a connected region $I \subset R^3$. Analytic representation of a solid is given by vector equation

$$\mathbf{P}(u, v, t) = \big(x(u, v, t), y(u, v, t), z(u, v, t)\big), \ (u, v, t) \in I, \tag{1.11}$$

where $\mathbf{P}(u, v, t)$ is vector function of three real variables. This function is defined, continuous and at least once differentiable on the region I. □

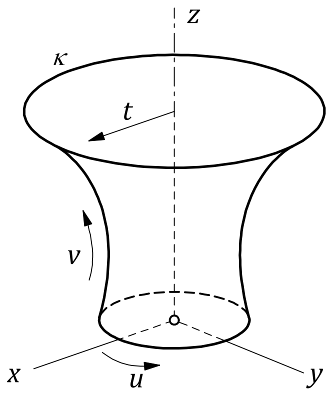


Figure 1.6: Solid

Example of a solid is given in fig. 1.6.

■ **Definition 1.28 – Solid point.** A *solid point* is the function value of vector function (1.11) for $(\alpha, \beta, \gamma) \in I$

$$\mathbf{P}(\alpha, \beta, \gamma) = \big(x(\alpha, \beta, \gamma), y(\alpha, \beta, \gamma), z(\alpha, \beta, \gamma)\big).$$

The parameter values $u = \alpha$, $v = \beta$, $t = \gamma$ that unambiguously define the position of a point located in the solid, are called parametric (curvilinear) coordinates of the solid point. □

For constant value of one variable in vector function of three variables, we obtain vector function of two variables representing a surface located in the solid. This surface is called *parametric surface of the solid.*

■ **Definition 1.29 – Parametric surfaces of a solid.** Let $\mathbf{P}(u, v, t)$, $(u, v, t) \in I$ be a vector equation of a solid and α, β and γ parameter values from I. Then the surface

$$\mathbf{P}(u, v, \gamma) = \big(x(u, v, \gamma), y(u, v, \gamma), z(u, v, \gamma)\big)$$

is called *uv-parametric surface of the solid* $\mathbf{P}(u, v, t)$, the surface

$$\mathbf{P}(u, \beta, t) = \big(x(u, \beta, t), y(u, \beta, t), z(u, \beta, t)\big)$$

is called *ut-parametric surface of the solid* $\mathbf{P}(u, v, t)$ and the surface

$$\mathbf{P}(\alpha, v, t) = \big(x(\alpha, v, t), y(\alpha, v, t), z(\alpha, v, t)\big)$$

is called *vt-parametric surface of the solid* $\mathbf{P}(u, v, t)$. □

Parametric surfaces form three systems of surfaces located in the solid, see example in fig. 1.7. Each parametric surface from one system intersects all parametric surfaces from the other two systems. Arbitrary three parametric surfaces (each from one system) are intersecting at a common solid point.

For constant values of two variables in vector function of three variables, we obtain vector function of one variable representing a curve located in the solid. This curve is called *parametric curve of the solid.*

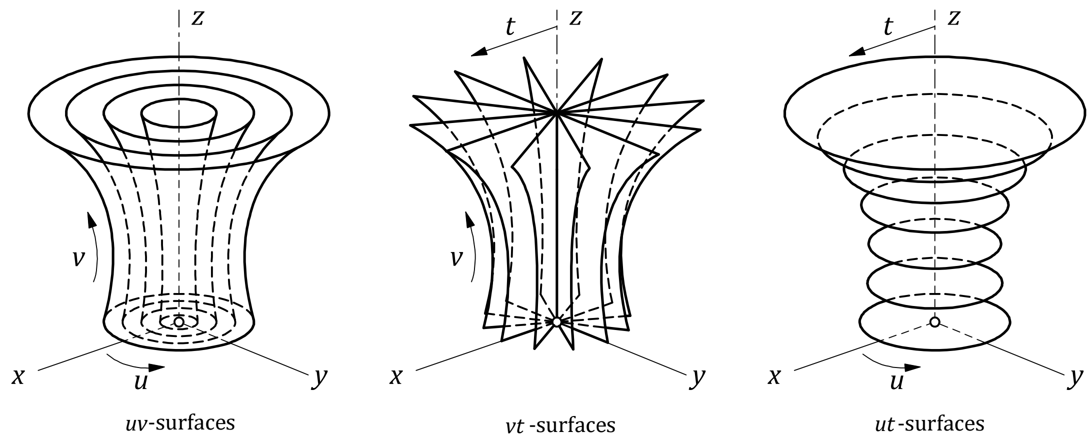


Figure 1.7: Parametric surfaces of a solid

■ **Definition 1.30 – Parametric curves of a solid.** Let $\mathbf{P}(u, v, t)$, $(u, v, t) \in I$ be a vector equation of a solid and α, β and γ are parameter values from I. Then the curve

$$\mathbf{P}(u, \beta, \gamma) = \big(x(u, \beta, \gamma), y(u, \beta, \gamma), z(u, \beta, \gamma)\big)$$

is called *u-parametric curve of the solid* $\mathbf{P}(u, v, t)$, the curve

$$\mathbf{P}(\alpha, v, \gamma) = \big(x(\alpha, v, \gamma), y(\alpha, v, \gamma), z(\alpha, v, \gamma)\big)$$

is called *v-parametric curve of the solid* $\mathbf{P}(u, v, t)$ and the curve

$$\mathbf{P}(\alpha, \beta, t) = \big(x(\alpha, \beta, t), y(\alpha, \beta, t), z(\alpha, \beta, t)\big)$$

is called *t-parametric curve of the solid* $\mathbf{P}(u, v, t)$. □

Parametric curves form three systems of curves in the solid, see example in fig. 1.8. One parametric curve from each system passes through each solid point.

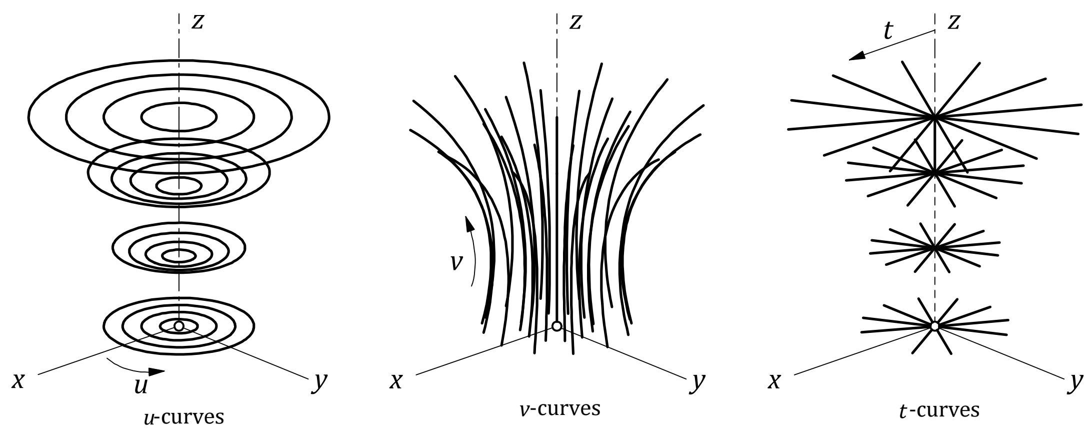


Figure 1.8: Parametric curves of a solid

■ **Definition 1.31 – Tangent vectors of parametric curves of a solid.** The first partial derivative of vector function (1.11)

$$\mathbf{P}^u(u,v,t) = \frac{\partial \mathbf{P}(u,v,t)}{\partial u} = \big(x^u(u,v,t), y^u(u,v,t), z^u(u,v,t)\big) \tag{1.12}$$

is a vector function which determines for $(\alpha,\beta,\gamma) \in I$ *tangent vector* $\mathbf{P}^u(\alpha,\beta,\gamma)$ *of u-parametric curve* $\mathbf{P}(u,\beta,\gamma)$ at point $\mathbf{P}(\alpha,\beta,\gamma)$ of the solid $\mathbf{P}(u,v,t)$.
The first partial derivative of vector function (1.11)

$$\mathbf{P}^v(u,v,t) = \frac{\partial \mathbf{P}(u,v,t)}{\partial v} = \big(x^v(u,v,t), y^v(u,v,t), z^v(u,v,t)\big) \tag{1.13}$$

is a vector function which determines for $(\alpha,\beta,\gamma) \in I$ *tangent vector* $\mathbf{P}^v(\alpha,\beta,\gamma)$ *of v-parametric curve* $\mathbf{P}(\alpha,v,\gamma)$ at point $\mathbf{P}(\alpha,\beta,\gamma)$ of the solid $\mathbf{P}(u,v,t)$.
The first partial derivative of vector function (1.11)

$$\mathbf{P}^t(u,v,t) = \frac{\partial \mathbf{P}(u,v,t)}{\partial t} = \big(x^t(u,v,t), y^t(u,v,t), z^t(u,v,t)\big) \tag{1.14}$$

is a vector function which determines for $(\alpha,\beta,\gamma) \in I$ *tangent vector* $\mathbf{P}^t(\alpha,\beta,\gamma)$ *of t-parametric curve* $\mathbf{P}(\alpha,\beta,t)$ at point $\mathbf{P}(\alpha,\beta,\gamma)$ of the solid $\mathbf{P}(u,v,t)$.
Orientation of tangent vectors of parametric curves and orientation of parametric curves is identical.
Point $\mathbf{P}(\alpha,\beta,\gamma)$ and tangent vector $\mathbf{P}^u(\alpha,\beta,\gamma)$ determine *tangent line to u-parametric curve* of the solid $\mathbf{P}(u,v,t)$. Point $\mathbf{P}(\alpha,\beta,\gamma)$ and tangent vector $\mathbf{P}^v(\alpha,\beta,\gamma)$ determine *tangent line to v-parametric curve* of the solid $\mathbf{P}(u,v,t)$. Point $\mathbf{P}(\alpha,\beta,\gamma)$ and tangent vector $\mathbf{P}^t(\alpha,\beta,\gamma)$ determinate *tangent line to t-parametric curve* of the solid $\mathbf{P}(u,v,t)$. □

■ **Definition 1.32 – Tangent plane of parametric surface of a solid.** The plane given by the pair of tangent vectors $\mathbf{P}^u(u,v,t)$ and $\mathbf{P}^v(u,v,t)$ at the regular solid point $\mathbf{P}(\alpha,\beta,\gamma)$ is called *tangent plane of uv-parametric surface of the solid* $\mathbf{P}(u,v,t)$ at point $\mathbf{P}(\alpha,\beta,\gamma)$. The plane given by the pair of tangent vectors $\mathbf{P}^u(u,v,t)$ and $\mathbf{P}^t(u,v,t)$ at the regular solid point $\mathbf{P}(\alpha,\beta,\gamma)$ is called *tangent plane of ut-parametric surface of the solid* $\mathbf{P}(u,v,t)$ at point $\mathbf{P}(\alpha,\beta,\gamma)$. The plane given by the pair of tangent vectors $\mathbf{P}^v(u,v,t)$ and $\mathbf{P}^t(u,v,t)$ at the regular solid point $\mathbf{P}(\alpha,\beta,\gamma)$ is called *tangent plane of vt-parametric surface of the solid* $\mathbf{P}(u,v,t)$ at point $\mathbf{P}(\alpha,\beta,\gamma)$. □

Chapter 2

Planar kinematic geometry

Planar kinematic geometry studies geometrical properties of curves generated by motion of *moving system* in the plane of *fixed system*. Physical properties such as velocity and acceleration of the motion are not taken into consideration. In this chapter, only synthetic representation of all solved problems is given and the following denotation of moving and fixed figures is used.

$A, B, C, \dots$	points (capital letters)
$A^0, A^1, A^2, \dots$	individual positions of moving point A
τ^A	trajectory of point A
$t^{A^0}, t^{A^1}, t^{A^2}, \dots$	tangent line to trajectory τ^A at individual positions of moving point A
$n^{A^0}, n^{A^1}, n^{A^2}, \dots$	normal line to trajectory τ^A at individual positions of moving point A
$a, b, c, \dots$	curves (small letters)
$a^0, a^1, a^2, \dots$	individual positions of moving curve a
(a)	envelope of moving curve a
$n^{a^0}, n^{a^1}, n^{a^2}, \dots$	normal line to moving curve a at individual positions of moving curve a
$T^{a^0}, T^{a^1}, T^{a^2}, \dots$	individual positions of point of contact between the moving curve a and its envelope (a)
$t^{a^0}, t^{a^1}, t^{a^2}, \dots$	tangent line of envelope (a) at point of contact $T^{a^0}, T^{a^1}, T^{a^2}, \dots$
p	fixed centrode
h	moving centrode
$h^0, h^1, h^2, \dots$	individual positions of moving centrode h
$S_0, S_1, S_2, \dots$	instantaneous centres of rotation on fixed centrode p
$S_0^i, S_1^i, S_2^i, \dots$	instantaneous centres of rotation on moving centrode h^i

Moving system is a set of planar figures subjected to the continuous motion in the plane of fixed system. The shape of the moving system does not change during the motion. Therefore, all the individual positions of the moving figures are mutually congruent. Example of this situation is drawn in fig. 2.1, where moving system is represented by triangle $\triangle ABC$ and fixed system is represented by trajectories τ^A, τ^B and τ^C. Thus, the following congruences are valid.

$$\triangle A^0B^0C^0 \cong \triangle A^1B^1C^1 \cong \triangle A^2B^2C^2 \cong \dots$$

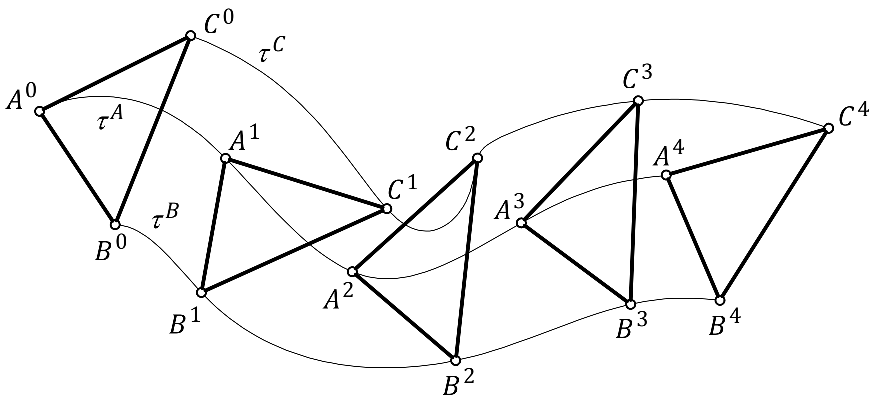


Figure 2.1: Example of planar motion

There are two types of curves generated by planar motion – *trajectories of moving points* and *envelopes of moving curves*. In the case of trajectory of moving point investigation, it is necessary to construct the moving point in sufficient number of ordered positions, see positions $A^0, A^1, \ldots, A^4$ in fig. 2.2 a).

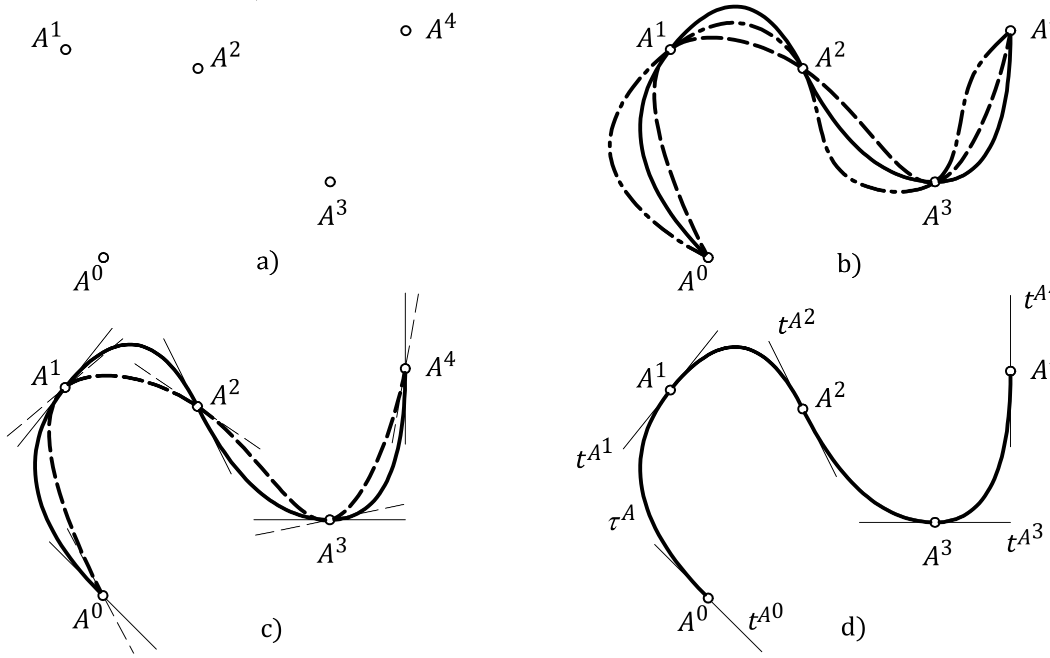


Figure 2.2: Example of trajectory generated by motion of point A

Obviously, it is possible to associate infinitely many trajectories passing through the sequence $A^0, A^1, \ldots, A^4$, see fig. 2.2 b). The shape of individual trajectories depends not only on the positions of moving point but also on the direction of tangent lines to the trajectory, see fig. 2.2 c). To preserve the readability of the picture, only two of the trajectories and their tangent lines are drawn here. Different directions of tangent lines determine different shape of the resulting trajectory. Thus, it is necessary to construct *a pair of point and tangent line* at each considered instant of the motion first. Then, it is possible to estimate the shape of the trajectory which passes through all positions of moving point and follows all tangent lines, see fig. 2.2 d).

Similar situation arises when investigating envelopes of moving curves, see fig. 2.3 a). Here, the individual positions $a^0, a^1, \ldots, a^4$ of a moving straight line a represent tangent lines to the generated envelope (a). Again, there are infinitely many envelopes which can be inscribed into this sequence of tangent lines. Three possible envelopes are shown in fig. 2.3 b). To estimate the shape of the envelope more precisely, it is necessary to determine points of contact between the moving curve and its envelope, see fig. 2.3 c). Thus, a pair of point and tangent line has to be known at each considered instant first. Then, the shape of the envelope can be estimated and drawn as a curve passing through all points of contact and following the direction of the moving straight line as is shown in fig. 2.3 d).

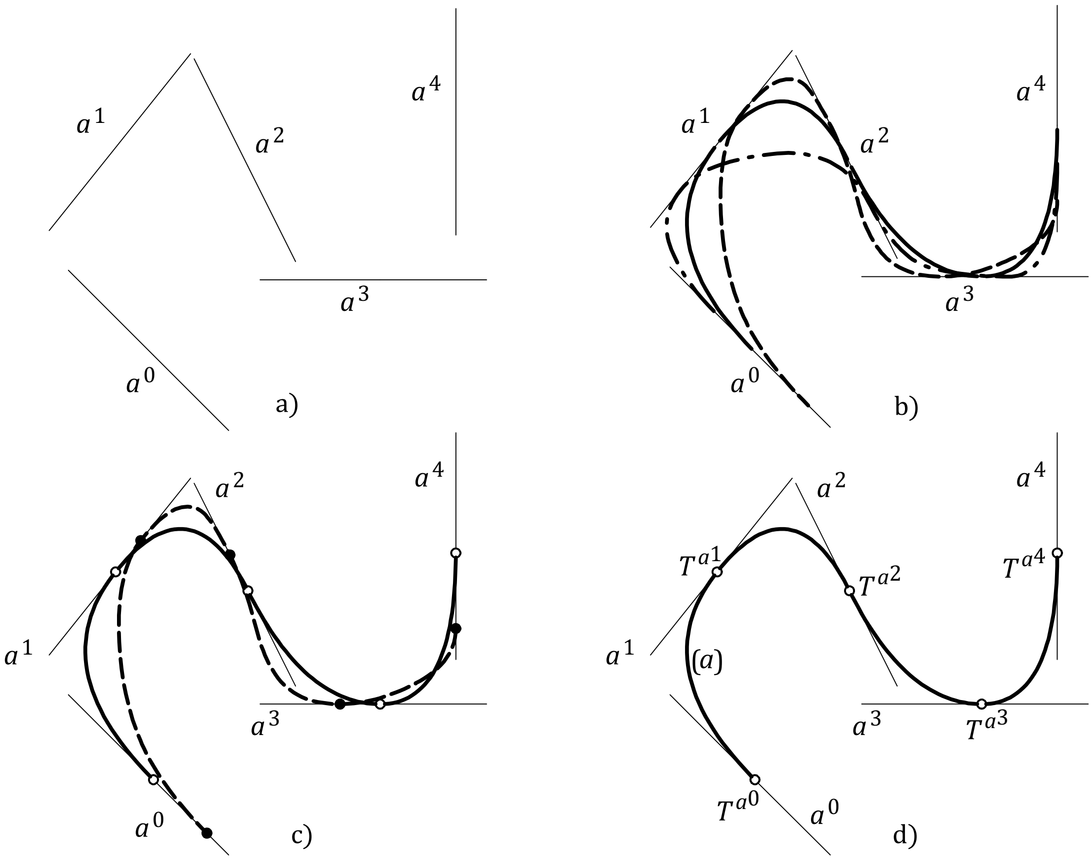


Figure 2.3: Example of envelope generated by motion of straight line a

2.1 Elementary and general planar motion

To solve the problem of construction of a pair of point and tangent line at arbitrary instant of general planar motion, elementary planar motions – *rotation* and *translation* will be investigated first. After that, approximation of general motion by means of a set of instantaneous rotations will be explained.

Rotation is given by centre S. Trajectories of all rotating points are circles with centre S, see fig. 2.4 a). Normal lines to each trajectory pass through the centre S and the rotating point. Tangent line passes through the rotating point perpendicularly to the normal line. Similarly, envelopes of all rotating curves are circles with the centre S, see fig. 2.4 b). Normal line to each envelope passes through the centre S perpendicularly to the rotating curve and intersects the rotating curve at the point of contact. The tangent line is identical to the tangent line of rotating curve constructed at the point of contact.

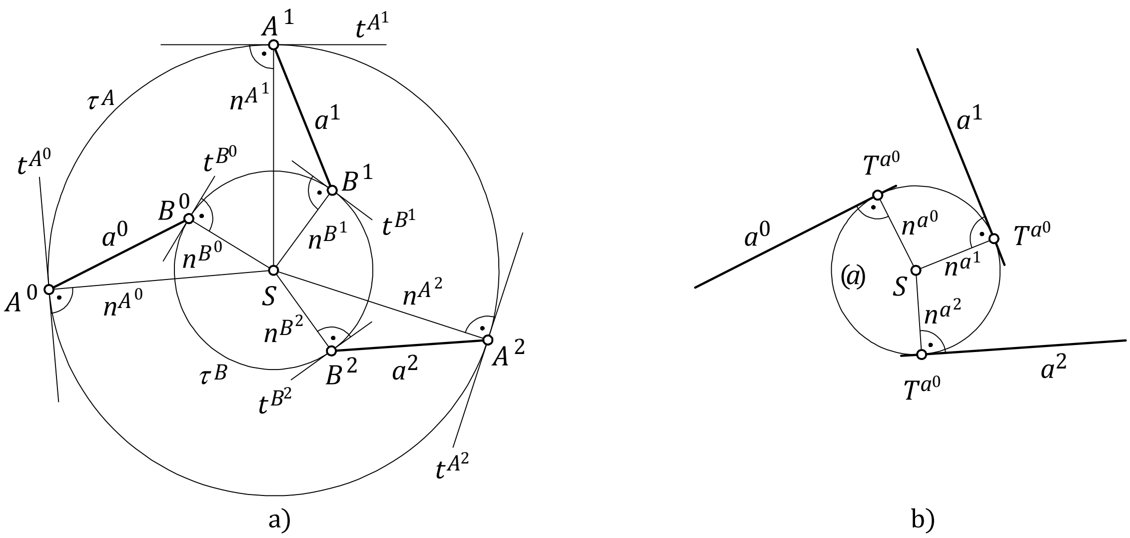


Figure 2.4: Rotation

Translation is given by direction s. Trajectories of all translated points, see fig. 2.5 a), and envelopes of all translated curves, see fig. 2.5 b), are straight lines parallel to the direction of translation. Tangent lines are identical to trajectories and envelopes generated by translation. Normal lines are perpendicular to the direction of translation, thus, they are mutually parallel and their intersection lies at infinity. Obviously, translation can be considered as a special case of rotation with centre of rotation at infinity.

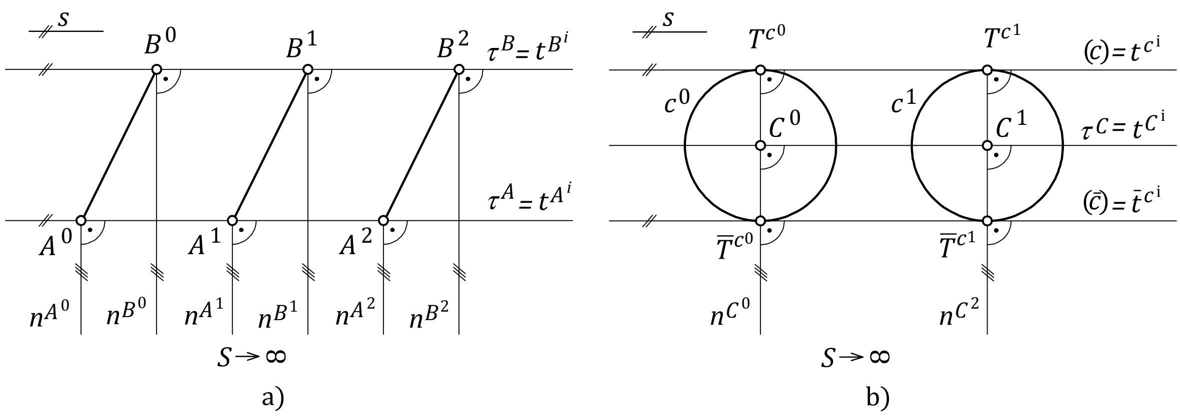


Figure 2.5: Translation

In general, it is possible to find elementary motion which maps any position of moving system to another. Let us consider two congruent positions of straight line segment $A^iB^i \cong A^jB^j$, see fig. 2.6, and construct bisectors $o^{A^{ij}}$ and $o^{B^{ij}}$ of straight line segments A^iA^j and B^iB^j. Depending on mutual position of these bisectors, the elementary motion can be identified as follows.

1. Intersecting bisectors – elementary motion is rotation given by centre at intersection of both bisectors: $S_{ij} = o^{A^{ij}} \cap o^{B^{ij}}$, see fig. 2.6 a).

2. Parallel bisectors – elementary motion is translation given by direction parallel with straight line segment A^iA^j (or B^iB^j): $s_{ij} \parallel A^iA^j$, $s_{ij} \parallel B^iB^j$, see fig. 2.6 b).

3. Identical bisectors – in the case of intersecting straight line segments, the elementary motion is rotation given by centre $S_{ij} = A^iB^i \cap A^jB^j$, see fig. 2.6 c). In the case of parallel straight line segments, the elementary motion is translation given by direction $s_{ij} \perp A^iB^i$ (or $s_{ij} \perp A^jB^j$), see fig. 2.6 d).

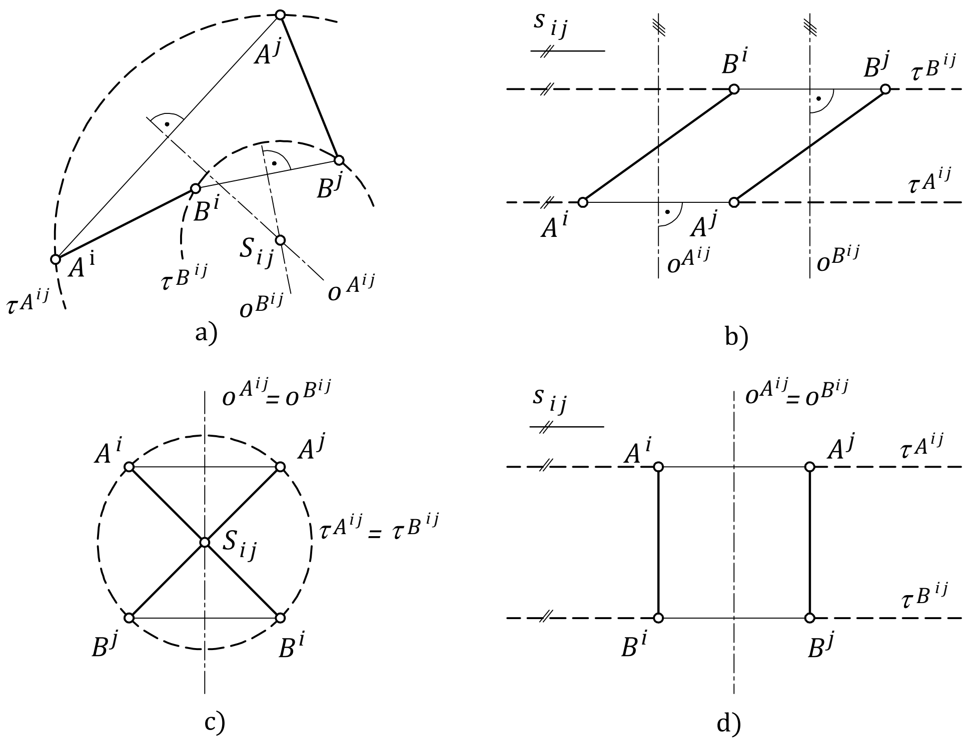


Figure 2.6: Properties of elementary planar motions

Using these properties of elementary motion, it is possible to approximate any planar motion between two consecutive positions of moving system by replacing rotation with either real centre (rotation) or centre at infinity (translation). Example of this situation is drawn in fig. 2.7 a), where the centre of replacing rotation $S_{ij} = o^{A^{ij}} \cap o^{B^{ij}}$ is constructed firstly. After that, the original trajectories τ^A, τ^B and τ^C between i-th and j-th position are replaced by circles $\tau^{A^{ij}}$, $\tau^{B^{ij}}$ and $\tau^{C^{ij}}$.

Since A^jB^j approaches A^iB^i, the secant lines A^iA^j and B^iB^j approximate the tangent lines to the corresponding trajectory. The limiting position of secant lines A^iA^j and B^iB^j is the tangent line t^{A^i} and t^{B^i} and the limiting position of S_{ij} is S_i, see fig. 2.7 b). Replacing rotation which approximates the motion between two infinitely close positions of moving system is called *instantaneous rotation* and centre S_i is called *instantaneous centre of rotation* or *pole of the motion.*

■ **Theorem 2.1 – Instantaneous centre of rotation.** *Instantaneous centre of rotation* S_i is the intersection of all normal lines to all trajectories of all moving points and to all envelopes of all moving curves at the i-th instant.

Proof. This statement follows from above mentioned considerations. □

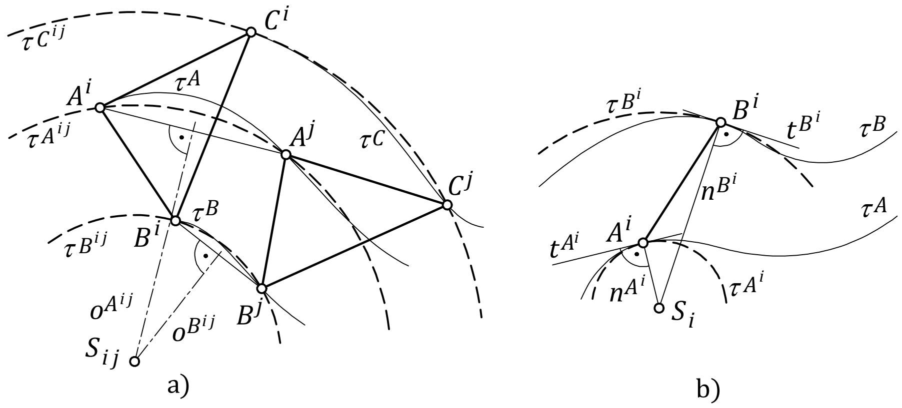


Figure 2.7: Approximation of general motion by instantaneous rotationat i-th instant

Instantaneous centre of rotation is useful for construction of tangent lines to the trajectories of moving points and points of contact between moving curves and their envelopes at the considered instant. Since the instantaneous centre of rotation lies at the intersection of all normal lines to trajectories and envelopes at the given instant, it is necessary to know two of them to be able to find this intersection. It follows that the motion in the plane is fully determined if the following figures are given.

1. Two trajectories of two moving points.
2. One trajectory of moving point and one envelope of moving curve.
3. Two envelopes of two moving curves.
4. Fixed and moving centrodes (see section 2.2).

General planar motion can be approximated by a set of instantaneous rotations given by a set of instantaneous centres. Then, the generated trajectories and envelopes can be approximated by a set of circular arcs.

2.2 Centrodes of the motion

Any planar motion can be described by two curves in the plane, called *centrodes*. The *fixed centrode* p is the locus of instantaneous centres of rotation in the plane of the fixed system. The *moving centrode* h is the locus of instantaneous centres of rotation in the plane of the moving system. The motion is realized by rolling the moving centrode on the fixed one. The moving system is fixtly connected to the moving centrode. Since the moving centrode h^j rolls without slipping on the fixed centrode p, the instantaneous centre of rotation $S_i = S_i^j$ is the point of contact between the two centrodes p and h^j. Thus, the fixed and moving centrodes have a common tangent line at the instantaneous centre of rotation $S_i = S_i^j$, see fig. 2.8.

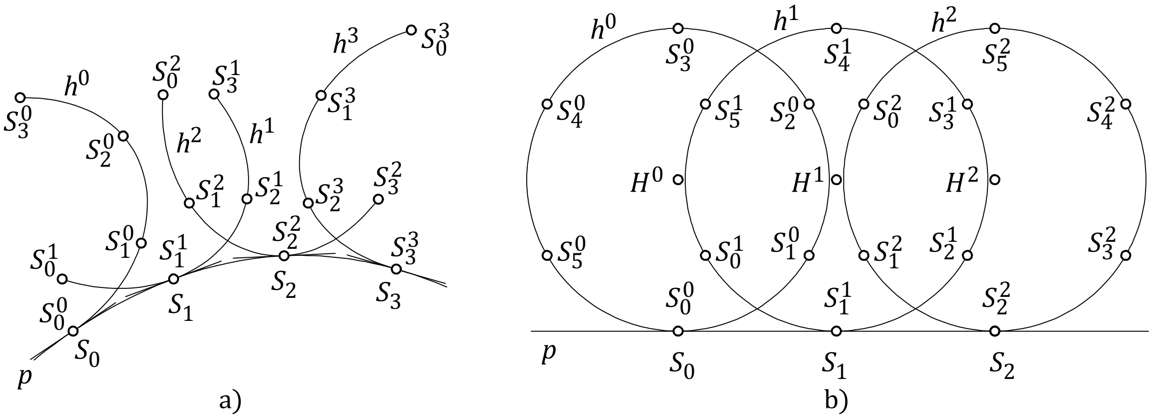


Figure 2.8: Motion given by fixed and moving centrodes

In synthetic representation, the fixed centrode is constructed as a curve passing through sufficient number of instantaneous centres of rotation. The moving centrode is constructed by means of transformation of instantaneous centres of rotation into the plane of moving system. The transformation from i-th instant into the required instant (usually the given 0-th instant) is realized by construction of two congruent positions of a suitable figure (usually triangle) containing the moving system and corresponding instantaneous centre of rotation, see fig. 2.14 in example 2.1, fig. 2.19 in example 2.2 and fig. 2.25 in example 2.3.

Another approach to the construction of moving centrode based on inverse motion is explained in the next section.

2.3 Inverse motion

Mutual motion of fixed and moving systems is relative, thus, it is possible to interchange their roles. The motion in which the original fixed system moves and the original moving system is stationary is called *inverse motion*. The role of centrodes is interchanged, too. The fixed centrode of the original motion becomes the moving centrode of the inverse motion and vice versa, see fig. 2.9.

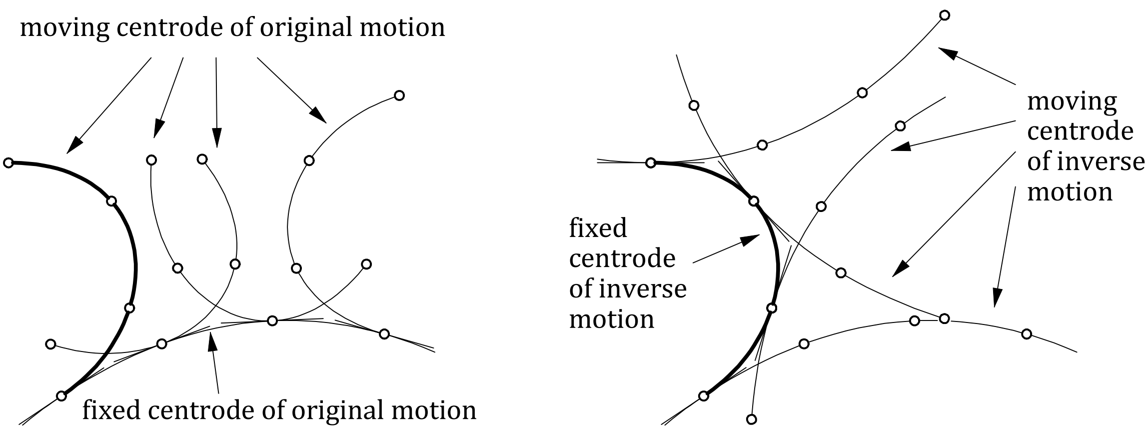


Figure 2.9: Inverse motion

Inverse motion has the following properties.

- If the moving point A generates the trajectory τ^A during the original motion, then the trajectory τ^A envelopes the point A^i at the selected i-th instant during the inverse motion.
- If the moving curve a generates the envelope (a) during the original motion, then the envelope (a) envelopes the curve a^i at the selected i-th instant during the inverse motion.
- Motion obtained by interchanging centrodes is the inverse motion to the original motion.

2.4 Example problems – motion given by trajectories and envelopes

This section is focused on examples of planar motion given by the first three ways mentioned on page 24. Only straight lines or circles are considered to be the given trajectories of moving points or moving curves. Similarly, only straight lines, circles and points are considered to be the given envelopes of moving curves.

At each of the following examples, the construction of a new position of the moving system and fixed centrode p as the set of instantaneous centres of rotation is presented first. Then, the trajectory τ^C of moving point C, envelope (c) of moving circle $c = (C, r)$ and envelope (d) of moving straight line d is described, including the construction of tangent lines to the generated trajectory and points of contact between the generated envelope and moving curve. Finally, the construction of moving centrode h^0 by means of transformation of instantaneous centres of rotation into the plane of moving system at the given instant is shown.

■ Example 2.1 – Motion given by two linear trajectories

Given

Linear trajectory τ^A, linear trajectory τ^B, points A, B, C, circle $c = (C, r)$ and straight line $d = AB$ at initial position, see fig. 2.10 a).

Required

Construct fixed centrode p, trajectory τ^C of the given point C, envelope (c) of the given circle c, envelope (d) of the given straight line d and moving centrode h at the given instant.

Analysis

The fixed system of the motion is represented by trajectories τ^A and τ^B. The given moving system is represented by moving points A and B. The following rules apply to this motion.

- Point A is located on trajectory τ^A at each instant.
- Point B is located on trajectory τ^B at each instant.
- The distance $\| AB \|$ of points A and B does not change.

This motion is inverse to the motion given by two point envelopes of two moving straight lines, circular and point envelopes of moving straight lines or two circular envelopes of moving straight lines. Since the motion given by two moving straight lines is solved in example 2.3, the configuration of individual figures in this example and in example 2.3 has been chosen so that the moving centrode h^0 in fig. 2.14 and fixed centrode p in fig. 2.25 are identical.

Moreover, the motion given by two linear trajectories can be realized by rolling the constructed moving centrode h on the fixed centrode p, i.e. by hypocycloidal motion (see section 2.5).

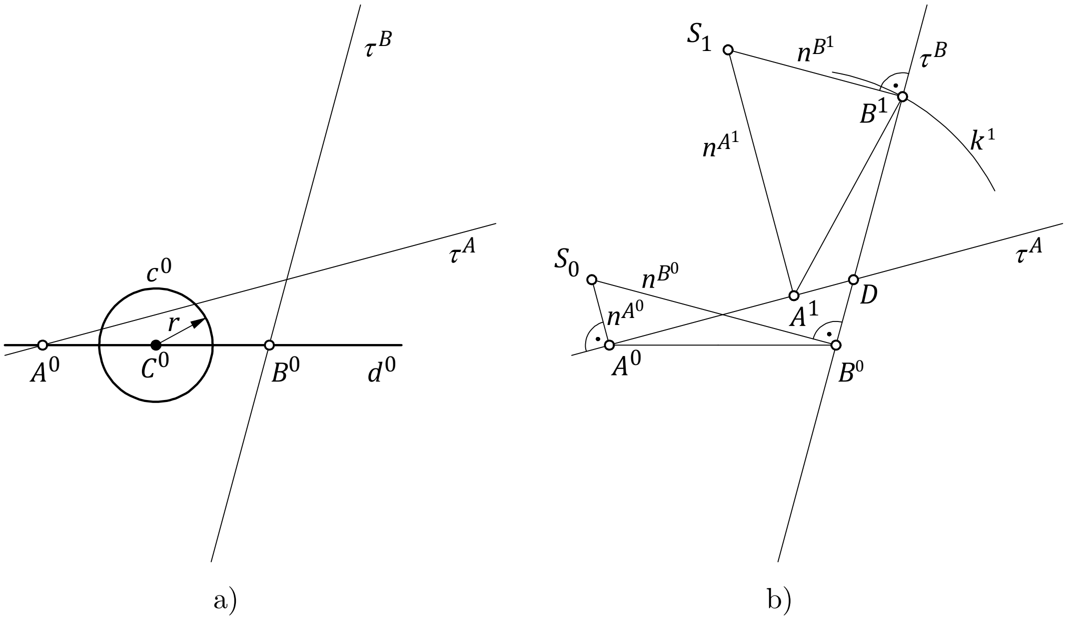


Figure 2.10: Motion given by two linear trajectories

Graphical solution

a) New position of the given moving system

1. Choose new position $A^1 \in \tau^A$, see fig. 2.10 b).
2. Construct new position $B^1 \in \tau^B$ so that $||A^1B^1|| = ||A^0B^0||$, i.e. point B^1 lies at intersection of circle $k^1 = (A^1, r = ||A^0B^0||)$ and trajectory τ^B: $B^1 = k^1 \cap \tau^B$. Note that it does not matter if we choose new position $B^1 \in \tau^B$ and construct new position $A^1 \in \tau^A$ so that $||A^1B^1|| = ||A^0B^0||$ (not drawn in fig. 2.10 b). Both approaches are equal.
3. Draw straight line segment A^1B^1.
4. Continue in a similar way to obtain a sufficient number of instants. Do not forget special limit positions of points A and B with respect to the intersection D of trajectories τ^A and τ^B: $A = D$ or $B = D$ and positions where the distance $||AD||$ or $||BD||$ is maximal.

b) Instantaneous centre of rotation and fixed centrode p

1. Construct normal line $n^{A^0} \perp \tau^A$, $A^0 \in n^{A^0}$, see fig. 2.10 b).
2. Construct normal line $n^{B^0} \perp \tau^B$, $B^0 \in n^{B^0}$.
3. Centre of instantaneous rotation $S_0 = n^{A^0} \cap n^{B^0}$.

4. Continue in a similar way at each instant. Draw the fixed centrode p as a curve passing through all positions of instantaneous centre of rotation. In this case, the fixed centrode is the circle $p = (D, r = ||DS_i||)$, drawn in fig. 2.11.

c) Trajectory of moving point

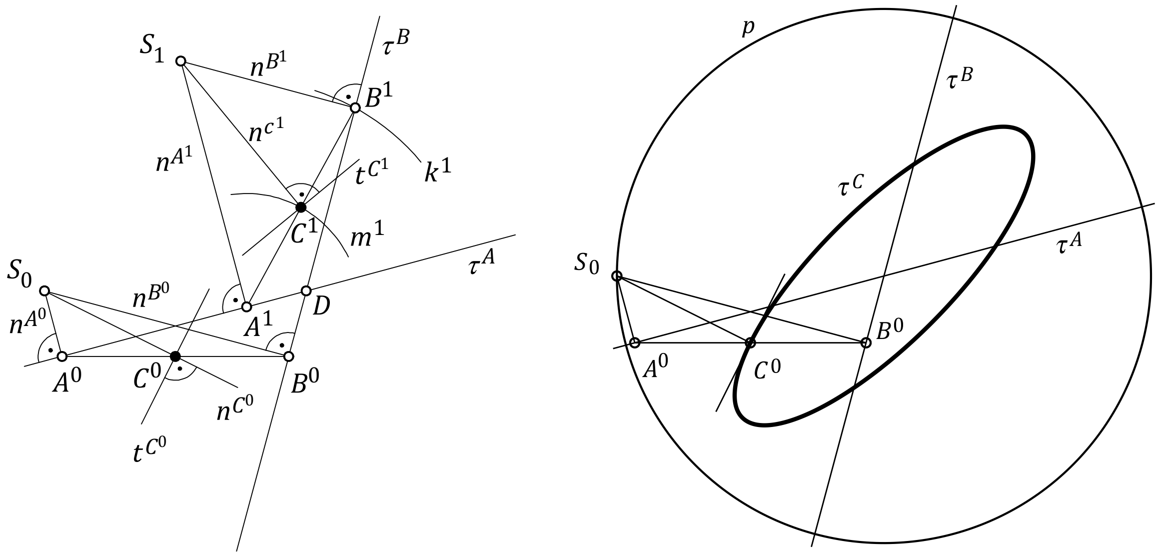
The position of moving point C with respect to the given moving system does not change, i.e. the distances $||AC||$ and $||BC||$ are constant during the whole motion. To draw the trajectory τ^C, it is necessary to construct point C at a sufficient number of instants and at each instant construct the tangent line to the trajectory τ^C.

1. Construct new position of moving point $C^1 = m^1 \cap A^1B^1$, $m^1 = (A^1, r = ||A^0C^0|)$, see fig. 2.11 a).

2. Draw normal line $n^{C^1} = S_1C^1$.

3. Construct tangent line $t^{C^1} \perp n^{C^1}$, $C^1 \in t^{C^1}$.

4. Continue in a similar way at each instant. Finally, draw the trajectory as a curve passing through all positions of the moving point and following the direction of tangent lines. The whole trajectory τ^C (ellipse in this case) is drawn in fig. 2.11 b).



a) b)

Figure 2.11: Trajectory τ^C of moving point C (motion given by two linear trajectories)

d) Envelope of moving circle

The position of moving circle $c = (C, r)$ with respect to the moving system given by points A and B does not change, i.e. the distances $||AC||$ and $||BC||$ and the radius r of the circle c are constant during the whole motion. Envelope of moving circle c always has two branches designated by (c) and $(\bar{c})$ in figures. To draw the envelope, it is necessary to construct this circle at a sufficient number of instants first. After that, the points of contact between the moving circle and its envelope and tangent lines to the envelope at each instant are constructed.

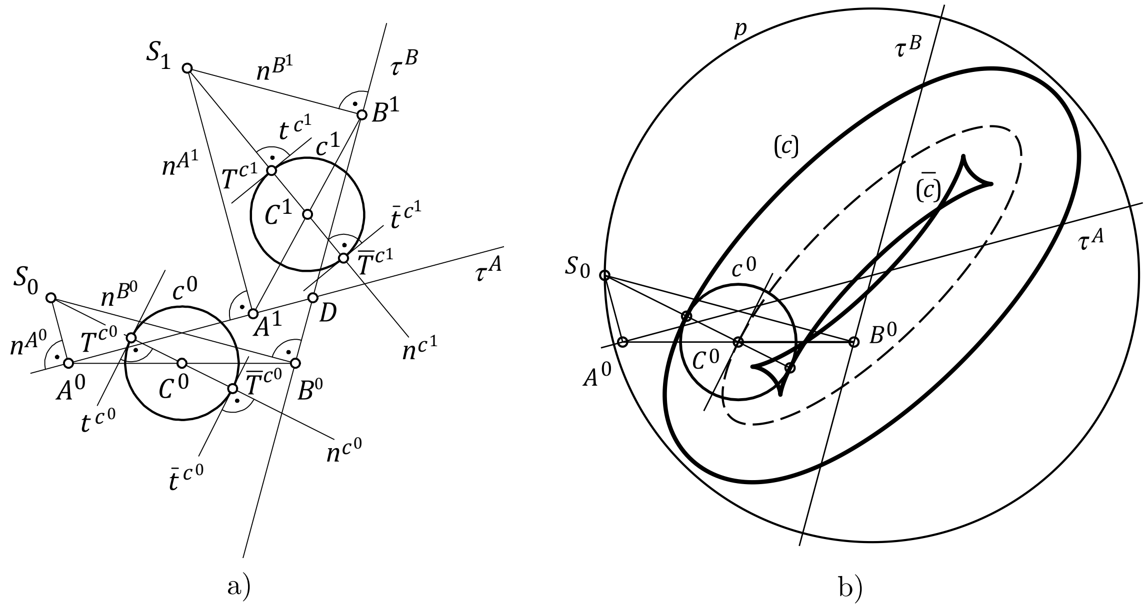


Figure 2.12: Envelope (c) of moving circle c (motion given by two linear trajectories)

1. Construct new position of moving circle $c^1 = (C^1, r)$, see fig. 2.12 a).
2. Construct normal line $n^{c^1} = S_1C^1$.
3. Points of contact $T^{c^1}, \overline{T}^{c^1} = c^1 \cap n^{c^1}$.
4. Construct tangent lines $t^{c^1} \perp n^{c^1}$, $T^{c^1} \in t^{c^1}$ and $\bar{t}^{c^1} \perp n^{c^1}$, $\overline{T}^{c^1} \in t^{c^1}$.
5. Continue in a similar way at each instant. Finally, draw each branch of the envelope as a curve passing through all corresponding points of contact and following the direction of corresponding tangent lines. The whole envelope $(c), (\bar{c})$ is drawn in fig. 2.12 b). In this case, the envelope branches are offset curves of the ellipse – trajectory τ^C of the centre C. The trajectory τ^C is drawn by dashed line in fig. 2.12 b), too.

e) Envelope of moving straight line

In general, the position of moving straight line d with respect to the moving system given by points A and B does not change, i.e. the angle formed by straight lines d and AB is constant during the whole motion. Here, this angle is equal to zero, because $d = AB$. The moving line is tangent line to its envelope at each instant. To draw the envelope, it is necessary to construct the moving line at a sufficient number of instants and at each instant construct point of contact between the moving line and its envelope.

1. Construct normal line $n^{d^1} \perp d^1$, $S_1 \in n^{d^1}$, see fig. 2.13 a).
2. Point of contact $T^{d^1} = d^1 \cap n^{d^1}$.
3. Continue in a similar way at each instant. Finally, draw the envelope as a curve passing through all positions of the point of contact and following the direction of tangent lines, i.e. individual positions of the moving straight line. The whole envelope (d) (irregular asteroid in this case) is drawn in fig. 2.13 b).

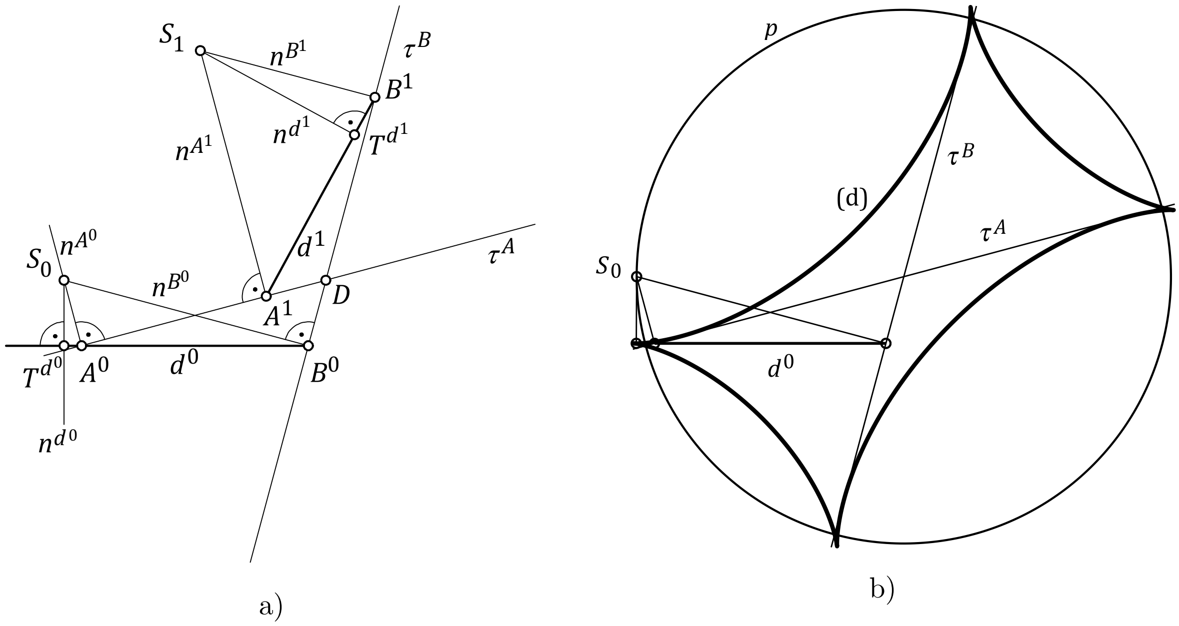


Figure 2.13: Envelope (d) of moving straight line d (motion given by two linear trajectories)

f) Moving centrode

The transformation from i-th instant into the 0-th instant is realized by construction of two congruent positions of triangle $\triangle A^i B^i S_i \cong \triangle A^0 B^0 S_i^0$. In this case, the moving centrode is the circle $h^0 = (H^0, r = H^0 D)$, where H^0 is the center of straight line segment $S_0 D$, see fig. 2.14.

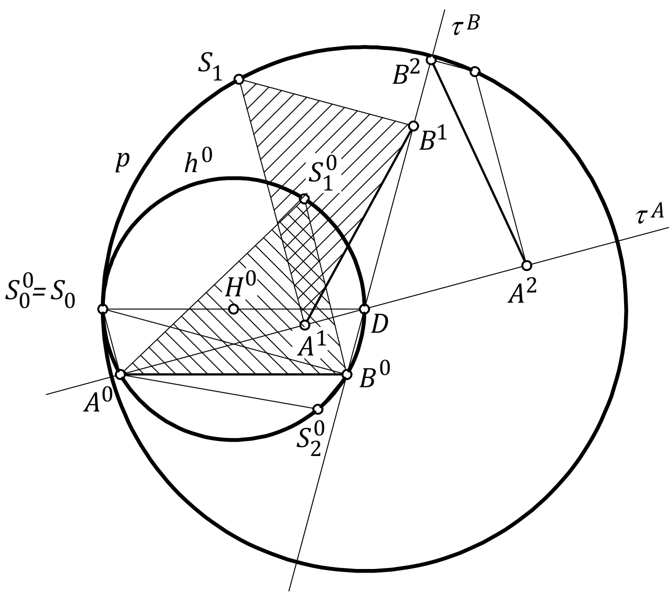


Figure 2.14: Moving centreode h^0 (motion given by two linear trajectories)

□

■ Example 2.2 – Motion given by circular and linear trajectories

Given

Circular trajectory τ^A, linear trajectory τ^B, points A, B, C, circle $c = (C, r)$ and straight line $d = AB$ at initial position, see fig. 2.15.

Required

Construct fixed centrode p, trajectory τ^C of the given point C, envelope (c) of the given circle c, envelope (d) of the given straight line d and moving centrode h at the given instant.

Analysis

The fixed system of this motion is represented by trajectories τ^A and τ^B. The given moving system is represented by moving points A and B. The following rules apply to this motion.

- Point A is located on trajectory τ^A at each instant.
- Point B is located on trajectory τ^B at each instant.
- The distance $||AB||$ of points A and B does not change.

This motion is inverse to the motion given by circular trajectory of moving point and point envelope of moving straight line or circular trajectory of moving point and circular envelope of moving straight line.

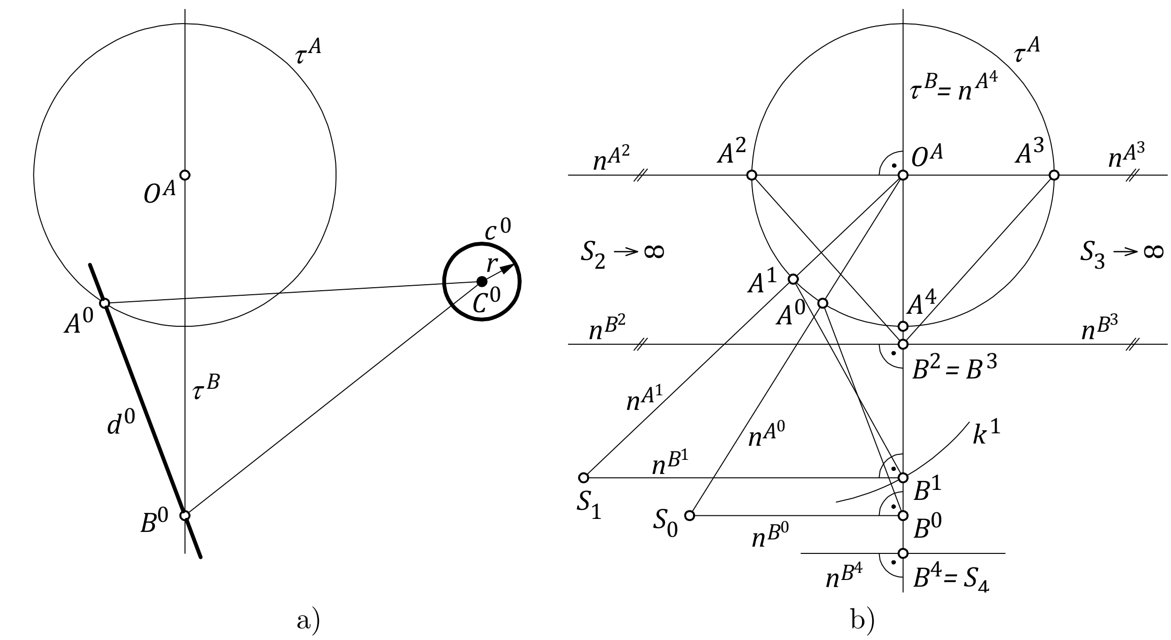


Figure 2.15: Motion given by circular and linear trajectories

Graphical solution

a) New position of the given moving system

1. Choose new position $A^1 \in \tau^A$, see fig. 2.15 b).
2. Construct $B^1 \in \tau^B$ so that $||A^1B^1|| = ||A^0B^0||$, i.e. $B^1 = k^1 \cap \tau^B$, $k^1 = (A^1, r = ||A^0B^0||)$. Note that it does not matter if we choose new position $B^1 \in \tau^B$ and construct A^1 so that $||A^1B^1|| = ||A^0B^0||$ (not drawn in fig. 2.15 b). Both approaches are equal.
3. Draw straight line segment A^1B^1.
4. Continue in a similar way to obtain a sufficient number of instants. Do not forget special positions of point A with respect to the trajectories τ^A or τ^B denoted by A^2, A^3 and A^4 in fig. 2.15 b), where $O^AA^2 \perp \tau^B$, $O^AA^3 \perp \tau^B$ and $A^4 \in \tau^B$.

b) Instantaneous centre of rotation

1. Construct normal line $n^{A^0} = O^AA^0$, $A^0 \in n^{A^0}$, see fig. 2.15 b).
2. Construct normal line $n^{B^0} \perp \tau^B$, $B^0 \in n^{B^0}$.
3. Instantaneous centre of rotation $S_0 = n^{A^0} \cap n^{B^0}$.
4. Continue in a similar way at each instant. In the special cases where both normal lines to the given trajectories are parallel, i.e. $n^{A^2} \parallel n^{B^2}$ and $n^{A^3} \parallel n^{B^3}$, the instantaneous centres of rotation lie at infinity: $S_2 \to \infty$ and $S_3 \to \infty$. At the special fourth instant $n^{A^4} = \tau^B$ and $S_4 = B^4$. The fixed centrode p is an open curve with two branches containing points at infinity, see fig. 2.16 b).

c) Trajectory of moving point

The position of moving point C with respect to the given moving system does not change, i.e. the distances AC and BC are constant during the whole motion. Since the point C does not lie on straight line AB, the moving system can be represented by triangle $\triangle ABC$ drawn in dot-and-dash line in fig. 2.16.

1. Construct new position of moving point $C^1 = l^1 \cap m^1$, $l^1 = (A^1, r = ||A^0C^0||)$, $m^1 = (B^1, r = ||B^0C^0||)$, see fig. 2.16 a).
2. Draw normal line $n^{C^1} = S_1C^1$.
3. Construct tangent line $t^{C^1} \perp n^{C^1}$, $C^1 \in t^{C^1}$.
4. Continue in a similar way at each instant. Finally, draw the trajectory as a curve passing through all positions of the moving point and following the direction of tangent lines. The whole trajectory τ^C is drawn in fig. 2.16 b).

d) Envelope of moving circle

The position of moving circle $c = (C, r)$ with respect to the moving system and its radius does not change during the whole motion.

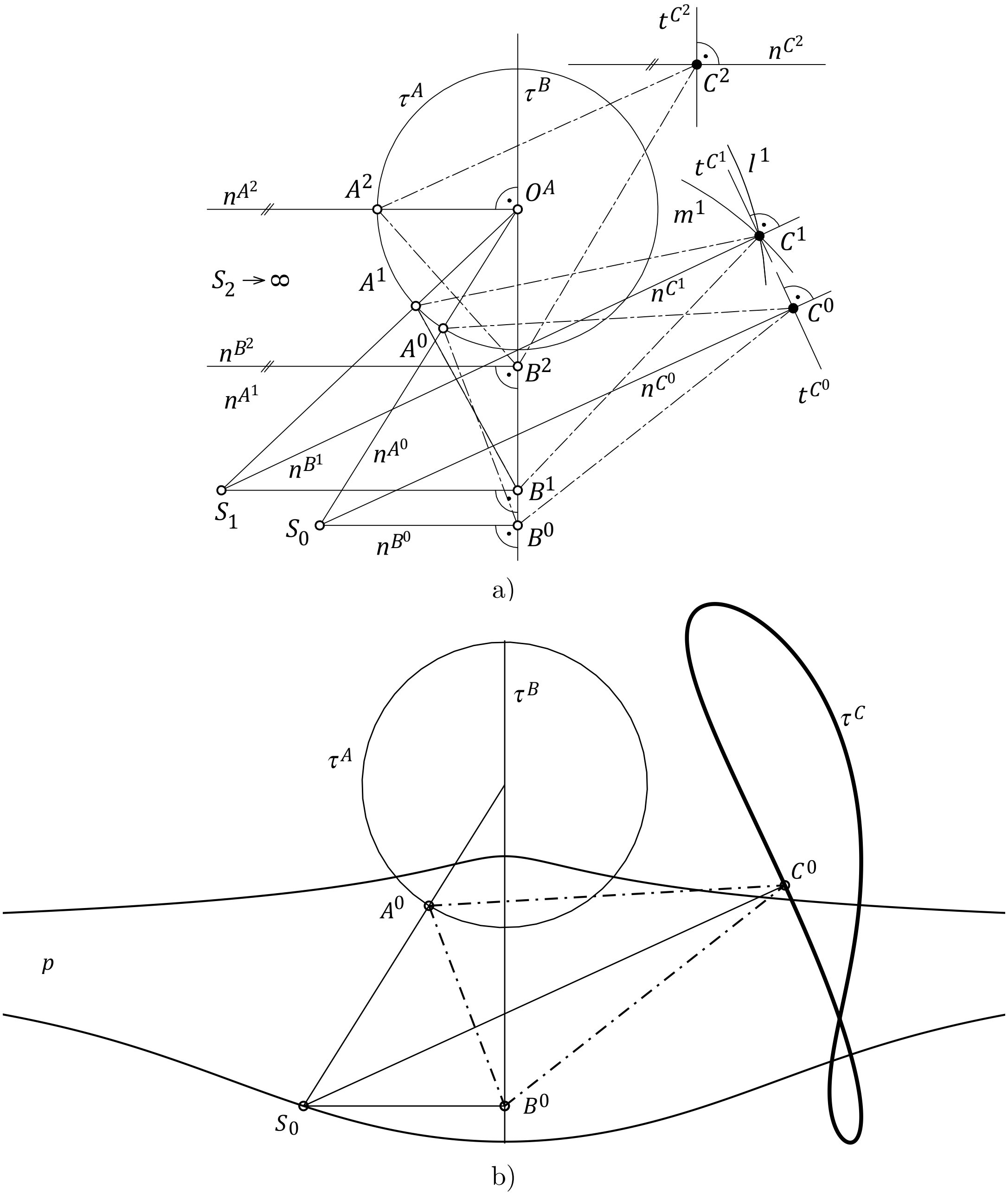


Figure 2.16: Trajectory τ^C of moving point C (motion given by circular and linear trajectories)

1. Construct new position of moving circle: $c^1 = (C^1, r)$, see fig. 2.17 a).
2. Construct normal line $n^{c^1} = S_1C^1$.
3. Points of contact $T^{c^1}, \overline{T}^{c^1} = c^1 \cap n^{c^1}$.
4. Construct tangent lines $t^{c^1} \perp n^{c^1}$, $T^{c^1} \in t^{c^1}$ and $\overline{t}^{c^1} \perp n^{c^1}$, $\overline{T}^{c^1} \in \overline{t}^{c^1}$.
5. Continue in similar a way at each instant. Finally, draw each branch of the envelope as a curve passing through all corresponding points of contact and following the direction of corresponding tangent lines. The whole envelope $(c), (\overline{c})$ is drawn in fig. 2.17 b).

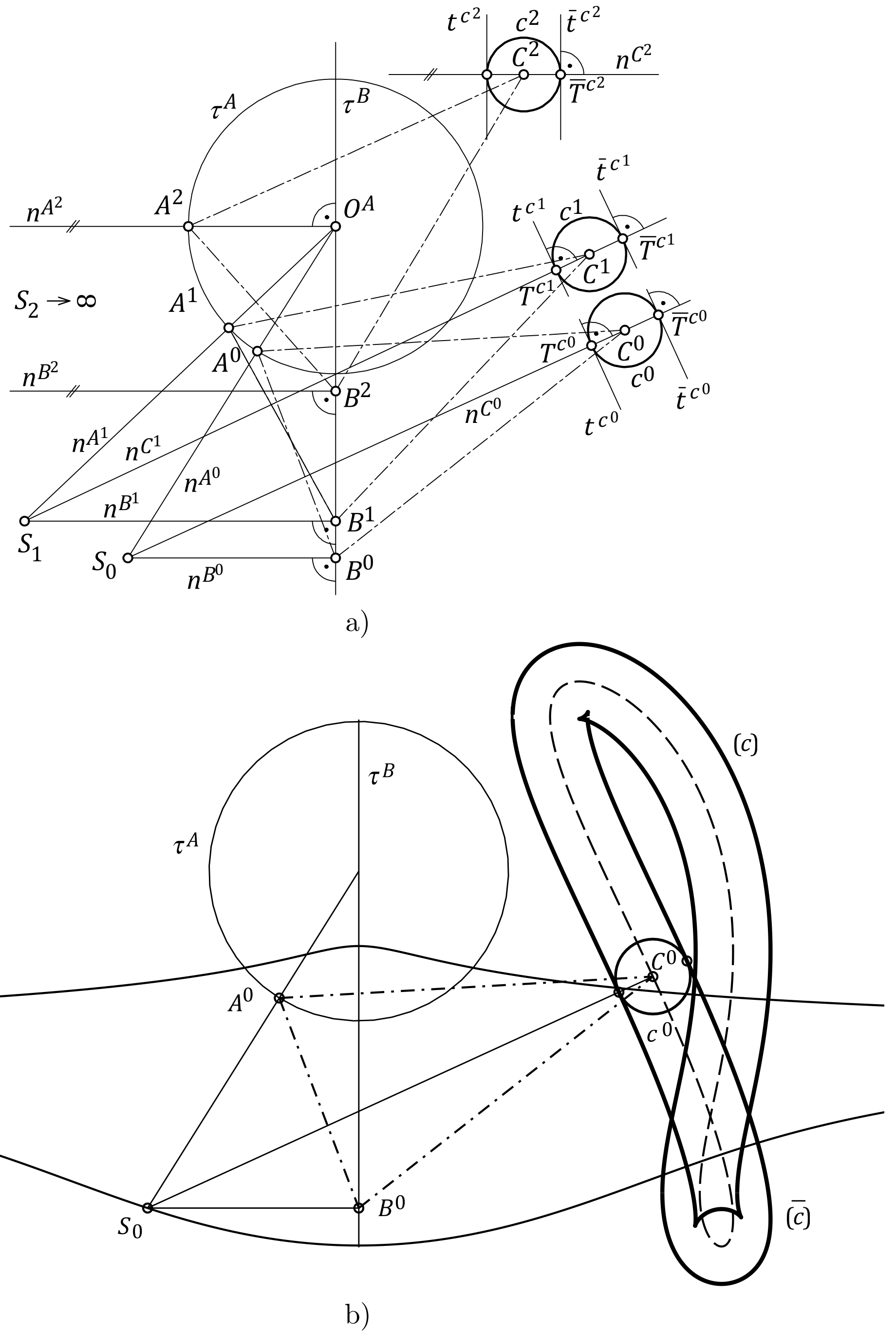


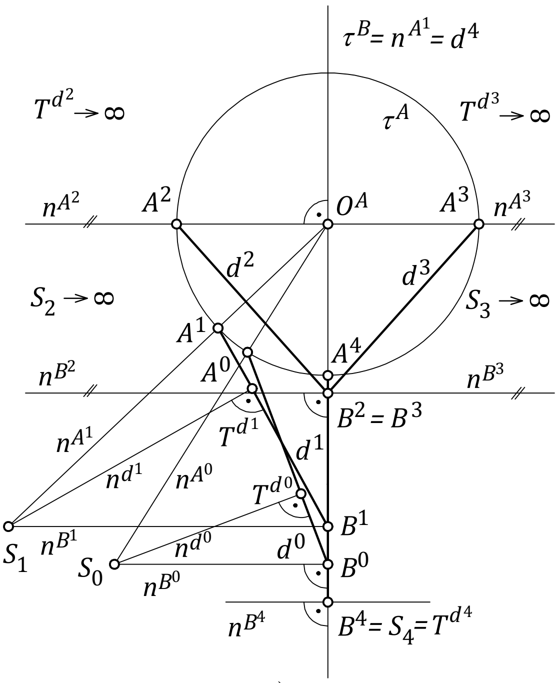
Figure 2.17: Envelope (c) of moving circle c (motion given by circular and linear trajectories)

e) Envelope of moving straight line

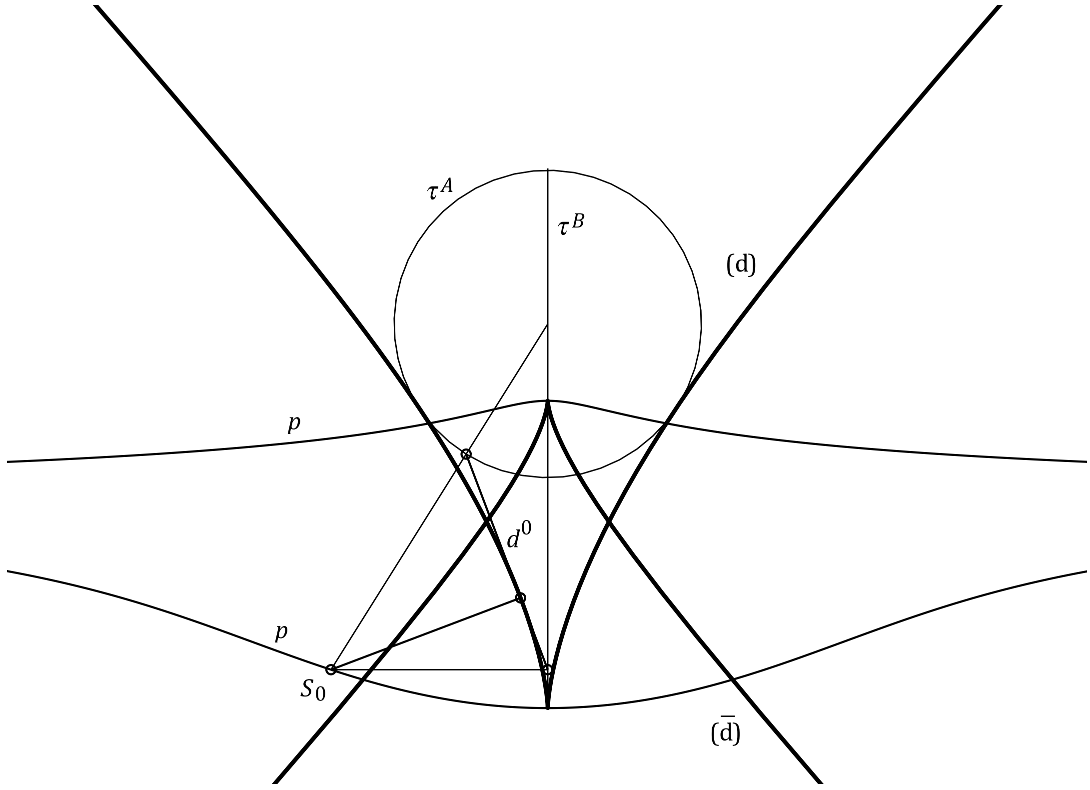
The position of moving straight line d with respect to the moving system given by points A and B is constant, because $d = AB$.

1. Construct normal line $n^{d^1} \perp d^1$, $S_1 \in n^{d^1}$, see fig. 2.18 a).

2. Point of contact $T^{d^1} = d^1 \cap n^{d^1}$.



a)



a)

Figure 2.18: Envelope (d) of moving straight line d (motion given by circular and linear trajectories)

3. Continue in a similar way at each instant. Finally, draw the envelope as a curve passing through all positions of the point of contact and following the direction of tangent lines, i.e. individual positions of the moving straight line. The important part of envelope (d) is drawn in fig. 2.13 b).

f) Moving centrode

The transformation from i-th instant into the 0-th instant is realized by construction of two congruent positions of triangle $\triangle A^i B^i S_i \cong \triangle A^0 B^0 S_i^0$. The moving centrode h^0 is open curve with two branches, see fig. 2.19.

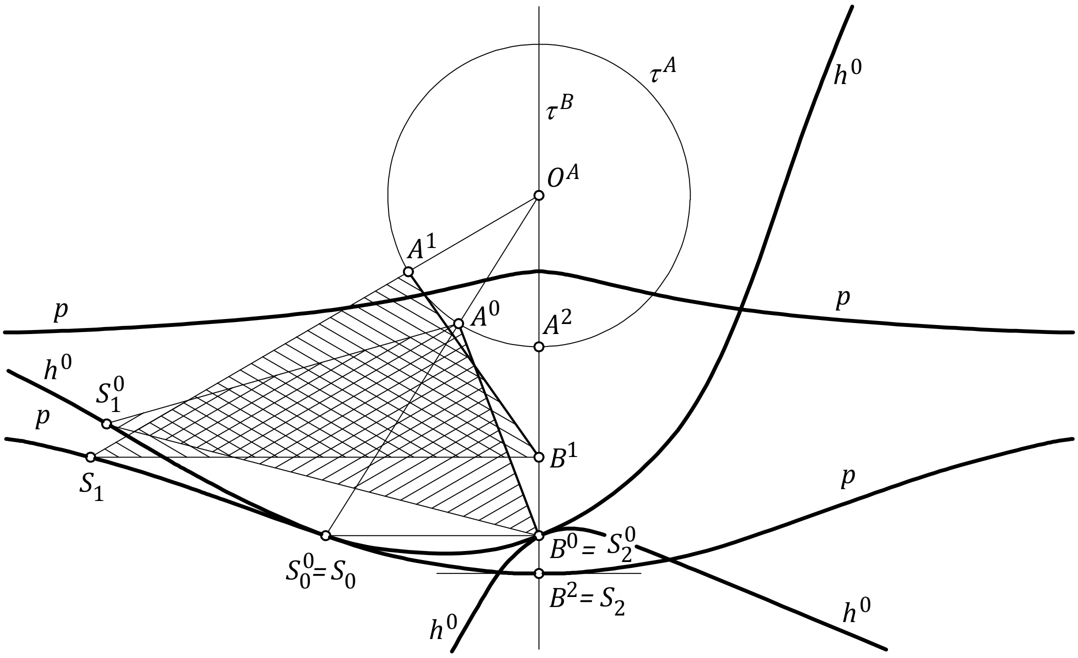


Figure 2.19: Moving centreode h^0 (motion given by circular and linear trajectories)

□

■ Example 2.3 – Motion given by two point envelopes of moving straight lines

Given

Point envelope (a) of straight line a, point envelope (b) of straight line b, point C, circle $c = (C, r)$, straight lines a, b and d at initial position, see fig. 2.20 a).

Required

Construct fixed centrode p, trajectory τ^C of the given point C, envelope (c) of the given circle c, envelope (d) of the given straight line d and moving centrode h at the given instant.

Analysis

The given fixed system is represented by point envelopes (a) and (b). The given moving system is represented by moving straight lines a and b. The following rules apply to this motion.

- Straight line a passes through point envelope (a) at each instant.
- Straight line b passes through point envelope (b) at each instant.
- The angle α formed by straight lines a and b does not change.

This motion is inverse to the motion given by two linear trajectories, thus, the configuration of individual figures in this example and in example 2.1 has been chosen so that the moving centrode h^0 in fig. 2.25 and fixed centrode p in fig. 2.14 are identical.

Moreover, the motion given by two point envelopes can be realized by rolling the constructed moving centrode h on the fixed centrode p, i.e. by pericycloidal motion (see section 2.5).

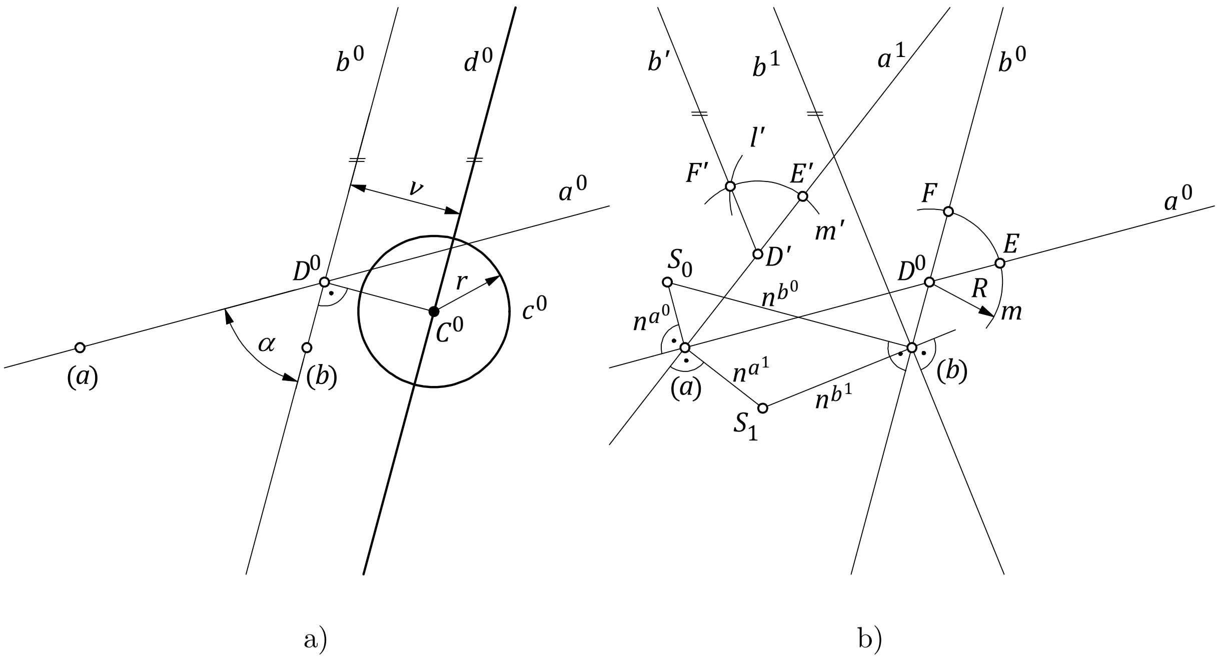


Figure 2.20: Motion given by two point envelopes

Graphical solution

a) New position of the given moving system

1. Choose new position a^1, $(a) \in a^1$, see fig. 2.20 b).
2. Construct new position $b^1, (b) \in b^1$ so that the angle $\angle a^1b^1 = \angle a^0b^0$: draw circle $m = (D^0, r = R)$, radius R is suitably chosen, $D^0 = a^0 \cap b^0$. Choose point $D' \in a^1$ and draw circle $m' = (D', r = R)$. Draw circle $l' = (E', r = ||EF||)$, $E' = m' \cap a^1$, $E = m \cap a^0$, $F = m \cap b^0$. Construct straight line $b' = D'F'$, $F' = l' \cap m'$. Construct $b^1||b'$, $b^1 \in (b)$. Note that it does not matter if we choose new position of b^1, $b^1 \in (b)$ and construct new position a^1 so that $\angle a^1b^1 = \angle a^0b^0$ (not drawn in fig. 2.20 b). Both approaches are equal.

3. Continue in a similar way to obtain a sufficient number of instants. Do not forget special positions of moving straight lines $a \in (b)$ and $b \in (a)$.

b) Instantaneous centre of rotation

Point envelope can be considered a circle with zero radius and the corresponding straight line a tangent line to this circle. Thus, normal line to the point envelope is a line perpendicular to the moving straight line passing through the point envelope.

1. Construct normal line $n^{a^0} \perp a^0$, $(a) \in n^{a^0}$.
2. Construct normal line $n^{b^0} \perp b^0$, $(b) \in n^{b^0}$.
3. Instantaneous centre of rotation $S_0 = n^{a^0} \cap n^{b^0}$.
4. Continue in a similar way at each instant. Draw the fixed centrode p as a curve passing through all positions of instantaneous centre of rotation. In this case, the fixed centrode is the circle $p = (P, r = ||D^0P||)$, where P is the centre of D^0S_0. The fixed centrode is drawn in fig. 2.22.

c) Trajectory of moving point

In general, the position of a point with respect to two intersecting straight lines can be expressed by two (oriented) distances obtained by orthogonal projection of the point onto one of the two lines, see fig. 2.21. Here, the orthogonal projection of point C onto straight line b is drawn. The position of point C with respect to the straight lines a and b is given by normal distance $v = ||CD||$, $CD \perp b$ and the distance $u = ||DE||$ in the direction of line b.

In this example, the distance $u = 0$ (see fig. 2.20), therefore, oriented distance v of moving point C from line b orthogonally measured from intersection D has to be preserved only. Note that it is possible to construct position of point C by means of orthogonal projection onto the line a, too. However, this approach is slightly more complicated because of non-zero distances u and v.

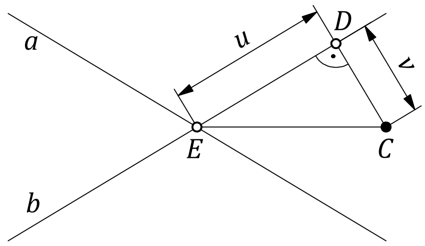
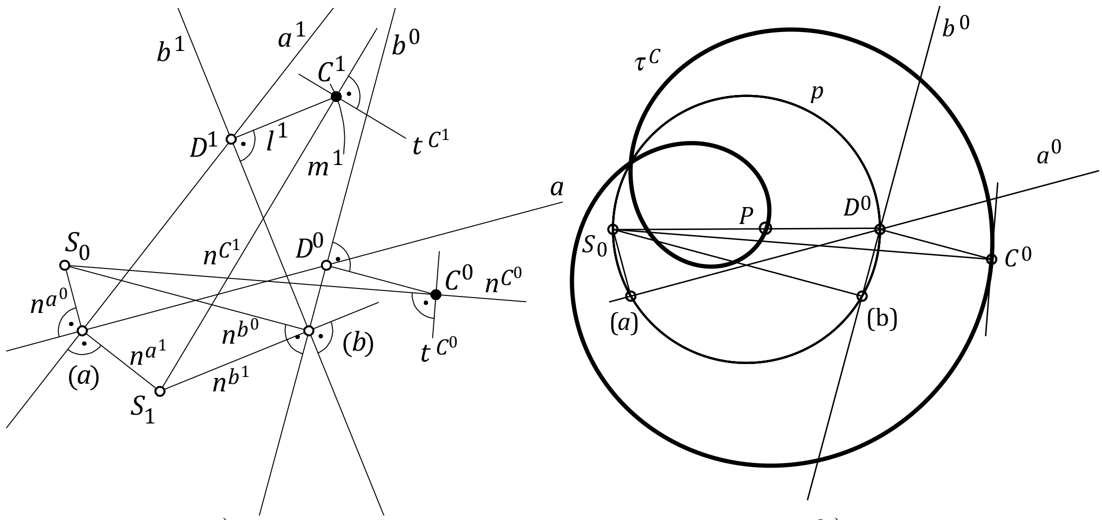


Figure 2.21: Position of point C with respect to two intersecting straight lines a and b

1. Construct new position C^1: construct line $l^1 \perp b^1$, $D^1 \in l^1$. Draw circle $m^1 = (D^1, r = ||D^0C^0||)$. Point $C^1 = l^1 \cap m^1$, see fig. 2.22 a).
2. Draw normal line $n^{C^1} = S_1C^1$.
3. Construct tangent line $t^{C^1} \perp n^{C^1}$, $C^1 \in t^{C^1}$.
4. Continue in a similar way at each instant. Finally, draw the trajectory as a curve passing through all positions of moving point and following the direction of tangent lines. The whole trajectory τ^C (epicycloidal curve, see section 2.5) is drawn in fig. 2.22 b).



a) b)

Figure 2.22: Trajectory τ^C of moving point C (motion given by two point envelopes)

d) Envelope of moving circle

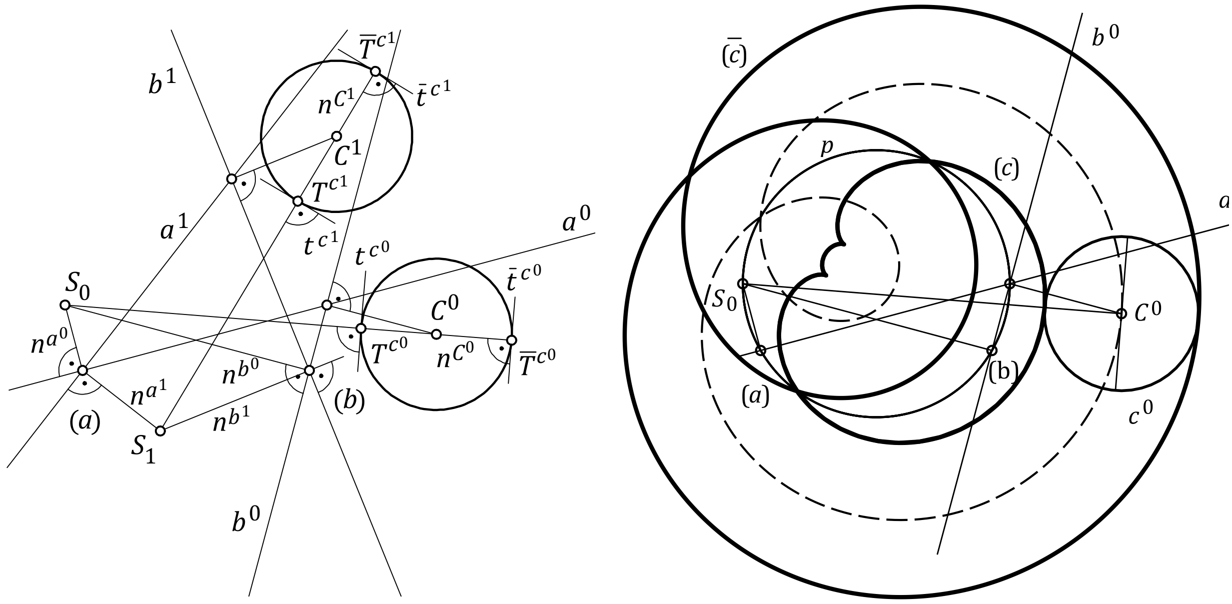
The position of moving circle $c = (C, r)$ with respect to the moving system and its radius does not change during the whole motion.

1. Construct new position $c^1 = (C^1, r)$, see fig. 2.23 a).
2. Construct normal line $n^{c^1} = S_1C^1$.
3. Points of contact $T^{c^1}, \overline{T}^{c^1} = c^1 \cap n^{c^1}$.
4. Construct tangent lines $t^{c^1} \perp n^{c^1}$, $T^{c^1} \in t^{c^1}$ and $\overline{t}^{c^1} \perp n^{c^1}$, $\overline{T}^{c^1} \in t^{c^1}$.
5. Continue in a similar way at each instant. Finally, draw each branch of the envelope as a curve passing through all corresponding points of contact and following the direction of corresponding tangent lines. The whole envelope (c), $(\overline{c})$ is drawn in fig. 2.23 b).

e) Envelope of moving straight line

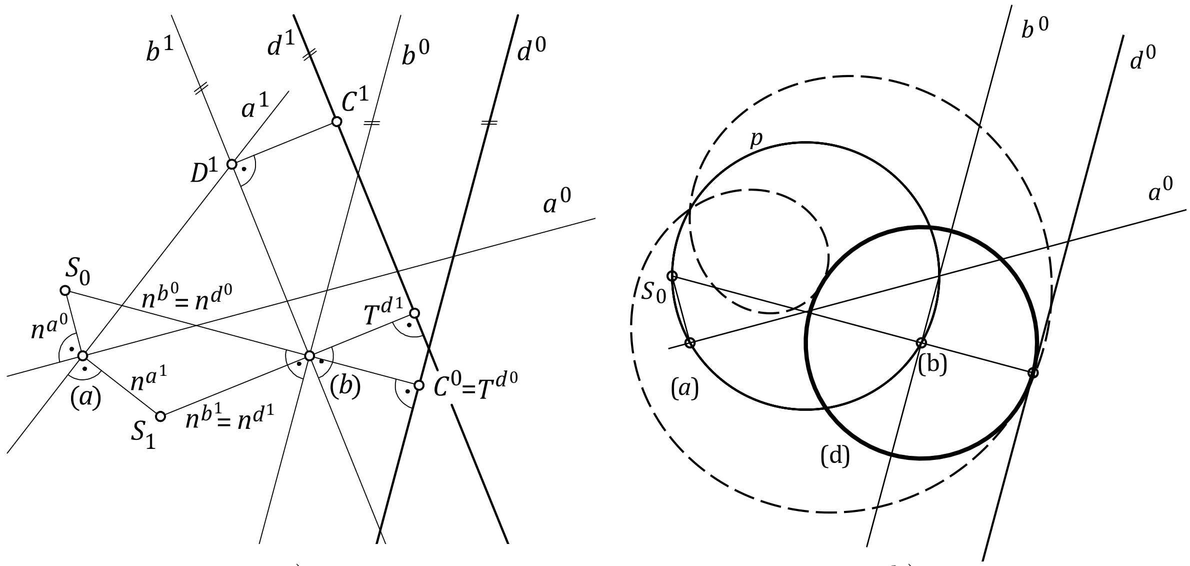
The position of moving straight line d with respect to the moving system is given by parallelism $d \parallel b$ and coincidence $C \in d$. This relationship has to be preserved during the whole motion.

1. Construct new position $d^1 \parallel b^1$, $C^1 \in d^1$, see fig. 2.24 a).
2. Construct normal line $n^{d^1} \perp d^1$, $S_1 \in n^{d^1}$.
3. Point of contact $T^{d^1} = d^1 \cap n^{d^1}$.
4. Continue in a similar way at each instant. Finally, draw the envelope as a curve passing through all positions of the point of contact and following the direction of tangent lines, i.e. individual positions of the moving straight line. The whole envelope (d) (circle $(d) = ((b), r = v)$ in this case) is drawn in fig. 2.24 b).



a) b)

Figure 2.23: Envelope (c) of moving circle c (motion given by two point envelopes)



a) b)

Figure 2.24: Envelope (d) of moving straight line d (motion given by two point envelopes)

f) Moving centrode

The transformation from i-th instant into the 0-th instant is realized by construction of two congruent positions of triangle $\triangle(b)D^iS_i \cong \triangle E^iD^0S_i^0$, $E^i \in b^0$. The moving centrode is the circle $h^0 = (D^0, r = ||D^0S_0||)$, see fig. 2.25.

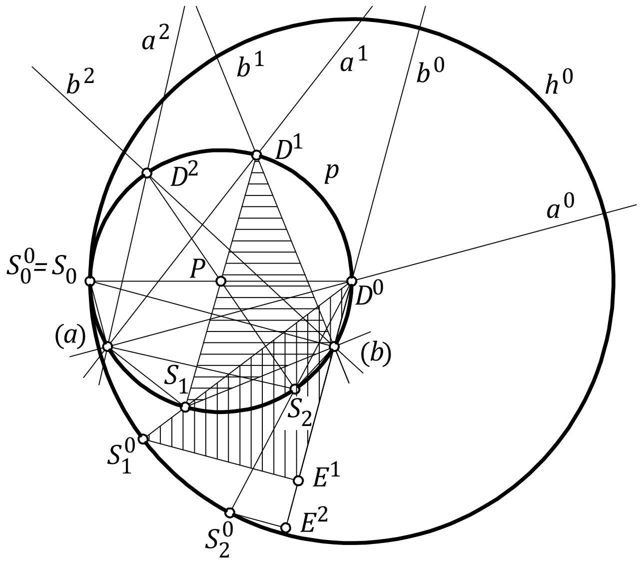


Figure 2.25: Moving centrode h^0 (motion given by two point envelopes)

□

2.5 Cyclic motions

The motion determined by fixed and moving centrodes where both centrodes are circles or one centrode is a straight line and the second one is a circle is called *cyclic motion*. Trajectories of moving points and envelopes of moving curves generated during cyclic motion are called *cycloids*. According to the shape of centrodes and their mutual configuration, the cyclic motions can be classified as follows.

- *Cycloidal motion* – the fixed centrode p is a straight line, the moving centrode h is a circle $h = (H, r)$, see fig. 2.26. Cycloidal motion is inverse to involute motion.

- *Involute motion* – the fixed centrode p is a circle $p = (P, r)$, the moving centrode h is a straight line, see fig. 2.27. Involute motions is inverse to cycloidal motion.

- *Epicycloidal motion* – both centrodes are circles, the moving centrode $h = (H, r_h)$ is rolling by its external circumference along the external circumference of the fixed centrode $p = (P, r_p)$, see fig. 2.28. Epicycloidal motion is inverse to itself.

- *Hypocycloidal motion* – both centrodes are circles, the moving centrode $h = (H, r_h)$ is rolling by its external circumference along the internal circumference of the fixed centrode $p = (P, r_p)$, see fig. 2.29. Hypocycloidal motion is inverse to pericycloidal motion.

- *Pericycloidal motion* – both centrodes are circles, the moving centrode $h = (H, r_h)$ is rolling by its internal circumference along the external circumference of the fixed centrode $p = (P, r_p)$, see fig. 2.30. Pericycloidal motion is inverse to hypocycloidal motion.

Examples of trajectories of moving points, envelopes of moving circles and envelopes of moving straight lines generated by cyclic motions are given in section 2.5. Here, the fixed centrode is drawn by dot line, the moving centrode is drawn by dash line, generated trajectories

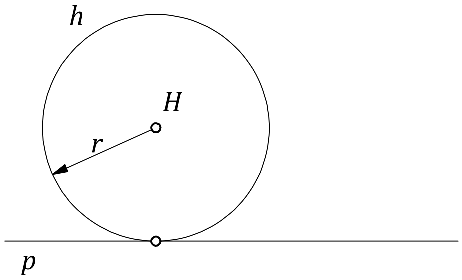


Figure 2.26: Cycloidal motion

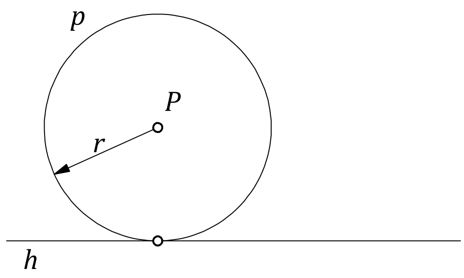


Figure 2.27: Involute motion

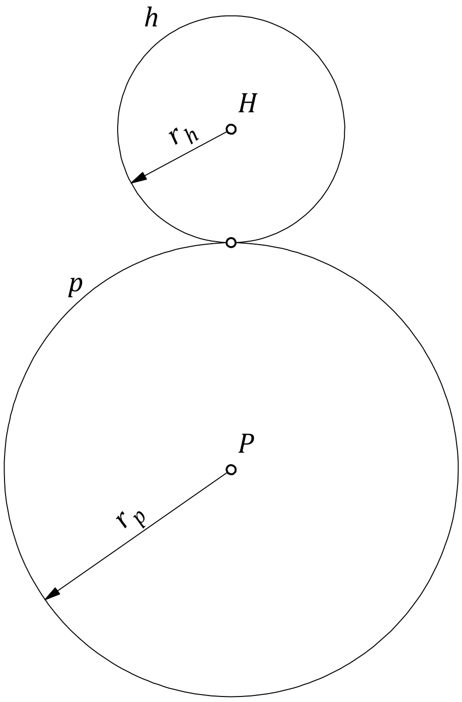


Figure 2.28: Epicycloidal motion

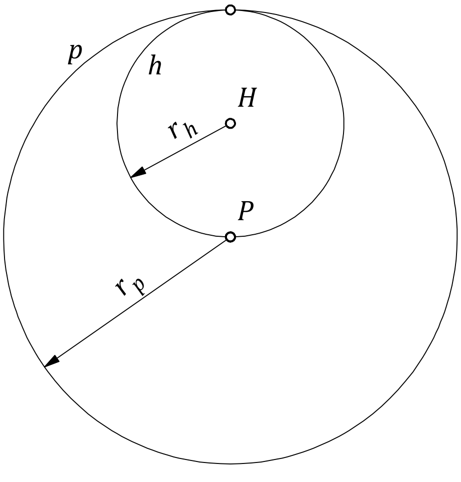


Figure 2.29: Hypocycloidal motion

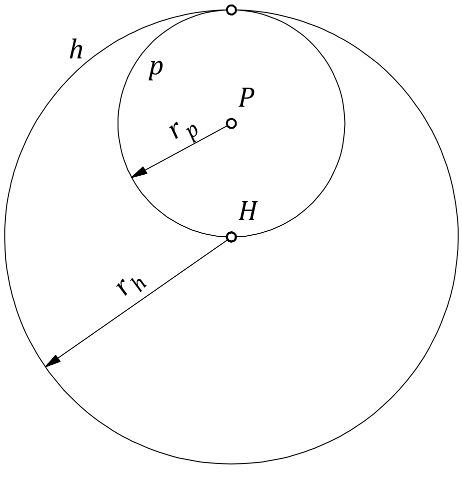
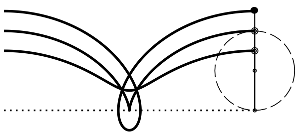
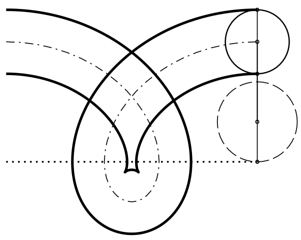
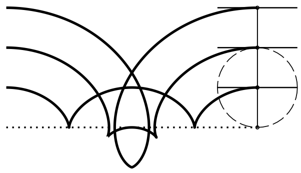
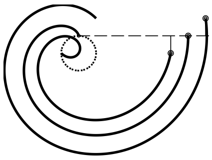
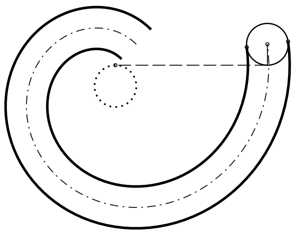
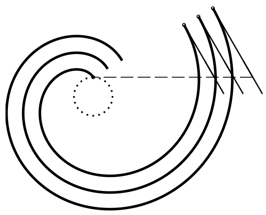
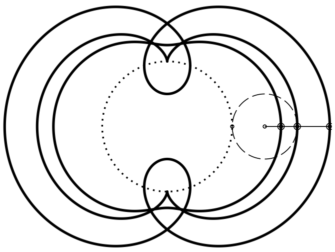
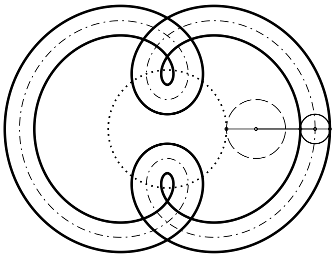
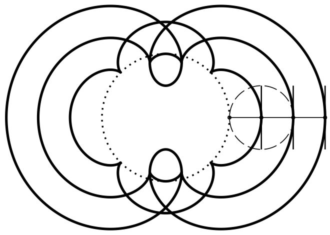
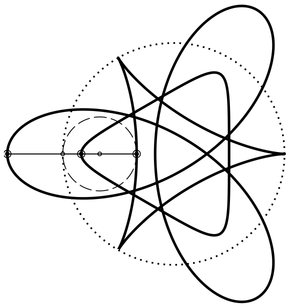
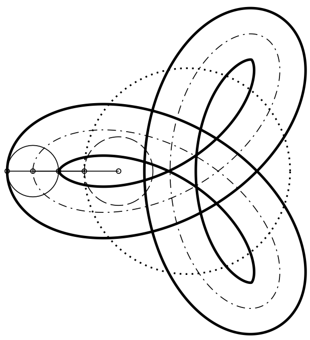
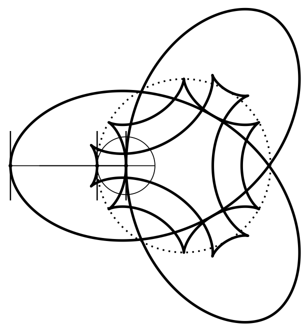
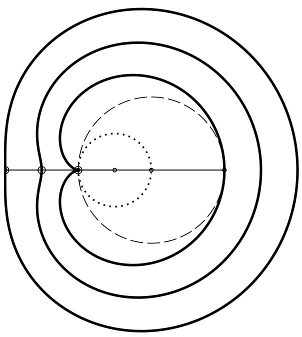
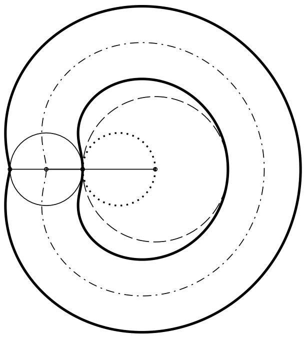
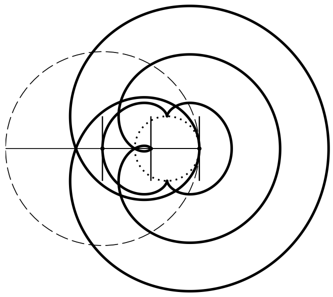


Figure 2.30: Pericycloidal motion

Table 2.1: Example of cyclic motions

	Trajectory of moving points	Envelope of moving circles	Envelope of moving lines
Cycloidal motion			
Involute motion			
Epicycloidal motion			
Hypocycloidal motion			
Pericycloidal motion			

and envelopes are drawn by thick continuous line and the trajectories of the centres of moving circles are drawn by dot-and-dash line.

Depending on mutual position of moving centrode h and a moving figure, the cycloid can be *prolate* (e.g. the moving point lies outside the moving centrode), *simple* (e.g. the moving point lies on the moving centrode) or *curtate* (e.g. the moving point is located inside the moving centrode).

If r_p/r_h is an integer or rational number expressed in lowest terms, the curves generated by epicycloidal, hypocycloidal or pericycloidal motion are closed. Otherwise, the generated curves never close.

2.6 Example problems – cyclic motion

In this section, the construction of curves generated by cyclic motion is shown in examples. Firstly, the construction of new position of moving centrode h is presented. Then, the construction of trajectory τ^C of moving point C, envelope (c) of moving circle $c = (C, r)$ and envelope (d) of moving straight line d is described. This description includes determination of mutual relation between the moving centrode h and the moving figure (point, circle, straight line) which does not change during the whole motion, construction of tangent lines to the generated trajectory and construction of points of contact between the generated envelope and moving curve.

■ **Example 2.4 – Cycloidal motion**

Given

Fixed centrode p, moving centrode $h = (H, r)$, point C, circle $c = (C, r = r_c)$ and straight line d at initial position, see fig. 2.31.

Figure 2.31: Cycloidal motion – example

Required

Construct trajectory τ^C of the given point C, envelope (c) of the given circle c and envelope (d) of the given straight line d.

Analysis

The fixed system is represented by the fixed centrode p. The given moving system is represented by moving centrode h. At j-th instant, the moving centrode h^j touches the fixed centrode p at point of contact $S_i^j = S_j$, $i = j$. This motion is inverse to the involute motion, see example 2.5.

Graphical solution

a) New position of the moving centrode

To be able to construct the moving centrode h at a sufficient number of instants h^j, $j = 0, 1, \ldots, n$, it is necessary to construct a sufficient number of instantaneous centres of rotation $S_i^0 \in h^0$, $i = 0, 1, \ldots, n$ and the same number of instantaneous centres of rotation $S_j \in p$, first. The rolling is realized without slipping, therefore the distance $||S_j S_{j+1}||$ on the fixed centrode p has to be equal to the length of arc $\widehat{S_i^0 S_{i+1}^0}$ on the moving centrode h. Thus, the procedure of construction can be made as follows.

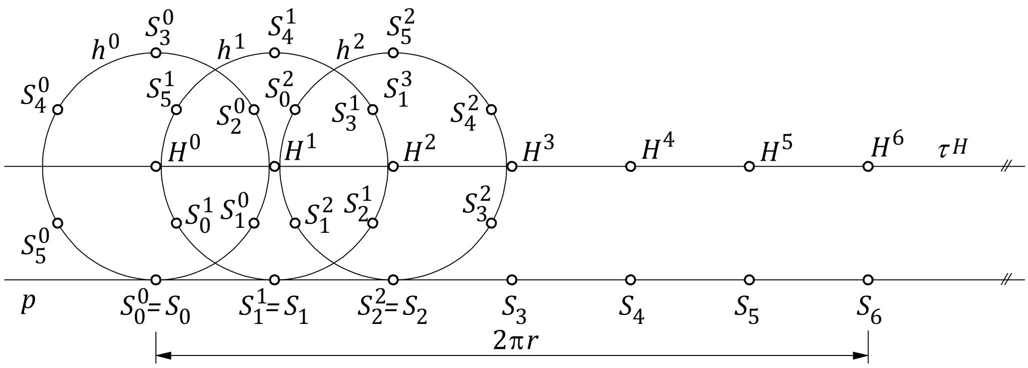


Figure 2.32: Construction of new positions of the moving centrode h (cycloidal motion)

1. Divide the moving centrode h_0 into a sufficient number of n (at least $n = 6$) equal parts to obtain $S_i^0 \in h_0$, $i = 0, 1, \ldots, n$, see fig. 2.32.
2. Calculate the length of arc
$$\widehat{S_i^0 S_{i+1}^0} = \frac{1}{n} 2\pi r, \tag{2.1}$$
take it in compass and mark it n-times from S_0 on the fixed centrode p to obtain instantaneous centres of rotation S_j. Note that it is possible to use any synthetic rectification method to approximate the length of an arc. For example, simple approximation of the length of arc $\widehat{AB}$ by the length of polygon $ACDEFB$ is shown in fig. 2.33.
3. Draw trajectory τ^H of the centre H of moving centrode h: $\tau^H \parallel p$, $H^0 \in \tau^H$.
4. Construct centre H^j at each instant: take the length given by eq. (2.1) in compass and mark it n-times from H^0 on the trajectory τ^H.
5. Draw moving centrode $h^j = (H^j, r)$ at each instant.
6. Construct instantaneous centres of rotation $S_i^j \in h^j$, $i, j = 0, 1, \ldots n$. Note that depending on actual configuration of moving figures attached to the moving centrode, it is not necessary to construct all positions h^j of the moving centrode, neither it is necessary to construct all instantaneous centres of rotation S_i^j at each position h^j.

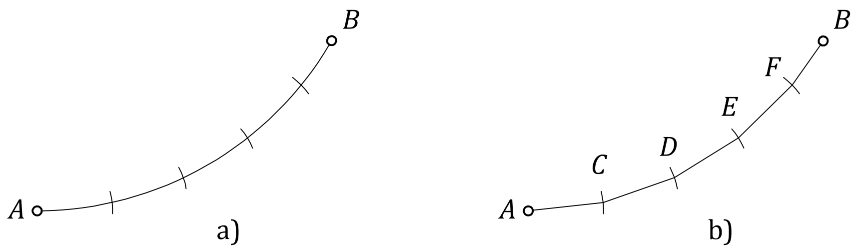


Figure 2.33: Approximation of the length of arc $\widehat{AB}$ by the length of polygon $ACDEFB$

b) Trajectory of moving point

The position of point C with respect to the moving centrode h has to be determined by two different parameters – for example by the distances of point C from two different points located on h. Here, the distances $||C^0S_3^0||$ and $||C^0S_4^0||$ are chosen, see fig. 2.34. Thus, the construction of trajectory of moving point includes the following steps.

1. Construct new position $C^j = l^j \cap m^j$, circle $l^j = (S_3^j, r = ||C^0S_3^0||)$, circle $m^j = (S_4^j, r = ||C^0S_4^0||)$.

2. Construct normal line $n^{C^j} = C^jS_j$.

3. Construct tangent line $t^{C^j} \perp n^{C^j}$, $C^j \in t^{C^j}$.

4. Draw the trajectory τ^C as a curve passing through all positions of the moving point and following the direction of corresponding tangent lines.

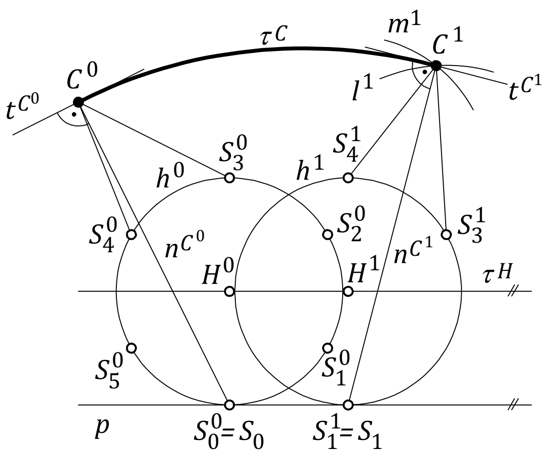


Figure 2.34: Trajectory τ^C of moving point C
(cycloidal motion)

c) Envelope of moving circle

The position of moving circle c with respect to the moving centrode h is given by position of the centre C of the circle c. The radius r_c does not change during the whole motion.

1. Construct new position of moving circle $c^j = (C^j, r = r_c)$.
2. Draw normal line $n^{cj} = C^j S_j$.
3. Points of contact $T^{cj}, \overline{T}^{cj} = c^j \cap n^{cj}$.
4. Construct tangent lines $t^{cj} \perp n^{cj}$, $T^{cj} \in t^{cj}$ and $\overline{t}^{cj} \perp n^{cj}$, $\overline{T}^{cj} \in \overline{t}^{cj}$.
5. Draw each branch of envelope $(c), (\overline{c})$ as a curve passing through all corresponding points of contact and following the direction of corresponding tangent lines, see fig. 2.35.

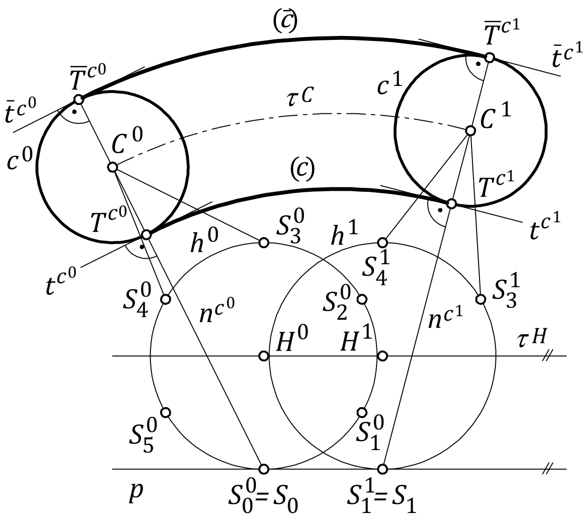


Figure 2.35: Envelope (c) of moving circle c (cycloidal motion)

d) Envelope of moving straight line

The position of moving straight line d with respect to the moving centrode h has to be determined by two parameters – for example by the distance between straight line d and the centre H, and by angle formed by line m passing through H perpendicularly to the moving line d and suitably chosen reference radius of h. Here, the radius $H^j S_0^j$ is chosen as the reference radius, see fig. 2.36. Thus, the distance $||H^j E^j||$ and the angle $\angle S_0^j H^j D^j$ do not change during the whole motion.

1. Construct straight line $m \perp d^0$, $H^0 \in m$.
2. Point $D^j = k^j \cap h^j$, circle $k^j = (S_0^j, r = ||S_0^0 D^0||)$, $D^0 = h^0 \cap m$.
3. Draw straight line $H^j D^j$.
4. Point $E^j = l^j \cap H^j D^j$, circle $l^j = (H^j, r = ||H^0 E^0||)$.

5. Construct new position $d^j \perp H^j D^j$, $E^j \in d^j$.
6. Construct normal line $n^{d^j} \perp d^j$, $n^{d^j} \in S_j$.
7. Point of contact $T^{d^j} = n^{d^j} \cap d^j$.
8. Draw the envelope as a curve passing through all points of contact and following the direction of tangent lines, i.e. individual positions of moving straight line d.

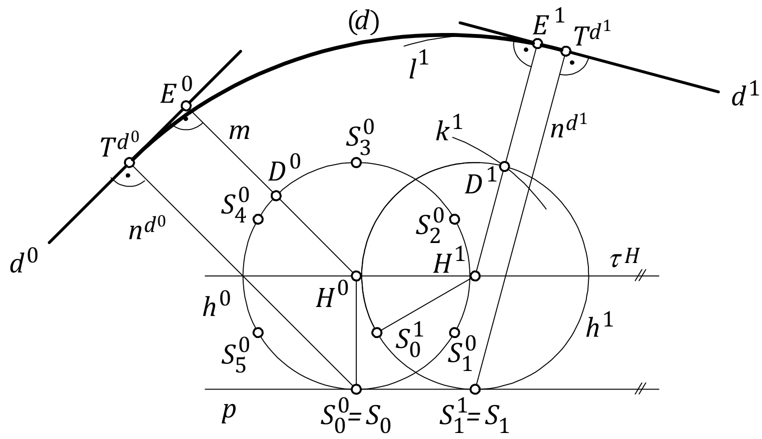


Figure 2.36: Envelope (d) of moving straight line d (cycloidal motion)

□

■ Example 2.5 – Involute motion

Given

Fixed centrode $p = (P, r)$, moving centrode h, point C, circle $c = (C, r_c)$ and straight line d at initial position, see fig. 2.37.

Figure 2.37: Involute motion – example

Required

Construct trajectory τ^C of the given point C, envelope (c) of the given circle c and envelope (d) of the given straight line d.

Analysis

The fixed system is represented by the fixed centrode p. The given moving system is represented by moving centrode h. At j-th instant, the moving centrode h^j touches the fixed centrode p at point of contact $S_i^j = S_j$, $i = j$. This motion is inverse to the cycloidal motion, see example 2.4.

Graphical solution

a) New position of the moving centrode

In the case of involute motion, the moving centrode h is a tangent line to the fixed centrode p at each instant.

1. Divide the fixed centrode p into a sufficient number of n (at least $n = 6$) equal parts to obtain $S_j \in p$, $j = 0, 1, \ldots, n$, see fig. 2.38.

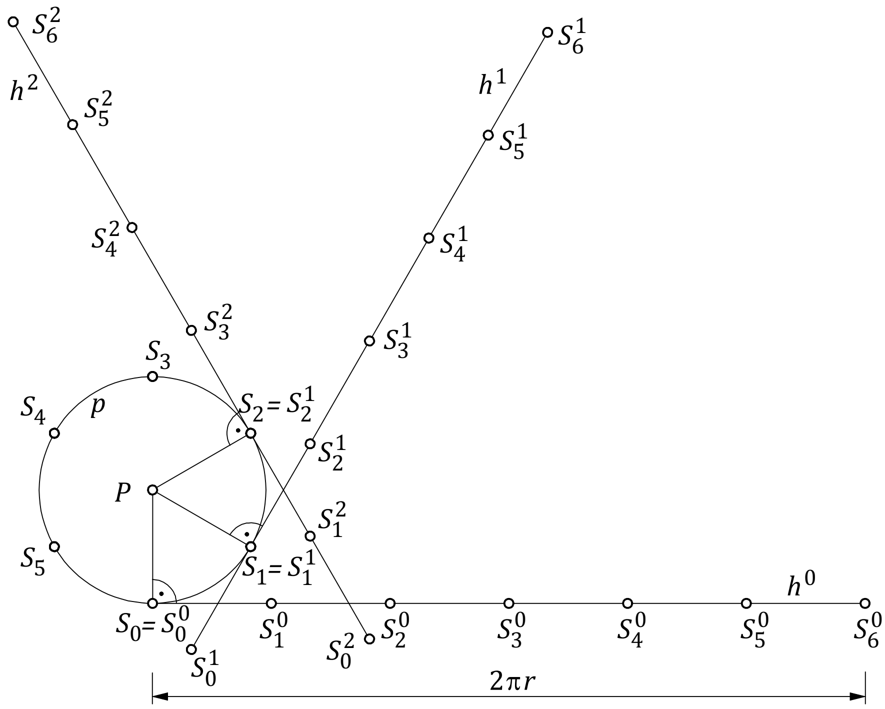


Figure 2.38: New positions of moving centrode h (involute motion)

2. Calculate the length of arc

$$\widehat{S_j S_{j+1}} = \frac{1}{n} 2\pi r \tag{2.2}$$

(or approximate it by the length of a polygon, see fig. 2.33), take it in compass and mark it n-times from S_0^0 on the moving centrode h^0 to obtain instantaneous centres of rotation S_i^j, $i = 0, 1, \ldots, n$.

3. Draw radius PS_j of the fixed centrode p.

4. Construct new position $h^j \perp PS_j$, $S_j \in h^j$.

5. Construct instantaneous centres of rotation $S_i^j \in h^j$, see fig. 2.38.

b) Trajectory of moving point

The position of moving point C with respect to the moving centrode h^j is given by two distances according to the fig. 2.21. Here, the normal distance $||C^j D^j||$ of point C from the moving centrode h and the distance $||D^j S_0^j||$ along the moving centrode h^j have to be constant during the whole motion.

1. Construct $m^0 \perp h^0$, $C^0 \in m^0$, see fig. 2.39.

2. Point $D^j = h^j \cap l^j$, circle $l^j = (S_0^j, r = ||S_0^0 D^0||)$.

3. Construct $m^j \perp h^j$, $D^j \in m^j$.

4. New position $C^j = m^j \cap k^j$, circle $k^j = (D^j, r = ||D^0 C^0||)$.

5. Draw normal line $n^{Cj} = C^j S_j$.

6. Construct tangent line $t^{Cj} \perp n^{Cj}$, $C^j \in t^{Cj}$.

7. Draw trajectory τ^C as a curve passing through all positions of the moving point and following the direction of tangent lines.

c) Envelope of moving circle

Position of moving circle c with respect to the moving centrode h is given by position of its centre C. The radius r_c does not change during the whole motion.

1. Construct new position of moving circle $c^j = (C^j, r = r_c)$, see fig. 2.40.

2. Draw normal line $n^{cj} = C^j S_j$.

3. Points of contact $T^{cj}, \overline{T}^{cj} = c^j \cap n^{cj}$.

4. Construct tangent lines $t^{cj} \perp n^{cj}$, $T^{cj} \in t^{cj}$ and $\overline{t}^{cj} \perp n^{cj}$, $\overline{T}^{cj} \in \overline{t}^{cj}$.

5. Draw each branch of envelope $(c), (\overline{c})$ as a curve passing through all corresponding points of contact and following the direction of corresponding tangent lines, see fig. 2.40.

d) Envelope of moving straight line

The position of moving straight line d with respect to the moving centrode h is determined by angle $\alpha = \angle h^j d^j$, see fig. 2.41, and the distance of the intersection of moving straight line d and the moving centrode h from a suitable chosen reference point located on h. According to fig. 2.41, the instantaneous centre of rotation S_0^j has been chosen as the reference point. Thus, the angle α and the distance $||S_0^j D^j||$ have to be constant during the whole motion.

1. Construct new position d^j. Use construction given in fig. 2.20 b), for example.

2. Construct normal line $n^{d^j} \perp d^j$, $S_j \in n^{d^j}$.

3. Point of contact $T^{d^j} = d^j \cap n^{d^j}$.

4. Draw the envelope (d) as a curve passing through all points of contact and following direction of tangent lines.

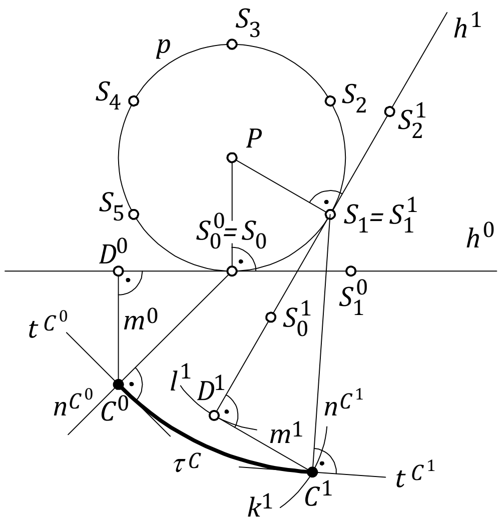


Figure 2.39: Trajectory τ^C
of moving point C (involute motion)

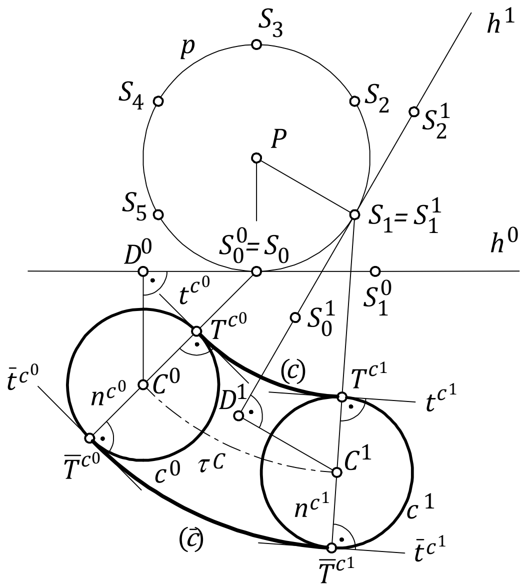


Figure 2.40: Envelope (c)
of moving circle c (involute motion)

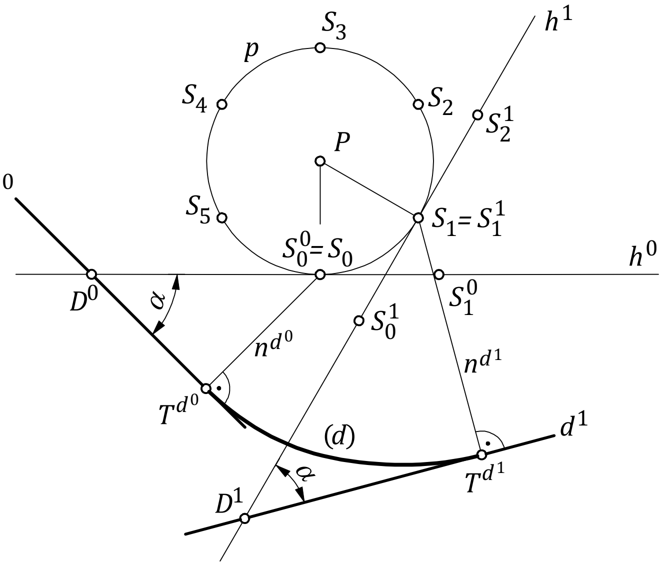


Figure 2.41: Envelope (d) of moving straight line d (involute motion)

□

■ Example 2.6 – Epicycloidal motion

Given

Fixed centrode $p = (P, r_p)$, moving centrode $h = (H, r_h)$, $r_p = 2r_h$, point C, circle $c = (C, r_c)$ and straight line d at initial position, see fig. 2.42 a).

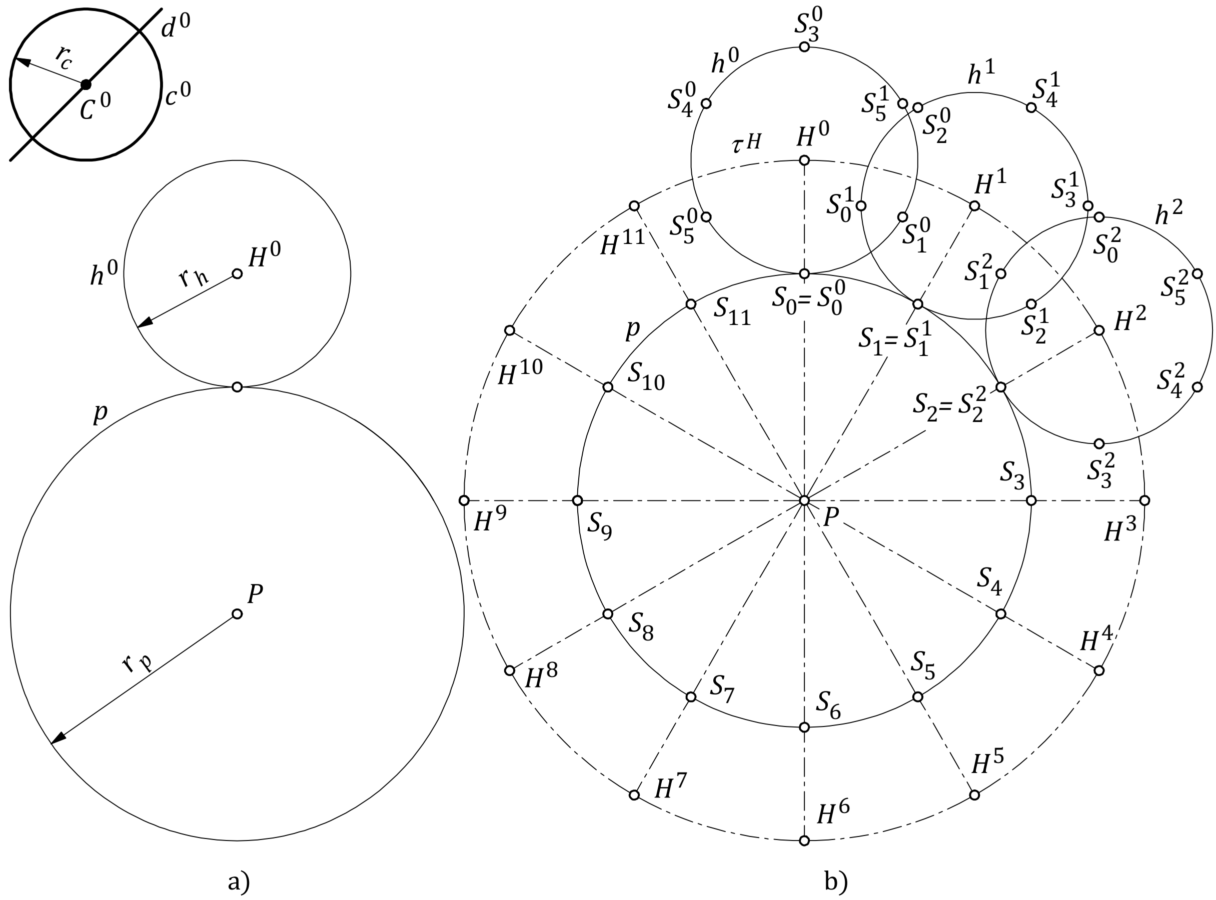


Figure 2.42: Epicycloidal motion

Required

Construct trajectory τ^C of the given point C, envelope (c) of the given circle c and envelope (d) of the given straight line d.

Analysis

The fixed system is represented by the fixed centrode p. The given moving system is represented by moving centrode h. At j-th instant, the moving centrode h^j touches the fixed centrode p at point of contact $S_i^j = S_j$, $i = j$. This motion is inverse to itself.

Graphical solution

a) New position of the moving centrode

The construction of instantaneous centres of rotation on both centrodes is very easy because $r_p = 2r_h$. Thus, the fixed centrode p can be divided into n equal parts and the moving centrode h into $\frac{n}{2}$ equal parts, to obtain equal length of arcs

$$\overset{\frown}{S_j S_{j+1}} = \tfrac{1}{n} 2\pi r_p$$

and

$$\overset{\frown}{S_i^j S_{i+1}^j} = \frac{1}{\frac{n}{2}} 2\pi r_h = \tfrac{1}{n} 2\pi r_p.$$

In fig. 2.42 b) the fixed centrode is divided into 12 parts by 30° and the moving centrode is divided in 6 parts by 60°. It is obvious that for $i > 5$ the instantaneous centres of rotation coincide with the instantaneous centres already drawn (in fig. 2.42 b) the coinciding instantaneous centres of rotation are not designated).

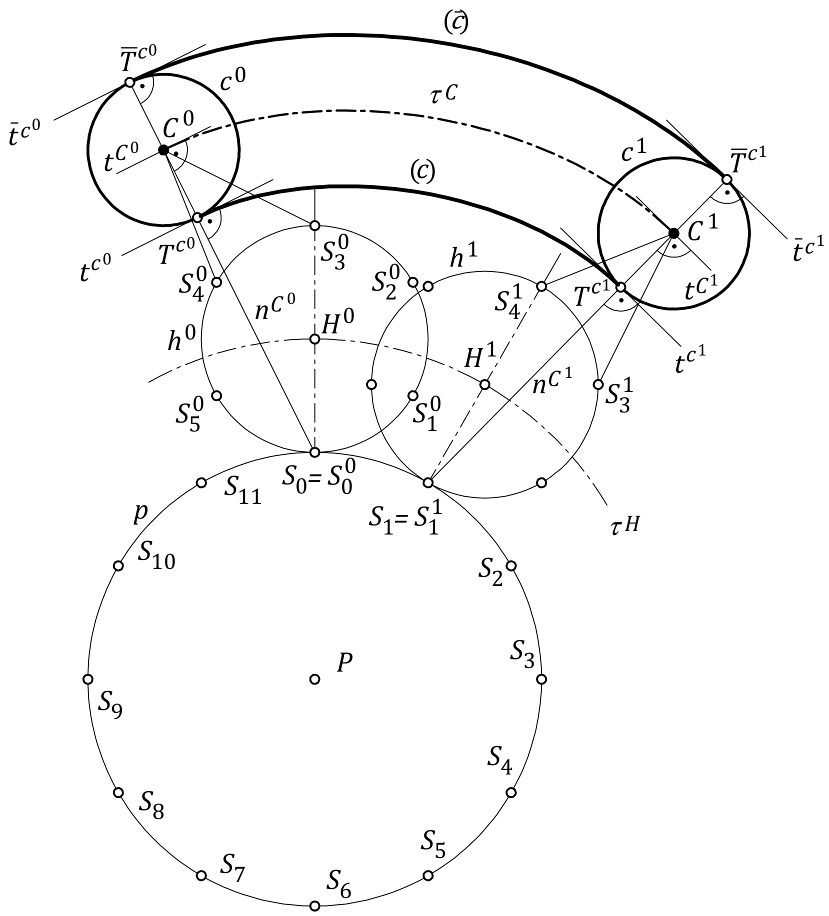


Figure 2.43: Trajectory τ^C of moving point C and envelope (c) of moving circle c (epicycloidal motion)

After determination of instantaneous centres of rotation on the fixed centrode p and the moving centrode h^0, the procedure of construction is as follows.

1. Draw trajectory τ^H of the centre H of the moving centrode h: $\tau^H = (P, r = ||PH^0||)$.
2. Centres $H^j = \tau^H \cap PS_j$.
3. Draw moving centrode $h^j = (H^j, r = r_h)$ at each instant.
4. Construct instantaneous centres of rotation $S_i^j \in h^j$, $i, j = 0, 1, \ldots, n$.

b) Curves generated by epicycloidal motion

Since the moving centrode is a circle, the determination of position of moving point C with respect to the moving centrode h, the construction of trajectory τ^C of moving point C, the construction of envelope (c), $(\bar{c})$ of moving circle c as well as the construction of envelope (d) of moving straight line d is given by procedures described in example 2.4.

A part of the trajectory of the moving point and the envelope of the moving circle generated by epicycloidal motion is drawn in fig. 2.43. A part of the envelope of the moving straight line is drawn in fig. 2.44.

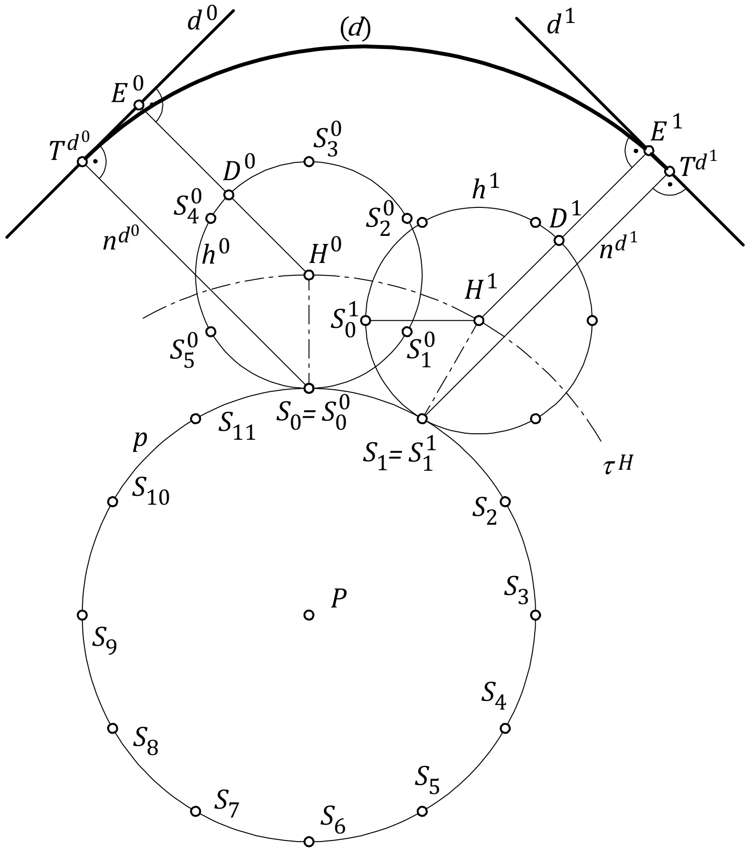


Figure 2.44: Envelope (d) of moving straight line d (epicycloidal motion)

□

■ Example 2.7 – Hypocycloidal motion

Given

Fixed centrode $P = (P, r_p)$, moving centrode $h = (H, r_h)$, $r_p = 2r_h$, point C, circle $c = (C, r_c)$ and straight line d at initial position, see fig. 2.45 a).

Required

Construct trajectory τ^C of the given point C, envelope (c) of the given circle c and envelope (d) of the given straight line d.

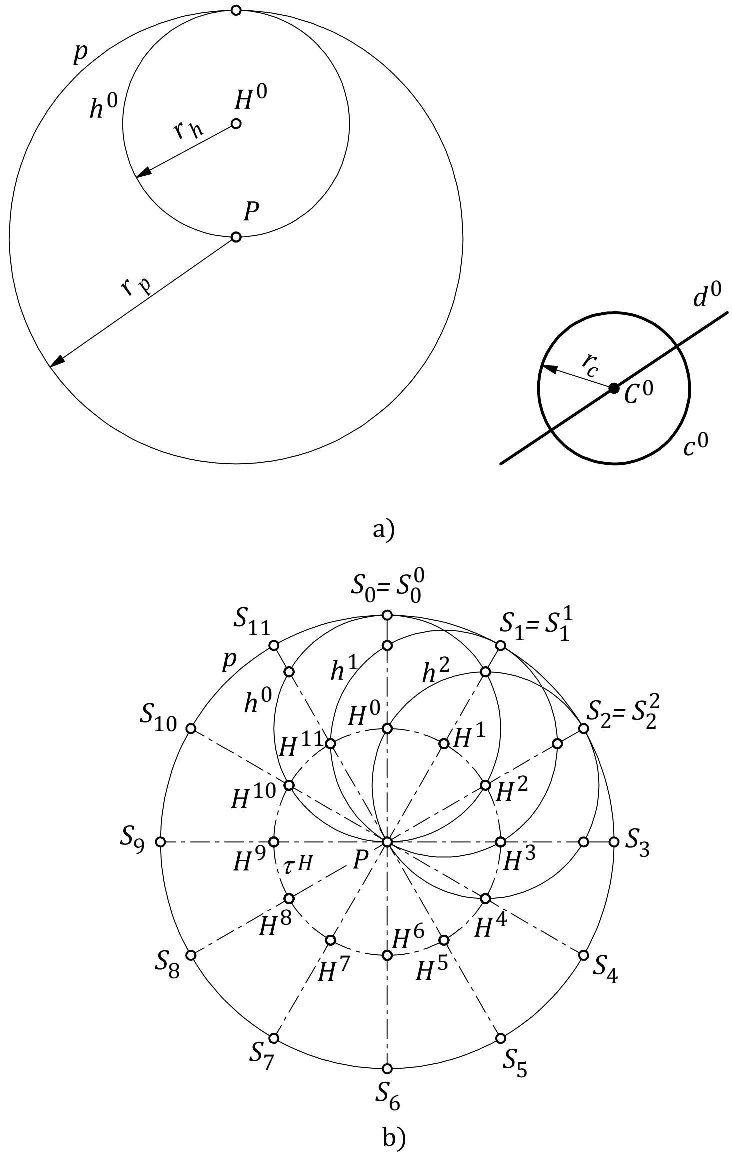


Figure 2.45: Hypocycloidal motion – example

Analysis

The fixed system is represented by the fixed centrode p. The given moving system is represented by moving centrode h. At j-th instant, the moving centrode h^j touches the fixed centrode p at the point of contact $S_i^j = S_j$. This motion is inverse to pericycloidal motion, see example 2.8.

Graphical solution

a) New position of the moving centrode

Since $r_p = 2r_h$, it is possible to follow instructions given in example 2.6 to construct new positions of moving centrode h, see fig. 2.45 b).

b) Curves generated by hypocycloidal motion

Since the moving centrode is a circle, the determination of position of moving point C with respect to the moving centrode h, the construction of trajectory τ^C of moving point C, the construction of envelope (c), $(\bar{c})$ of moving circle c as well as the construction of envelope (d) of moving straight line d is given by procedures described in example 2.4. The only difference is the choice of reference instantaneous centres of rotation located on the moving centrode when determining the position of moving point C with respect to the moving centrode h. Here, the instantaneous centres of rotation S_2^j and S_3^j have been chosen as the reference points. Thus, the distances $||C^j S_2^j||$ and $||C^j S_3^j||$ have to remain constant during the whole motion.

A part of the trajectory of the moving point and the envelope of the moving circle generated by hypocycloidal motion is drawn in fig. 2.46. A part of the envelope of the moving straight line is drawn in fig. 2.47.

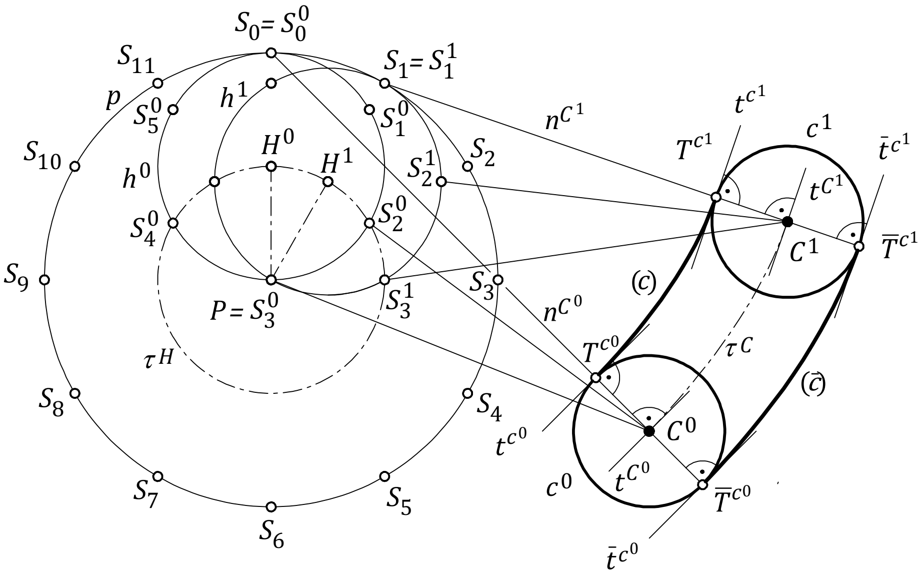


Figure 2.46: Trajectory τ^C of moving point C and envelope (c) of moving circle c (hypocycloidal motion)

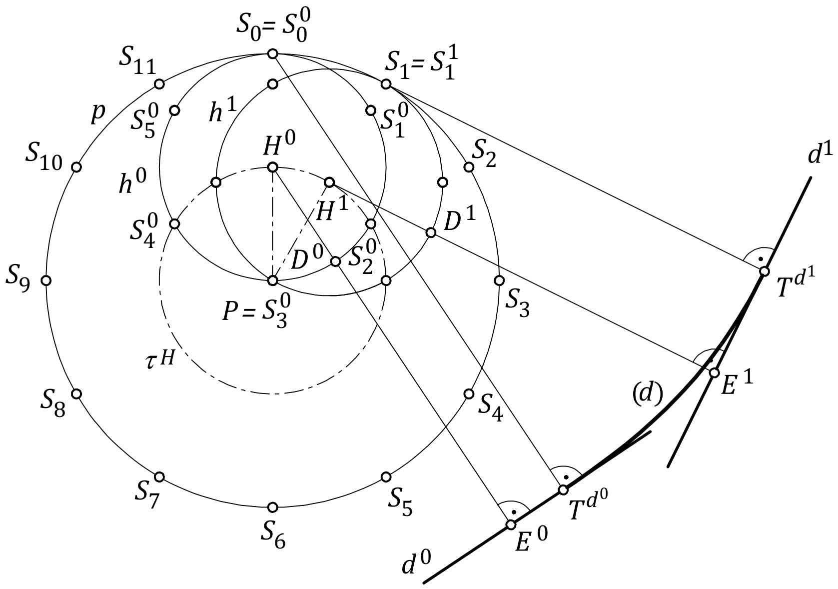


Figure 2.47: Envelope (d) of moving straight line d (hypocycloidal motion)

□

■ Example 2.8 – Pericycloidal motion

Given

Fixed centrode $P = (P, r_p)$, moving centrode $h = (H, r_h)$, $r_p = 2r_h$, point C, circle $c = (C, r_c)$ and straight line d at initial position, see fig. 2.48 a).

Required

Construct trajectory τ^C of the given point C, envelope (c) of the given circle c and envelope (d) of the given straight line d.

Analysis

The fixed system is represented by the fixed centrode p. The given moving system is represented by moving centrode h. At j-th instant, the moving centrode h^j touches the fixed centrode p at point of contact $S_i^j = S_j$, $i = j$. This motion is inverse to hypocycloidal motion, see example 2.7.

Graphical solution

a) New position of the moving centrode

Since $r_h = 2r_p$, the situation in this example is similar to the case of epicycloidal or hypocycloidal motions described in examples 2.6 and 2.7. Here, the moving centrode h can be divided into n equal parts and the fixed centrode p into $\frac{n}{2}$ equal parts, to obtain arcs of equal length

$$\widehat{S_i^j S_{i+1}^j} = \frac{1}{n} 2\pi r_h = \frac{1}{n} 4\pi r_p$$

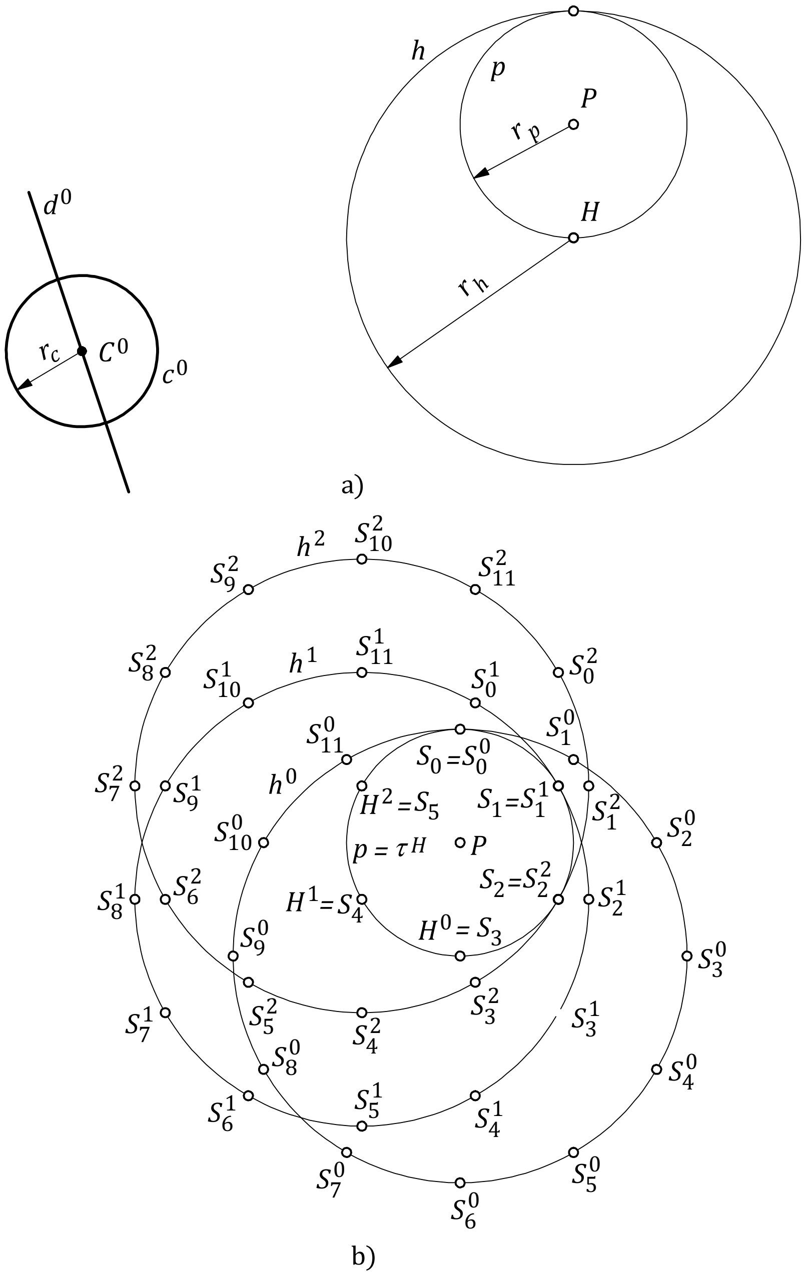


Figure 2.48: Pericycloidal motion – example

and

$$\widehat{S_j S_{j+1}} = \frac{1}{\frac{n}{2}} 2\pi r_p = \frac{1}{n} 4\pi r_p.$$

In fig. 2.48 b) the fixed centrode is divided into 6 parts by 60° and the moving centrode is divided into 12 parts by 30°, the coinciding instantaneous centres of rotation on the fixed centrode are not designated. The trajectory $\tau^H = p$ and centres $H^j = S_{j+3}$.

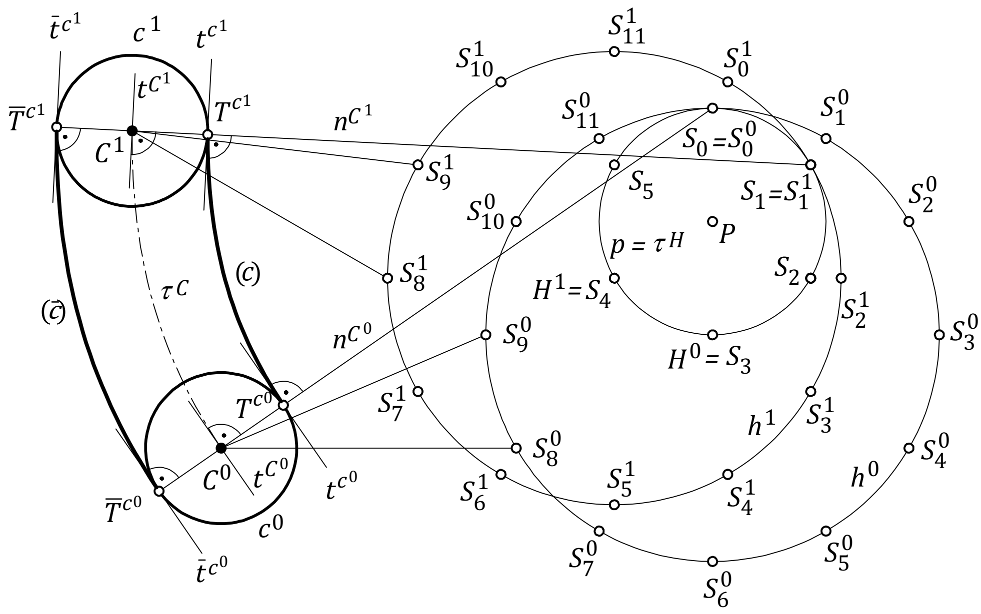


Figure 2.49: Trajectory τ^C of moving point C and envelope (c) of moving circle c (pericycloidal motion)

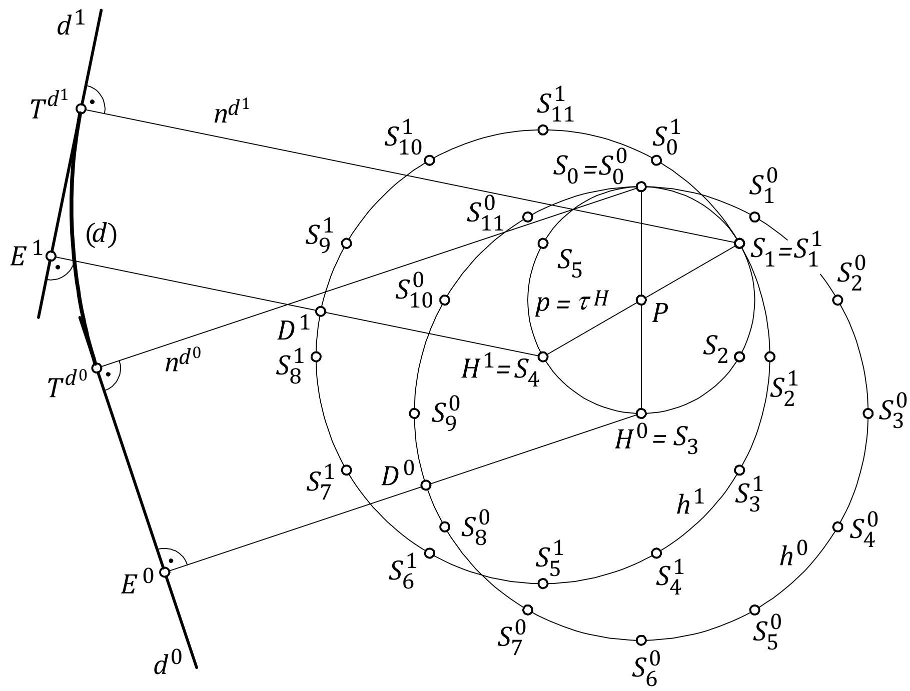


Figure 2.50: Envelope (d) of moving straight line d (pericycloidal motion)

b) Curves generated by pericycloidal motion

Since the moving centrode is a circle, the determination of the position of moving point C with respect to the moving centrode h, the construction of trajectory τ^C of moving point C, the construction of envelope (c), $(\bar{c})$ of moving circle c as well as the construction of envelope (d) of moving straight line d is given by procedures described in example 2.4. The only difference is the choice of reference instantaneous centres of rotation located on the moving centrode when determining the position of moving point C with respect to the moving centrode h. Here, the instantaneous centres of rotation S_8^j and S_9^j have been chosen as the reference points. Thus, the distances $||C^j S_8^j||$ and $||C^j S_9^j||$ have to remain constant during the whole motion.

A part of trajectory of moving point and envelope of moving circle generated by pericycloidal motion is drawn in fig. 2.49. A part of envelope of moving straight line is drawn in fig. 2.50. □

Chapter 3

Methods of projection

Projection is a special type of mapping of 3D objects on a 2D medium (technical drawing, display) developed to represent geometrical shape and graphical information. Two basic types of projection methods can be distinguished – *central projection* and *parallel projection*.

3.1 Central projection

Central projection (perspective projection) is given by a real point – *centre of projection* S and *plane of projection* ρ, $S \notin \rho$. The *central projection* (*central view*) of an arbitrary point A, $A \neq S$ is the intersection point $A' \in \rho$ of *projecting line* $a = AS$ and plane of projection ρ: $A' = a \cap \rho$, see fig. 3.1, where an example of central projection of points $A, B \notin \rho$ and $C \in \rho$ is depicted (the intersections A', B' and C' are estimated). The centre of projection has no image. The central projection of an object is a figure obtained as a set of central projections of all points of the projected object.

Application of central projection in mechanical engineering is not appropriate due to distortion of drawn objects, see example of corner brace in fig. 3.2. Special case of central projection – linear perspective – is widely used in art, architecture and civil engineering because the result picture closely approximates the view obtained by the human eye.

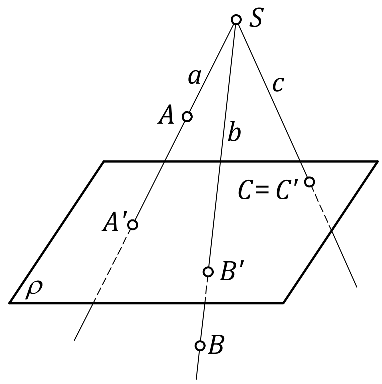


Figure 3.1: Central projection

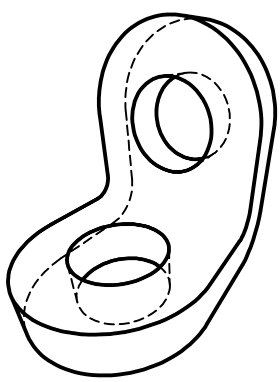


Figure 3.2: Corner brace in central projection

3.2 Parallel projection

Parallel projection is given by *direction of projection* s and *plane of projection* ρ, $s \nparallel \rho$. The direction of projection is represented by a pencil of straight lines passing through the same *point at infinity* (*ideal point*) and intersecting the plane of projection at a real point. Parallel projection can be considered a special type of central projection with the centre of projection at infinity. The *parallel projection* (*parallel view*) of an arbitrary point lies at the intersection of the line parallel with the direction of projection passing through the point and plane of projection.

Straight line parallel with the direction of projection is called *projecting line* and straight line parallel with the plane of projection is called *principal line*. Intersection of line and the plane of projection is called *piercing point* of the line. Plane parallel with the direction of projection is called *projecting plane* and plane parallel with the plane of projection is called *principal plane*. The intersection of a plane and plane of projection is called *trace* of the plane.

Parallel view of an object is a figure obtained as a set of parallel views of all points of the projected object. Specially, parallel view of a point, line and plane are obtained as follows.

- Parallel view of point A is point A' at intersection of projecting line $a \parallel s$, $A \in a$ and plane of projection ρ: $A' = a \cap \rho$, see fig. 3.3 a), where an example of parallel view of points A, B and C is depicted (the intersections A', B' and C' are estimated).
- Parallel view of projecting line $a \parallel s$ is point a' at the intersection of projecting line a and plane of projection ρ: $a' = a \cap \rho$. Parallel view of line b which is not projecting line $b \nparallel s$ is the intersection line b' of projecting plane $\beta \parallel s$, $b \subset \beta$ and plane of projection ρ: $b' = \beta \cap \rho$, see fig. 3.3 b).
- Parallel view of projecting plane $\alpha \parallel s$ is the intersection line α' of projecting plane α and plane of projection ρ: $\alpha' = \alpha \cap \rho$. Parallel view β' of plane β which is not projecting plane $\beta \nparallel \rho$ is the whole plane of projection ρ: $\beta' = \rho$, see fig. 3.3 c).

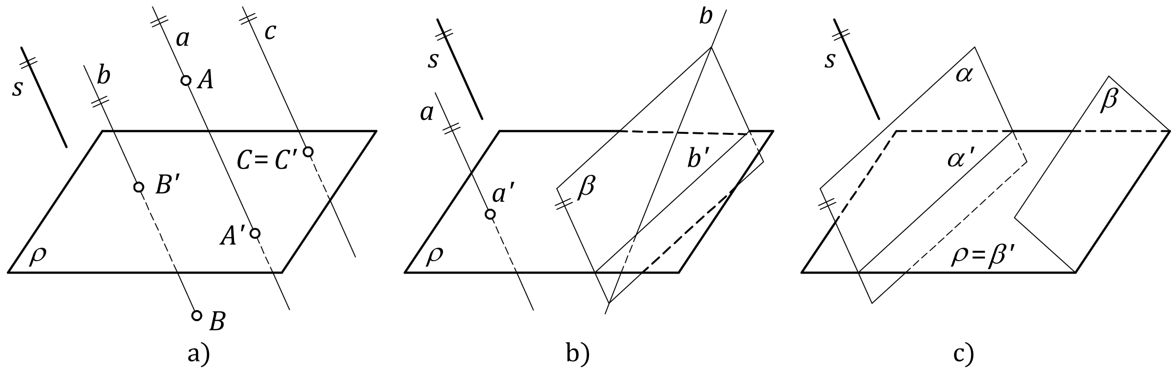


Figure 3.3: Parallel projection

According to the angle φ between the direction of parallel projection and the plane of projection ρ, two types of parallel projection are distinguished – *oblique projection* when $\varphi \neq 90°$ and *orthogonal projection* when $\varphi = 90°$. Both types of parallel projection are widely used in mechanical engineering due to the following important properties.

- The planar figure U located in a principal plane and its parallel view U' are congruent figures $U \cong U'$ because there exists a translation defined by the direction of projection s which maps figure U into U'.

- Parallelism is invariant property of parallel projection. It follows that parallelism of parallel straight lines which are not projecting lines is preserved due to the parallelism of their projecting planes. Consequently, parallel planes have parallel traces as well as views of principal lines.

- Parallel projection preserves dividing ratio of three different collinear points A, B and C: $||AB|| : ||BC|| = ||A'B'|| : ||B'C'||$. It follows, for example, that the centre of a figure is projected to the centre of its parallel view.

- Circle c which does not lie in a projection plane is projected as an ellipse c' – intersection curve of the projecting cylindrical surface χ created by projecting lines of all points on the circle and the plane of projection, see fig. 3.4. Two mutually perpendicular diameters $MN \perp PQ$ of the circle are projected as conjugated diameters $M'N'$ and $P'Q'$ of the ellipse.

  Conjugated diameters MN and PQ of an ellipse are drawn in fig. 3.5 left. A *diameter* of an ellipse is any straight line segment with the centre at the centre of the ellipse and end points on the ellipse. Two diameters are called *conjugated* if tangent lines at end points of one diameter are parallel with the other one, i.e. $MN \parallel EF \wedge MN \parallel HG$ and $PQ \parallel EH \wedge PQ \parallel FG$. The parallelogram $EFGH$ is called *tangent parallelogram of the ellipse*. The ellipse is inscribed into its tangent parallelogram. Major and minor axis of the ellipse represent a special case of perpendicular conjugated diameters, where the major axis has maximal length and the minor axis has minimal length.

  In the case of a circle (a special case of an ellipse), all diameters have the same length, tangent parallelogram becomes tangent square and conjugated diameters are always perpendicular, see fig. 3.5 right.

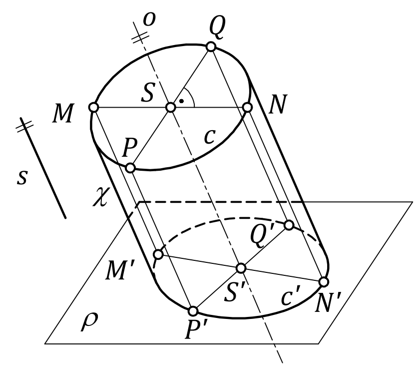


Figure 3.4: Parallel projection of a circle

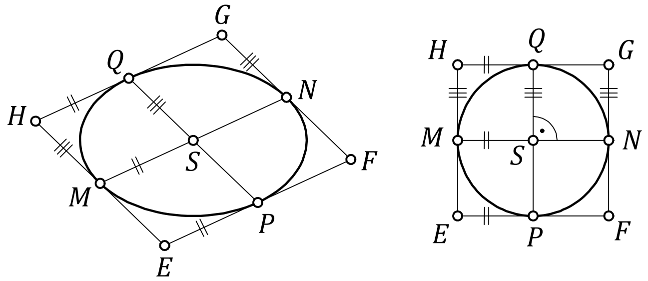


Figure 3.5: Conjugated diameters and tangent parallelogram of an ellipse and circle

- Sphere σ is projected as an ellipse c' and its interior area. Ellipse c' is the intersection curve of the projecting cylinder χ created by projecting lines of all points on the sphere and the plane of projection, see fig. 3.6 a). In fact, ellipse c' is a view of principal circle c of sphere σ lying in the plane perpendicular to the direction of projection s. Since $s \not\perp \rho$, the circle c is projected as ellipse c', see section 4.2.2.

Additionally, the following special properties are valid for the parallel orthogonal projection.

- Parallel orthogonal projection shortens the length of straight line segments that are not parallel with the plane of projection. It follows from $||A'B'|| = ||AB|| \cos\alpha$, where α is the angle formed by straight line segment AB and its parallel orthogonal view $A'B'$.

- Two perpendicular lines are projected as perpendicular lines if at least one of the lines is parallel to the plane of projection and none of them is perpendicular to the plane of projection. To demonstrate this property, consider lines $a \perp b$, $a \nparallel s$, $b \nparallel s$, $b \parallel \rho$ and projecting plane $\alpha = (a, a')$. Since $a \perp b \wedge b \perp s \Rightarrow b \perp \alpha$, $b \perp \alpha \wedge a' \subset \alpha \Rightarrow b \perp a'$, and, finally, as $b \perp a' \wedge b \parallel b' \Rightarrow b' \parallel a'$.

- Orthogonal view of circle $c = (S, r)$ which does not lie in a projection plane is ellipse $c' = (S', a = 2r, b)$ with major axis $a = 2r$ lying on the view of principal line passing through the centre S of the circle.

- Sphere σ is projected as circle c' and its interior area. Circle c' is the intersection curve of the projecting cylinder χ created by projecting lines of all points on the sphere and plane of projection, see fig. 3.6 b). In fact, circle c' is a view of principal circle c of sphere σ lying in the plane perpendicular to the direction of projection s. Since $s \perp \rho$, circle c is projected as circle c', see section 4.2.2.

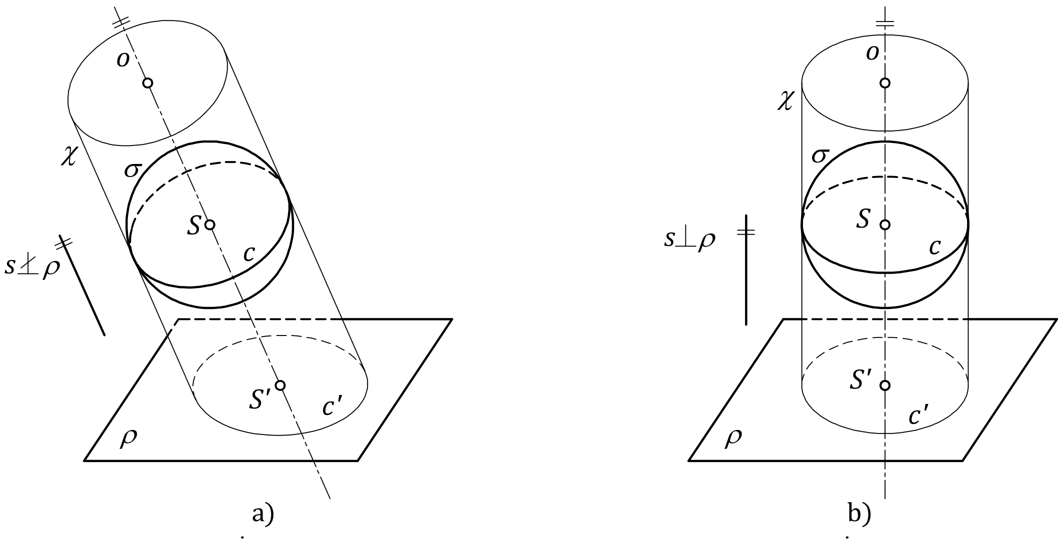


Figure 3.6: Parallel projection of a sphere

To obtain unambiguous parallel view of a three-dimensional object, it is necessary to determine the position of the plane of projection with respect to the chosen coordinate system and give the rules of mapping. Usually, Cartesian coordinate system (O, x, y, z) with unit coordinate vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ is considered. Three mutually perpendicular coordinate axes x, y and z intersect at origin O. Coordinate planes are determined by axes of coordinate system: *horizontal plane* $\pi = (x, y)$, *frontal plane* $\nu = (x, z)$ and *profile plane* $\mu = (y, z)$. Point $A = (x_A, y_A, z_A)$ is orthogonally projected onto coordinate planes and the *top view* $A_1 \in \pi$, *front view* $A_2 \in \nu$ and *profile view* $A_3 \in \mu$ are obtained as intersections of projecting lines $AA_1 \perp \pi$, $AA_2 \perp \nu$ and $AA_3 \perp \mu$ with the corresponding coordinate planes.

Due to the parallelism preserving property, the projecting lines are parallel with coordinate axes, i.e. $AA_1 \parallel z$, $AA_2 \parallel y$ and $AA_3 \parallel x$, see fig. 3.7. Here, a sketch of the *coordinate box* of point $A = (x_A, y_A, z_A)$ is drawn. Coordinate box is axes-aligned box in the first octant with one vertex at origin. The lengths of edges of a real coordinate box are equal to Cartesian coordinates of the point. Depending on the type of projection, the lengths of projected edges can be distorted.

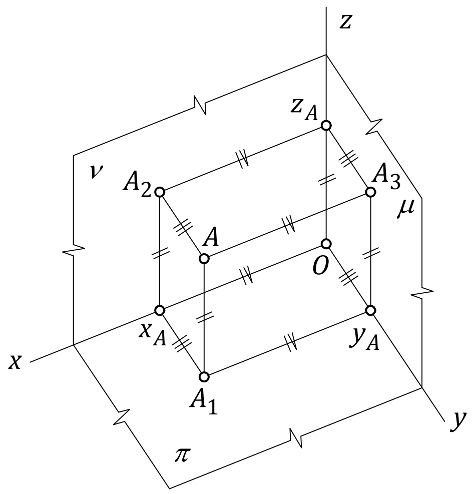


Figure 3.7: Coordinate box of point $A = (x_A, y_A, z_A)$

Various types of projection can be distinguished according to the position of the plane of projection with respect to the coordinate system, number of planes of projection and the angle of the direction of projection. Here, the following types of parallel projection are briefly mentioned: Monge projection, orthogonal axonometry and its special variants isometry and technical isometry, oblique projection and its special variant military perspective. To study these methods in detail, see [1-4], for example.

3.2.1 Monge projection

Monge projection is parallel orthogonal projection onto two coordinate planes – horizontal plane $\pi = (x, y)$ and frontal plane $\nu = (x, z)$. Each point $A = (x_A, y_A, z_A)$ is represented by a pair of *adjacent views* – the top view $A_1 = (x_A, y_A)$, i.e. the orthogonal projection of point A onto the horizontal plane π and the front view $A_2 = (x_A, z_A)$, i.e. the orthogonal projection of point A onto the frontal plane ν.

To obtain a two-dimensional drawing, one plane of projection is rotated about the common intersection line (x-axis) by 90° to identify both planes with picture plane. Thus, a pair of adjacent views in one picture plane is obtained, see fig. 3.8 left. The projection of x-axis is called *folding line*. Since the top view x_1 and the front view x_2 of x-axis are identical, the folding line is designated by x_{12}. The coordinate system is created by folding line x_{12}, the top view y_1 of y-axis and the front view z_2 of z-axis. Since $y \perp \nu$ and $z \perp \pi$, the front view y_2 as well as the top view z_1 are identical to origin O_{12}. The orientation of axes of the right-handed coordinate system is obvious from the figure.

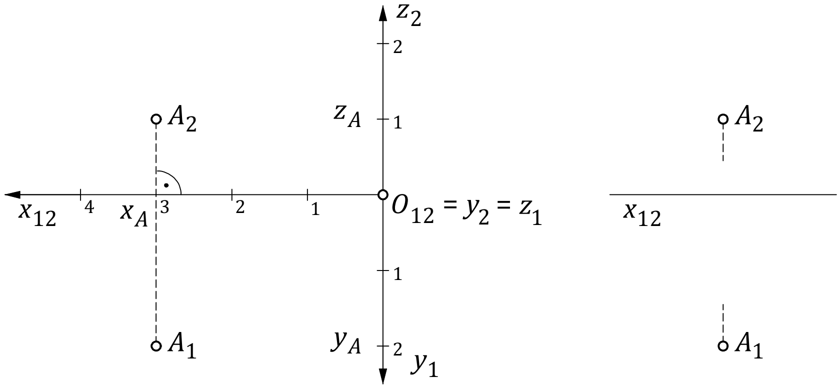


Figure 3.8: Point $A = (3, 2, 1)$ in Monge projection

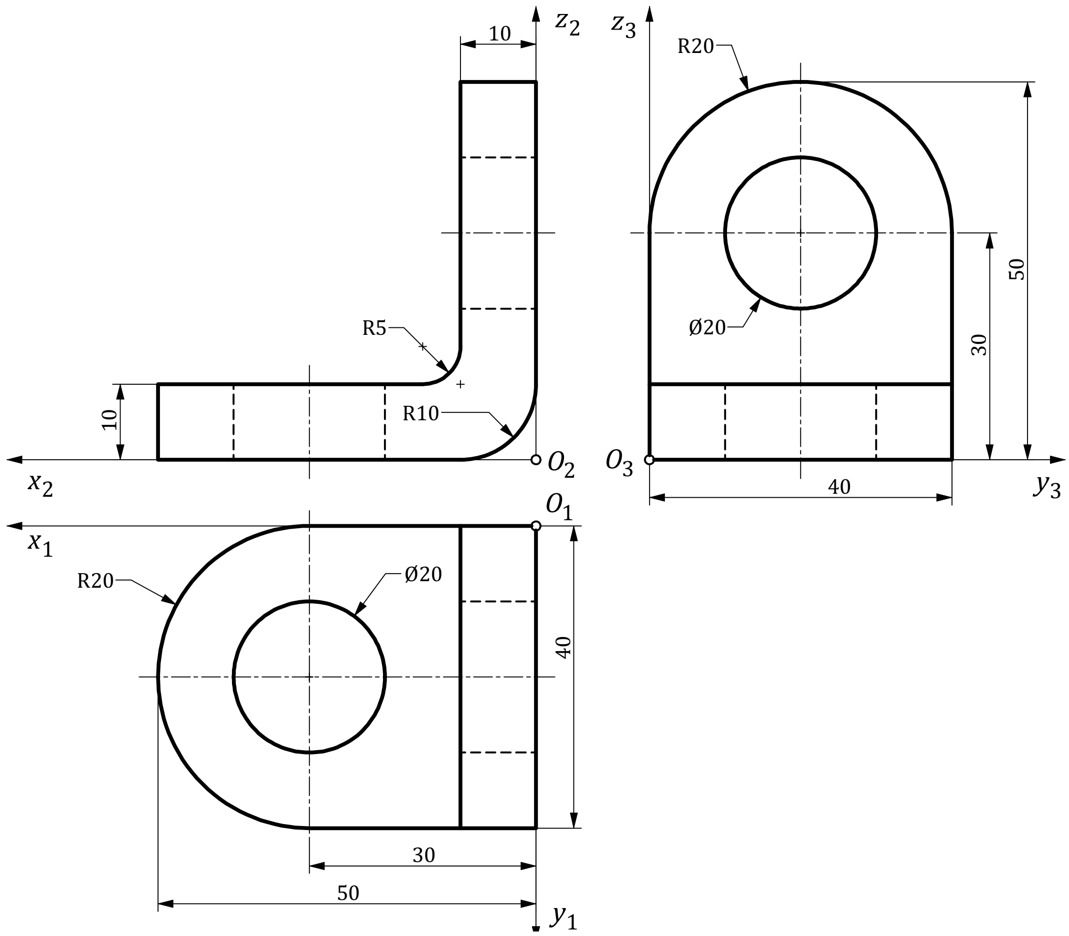


Figure 3.9: Top, front and profile views of corner brace

The plane given by point A, its top view A_1 and front view A_2 is perpendicular to x-axis, see fig. 3.7. Therefore, it is projected as connection line A_1A_2 perpendicular to the folding line. This connection line A_1A_2 is called *ordinate*.

Depending on the problem which has to be solved in Monge projection, it is not necessary to draw all axes of the coordinate system, i.e. x_{12}, y_1 and z_2. Usually, only folding line x_{12} is drawn, see fig. 3.8 right.

Depending on the shape of the drawn object, more planes of projection can be used. An example of top, front and profile view of corner brace from fig. 3.2 is drawn in fig. 3.9. Monge projection is widely used in constructive geometry (see chapter 5 – chapter 7) and in technical drawings due to the excellent properties of multiview orthogonal projection.

3.2.2 Orthogonal axonometry

Orthogonal axonometry (axonometry) is parallel orthogonal projection onto one *axonometric plane of projection* ρ in general position with respect to the Cartesian coordinate system (O, x, y, z). Axonometric plane of projection ρ intersects coordinate axes x, y and z at points

$$X = \rho \cap x,\ Y = \rho \cap y \text{ and } Z = \rho \cap z,$$

vertices of *axonometric triangle* $\triangle XYZ$, and coordinate planes $\pi = (x, y)$, $\mu = (y, z)$ and $\nu = (x, z)$ at straight lines

$$XY = \rho \cap \pi,\ YZ = \rho \cap \mu \text{ and } ZX = \rho \cap \nu,$$

edges of axonometric triangle $\triangle XYZ$, see fig. 3.10. The projection of origin O of coordinate system is point $O' \in \rho$ at orthocentre of axonometric triangle $\triangle XYZ$. Projection of coordinate axes x, y and z are straight lines $x' = O'X$, $y' = O'Y$ and $z' = O'Z$ at altitudes of axonometric triangle $\triangle XYZ$. The lengths u_x, u_y and u_z of projected unit coordinate vectors are called *axonometric units*. In general, axonometric units are not equal.

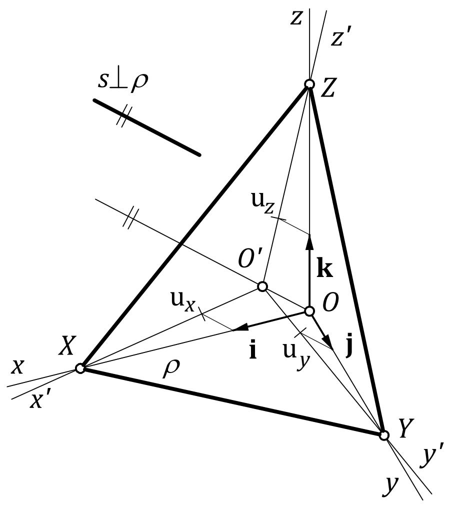


Figure 3.10: Orthogonal axonometry

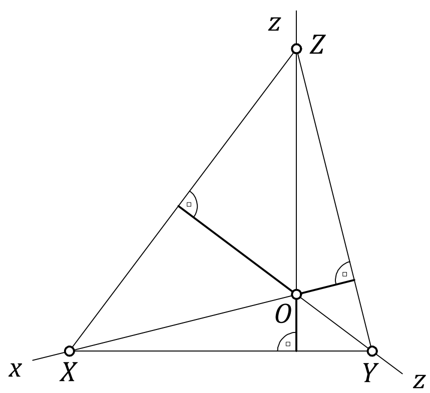


Figure 3.11: Axonometric triangle and coordinate system

Orthogonal axonometry has the following properties.

1. Axonometric triangle $\triangle XYZ$ is a triangle with all acute angles. It follows from the general position of axonometric plane of projection ρ with respect to the Cartesian coordinate system (O, x, y, z).

2. Axonometric views of coordinate axes x', y' and z' are lines at the altitudes of axonometric triangle XYZ, i.e. $x' \perp YZ$, $y' \perp ZX$ and $z' \perp XY$. To demonstrate this property, consider x' in fig. 3.10, for example. Now $YZ \perp OO'$ (because $OO' \parallel s$, $s \perp \rho$ and $YZ \subset \rho$) and at the same time $YZ \perp XO$ (because $XO \subset x$, $x \perp (y, z)$ and $YZ \subset (y, z)$). It follows that YZ is perpendicular to the plane XOO' and, consequently, it is perpendicular to x' because $x' \subset XOO'$. Similarly, it is possible to show that $y' \perp ZX$ and $z' \perp XY$.

3. The angles formed by positive semi-axes x', y' and z' are obtuse. This is the consequence of the two previous properties.

To construct axonometric views of projected objects, axonometric plane of projection is identified with the picture plane, see fig. 3.11. Orthogonal axonometry is uniquely defined by acute angled axonometric triangle $\triangle XYZ$ (axes x, y and z are constructed as altitudes of this triangle without any special denotation) or by coordinate axes x, y and z forming obtuse angles (edges of axonometric triangle are perpendicular to these axes).

After construction of coordinate system and axonometric triangle, it is necessary to determine axonometric units u_x, u_y and u_z. The procedure of axonometric units construction is based on revolution of the coordinate plane into the axonometric plane of projection about the common intersection line (the corresponding edge of axonometric triangle). After this revolution, all figures in the rotated plane appear in true size. The trajectory of each point located in the rotated plane is a circle with the centre on axis of revolution lying in the plane perpendicular to the axis of revolution. Therefore, this trajectory is projected as a line perpendicular to the corresponding edge of axonometric triangle. In particular, the procedure of construction of axonometric units u_x and u_y consists of the following steps, see fig. 3.12.

1. Construct circle c given by diameter XY.

2. Construct rotated origin (O), i.e. construct line perpendicular to XY passing through O (extension of z-axis). Rotated origin (O) lies at the intersection of circle c and the extended z-axis.

3. Construct rotated axes $(x) = (O)X$ and $(y) = (O)Y$ and starting at (O), draw straight line segments of unit length u along (x) and (y).

4. Rotate terminate points of unit segments back into the plane (x, y), i.e. construct lines perpendicular to XY passing through terminate points of unit segments. Axonometric units u_x and u_y are given by intersections of these perpendicular lines and axes x and y.

5. To obtain axonometric unit u_z, it is necessary to rotate plane (x, z) or (y, z) into the axonometric plane of projection, see fig. 3.12.

After construction of axonometric units, it is possible to draw equidistant tickmarks along individual axes of axonometric coordinate system with the distance given by the corresponding axonometric unit. Then, the axonometric view of any point given by Cartesian coordinates can be simply drawn by means of scales on the projected coordinate axes, see fig. 3.13. Here, the axonometric view of the coordinate box of point $A = (x_A, y_A, z_A) = (2, 3, 1)$ is drawn.

To obtain one to one correspondence between point A in three-dimensional space and its axonometric view A', it is not necessary to construct the whole coordinate box. The construction of axonometric view A' of the point itself and axonometric view of one of its orthogonal projections onto coordinate planes is sufficient. Usually, axonometric view of a point is complemented by the axonometric view of its top view without using any special denotation, see fig. 3.14.

Corner brace given by orthogonal views in fig. 3.9 in axonometry (defined by axonometric triangle $\triangle XYZ$) is drawn in fig. 3.15. The dimensions parallel with coordinate axes are shortened according to the corresponding axonometric units which have to be constructed. The circles lying in coordinate planes (or in the planes parallel with coordinate planes) are projected as ellipses. The length of major axis of each ellipse is equal to the diameter of the corresponding circle. The major axis of each ellipse is axonometric view of the diameter of the circle parallel with axonometric plane of projection, i.e. parallel with the corresponding edge of axonometric triangle $\triangle XYZ$, and, consequently, perpendicular to the corresponding coordinate axis.

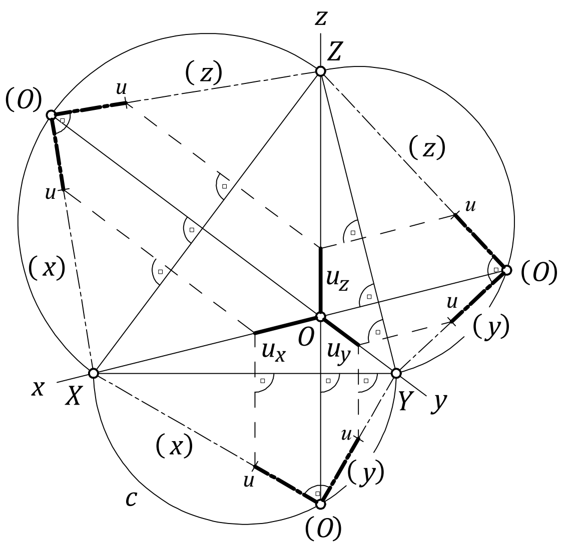


Figure 3.12: Construction of axonometric units

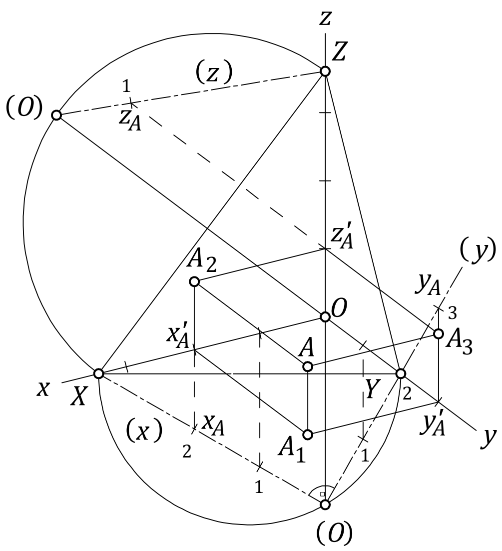


Figure 3.13: Coordinate box of point $A = (2, 3, 1)$ in orthogonal axonometry

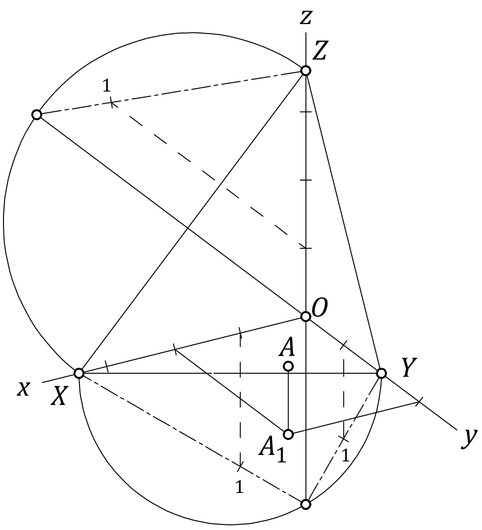


Figure 3.14: Point $A = (2, 3, 1)$ in axonometry

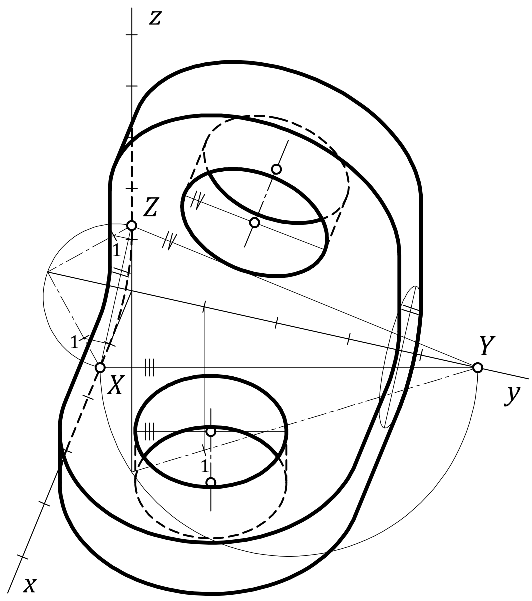


Figure 3.15: Corner brace in axonometry

Axonometry is widely used for 3D modelling of technical details in CAD (Computer Aided Design) and CAM (Computer Aided Manufacturing) systems. The fact that the axonometric units are generally not equal can be considered a disadvantage when drawing technical details in axonometry by hand. This disadvantage is eliminated when the axonometric plane of projection is placed in a special position with respect to the coordinate system, as is described in the following section.

Isometry

Isometry is a special type of orthogonal axonometry where the axonometric plane of projection ρ intersects axes of coordinate system at the same distances from origin. Consequently, isometry has the following properties, see fig. 3.16.

1. Axonometric triangle $\triangle XYZ$ is equilateral. It follows from the special position of axonometric plane of projection with respect to the coordinate system

2. The angles formed by positive semi-axes x', y' and z' are equal to $120°$. It follows from geometrical properties of an equilateral triangle.

3. All dimensions parallel with coordinate axes are shortened by the same ratio given by *isometry scale coefficient*

$$k = \sqrt{\frac{2}{3}}. \tag{3.1}$$

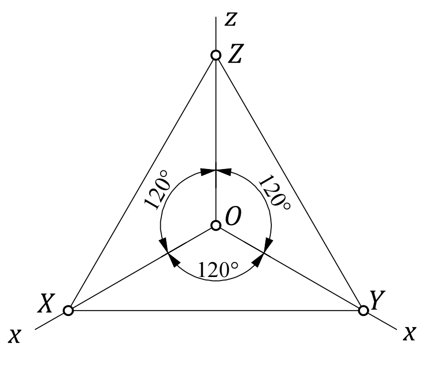


Figure 3.16: Isometry

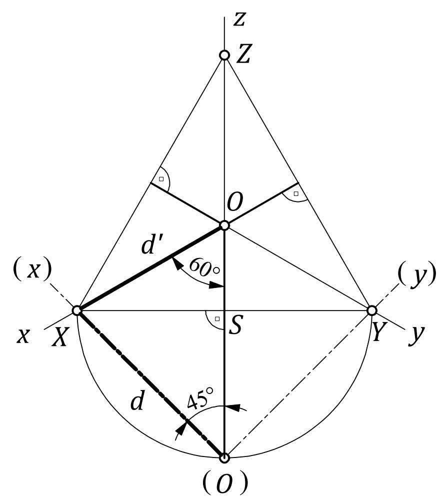


Figure 3.17: To the derivation of isometric scaling coefficient

To derive the value of isometry scaling coefficient, consider situation depicted in fig. 3.17. Here

$$k = \frac{d'}{d}. \tag{3.2}$$

From the right angled triangle $\triangle XOS$ we have

$$\sin 60° = \frac{\| XS \|}{d'} \Longrightarrow d' = \frac{\| XS \|}{\sin 60°} \tag{3.3}$$

and from the right angled triangle $\triangle XS(O)$ we have

$$\sin 45^\circ = \frac{\| XS \|}{d} \Longrightarrow d = \frac{\| XS \|}{\sin 45^\circ}. \tag{3.4}$$

After substitution eq. (3.3) and eq. (3.4) in eq. (3.2) and knowing that $\sin 60^\circ = \frac{\sqrt{3}}{2}$ and $\sin 45^\circ = \frac{\sqrt{2}}{2}$, we get eq. (3.1).

It is not necessary to construct axonometric units when drawing isometric views of objects in isometry. Due to the same scaling coefficient, it is possible to multiply all Cartesian coordinates of points by $k = \sqrt{\frac{2}{3}}$ (approximately $k \doteq 0.8$) and measure shortened dimensions along axes of isometric coordinate system. Thus, point $A = (x_A, y_A, z_A)$ is projected into

$$A' = (x'_A, y'_A, z'_A) = (x_A\sqrt{\tfrac{2}{3}}, y_A\sqrt{\tfrac{2}{3}}, z_A\sqrt{\tfrac{2}{3}}), \tag{3.5}$$

see example in fig. 3.18 a), where isometric view of point $A = (2, 3, 1)$ is drawn.

Figures lying in planes parallel with isometric plane of projection are projected in true size as follows from properties of parallel projection. Thus, a sphere given by centre $S = (x_S, y_S, z_S)$ and radius r is projected into a circle with the centre

$$S' = (x_S\sqrt{\tfrac{2}{3}}, y_S\sqrt{\tfrac{2}{3}}, z_S\sqrt{\tfrac{2}{3}})$$

and radius $r' = r$. In fact, the projected circle is projection of the principal circle of the sphere lying in the plane parallel with isometric plane of projection. Therefore, it is projected in true size. Example of isometric view of the sphere with centre $S = (0, 0, 2)$ and radius $r = 2$ is drawn in fig. 3.18 b). Note that the distances between the tickmarks on coordinate axes are not shortened by $\sqrt{\frac{2}{3}}$ in this picture.

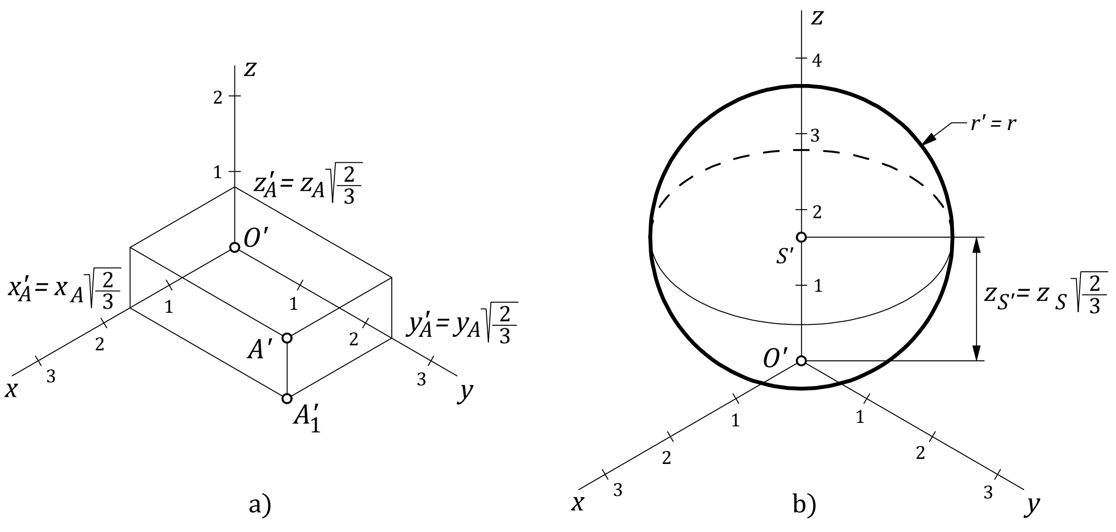


Figure 3.18: Coordinate box of point $A = (2, 3, 1)$ (left)
and sphere with centre $S = (0, 0, 2)$ and radius $r = 2$ (right) in isometry

Technical isometry

Technical isometry or *isometric drawing* is an alternative of isometry when scaling coefficient $k = 1$. This alternative is widely used in mechanical engineering because the inconvenient multiplication of coordinates by $\sqrt{\frac{2}{3}}$ is eliminated. Thus, point $A = (x_A, y_A, z_A)$ is projected into

$$A' = (x'_A, y'_A, z'_A) = (x_A, y_A, z_A), \tag{3.6}$$

i.e. the coordinates are projected in true size. An example of point $A = (2, 3, 1)$ drawn in technical isometry is given in fig. 3.19 a).

Figures lying in planes parallel with isometric plane of projection are projected in larger size, i.e. multiplied by $\sqrt{\frac{3}{2}}$ (approximately 1.2), see example of technical isometry of a sphere with centre $S = (0, 0, 2)$ and radius $r = 2$ in fig. 3.19 b).

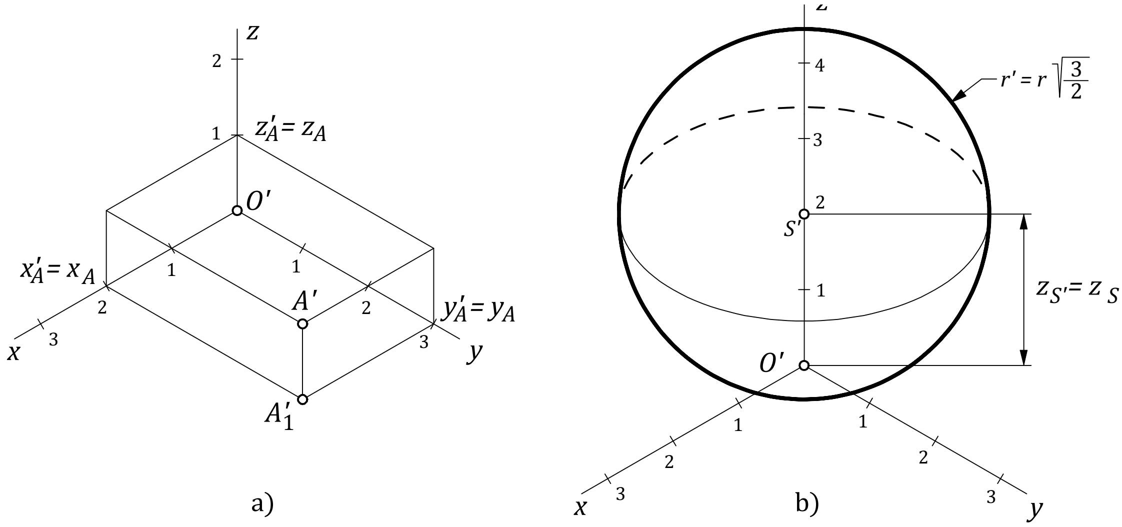


Figure 3.19: Coordinate box of point $A = (2, 3, 1)$ (left)
and sphere with centre $S = (0, 0, 2)$ and radius $r = 2$ (right) in technical isometry

Corner brace given by orthogonal views in fig. 3.9 in technical isometry is drawn in fig. 3.20. The dimensions parallel with coordinate axes are projected in true sizes. The circles lying in coordinate planes (or in planes parallel with coordinate planes) are projected as ellipses. The length of major semi-axis of each ellipse is given by $r\sqrt{\frac{3}{2}}$, where r is true size of the radius of the corresponding circle. The major axis of each ellipse is perpendicular to the corresponding coordinate axis (or parallel with the corresponding edge of isometric triangle which does not have to be drawn).

Many other examples of objects drawn in isometry or technical isometry can be found in this textbook in chapters describing problems of spatial analytic geometry and surfaces (quadratic surfaces, surfaces of revolution, helicoidal surfaces, developable and transition surfaces and envelope surfaces).

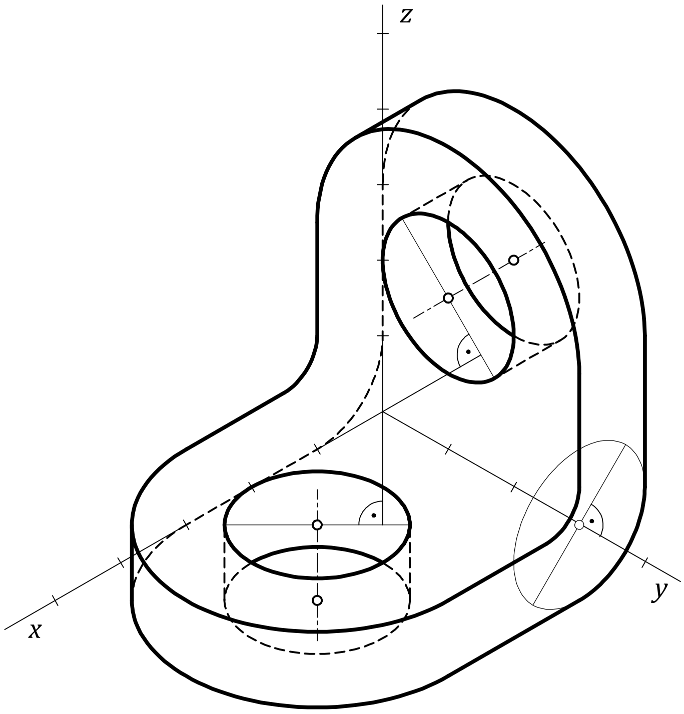


Figure 3.20: Corner brace in technical isomentry

3.2.3 Oblique projection

Oblique projection is a parallel projection onto profile plane $\mu = (y, z)$ where direction of projection s is not perpendicular to the profile plane of projection $s \not\perp \mu$. The oblique view of coordinate system (O', x', y', z') is constructed according to the following rules. Usually, the direction of z'-axis is vertical. Since the plane of projection is identical to the profile plane μ, right angle formed by y- and z-axes is preserved, i.e. $y' \perp z'$. Obtuse angle $\omega = \angle x'y'$ is chosen, see fig. 3.21 a).

The length of y- and z-coordinates of point $A = (x_A, y_A, z_A)$ is preserved, i.e. $y'_A = y_A$ and $z'_A = z_A$. Since $x \nparallel \mu$, x-coordinate of point A is distorted. Depending on the chosen *quotient* q of oblique projection, the projected x'-coordinate can be longer $q > 1$, equal $q = 1$ or shorter $q < 1$ than the true length, i.e. $x'_A = qx_A$.

To be able to construct oblique view of any object, acute angle $\omega = \angle y'z'$ and quotient $q = x' : x$ have to be given. Example of coordinate box of point $A = (2, 3, 1)$ in oblique projection ($\omega = 135°, q = 4 : 5$) is drawn in fig. 3.21 b). Note that the distances between the tickmarks on x'-axis are not shortened by q in this picture.

It is not necessary to draw the whole coordinate box when constructing oblique view of a point. To obtain one to one correspondence between point A in three-dimensional space and its oblique view A', the oblique view A' of the point itself and oblique view of one of its orthogonal projections onto coordinate planes are necessary to construct. Usually, the oblique view of point is complemented by the oblique view of its top view and no special notation for oblique views of individual figures is used, see fig. 3.21 c) (the distances between the tickmarks on x-axis are not shortened by q in this picture).

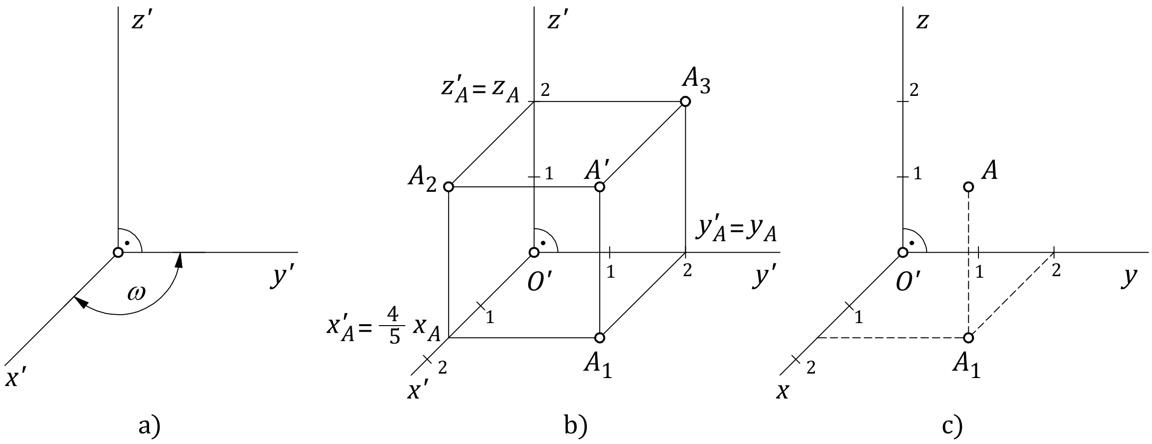


Figure 3.21: Oblique projection

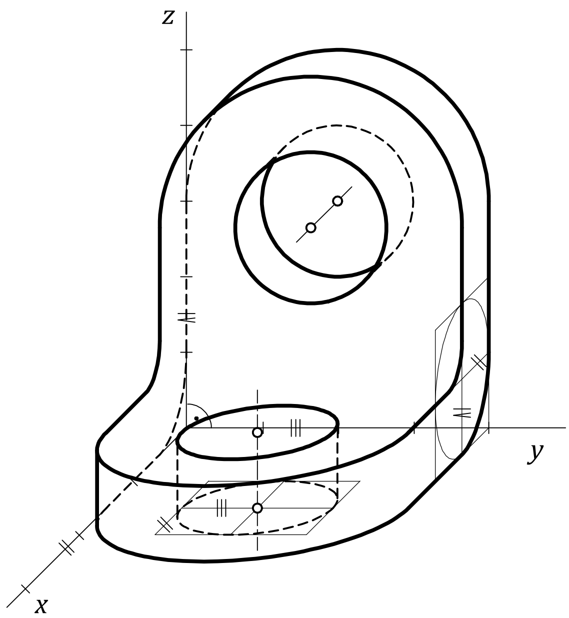


Figure 3.22: Corner brace in oblique projection ($\omega = 135°,\ q = 1:2$)

Corner brace given by orthogonal views in fig. 3.9 in oblique projection ($\omega = 135°, q = 1:2$) is drawn in fig. 3.22. The dimensions parallel with coordinate y- and z-axes are projected in true sizes. The dimensions parallel with coordinate x-axis are multiplied by $q = \frac{1}{2}$. The circles lying in the profile plane (or in the plane parallel with the profile plane) are projected as circles with true radii without any distortion. The circles lying in the horizontal or frontal planes (or in planes parallel with the horizontal or frontal planes) are projected as ellipses. Conjugated diameters of each ellipse parallel with corresponding coordinate axes are oblique view of a pair of mutually perpendicular diameters of the circle parallel with coordinate axes. The lengths of conjugated diameters parallel with y- and z-axes are equal to the diameter of the corresponding circle. The length of conjugated diameters parallel with x-axis are equal to $2qr$, where r is radius of the corresponding circle.

Oblique projection is useful for hand drawn sketches of many objects with the exception of sphere, because sphere is projected as an ellipse and its interior area. Oblique projection is not used in CAD/CAM systems just due to the distortion of spherical surfaces which could be felt user-unfriendly in case the user is not familiar with properties of oblique projection.

Military perspective

Military perspective is a special type of oblique projection where the plane of projection is horizontal plane $\pi = (x, y)$ and the direction of projection s forms angle 45° with the plane of projection $\angle s\pi = 45°$. Since the plane of projection is identical to the horizontal plane π, x- and y-coordinates of points are projected in true sizes. Moreover, as the angle of direction s is 45°, the z-coordinates are projected in true size, too. This property is obvious from fig. 3.23: $z_A = ||AA_1|| = ||A_1A'||$, where A' is the military perspective of point $A = (x_A, y_A, z_A)$.

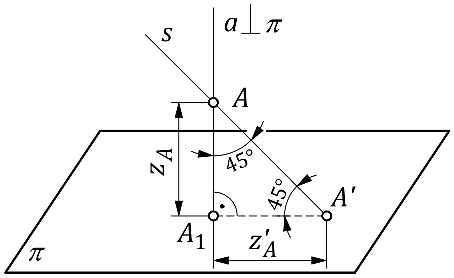


Figure 3.23: Projection of z-coordinates in military perspective

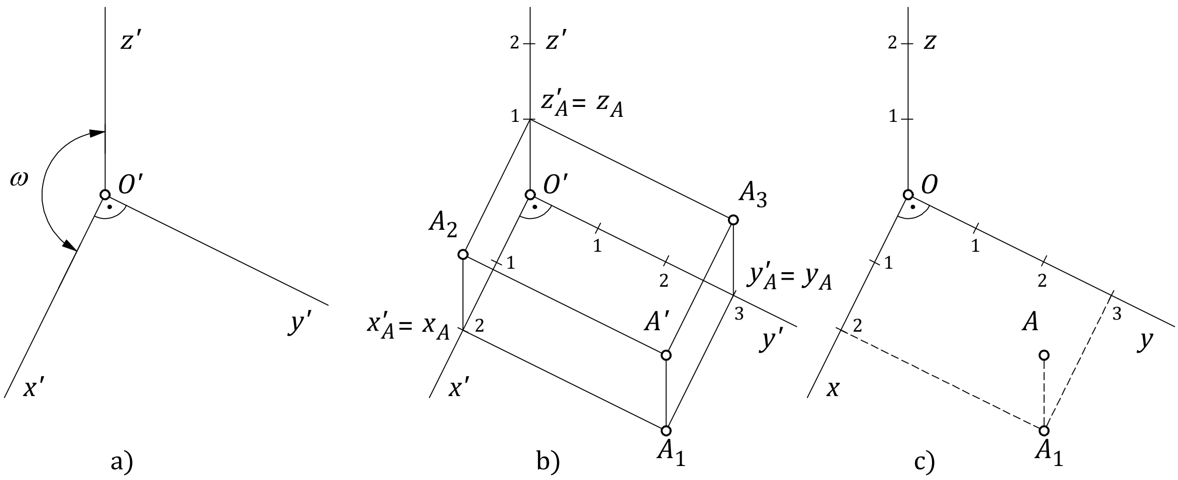


Figure 3.24: Military perspective

The view of coordinate system (O', x', y', z') in military perspective is constructed according to the following rules. The direction of z'-axis is vertical, angle $\omega = \angle x'z'$ is chosen $(\omega \neq 0°, 90°, 180°, 270°)$, right angle formed by x- and y-axes is preserved, i.e. $x' \perp y'$, see fig. 3.24 a). Dimensions in the direction of all three coordinate axes are projected in true size, therefore the coordinate boxes of projected points can be easily drawn, see example in fig. 3.24 b) and its simplification in fig. 3.24 c).

Corner brace given by orthogonal views in fig. 3.9 in military perspective is drawn in fig. 3.25. The dimensions parallel with coordinate axes are projected in true sizes. The circles parallel with the horizontal plane of projection are projected as circles with true radii without any distortion. The circles lying in the frontal or profile planes (or in planes parallel with frontal and profile planes) are projected as ellipses. Since the military perspective belongs to the oblique projection, the properties of conjugated diameters of ellipses are determined by properties of the above mentioned oblique projection.

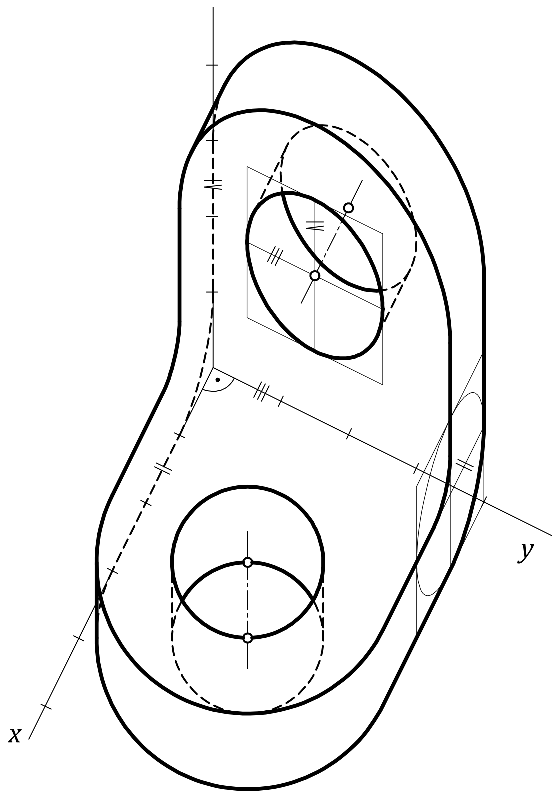


Figure 3.25: Corner brace in military perspective

3.2.4 Construction of parallel projection of a circle

In general, parallel projection of a circle is an ellipse, see properties of parallel projection on page 62. Here, circle $c = (S, r)$ lying in principal or projecting plane is considered only. Two mutually perpendicular diameters of the circle parallel with coordinate axes are projected into the conjugated diameters of the ellipse. Depending on the type of projection, the lengths of the projected figures are as follows.

- **Monge projection** – circles lying in the principal planes are drawn in fig. 3.26.

  The top view of circle $c = (S, r)$ parallel with the horizontal plane of projection π is circle $c_1 = (S_1, r)$. The front view is the straight line segment c_2 parallel with x_{12} of length equal to $2r$, see fig. 3.26 a).

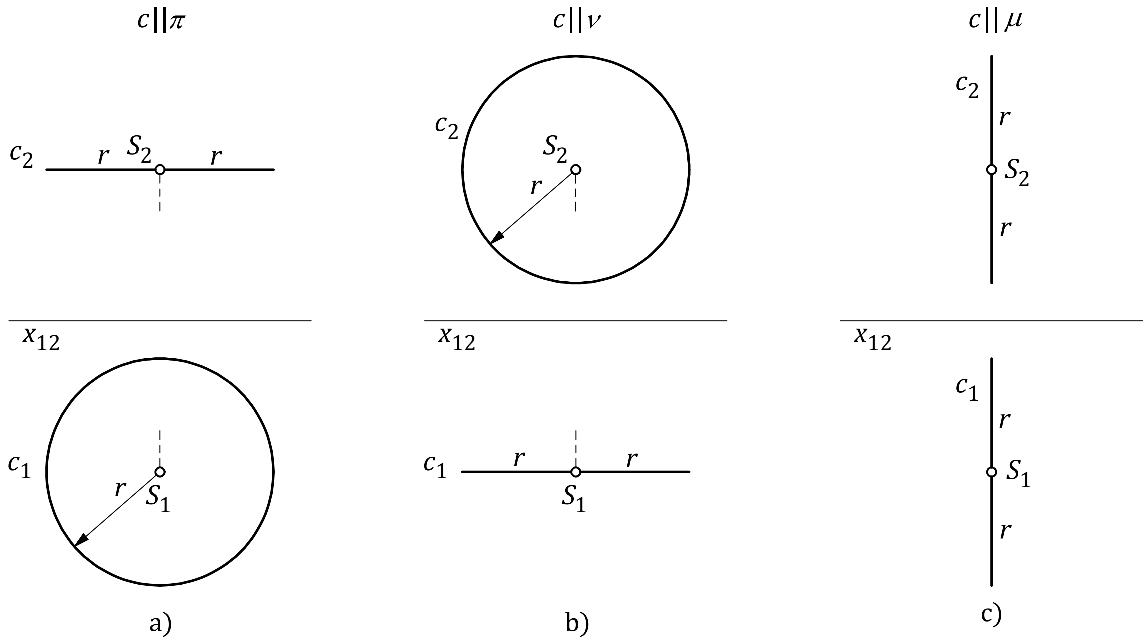


Figure 3.26: Circle in principal planes in Monge projection

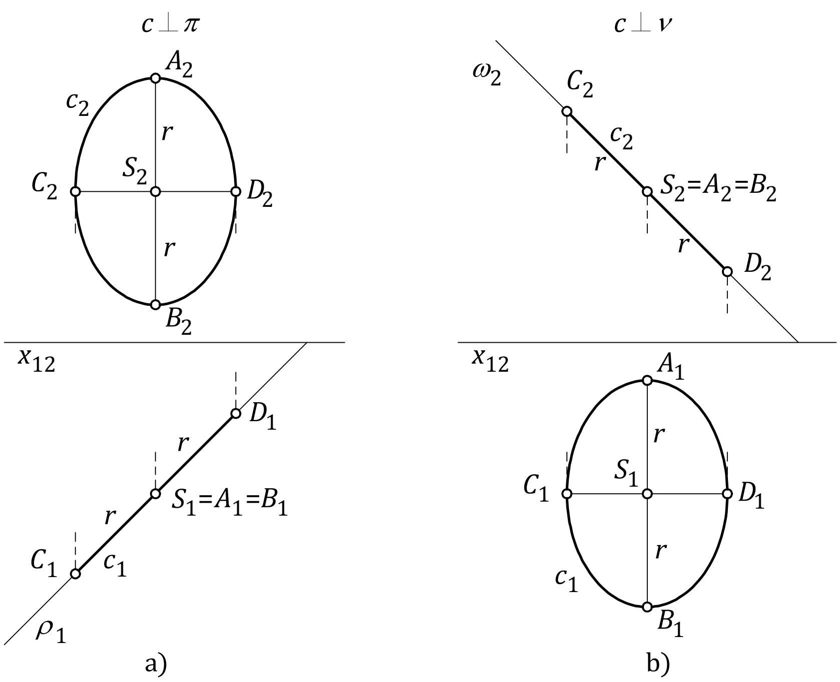


Figure 3.27: Circle in projecting planes in Monge projection

The top view of circle $c = (S, r)$ parallel with the frontal plane of projection ν is the straight line c_1 parallel with x_{12} of length equal to $2r$. The front view is circle $c_2 = (S_2, r)$, see fig. 3.26 b).

Both the front view and the top view of circle $c = (S, r)$ parallel with the profile plane of projection μ are straight line segments perpendicular to x_{12} of length equal to $2r$, see fig. 3.26 c).

Circles lying in the projecting planes $\rho \perp \pi$ and $\omega \perp \nu$ are drawn in fig. 3.27.

The top view of circle $c = (S, r)$ lying in the plane $\rho \perp \pi$ is straight line $C_1D_1 \subset \rho_1$ of length $2r$. The front view is the ellipse determined by major axis $A_2B_2 \perp x_{12}$ of length $2r$ and minor axis $C_2D_2 \perp A_2B_2$. The positions of minor vertices are given by ordinates constructed from top views C_1 and D_1, i.e. $C_1C_2, D_1D_2 \perp x_{12}$, see fig. 3.27 a). This ellipse can be approximated by means of osculation circles (see section 3.2.4) or constructed by means of parallelogram method (see section 3.2.4), for example.

The front view of circle $c = (S, r)$ lying in the plane $\omega \perp \nu$ is straight line $C_2D_2 \subset \omega_2$ of length $2r$. The top view is the ellipse determined by major axis $A_1B_1 \perp x_{12}$ of length $2r$ and minor axis $C_1D_1 \perp A_1B_1$. The positions of minor vertices are given by ordinates constructed from top views C_2 and D_2, i.e. $C_1C_2, D_1D_2 \perp x_{12}$, see fig. 3.27 b). This ellipse can be approximated by osculation circles (see section 3.2.4) or constructed by means of parallelogram method (see section 3.2.4), for example.

- **Axonometry** – conjugated diameters of the ellipse, i.e. the projection of two mutually perpendicular diameters of a circle parallel with coordinate axes are shortened according to the axonometric units. The diameter parallel with the axonometric plane of projection is projected in true size (i.e. the length of major axis of the ellipse is equal to $2r$). The ellipse can be constructed by means of parallelogram method (see section 3.2.4), for example.

- **Isometry** – conjugated diameters of the ellipse, i.e. the projection of two mutually perpendicular diameters of a circle parallel with coordinate axes are shortened by $\sqrt{\frac{2}{3}}$. The diameter parallel with the isometric plane of projection is projected in true size (i.e. the length of the major axis of the ellipse is equal to $2r$). The ellipse can be constructed by means of parallelogram method (see section 3.2.4), for example.

- **Technical isometry** – conjugated diameters of the ellipse, i.e. the projection of two mutually perpendicular diameters of the circle parallel with coordinate axes are projected in true size (i.e. their length is equal to $2r$), the diameter parallel with the isometric plane of projection is elongated by $\sqrt{\frac{3}{2}}$. The ellipse can be constructed by means of parallelogram method (see section 3.2.4), for example.

- **Oblique projection** – conjugated diameters of the ellipse, i.e. the projection of two mutually perpendicular diameters of a circle parallel with $y-$ or $z-$axes are projected in true size. If the diameter is parallel with x-axis, its projected length is equal to $2qr$, where q is the given quotient of oblique projection. The ellipse can be constructed by means of parallelogram method. Circles lying in planes parallel with profile plane μ are projected as circles without any distortion.

- **Military perspective** – conjugated diameters of the ellipse, i.e. the projection of two mutually perpendicular diameters of a circle parallel with all coordinate axes are projected in true size (i.e. their length is equal to $2r$). The ellipse can be constructed by means of parallelogram method. Circles lying in planes parallel with horizontal plane π are projected as circles without any distortion.

Approximation of an ellipse by means of osculation circles

Ellipse is approximated by four osculation circles with point of contact at vertices of the ellipse. Construction of osculation circles of the ellipse can be used when major axis AB and major axis CD of the ellipse are given. The procedure is as follows, see fig. 3.28.

1. Given axes AB, CD, construct rectangle $ASDV$, $S = AB \cap CD$.
2. Draw diagonal AD of rectangle $ASDV$.
3. Construct straight line $p \perp AD$, $V \in p$.
4. Centers of osculation circles $K = p \cap AB$ and $K' = p \cap CD$.
5. Draw osculation circles $k = (K, R = ||KA||)$ and $k' = (K', R' = ||K'D||)$.
6. Using symmetry with respect to the centre S of the ellipse, construct osculation circles $l = (L, R = ||KA||)$ and $l' = (L', R' = ||K'D||)$.
7. Approximate the ellipse by osculation circles in the neighbourhood of vertices and estimate the shape of ellipse in the neighbourhood of intersection of ellipse with the straight line p.

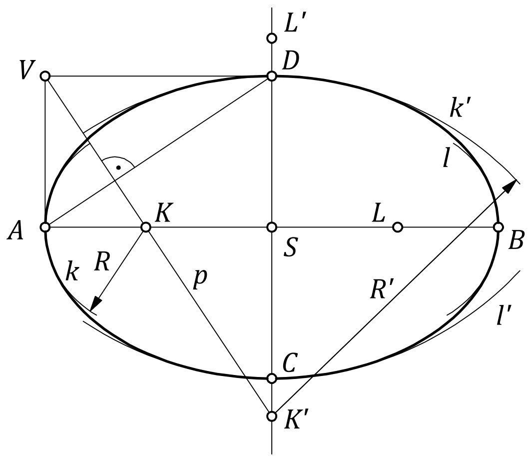


Figure 3.28: Approximation of an ellipse by means of osculation circles

Parallelogram construction of an ellipse

The parallelogram method (partition construction) can be used when conjugated diameters of the ellipse or its tangent parallelogram are given. In fig. 3.29 the parallelogram method (partition construction) is depicted. The procedure of construction is as follows.

1. If conjugated diameters MN and PQ (or the major and minor axes) of the ellipse are given, construct corresponding tangent parallelogram (or tangent rectangle) $EFGH$. If tangent parallelogram $EFGH$ of the ellipse is given, construct corresponding conjugated diameters MN and PQ.

2. Consider sub-parallelogram $MSQE$, for example. Divide the distance between EQ and SQ into the same number of equal parts. In fig. 3.29 left, the segments EQ and SQ are divided into 4 parts, therefore points 1, 2, 3 and 1', 2', 3' are obtained, respectively.
3. Connect point M with dividing points on EQ segment, i.e. draw lines $M1$, $M2$, $M3$.
4. Connect point N with dividing points on SQ segment, i.e. draw lines $N1'$, $N2'$, $N3'$.
5. Points on ellipse lie at intersections of the connecting lines, i.e. $L_1 = M1 \cap N1'$, $L_2 = M2 \cap N2'$ and $L_3 = M3 \cap N3'$.
6. Consider the other sub-parallelograms $SNHQ$, $FPSM$ and $PGNS$. Proceed in a similar way to obtain points along the whole circumference of the ellipse, see fig. 3.29 right.
7. Draw the ellipse as a curve passing through all constructed points.

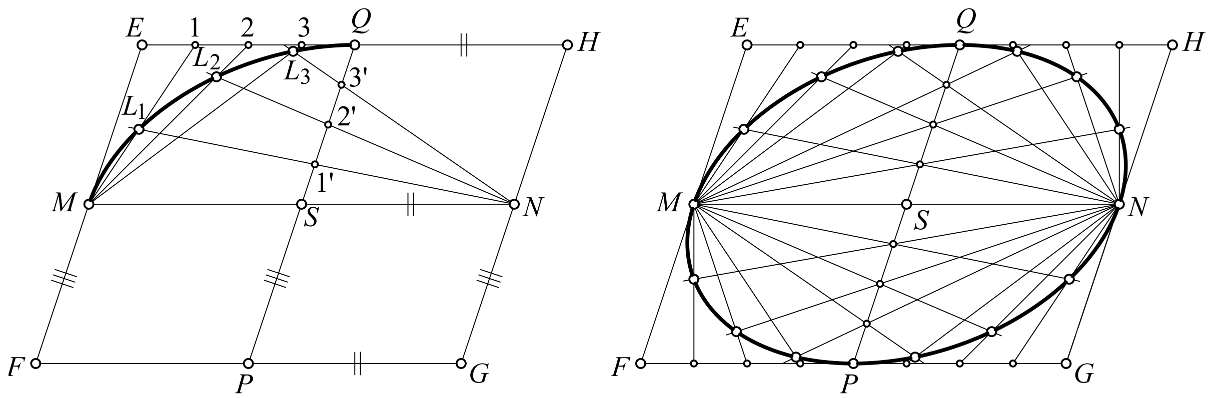


Figure 3.29: Construction of ellipse by means of parallelogram method

Chapter 4

Analytic geometry

Analytic geometry uses algebra to solve problems in geometry. Geometrical figures are represented by algebraic equations and their position is represented analytically by coordinates. A brief overview of basic vector operations is given in this chapter firstly. After that, analytic representations of straight line, conic sections, plane and quadratic surfaces is presented. Geometrical interpretation of analytic expression of individual figures is simultaneously demonstrated in examples.

Note that in following analytic representation, all points are considered in vector form and denoted as vectors, i.e. $\mathbf{A} = (1, 2, 3)$.

4.1 Vectors

Two points $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A}, z_\mathbf{A})$ and $\mathbf{B} = (x_\mathbf{B}, y_\mathbf{B}, z_\mathbf{B})$ define a *directed line segment* $\overrightarrow{\mathbf{AB}}$ extending from point $\mathbf{A}$ to point $\mathbf{B}$. A directed line segment corresponds to *vector* $\mathbf{u}$

$$\mathbf{u} = \overrightarrow{\mathbf{AB}} = (x_\mathbf{B} - x_\mathbf{A}, y_\mathbf{B} - y_\mathbf{A}, z_\mathbf{B} - z_\mathbf{A}) = (u_1, u_2, u_3) \tag{4.1}$$

which extends from point $\mathbf{A}$ to point $\mathbf{B}$. Point $\mathbf{A}$ is often called the *tail* of vector $\mathbf{u}$, point $\mathbf{B}$ is called the *head* of vector $\mathbf{u}$. The *radius vector* of point $\mathbf{A}$ is a vector with the tail at the origin and the head at point $\mathbf{A}$.

Magnitude $||\mathbf{u}||$ of vector $\mathbf{u}$ is equal to the distance of points $\mathbf{A}$ and $\mathbf{B}$

$$||\mathbf{u}|| = ||\mathbf{AB}|| = \sqrt{(x_\mathbf{B} - x_\mathbf{A})^2 + (y_\mathbf{B} - y_\mathbf{A})^2 + (z_\mathbf{B} - z_\mathbf{A})^2} = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are *linearly dependent* if one of the vectors is linear combination of the other

$$\mathbf{u} = k\mathbf{v}, \text{ i.e. } u_1 = kv_1,\ u_2 = kv_2,\ u_3 = kv_3,$$

where k is a real number.

Vectors $\mathbf{u}$ and $\mathbf{v}$ are called *opposite vectors* if $\mathbf{u} = -\mathbf{v}$.

Angle φ formed by vectors $\mathbf{u}$ and $\mathbf{v}$ is angle $0 \leqslant \varphi \leqslant \pi$ formed by oriented straight line segments by means of which the vectors $\mathbf{u}$ and $\mathbf{v}$ are depicted.

Dot product of vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\varphi. \tag{4.2}$$

Dot product is useful for calculation of angle φ formed by two vectors $\mathbf{u}$ and $\mathbf{v}$

$$\cos\varphi = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||} = \frac{u_1v_1 + u_2v_2 + u_3v_3}{||\mathbf{u}|| \cdot ||\mathbf{v}||}. \tag{4.3}$$

Two nonzero vectors are *orthogonal* (perpendicular to each other) if their dot product is equal to zero.

Cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = \mathbf{i}(u_2v_3 - u_3v_2) + \mathbf{j}(u_3v_1 - u_1v_3) + \mathbf{k}(u_1v_2 - u_2v_1).$$

It can be written in a shorthand notation that takes the form of a determinant

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is perpendicular to both vectors $\mathbf{u}$ and $\mathbf{v}$ with the orientation determined by the right-hand rule. Magnitude of cross product given by

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \sin\varphi$$

represents a number equal to the area of a parallelogram, sides of which are given by the vectors $\mathbf{u}$ and $\mathbf{v}$.

Scalar triple product (*mixed product*) of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is given by

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The volume of a parallelepiped whose sides are given by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is equal to the absolute value of the scalar triple product of these vectors.

$$V_{\text{parallelepiped}} = |[\mathbf{u}, \mathbf{v}, \mathbf{w}]|.$$

The volume of a tetrahedron whose sides are given by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is equal to one sixth of absolute value of the scalar triple product of these vectors.

$$V_{\text{tetrahedron}} = \tfrac{1}{6}|[\mathbf{u}, \mathbf{v}, \mathbf{w}]|.$$

4.2 Planar analytic geometry

4.2.1 Straight line in two-dimensional space

A straight line $\mathbf{AB}$ is unambiguously determined by two different points $\mathbf{A} \neq \mathbf{B}$. In E^2, a straight line can be analytically expressed by the following equations: *vector*, *parametric*, *slope*, *intercept* and *general*. Geometrical meaning of all these forms (see below) is obvious from fig. 4.1.

Vector equation

Vector equation of a straight line given by two points $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A})$ and $\mathbf{B} = (x_\mathbf{B}, y_\mathbf{B})$ has the following form

$$\mathbf{P}(t) = \mathbf{A} + \mathbf{u} \cdot t, \;\; t \in R,$$

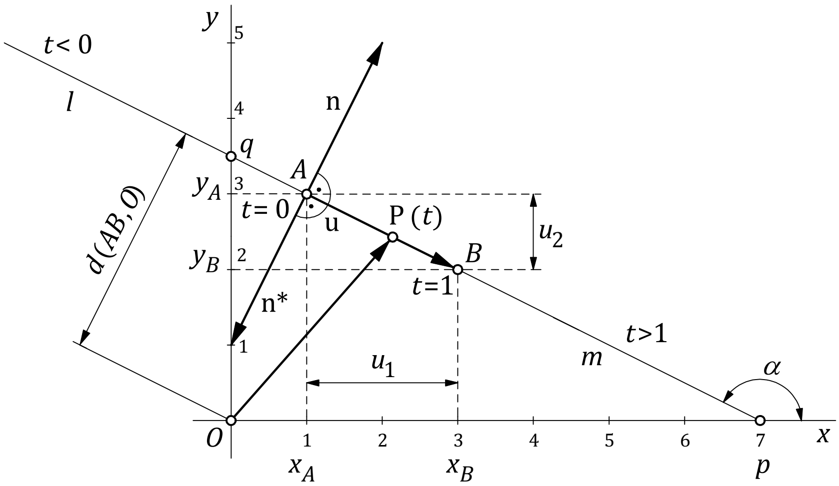


Figure 4.1: Straight line in two-dimensional space

where $\mathbf{P}(t)$ is univariate vector function, t is parameter and $\mathbf{u} = (u_1, u_2)$ is *direction vector* given by eq. (4.1). Coordinates of vector function $\mathbf{P}(t)$ are coordinate functions $x(t)$ and $y(t)$

$$\mathbf{P}(t) = (x(t), y(t)) = (x_{\mathbf{A}} + u_1 t, y_{\mathbf{A}} + u_2 t),\ \ t \in R. \tag{4.4}$$

By substituting $t \in R$ in eq. (4.4), we get any point on straight line $\mathbf{AB}$. In particular, $\mathbf{P}(0) = \mathbf{A}$, $\mathbf{P}(1) = \mathbf{B}$. Points located on half-line l with origin $\mathbf{A}$, $\mathbf{B} \notin l$, are obtained for $t < 0$ and points located on half-line m with origin $\mathbf{B}$, $\mathbf{A} \notin m$, are obtained for $t > 1$. It is possible to express any straight line in E^2 by means of vector equation.

Parametric equations

Parametric equations of straight line given by two points $\mathbf{A} = (x_{\mathbf{A}}, y_{\mathbf{A}})$ and $\mathbf{B} = (x_{\mathbf{B}}, y_{\mathbf{B}})$ are coordinate functions of vector function given by eq. (4.4), i.e

$$\begin{aligned} x(t) &= x_{\mathbf{A}} + u_1 t, \\ y(t) &= y_{\mathbf{A}} + u_2 t,\ \ t \in R, \end{aligned} \tag{4.5}$$

where $\mathbf{u} = (u_1, u_2)$ is direction vector given by eq. (4.1) and t is parameter. By substituting $t \in R$ in eq. (4.5), we get any point on straight line. It is possible to express any straight line in E^2 by means of parametric equations.

Slope equation

Slope equation of straight line given by two points $\mathbf{A} = (x_{\mathbf{A}}, y_{\mathbf{A}})$ and $\mathbf{B} = (x_{\mathbf{B}}, y_{\mathbf{B}})$ is as follows

$$y = kx + q,\ \ x \in R, \tag{4.6}$$

where x is the independent variable, y is the dependent variable, k is the *slope* of straight line given by

$$k = \tan(\alpha) = \frac{y_{\mathbf{B}} - y_{\mathbf{A}}}{x_{\mathbf{B}} - x_{\mathbf{A}}} = \frac{u_2}{u_1},\ \ u_1 \neq 0, \tag{4.7}$$

α is an angle formed by straight line and x-axis of the coordinate system and q is y-intercept (the point where straight line crosses y-axis). For $k = 0$, the straight line is parallel with x-axis. For $k < 0$, the straight line slants downward to the right. For $k > 0$, the straight line slants upward to the right. It is impossible to express straight lines parallel with y-axis by means of slope equation due to the condition $u_1 \neq 0$.

Intercept equation

Intercept equation of straight line given by two points $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A})$ and $\mathbf{B} = (x_\mathbf{B}, y_\mathbf{B})$ has the following form

$$\frac{x}{p} + \frac{y}{q} = 1, \; p \neq 0 \wedge q \neq 0, \tag{4.8}$$

where p is x-intercept (the point where straight line crosses x-axis) and q is y-intercept. A straight line passing through origin of coordinate system cannot be expressed in intercept form because although the intercepts exists ($p = q = 0$), it is impossible to substitute $p = q = 0$ into eq. (4.8).

General equation

General equation of straight line given by two points $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A})$ and $\mathbf{B} = (x_\mathbf{B}, y_\mathbf{B})$ has the following form

$$ax + by + c = 0, \tag{4.9}$$

where a and b are the coordinates of normal vector $\mathbf{n} = (a, b)$ of the straight line. Since $\mathbf{n} \perp \mathbf{u}$, dot product $\mathbf{u} \cdot \mathbf{n}$ given by eq. (4.2) can be applied to obtain the coordinates of two opposite normal vectors

$$\mathbf{n} = (-u_2, u_1) \quad \text{or} \quad \mathbf{n}^* = (u_2, -u_1). \tag{4.10}$$

By substituting eq. (4.10) in eq. (4.9), the value c or c^* can be determined

$$c = u_2 x_\mathbf{A} - u_1 y_\mathbf{A} \quad \text{or} \quad c^* = -u_2 x_\mathbf{A} + u_1 y_\mathbf{A}. \tag{4.11}$$

Geometrical meaning of c follows from

$$c = \frac{d(\mathbf{AB}, \mathbf{O})}{||\mathbf{n}||},$$

where $d(\mathbf{AB}, \mathbf{O})$ is the normal distance of straight line $\mathbf{AB}$ from the origin of coordinate system. It is possible to express any straight line by means of general equation in E^2.

4.2.2 Conic sections

Conic sections are second-order algebraic curves (the sets of roots of polynomial $f(x, y) = 0$) given by the following general equation

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0, \tag{4.12}$$

where $a_{ij} \in R$ and $(a_{11}, a_{22}, a_{12}) \neq 0$. Conic section is called regular if the matrix of coefficients

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

is regular. Otherwise, the conic section is called singular. Circle, ellipse, hyperbola and parabola belong to the regular conic sections. A pair of straight lines (intersecting or identical) or a point belong to the singular conic sections.

If $a_{12} = 0$, the conic section is in axes-aligned position, i.e. the axes of conic section are parallel with the axes of coordinate system. In this case, it is easy to turn the general eq. (4.12) into canonical form by completing the square and determine the type and characteristic features (Cartesian coordinates of centre (or vertex) and length of semiaxes) of conic section.

Conic sections can be obtained as intersection curves of a plane and cone of revolution. If the vertex of the cone does not lie at infinity and section plane does not pass through the vertex of the cone, the intersection curve is a regular conic section, see fig. 4.2. Otherwise, a singular conic section is obtained. Depending on the angle β formed by section plane and revolution axis of the cone, the individual types of conic sections are obtained: circle for $\beta = 90°$, ellipse for $\beta > \alpha$, parabola for $\beta = \alpha$ and hyperbola for $\beta < \alpha$, where α is a half of vertex angle of the cone.

In the following, a brief review of geometric definitions, canonical equations in general form and graphical examples of regular conic sections is given.

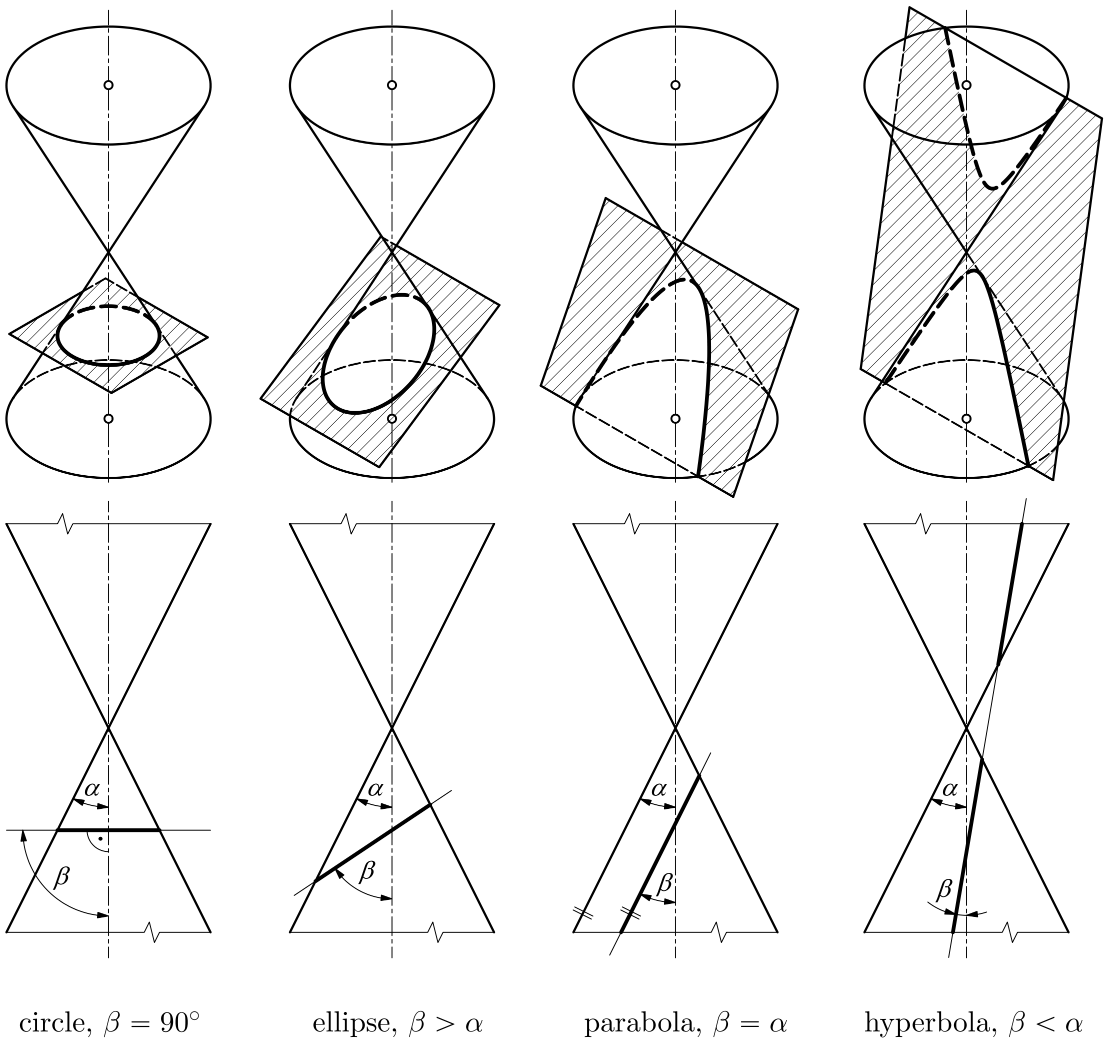


Figure 4.2: Conic sections

Circle

A circle is a set of points in a plane whose distance r (radius) from a fixed point S (centre) is constant, see fig. 4.3. If the centre is given by Cartesian coordinates in (x, y) plane $\mathbf{S} = (m, n)$, the equation of the circle is

$$(x - m)^2 + (y - n)^2 = r^2.$$

The lower and upper semicircle is given by

$$y = n + \sqrt{r^2 - (x - m)^2} \quad \text{and} \quad y = n - \sqrt{r^2 - (x - m)^2},$$

respectively. The right and left semicircle is given by

$$x = m + \sqrt{r^2 - (y - n)^2} \quad \text{and} \quad x = m - \sqrt{r^2 - (y - n)^2},$$

respectively.

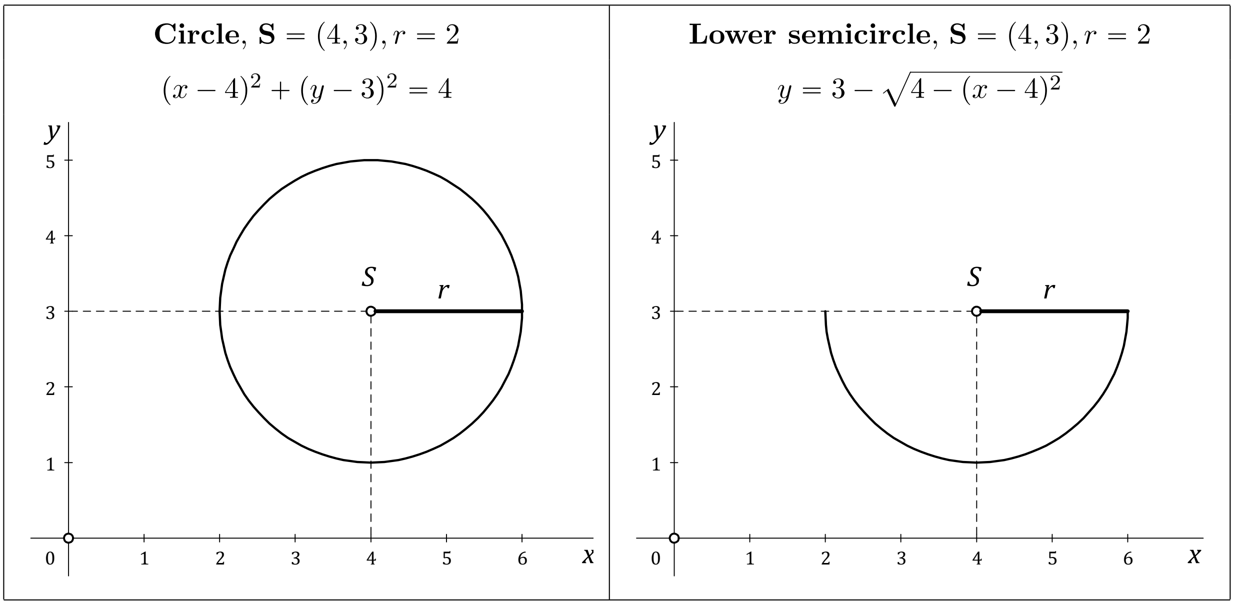


Figure 4.3: Circle

Ellipse

An ellipse is a set of points in a plane the sum of whose distances from two fixed points $\mathbf{F}_1$ and $\mathbf{F}_2$ (the foci) is equal to the length of major axis of the ellipse, see fig. 4.4.

The equation of an ellipse with axes of symmetry parallel with the axes of coordinate system is given by

$$\frac{(x - m)^2}{a^2} + \frac{(y - n)^2}{b^2} = 1, \tag{4.13}$$

where $\mathbf{S} = (m, n)$ is the centre of the ellipse, $a \parallel x$ and $b \parallel y$ are the semiaxes of the ellipse. Note that in eq. (4.13), semiaxis a is always supposed to be parallel with x-axis and semiaxis b parallel with y-axis, no matter which semiaxis is longer.

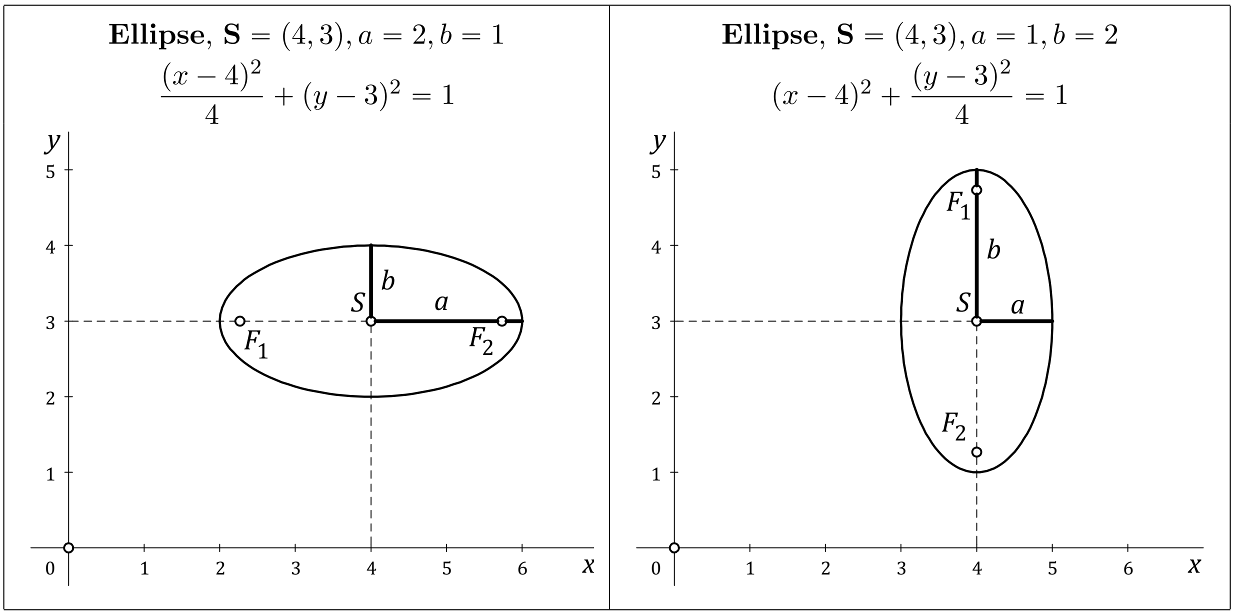


Figure 4.4: Ellipse

Hyperbola

A hyperbola is a set of points in a plane the difference of whose distances from two fixed points $\mathbf{F}_1$ and $\mathbf{F}_2$ (the foci) is equal to the length of the major axis of the hyperbola.

If the axes of symmetry of a hyperbola are parallel with the axes of coordinate system, see fig. 4.5, the equation of hyperbola is

$$\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} = 1, \ \mathbf{F}_1\mathbf{F}_2 \parallel x \tag{4.14}$$

or

$$-\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} = 1, \ \mathbf{F}_1\mathbf{F}_2 \parallel y, \tag{4.15}$$

where $\mathbf{S} = (m, n)$ is the centre of the hyperbola, $a \parallel x$ and $b \parallel y$ are the semiaxes of the hyperbola.

Note that in eq. (4.14) and eq. (4.15), semiaxis a is always supposed to be parallel with x-axis and semiaxis b parallel with y-axis, no matter which semiaxis is longer.

If axes of hyperbola are of the same length $a = b$ and the angle φ formed by $\mathbf{F}_1\mathbf{F}_2$ and x-axis is 45° or 135°, i.e. asymptotes of hyperbola are parallel with the axes of coordinate system, see fig. 4.6, the equation of hyperbola is

$$y - n = \frac{k}{x-m}, \ \varphi = 45^\circ,$$

or

$$y - n = -\frac{k}{x-m}, \ \varphi = 135^\circ,$$

where $k = \frac{1}{2}a^2$.

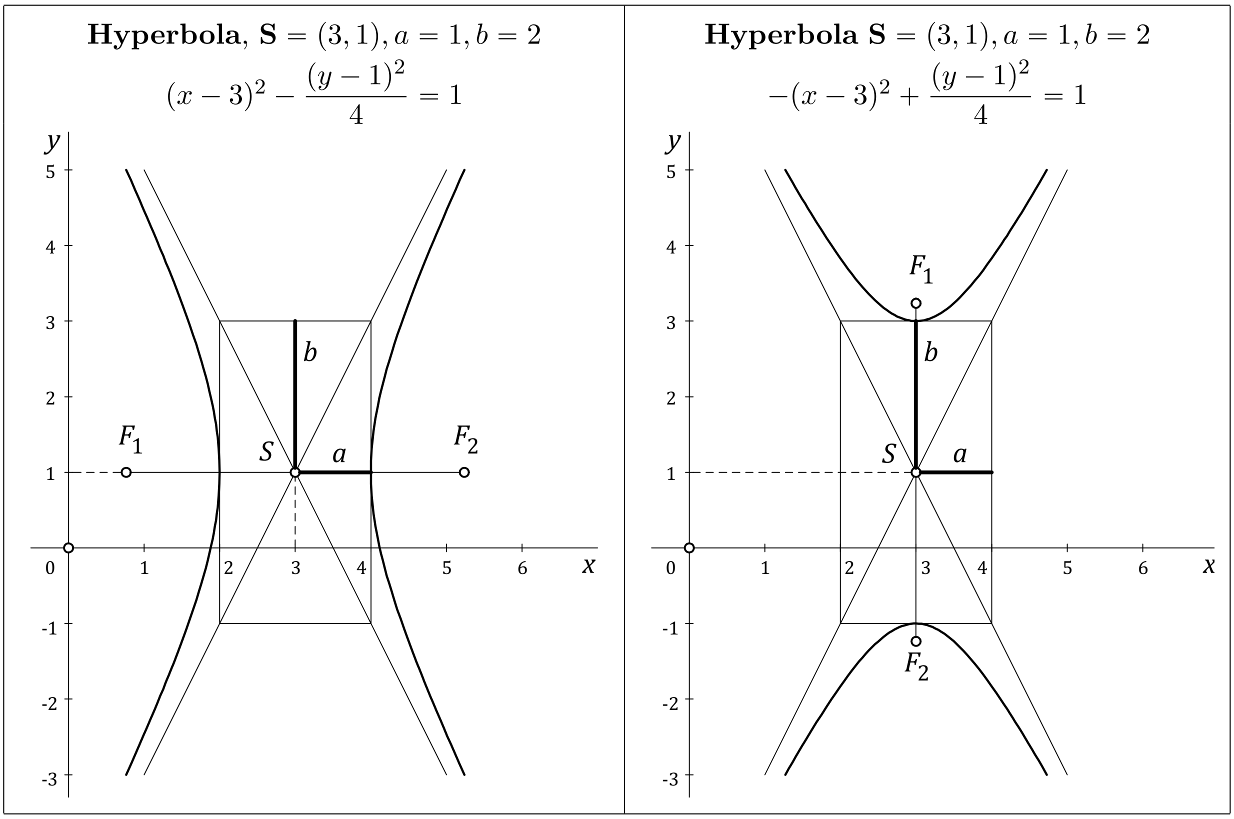


Figure 4.5: Hyperbola (axes of symmetry parallel with the axes of coordinate system)

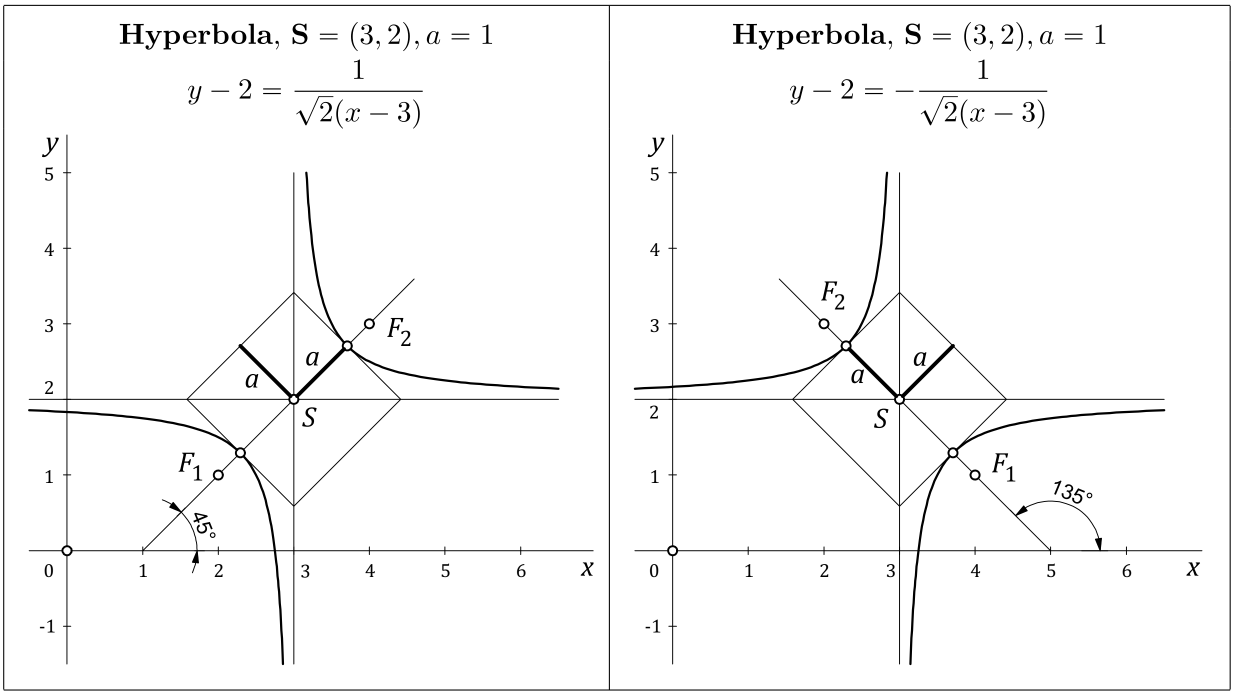


Figure 4.6: Hyperbola ($a = b$, asymptotes parallel with axes of the coordinate system)

Parabola

A parabola is a set of points in a plane that are equidistant from a fixed point **F** (focus) and a fixed straight line d (directrix), see fig. 4.7. The point halfway between the focus and the directrix is called vertex **V** od parabola. The equation of parabola with the axis of symmetry parallel with an axis of coordinate system is

$$(x-m)^2 = 2p(y-n),\ \mathbf{VF} \parallel +y,$$
$$(x-m)^2 = -2p(y-n),\ \mathbf{VF} \parallel -y,$$
$$(y-n)^2 = 2p(x-m),\ \mathbf{VF} \parallel +x,$$
$$(y-n)^2 = -2p(x-m),\ \mathbf{VF} \parallel -x,$$

where $\mathbf{V} = (m, n)$ and p is the distance between the focus and directrix.

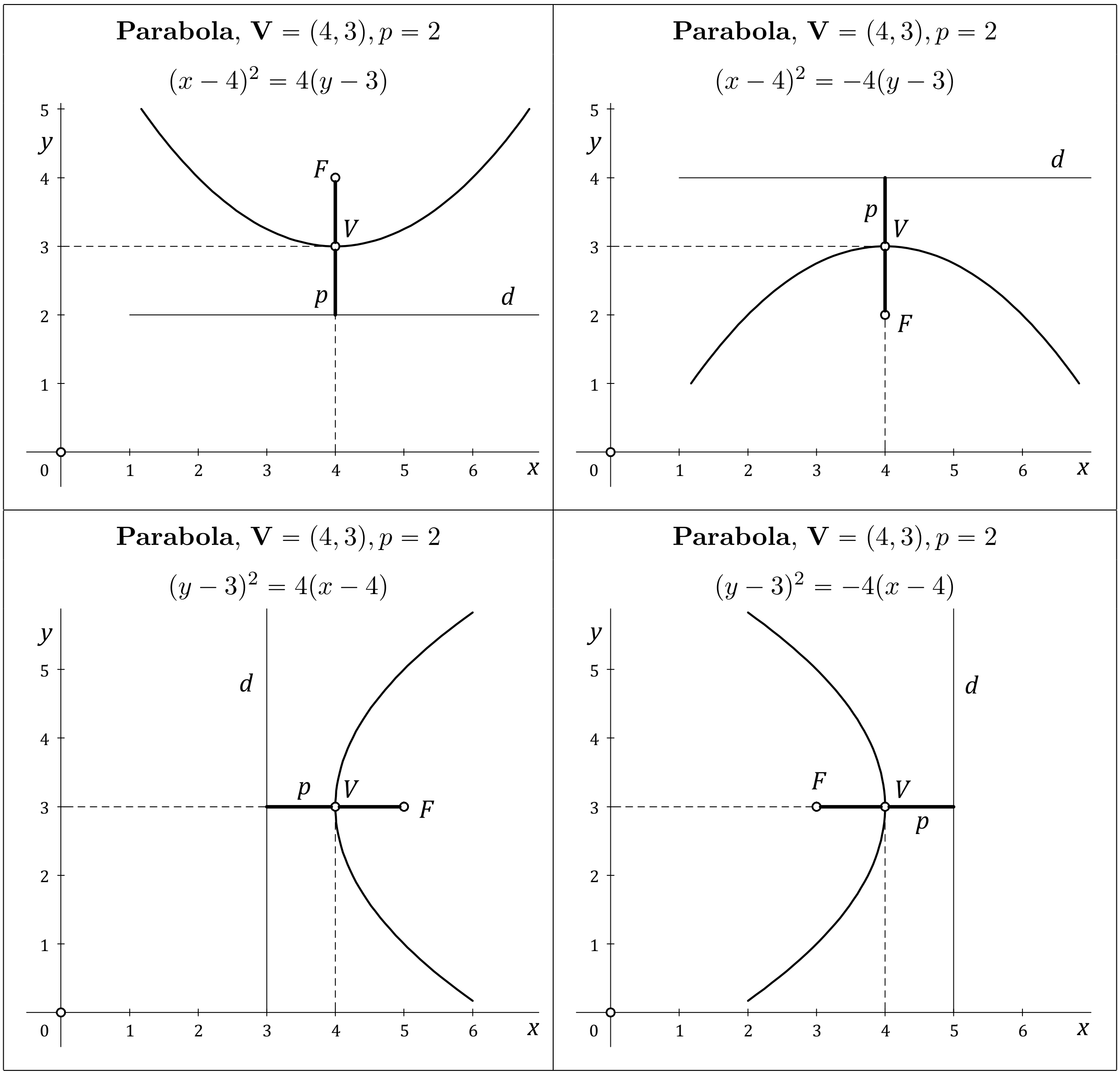


Figure 4.7: Parabola

4.3 Spatial analytic geometry

4.3.1 Straight line in three-dimensional space

In E^3, a straight line can be analytically expressed by a vector equation or parametric equations. Vector equation of straight line determined by two points $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A}, z_\mathbf{A})$ and $\mathbf{B} = (x_\mathbf{B}, y_\mathbf{B}, z_\mathbf{B})$ is given by

$$\mathbf{P}(t) = \mathbf{A} + \mathbf{u} \cdot t, \ \ t \in R,$$

where $\mathbf{P}(t)$ is univariate vector function, $\mathbf{u}$ is *direction vector* of straight line given by eq. (4.1) and t is parameter. Function value of $\mathbf{P}(t)$ is radius vector of a point located on the straight line. Coordinates of $\mathbf{P}(t)$ are univariate *coordinate functions* $x(t)$, $y(t)$ and $z(t)$

$$\mathbf{P}(t) = (x(t), y(t), z(t)) = (x_\mathbf{A} + u_1 \cdot t, y_\mathbf{A} + u_2 \cdot t, z_\mathbf{A} + u_3 \cdot t)\,, \ \ t \in R. \tag{4.16}$$

Parametric equations of a straight line are given by coordinate functions

$$\begin{aligned} x(t) &= x_\mathbf{A} + u_1 \cdot t, \\ y(t) &= y_\mathbf{A} + u_2 \cdot t, \\ z(t) &= z_\mathbf{A} + u_3 \cdot t, \ \ t \in R. \end{aligned} \tag{4.17}$$

After substitution $t = 0$ in eq. (4.16) or in eq. (4.17), we obtain Cartesian coordinates of point $\mathbf{A}$. i.e. $\mathbf{P}(0) = \mathbf{A}$. Similarly, $\mathbf{P}(1) = \mathbf{B}$. For $t < 0$ we obtain points on half-line l with origin $\mathbf{A}$, $\mathbf{B} \notin l$; for $t > 1$ we obtain points on half-line m with origin $\mathbf{B}$, $\mathbf{A} \notin m$, see example in fig. 4.8.

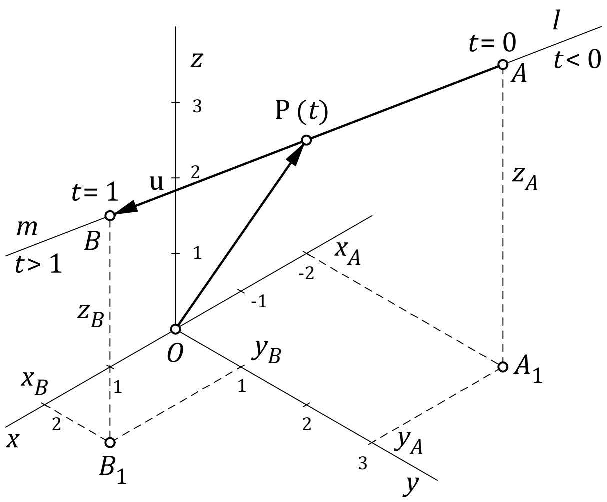


Figure 4.8: Straight line in three-dimensional space

4.3.2 Plane

A plane is unambiguously determined by three noncollinear points in E^3. In this section, the following forms of analytic expression of a plane are mentioned: vector equation, parametric equations, general equation and intercept equation.

Vector equation of a plane

Vector equation of a plane determined by three points $\mathbf{A} = (x_{\mathbf{A}}, y_{\mathbf{A}}, z_{\mathbf{A}})$, $\mathbf{B} = (x_{\mathbf{B}}, y_{\mathbf{B}}, z_{\mathbf{B}})$ and $\mathbf{C} = (x_{\mathbf{C}}, y_{\mathbf{C}}, z_{\mathbf{C}})$ is given by

$$\mathbf{P}(s,t) = \mathbf{A} + \mathbf{u} \cdot s + \mathbf{v} \cdot t, \ (s,t) \in R^2,$$

where $\mathbf{P}(s,t)$ is bivariate vector function, $\mathbf{u} = \overrightarrow{\mathbf{AB}}$ and $\mathbf{v} = \overrightarrow{\mathbf{AC}}$ are linearly independent *direction vectors* and t and s are parameters. Function value of $\mathbf{P}(s,t)$ is radius vector of a point located on the plane. Coordinates of $\mathbf{P}(s,t)$ are bivariate functions $x(s,t)$, $y(s,t)$ and $z(s,t)$.

$$\begin{aligned} \mathbf{P}(s,t) &= (x(s,t), y(s,t), z(s,t)) = \\ &= (x_{\mathbf{A}} + u_1 \cdot s + v_1 \cdot t, y_{\mathbf{A}} + u_2 \cdot s + v_2 \cdot t, z_{\mathbf{A}} + u_3 \cdot s + v_3 \cdot t), \ (x,t) \in R^2. \end{aligned} \tag{4.18}$$

It is possible to express any plane in E^3 by means of vector equation.

Parametric equations of a plane

Parametric equations are given by coordinate functions

$$\begin{aligned} x(s,t) &= x_{\mathbf{A}} + u_1 \cdot s + v_1 \cdot t, \\ y(s,t) &= y_{\mathbf{A}} + u_2 \cdot s + v_2 \cdot t, \\ z(s,t) &= z_{\mathbf{A}} + u_3 \cdot s + v_3 \cdot t, \ (s,t) \in R^2. \end{aligned} \tag{4.19}$$

After substitution $(s,t) = (0,0)$ in eq. (4.18) or in eq. (4.19), we obtain Cartesian coordinates of point $\mathbf{A}$, i.e. $\mathbf{P}(0,0) = \mathbf{A}$. Similarly, $\mathbf{P}(0,1) = \mathbf{B}$ and $\mathbf{P}(1,0) = \mathbf{C}$, see example in fig. 4.9. Here, the plane given by points $\mathbf{A} = (2,1,3)$, $\mathbf{B} = (4,2,1)$ and $\mathbf{C} = (1,4,2)$ is drawn in technical isometry. The plane is represented by intersecting lines (traces) with coordinate planes (x,y), (y,z) and (x,z) in the first octant of the space. Any point located on the plane can be obtained by substitution of suitable pair of parameter values. For example, point $\mathbf{D} = (3,5,0)$ whose coordinates are obtained as function value $\mathbf{P}(1,1)$ is depicted in fig. 4.9. It is possible to express any plane in E^3 by means of parametric equations.

General equation of a plane

General equation of a plane is given by

$$ax + by + cz + d = 0, \tag{4.20}$$

where

$$\mathbf{n} = (a,b,c) = \mathbf{u} \times \mathbf{v} \tag{4.21}$$

is *normal vector* of the plane, the distance between the plane and origin $\mathbf{O}$ is given by $\frac{d}{\|\mathbf{n}\|}$ and x, y and z are variables. It is possible to express any plane in E^3 by means of general equation.

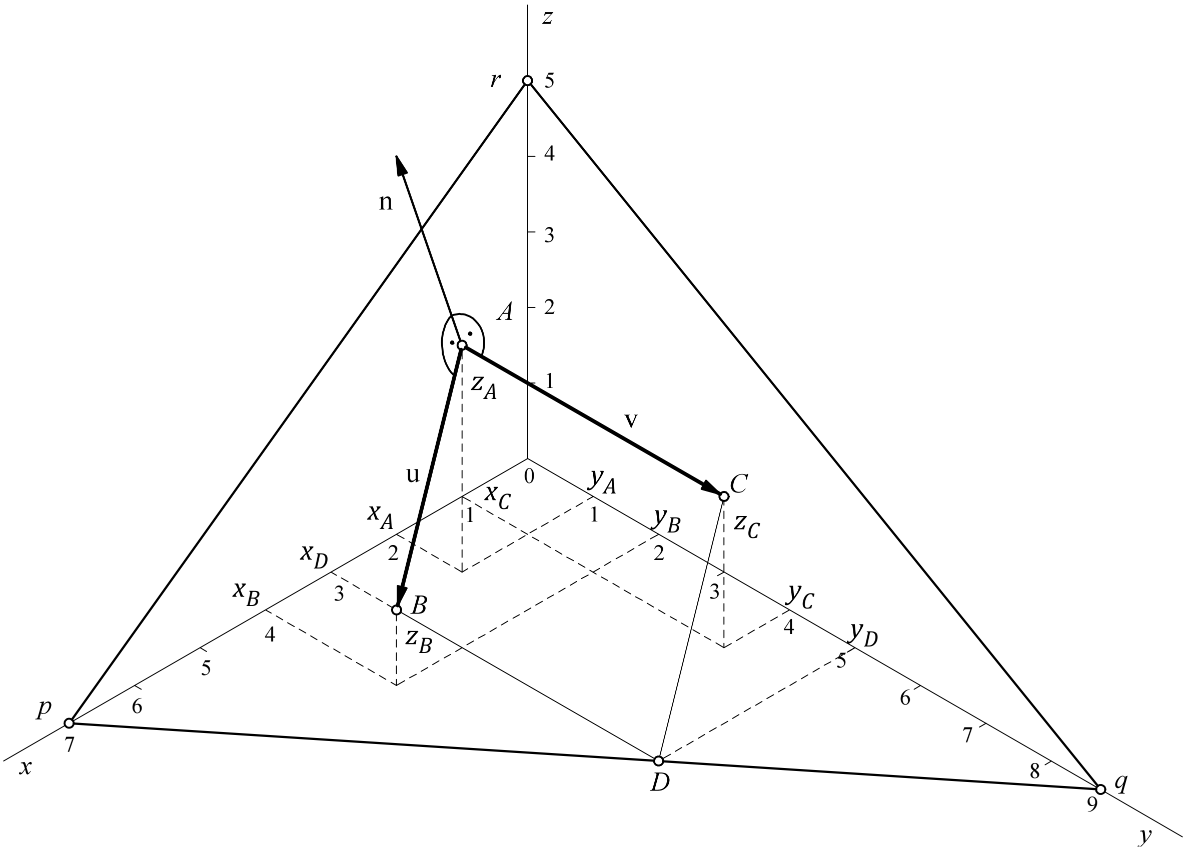


Figure 4.9: Plane in three-dimensional space

Intercept equation of a plane

Intercept equation of a plane is given by

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1, \tag{4.22}$$

where p, q and r is x-intercept, y-intercept and z-intercept of the plane in the given order, x, y and z are variables. Intercept equation is very useful for plane drawing by means of traces, see fig. 4.9. It is impossible to express a plane passing through origin by means of intercept equation, because $p = q = r = 0$.

4.4 Mutual relationship of points, lines and planes

From the point of view of applications, it is very important to know mutual relationship of geometrical figures in two or three-dimensional space, for example, to be able to detect unacceptable collisions of the figures analysed. In the following of this section, a brief list of rules based on geometrical properties of points, lines and planes which can be used to recognize the relationship among these figures is given.

Two points

Consider two points $\mathbf{A} = (x_\mathbf{A}, y_\mathbf{A}, z_\mathbf{A})$ and $\mathbf{B} = (x_\mathbf{B}, y_\mathbf{B}, z_\mathbf{B})$. These points can be

- *Identical:* $\mathbf{A} = \mathbf{B}$ if their corresponding coordinates are equal

$$\mathbf{A} = \mathbf{B} \;\Leftrightarrow\; x_{\mathbf{A}} = x_{\mathbf{B}} \;\wedge\; y_{\mathbf{A}} = y_{\mathbf{B}} \;\wedge\; z_{\mathbf{A}} = z_{\mathbf{B}}.$$

- *Different:* $\mathbf{A} \neq \mathbf{B}$ if at least one of the following inequalities is valid

$$x_{\mathbf{A}} \neq x_{\mathbf{B}},\; y_{\mathbf{A}} \neq y_{\mathbf{B}},\; z_{\mathbf{A}} \neq z_{\mathbf{B}}.$$

Point and line

Consider point $\mathbf{M} = (x_{\mathbf{M}}, y_{\mathbf{M}}, z_{\mathbf{M}})$ and straight line m: $\mathbf{P}(t) = \mathbf{A} + \mathbf{u} \cdot t$. Mutual position of these figures can be as follows.

- *Incidence:* $\mathbf{M} \in m$, if coordinates of point $\mathbf{M}$ satisfy the equation of line m

$$\mathbf{M} \in m \;\Leftrightarrow\; x_{\mathbf{M}} = x_{\mathbf{A}} + u_1 \cdot t \;\wedge\; y_{\mathbf{M}} = y_{\mathbf{A}} + u_2 \cdot t \;\wedge\; z_{\mathbf{M}} = z_{\mathbf{A}} + u_3 \cdot t.$$

  We can say that line m is incident to point $\mathbf{M}$ or point $\mathbf{M}$ belongs to line m or line m passes through point $\mathbf{M}$.

- *No incidence:* $\mathbf{M} \notin m$ if coordinates of point $\mathbf{M}$ do not satisfy the equation of line m.

The distance $d(\mathbf{M}, m)$ between point $\mathbf{M}$ and line m is given by

$$d(\mathbf{M}, m) = \frac{||\mathbf{u} \times \overrightarrow{\mathbf{AM}}||}{||\mathbf{u}||}.$$

Two lines

Consider two straight lines m: $\mathbf{P}(t) = \mathbf{A} + \mathbf{u} \cdot t$ and n: $\mathbf{R}(s) = \mathbf{B} + \mathbf{v} \cdot s$. These lines can be

- *Identical:* $m = n$ if their direction vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent and $\mathbf{A} \in n$ (or $\mathbf{B} \in m$).
- **Parallel:** $m \parallel n$ if their direction vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent and $\mathbf{A} \notin n$ (or $\mathbf{B} \notin m$).
- *Intersecting:* if their direction vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and $[\mathbf{u}, \mathbf{v}, \mathbf{AB}] = 0$.
- *Skew:* if their direction vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and $[\mathbf{u}, \mathbf{v}, \mathbf{AB}] \neq 0$.

The angle formed by lines m and n is given by the angle of their direction vectors eq. (4.3).

Point and plane

Consider point $\mathbf{M} = (x_{\mathbf{M}}, y_{\mathbf{M}}, z_{\mathbf{M}})$ and plane ρ: $\mathbf{P}(s, t) = \mathbf{A} + \mathbf{u} \cdot s + \mathbf{v} \cdot t$. Mutual position of these figures can be as follows.

- *Incidence:* $\mathbf{M} \in \rho$ if coordinates of point $\mathbf{M}$ satisfy the equation of plane ρ

$$\mathbf{M} \in \rho \;\Leftrightarrow\; x_{\mathbf{M}} = x_{\mathbf{A}} + u_1 \cdot s + v_1 \cdot t \;\wedge\; y_{\mathbf{M}} = y_{\mathbf{A}} + u_2 \cdot s + v_2 \cdot t \;\wedge\; z_{\mathbf{M}} = z_{\mathbf{A}} + u_3 \cdot s + v_3 \cdot t.$$

- *No incidence:* $\mathbf{M} \notin \rho$ if coordinates of point $\mathbf{M}$ do not satisfy the equation of plane ρ.

The distance $d(\mathbf{M}, \rho)$ between point $\mathbf{M}$ and plane ρ is given by

$$d(\mathbf{M}, \rho) = \frac{|\mathbf{n} \cdot \overrightarrow{\mathbf{AM}}|}{||\mathbf{n}||},$$

where $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ is normal vector of plane ρ. If the plane is expressed by general equation, it is possible to calculate the distance $d(\mathbf{M}, \rho)$ according to the formula

$$d(\mathbf{M}, \rho) = \frac{|ax_{\mathbf{M}} + by_{\mathbf{M}} + cz_{\mathbf{M}} + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Line and plane

Consider line m: $\mathbf{P}(t) = \mathbf{A} + \mathbf{u} \cdot t$ and plane ρ with normal vector $\mathbf{n}$. Mutual position of these figures can be as follows.

- *Incidence:* $m \subset \rho$ if vectors $\mathbf{u}$ and $\mathbf{n}$ are perpendicular $\mathbf{u} \perp \mathbf{n}$ and $\mathbf{A} \in \rho$.
- *Parallel:* $m \parallel \rho$ if vectors $\mathbf{u}$ and $\mathbf{n}$ are perpendicular $\mathbf{u} \perp \mathbf{n}$ and $\mathbf{A} \notin \rho$.
- *Intersecting:* if vectors $\mathbf{u}$ and $\mathbf{n}$ are not perpendicular $\mathbf{u} \not\perp \mathbf{n}$.

The angle formed by line m and plane ρ is given by the angle formed by line m and its orthogonal projection onto the plane ρ.

Two planes

Consider plane ρ with normal vector $\mathbf{n}$ and plane σ with normal vector $\mathbf{m}$. Mutual position of these figures can be as follows.

- *Identical:* $\rho = \sigma$ if vectors $\mathbf{n}$ and $\mathbf{m}$ are linearly dependent and $\mathbf{A} \in \sigma$ (or $\mathbf{B} \in \rho$).
- *Parallel:* $\rho \parallel \sigma$ if vectors $\mathbf{n}$ and $\mathbf{m}$ are linearly dependent and $\mathbf{A} \notin \sigma$ (or $\mathbf{B} \notin \rho$).
- *Intersecting:* if vectors $\mathbf{n}$ and $\mathbf{m}$ are linearly independent.

The angle formed by planes ρ and σ is given by the angle formed by their normal vectors.

4.5 Quadratic surfaces

Quadratic surfaces (quadrics) are second-order algebraic surfaces (the sets of roots of polynomial $f(x, y, z) = 0$) given by the following general equation

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2(a_{12}xy + a_{13}xz + a_{23}yz) + 2(a_{14}x + a_{24}y + a_{34}z) + a_{44} = 0, \quad (4.23)$$

where $a_{ij} \in R$ and $(a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23} \neq 0$. Quadratic surface is called regular if the matrix of coefficients

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

is regular. Otherwise, the quadratic surface is called singular. Sphere, ellipsoid, hyperboloid and paraboloid belong to the regular quadratic surfaces, cylinder, cone and a par of planes belong to the singular quadratic surfaces. Quadratic surface intersects every plane in conic section.

If $(a_{12}, a_{13}, a_{23}) = 0$, the quadratic surface is in axes-aligned position, i.e. the axes of quadratic surface are parallel with the axes of coordinate system. Similarly to the case of conic sections, it is easy to turn the general eq. (4.23) into canonical form by completing the square and determine the type and characteristic features of quadratic surfaces.

In the following, a brief review of canonical equations and graphical examples of sphere, ellipsoid, hyperboloid, paraboloid, cone and cylinder in axes-aligned position is given.

4.5.1 Sphere

The equation of a sphere given by centre $\mathbf{S} = (m, n, p)$ and radius r, has the following form

$$(x - m)^2 + (y - n)^2 + (z - p)^2 = r^2.$$

Intersection of a sphere and planes passing through the centre **S** parallel with the coordinate planes are principal circles with radius r, see fig. 4.10, where a sphere with centre at origin is drawn in top, front, profile and isometric views.

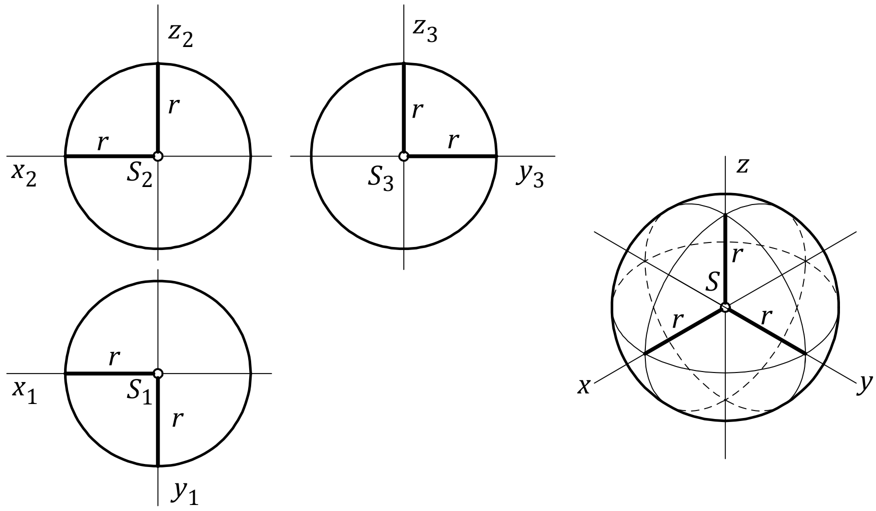


Figure 4.10: Sphere

4.5.2 Ellipsoid

The equation of an ellipsoid given by centre $\mathbf{S} = (m, n, p)$ and semiaxes $a \parallel x$, $b \parallel y$ and $c \parallel z$ is

$$\frac{(x - m)^2}{a^2} + \frac{(y - n)^2}{b^2} + \frac{(z - p)^2}{c^2} = 1.$$

There are the following types of ellipsoid.

- $a \neq b \neq c$: the ellipsoid is called *three-axial*. The intersection of three-axial ellipsoid and planes passing through the centre **S** parallel with the coordinate planes are ellipses, see fig. 4.11 a).

- $a = b$, $b = c$ or $a = c$: the ellipsoid is called *ellipsoid of revolution* with axis of revolution parallel with z-, x- or y-axis in the given order. The intersection of ellipsoid of revolution and plane passing through the centre **S** perpendicular to the axis of revolution is a circle. Example of ellipsoid of revolution is given in fig. 4.11 b).

- $a = b = c$: the ellipsoid becomes a sphere.

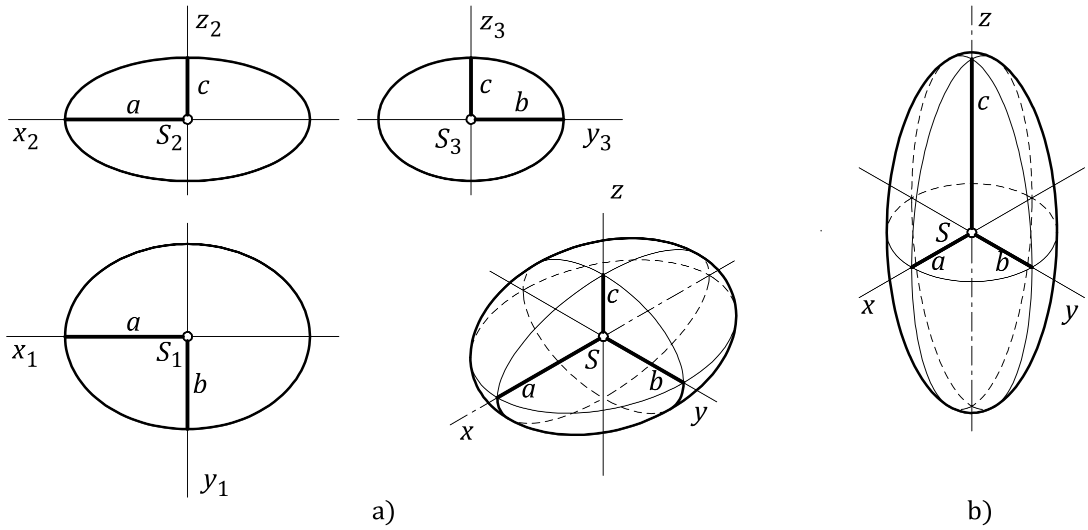


Figure 4.11: Three-axial ellipsoid and ellipsoid of revolution

4.5.3 Hyperboloid

A *one-sheeted hyperboloid* (*hyperboloid of one sheet*) is given by centre $\mathbf{S} = (m, n, p)$ and semiaxes $a \parallel x$, $b \parallel y$ and $c \parallel z$. If the axis of a one-sheeted hyperboloid is parallel with x, y or z-axis, the one-sheeted hyperboloid has the following equation

$$-\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} + \frac{(z-p)^2}{c^2} = 1,$$
$$\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} + \frac{(z-p)^2}{c^2} = 1,$$
$$\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} - \frac{(z-p)^2}{c^2} = 1,$$

see examples in fig. 4.12, fig. 4.13 a) or fig. 4.13 b) in the given order.

There are the following types of one-sheeted hyperboloid.

- $a \neq b \neq c$: the one-sheeted hyperboloid is called *one-sheeted elliptic hyperboloid*. Depending on position of axis of hyperboloid, intersections of one-sheeted hyperboloid and planes passing through the centre **S** parallel with the coordinate planes are two hyperbolas and one ellipse, see example in fig. 4.12.

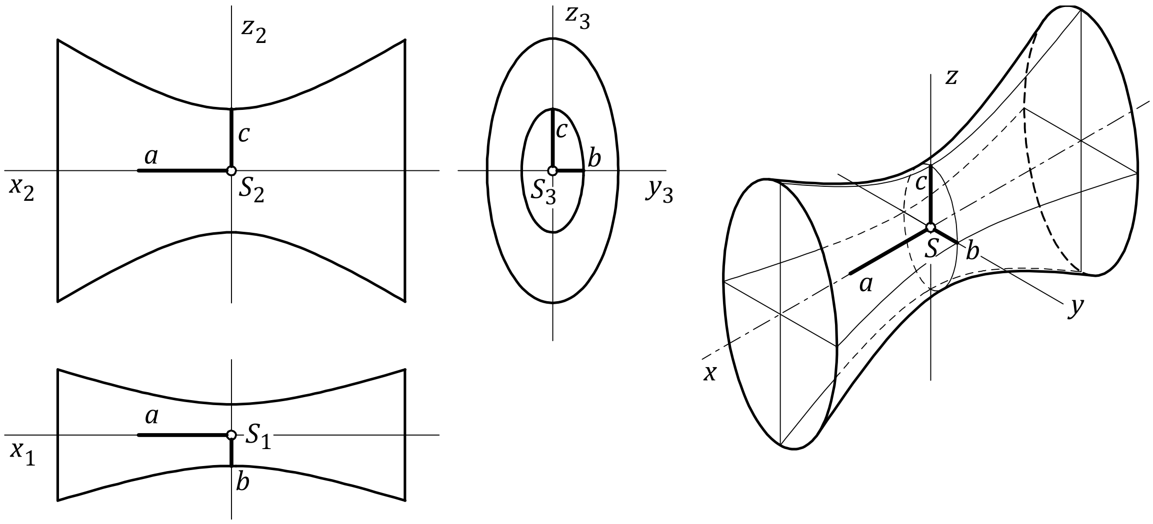


Figure 4.12: One-sheeted elliptic hyperboloid

- $a = b$, $b = c$ or $a = c$: the one-sheeted hyperboloid is called *one-sheeted hyperboloid of revolution* with axis of revolution parallel with z, x or y-axis in the given order. Example of one-sheeted hyperboloid of revolution with $a = b$ is drawn in fig. 4.13 c). One-sheeted hyperboloid of revolution can be generated by rotating a hyperbola about bisector of line $\mathbf{F}_1\mathbf{F}_2$ or by rotating a skew straight line about the line $\mathbf{F}_1\mathbf{F}_2$, see section 4.2.2.

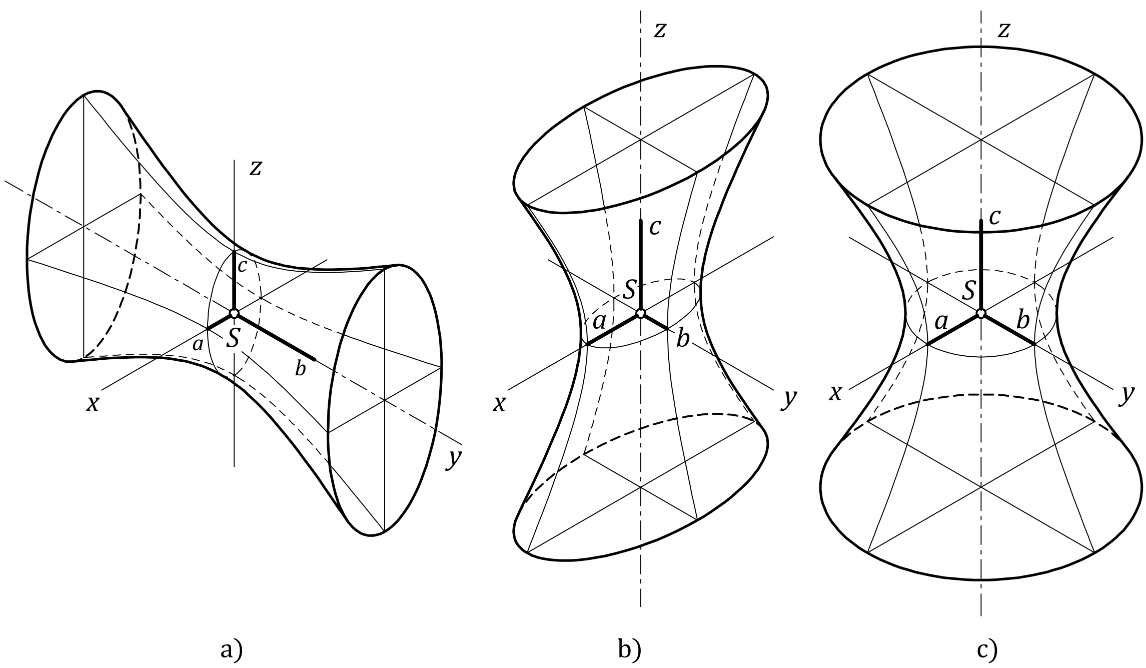


Figure 4.13: One-sheeted hyperboloids

A *two-sheeted hyperboloid* (*hyperboloid of two sheets*) is given by centre **S** and semiaxes $a \parallel x$, $b \parallel y$ and $c \parallel z$. If the axis of a two-sheeted hyperboloid is parallel with x-, y- or z-axis, the two-sheeted hyperboloid has the following equation

$$\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} - \frac{(z-p)^2}{c^2} = 1,$$
$$-\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} - \frac{(z-p)^2}{c^2} = 1,$$
$$-\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} + \frac{(z-p)^2}{c^2} = 1,$$

see examples in fig. 4.14, fig. 4.15 a) or fig. 4.15 b) in the given order.

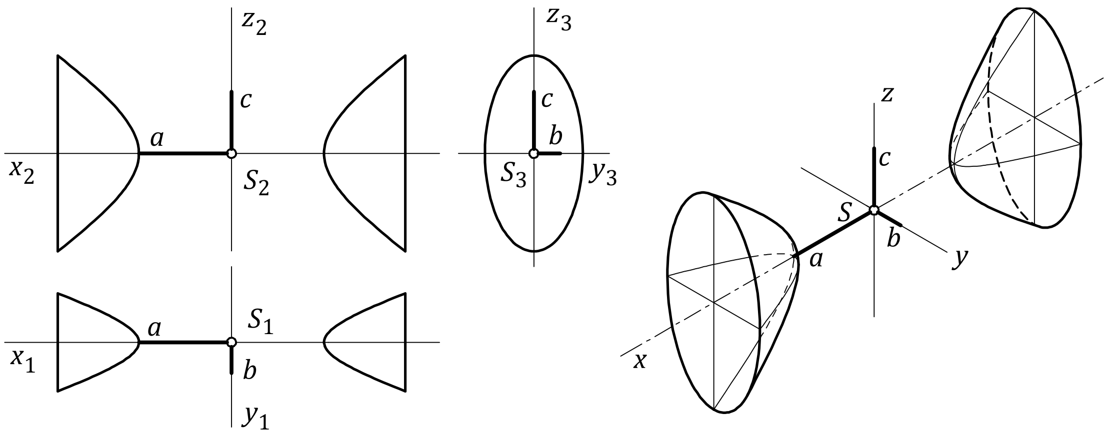


Figure 4.14: Two-sheeted elliptic hyperboloid

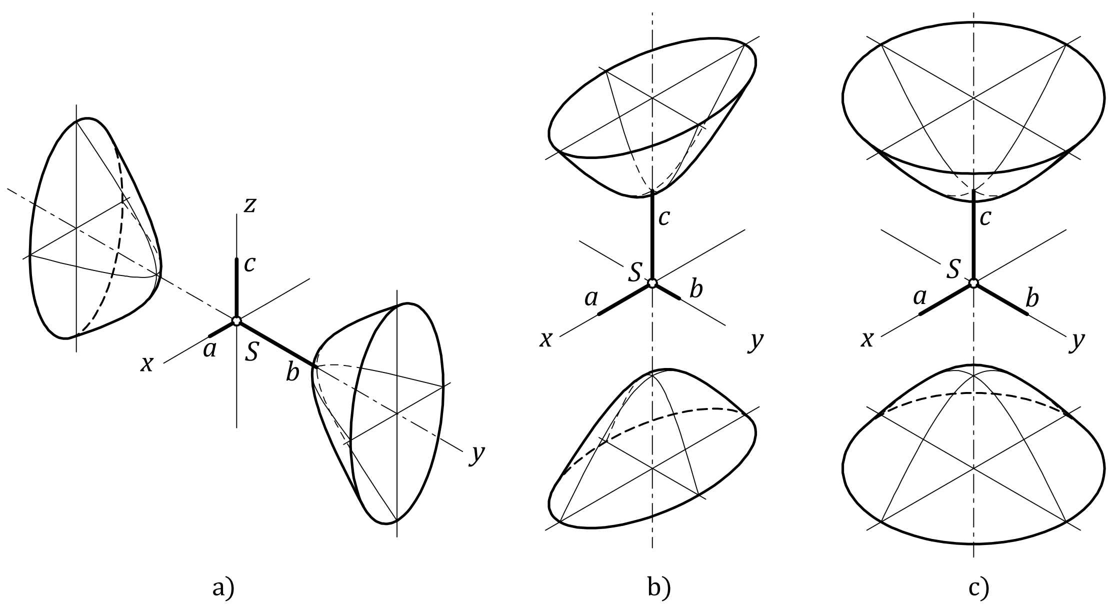


Figure 4.15: Two-sheeted hyperboloids

There are the following types of two-sheeted hyperboloid.

- $a \neq b \neq c$: the two-sheeted hyperboloid is called *two-sheeted elliptic hyperboloid*. Depending on position of axis of hyperboloid, intersections of two-sheeted hyperboloid and planes passing through the centre $\mathbf{S}$ parallel with the coordinate planes are two hyperbolas and one ellipse, see example in fig. 4.14.
- $a = b$, $b = c$ or $a = c$: the two-sheeted hyperboloid is called *two-sheeted hyperboloid of revolution* with axis of revolution parallel with z, x or y-axis in the given order. Example of two-sheeted hyperboloid of revolution with $a = b$ is drawn in fig. 4.15 c). Two-sheeted hyperboloid of revolution can be generated by rotating a hyperbola about the line $\mathbf{F}_1\mathbf{F}_2$, see section 4.2.2.

4.5.4 Cone

In general, a cone is a set of generating lines passing through vertex and every point of directing curve, see chapter 8. If the directing curve is an ellipse and the straight line given by the vertex and the centre of ellipse is perpendicular to the director plane of the ellipse, the cone is called *elliptic*, see examples in fig. 4.16, fig. 4.17 a) and fig. 4.17 b). Elliptic cone is given by vertex $\mathbf{V} = (m, n, p)$, height $\mathbf{VS}$ ($\mathbf{VS} = a$, $\mathbf{VS} = b$ or $\mathbf{VS} = c$) and two semiaxes (b and c, a and c or a and b). Height is the distance between the vertex $\mathbf{V}$ of the cone and the centre $\mathbf{S}$ of the ellipse with the given semiaxes. This ellipse is the intersection curve between the cone and the plane passing through the centre $\mathbf{S}$ perpendicularly to the axis of the cone.

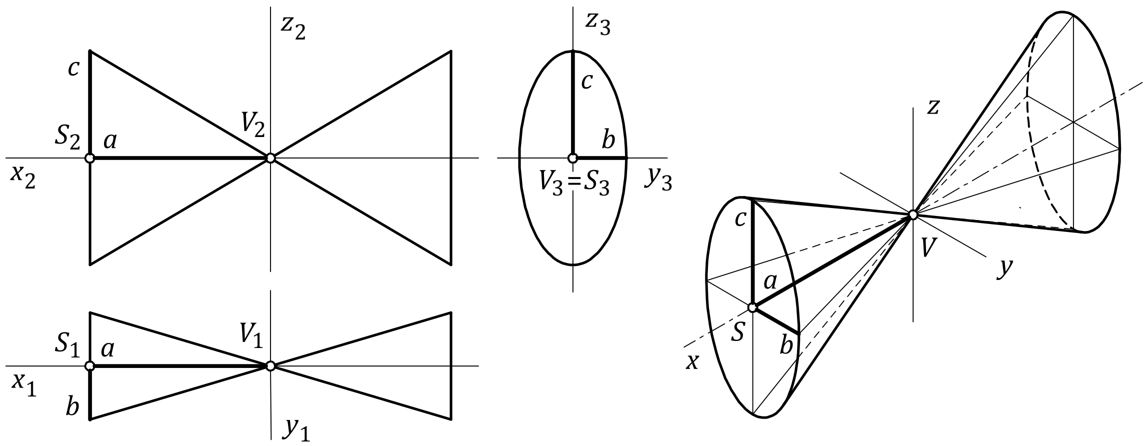


Figure 4.16: Elliptic cone

Depending on the position of the axis $\mathbf{VS} \subset o$ of the cone with respect to the coordinate axes, the equation of the cone is as follows

$$-\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} + \frac{(z-p)^2}{c^2} = 0, \ o \parallel x, \text{ semiaxes: } b, c,$$
$$\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} + \frac{(z-p)^2}{c^2} = 0, \ o \parallel y, \text{ semiaxes: } a, c,$$
$$\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} - \frac{(z-p)^2}{c^2} = 0, \ o \parallel z, \text{ semiaxes: } a, b,$$

see examples in fig. 4.16, fig. 4.17 a) and fig. 4.17 b) in the given order.

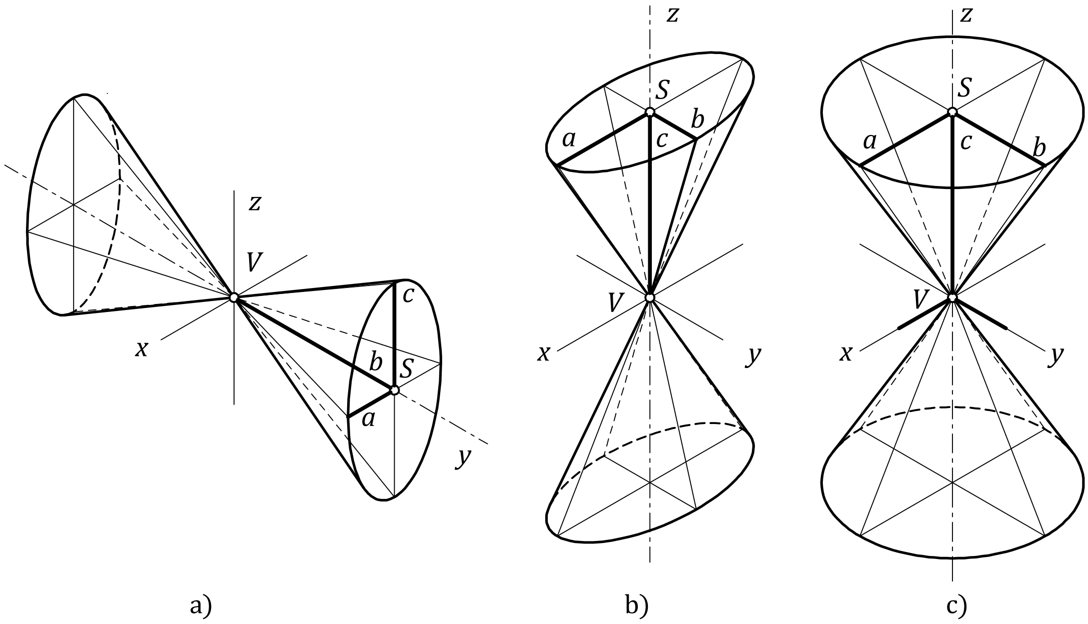


Figure 4.17: Cones

Intersections of the cone and planes passing through the vertex **V** parallel with coordinate planes are two straight lines intersecting at the vertex **V**. Intersection of the cone and plane $\rho \perp o$ is an ellipse if $\mathbf{V} \notin \rho$ or vertex **V** if $\mathbf{V} \in \rho$, see fig. 4.16.

Cone of revolution, see example in fig. 4.17 c), can be considered a limiting case of a hyperboloid generated by rotating the asymptotes of a hyperbola about bisector of the line $\mathbf{F}_1\mathbf{F}_2$ of the hyperbola (see section 4.2.2) or about the line $\mathbf{F}_1\mathbf{F}_2$, i.e. the generating line and axis of the cone are intersecting.

4.5.5 Paraboloid

Elliptic paraboloid is given by vertex $\mathbf{V} = (m, n, p)$, height **VS** ($\mathbf{VS} = a$, $\mathbf{VS} = b$ or $\mathbf{VS} = c$) and two semiaxes (b and c, a and c or a and b). Height is the distance between the vertex **V** of the paraboloid and the centre **S** of the ellipse with the given semiaxes. This ellipse is the intersection curve between the paraboloid and the plane passing through the centre **S** perpendicularly to the axis of the paraboloid. Depending on the position of the axis $\mathbf{VS} \subset o$ of the paraboloid with respect to the coordinate axes, the equation of the paraboloid is as follows

$$\frac{(y-n)^2}{b^2} + \frac{(z-p)^2}{c^2} = \frac{x-m}{a}, \quad o \parallel x, \text{ semiaxes: } b, c,$$
$$\frac{(x-m)^2}{a^2} + \frac{(z-p)^2}{c^2} = \frac{y-n}{b}, \quad o \parallel y, \text{ semiaxes: } a, c,$$
$$\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} = \frac{z-p}{c}, \quad o \parallel z, \text{ semiaxes: } a, b,$$

see examples in fig. 4.18, fig. 4.19 a) or fig. 4.19 b) in the given order.

Intersections of the paraboloid and planes passing through the vertex **V** parallel with coordinate planes are two parabolas. Intersection of the paraboloid and the plane $\rho \perp o$ is an ellipse if $\mathbf{V} \notin \rho$ or vertex **V** if $\mathbf{V} \in \rho$, see fig. 4.18.

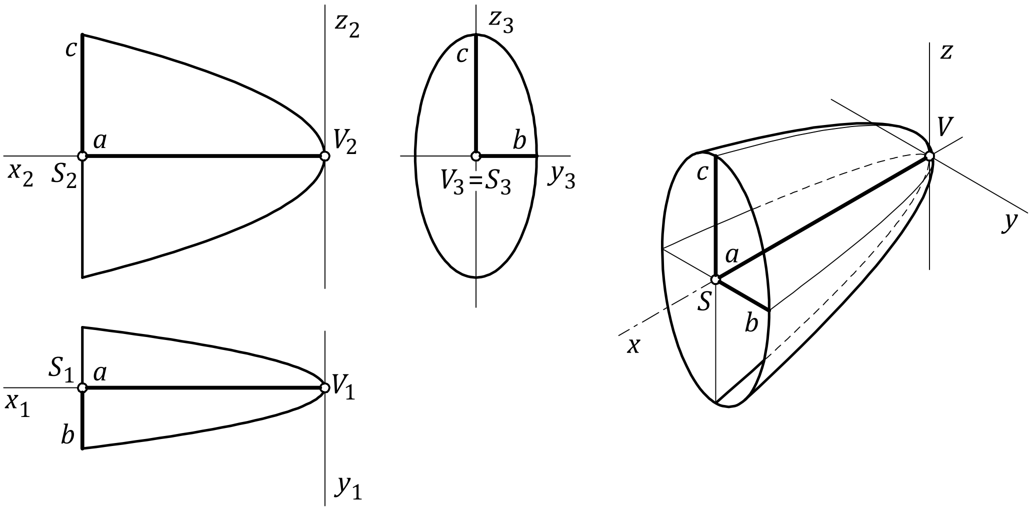


Figure 4.18: Elliptic paraboloid

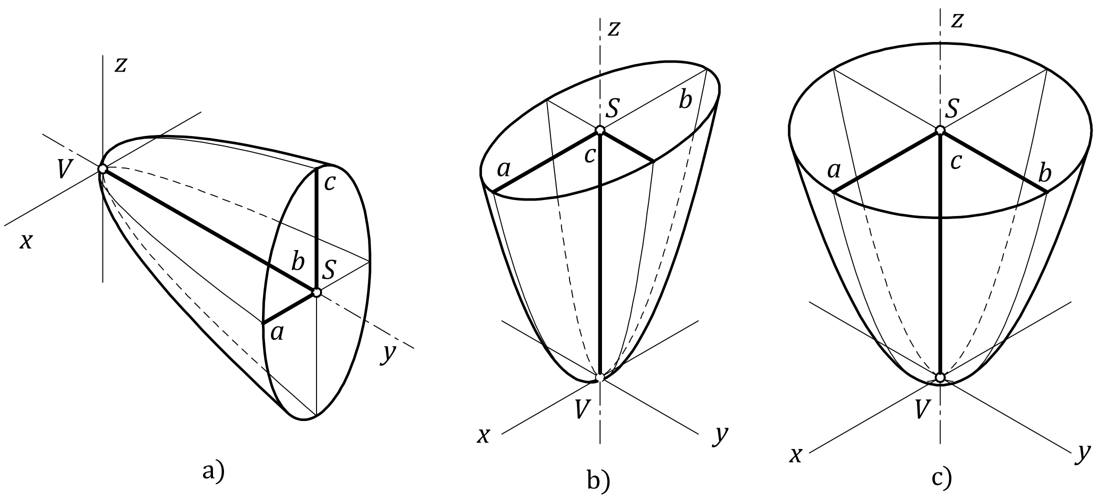


Figure 4.19: Elliptic paraboloids and paraboloid of revolution

If the length of the semiaxes of the intersection ellipse are not equal, the paraboloid is called *elliptic paraboloid*. Otherwise, the paraboloid is called *paraboloid of revolution*, see fig. 4.19 c).

Hyperbolic paraboloid is given by vertex $\mathbf{V} = (m, n, p)$, height $\mathbf{VS}$ ($\mathbf{VS} = a$, $\mathbf{VS} = b$ or $\mathbf{VS} = c$) and two semiaxes (b and c, a and c or a and b). Height is the distance between the vertex $\mathbf{V}$ of the paraboloid and the centre $\mathbf{S}$ of the hyperbola with the given semiaxes. This hyperbola is the intersection curve between the paraboloid and the plane passing through the centre $\mathbf{S}$ perpendicularly to the axis of the paraboloid. Depending on the position of the axis $\mathbf{VS} \subset o$ of the paraboloid with respect to the coordinate axes, the equation of the paraboloid is as follows

$$\frac{(y-n)^2}{b^2} - \frac{(z-p)^2}{c^2} = \frac{x-m}{a}, \;\; o \parallel x, \;\; \text{semiaxes: } b, c,$$

$$\frac{(x-m)^2}{a^2} - \frac{(z-p)^2}{c^2} = \frac{y-n}{b},\ o \parallel y,\ \text{semiaxes: } a, c,$$
$$\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} = \frac{z-p}{c},\ o \parallel z,\ \text{semiaxes: } a, b.$$

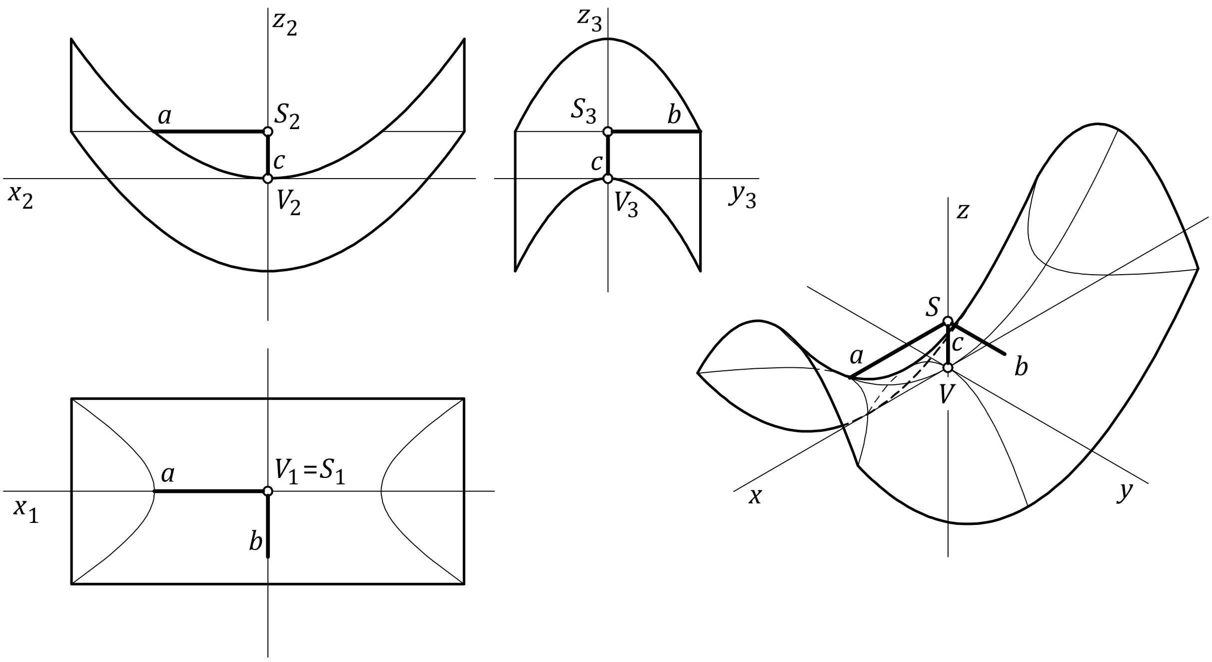


Figure 4.20: Hyperbolic paraboloid

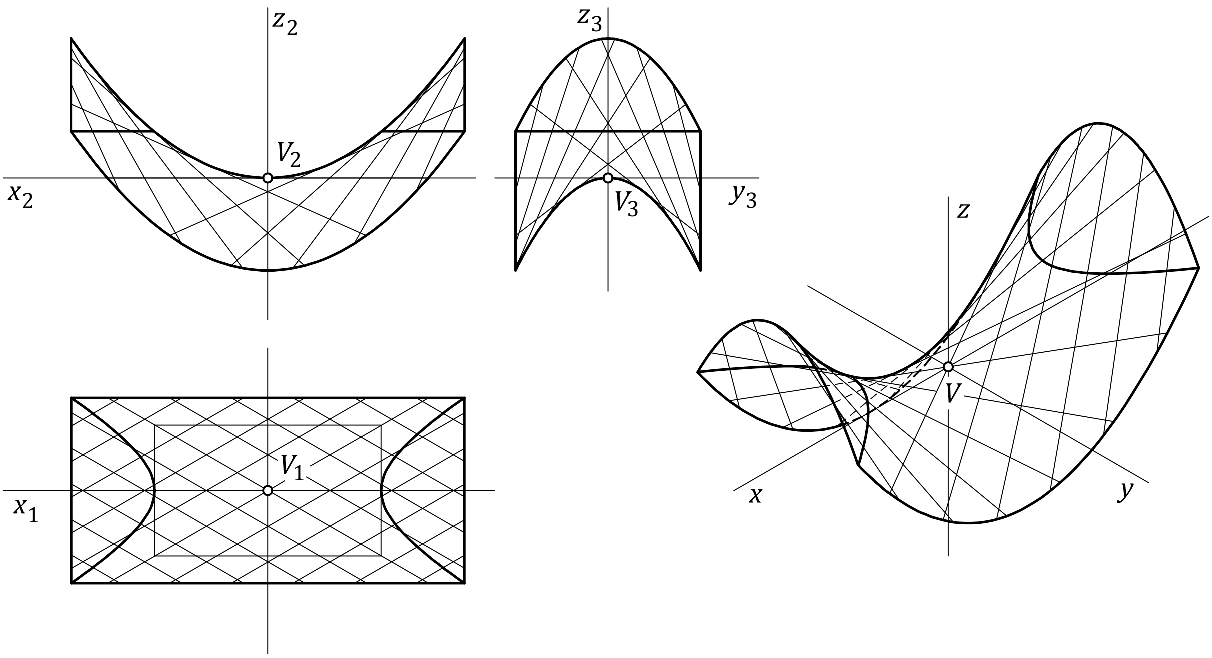


Figure 4.21: Hyperbolic paraboloid as double ruled surface

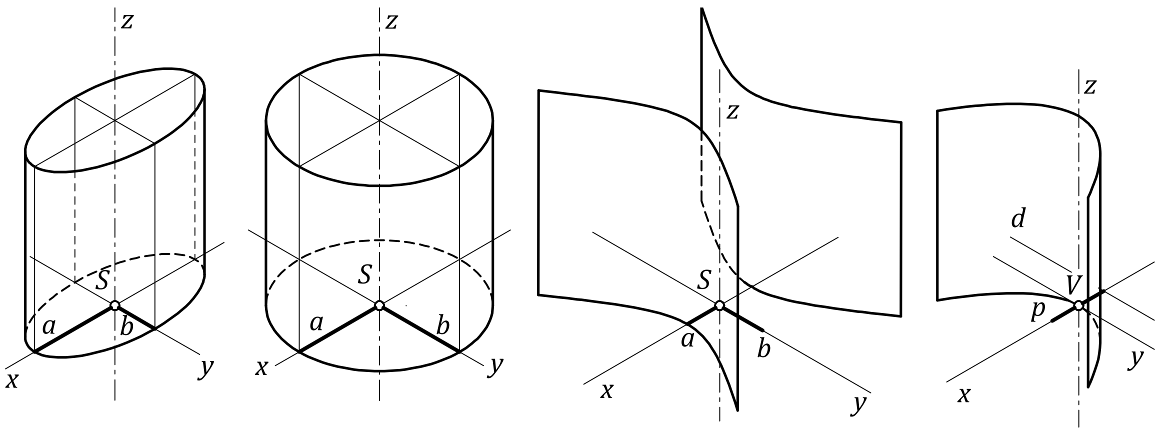
An example of hyperbolic paraboloid with the vertex at origin and its axis identical to z-axis is drawn in fig. 4.20.

Intersection curve between a hyperbolic paraboloid and any plane perpendicular to the axis of the paraboloid is a hyperbola. Specially, if the section plane passes through the vertex **V**, asymptotes of hyperbolas sections are obtained. Moreover, intersection curve between the hyperbolic paraboloid and any plane parallel with both asymptotes of the hyperbola and the axis of the hyperbolic paraboloid is a straight line. Considering a set of such section planes in both asymptotic directions, two sets of straight lines can be obtained as intersection curves. Thus, the hyperbolic paraboloid can be considered a set of straight lines and, therefore, it is called *double ruled surface*, see fig. 4.21.

4.5.6 Cylinder

In general, a cylinder is a set of generating straight lines parallel with the given direction and passing through every point of the directing curve, see chapter 8. If the directing curve is an ellipse, circle, hyperbola or parabola and the direction is perpendicular to the plane of the directing conic section, the cylinder is called *elliptic*, see fig. 4.22 a), *circular*, see fig. 4.22 b), *hyperbolic*, see fig. 4.22 c) or *parabolic*, see fig. 4.22 d). Circular cylinder is a *cylinder of revolution*.

A cylinder is given by the generating conic section, i.e. by the centre **S** and radius in the case of cylinder of revolution, by the centre **S** and two semiaxes in the case of elliptic or hyperbolic cylinder and by the vertex **V** and parameter p in the case of parabolic cylinder. Thus, the equation of cylinder is identical to the equation of the generating conic section.



a) b) c) d)

Figure 4.22: Cylinders

4.6 Example problems – analytic geometry

■ **Example 4.1 – Straight line in E^2**

Given

Points $\mathbf{A} = (1, 3)$ and $\mathbf{B} = (3, 2)$.

Required

Find vector, parametric, slope, intercept and general equations of straight line **AB**. Draw a graph of straight line **AB**.

Solution

The direction vector according to the eq. (4.1) is $\mathbf{u} = (2, -1)$. Then, the vector equation given by eq. (4.4) is

$$\mathbf{P}(t) = (1 + 2t, 3 - t),\ \ t \in R$$

and parametric equations according to eq. (4.5) are

$$\begin{aligned} x(t) &= 1 + 2t, \\ y(t) &= 3 - t,\ \ t \in R. \end{aligned} \tag{4.24}$$

The slope given by eq. (4.7) is $k = -\frac{1}{2}$ and q can be determined from eq. (4.24). The value of parameter t for which the condition $x(t) = 0$ is valid is

$$1 + 2t = 0 \ \Rightarrow \ t = -\frac{1}{2}.$$

By substituting $t = -\frac{1}{2}$ in y-coordinate function from eq. (4.24), we obtain $q = y(-\frac{1}{2}) = \frac{7}{2}$. Finally, the slope equation of straight line **AB** can be written

$$y = -\frac{1}{2}x + \frac{7}{2}.$$

Similarly, the x-intercept p can be determined by substituting the value of parameter t for which the condition $y(t) = 0$ is valid in x-coordinate function from eq. (4.24)

$$3 - t = 0 \ \Rightarrow \ t = 3.$$

Thus $p = x(3) = 7$ and intercept equation of straight line is given by

$$\frac{x}{7} + \frac{2}{7}y = 1.$$

General equation can be obtained by modification of intercept equation

$$x + 2y - 7 = 0 \tag{4.25}$$

or it is possible to calculate normal vectors according to eq. (4.10)

$$\mathbf{n} = (1, 2) \ \text{ or } \ \mathbf{n}^* = (-1, -2).$$

and substitute these normal vectors and coordinates of any point located on straight line **AB** in eq. (4.9) to obtain c and c^*. By substituting point **A**, we get $c = -7$ and $c^* = 7$. Thus, the general equation of straight line **AB** can be expressed by eq. (4.25) or by

$$-x - 2y + 7 = 0.$$

It is obvious that both general equations represent the same straight line.

The graph of straight line **AB** is drawn in fig. 4.1. □

■ Example 4.2 – Straight line in E^3

Given

Points $\mathbf{A} = (-2, 3, 4)$ and $\mathbf{B} = (2, 1, 3)$.

Required

Find vector and parametric equations of straight line **AB**. Using technical isometry, draw a graph of straight line **AB**.

Solution

The direction vector according to the eq. (4.1) is $\mathbf{u} = (4, -2, -1)$. Then, the vector equation according to eq. (4.16) is

$$\mathbf{P}(t) = (-2 + 4t, 3 - 2t, 4 - t),\ t \in R$$

and parametric equations according to eq. (4.17) are

$$\begin{aligned} x(t) &= -2 + 4t, \\ y(t) &= 3 - 2t, \\ z(t) &= 4 - t,\ t \in R. \end{aligned}$$

The graph of straight line **AB** in technical isometry is drawn in fig. 4.8. □

■ Example 4.3 – Plane

Given

Points $\mathbf{A} = (2, 1, 3)$, $\mathbf{B} = (4, 2, 1)$ and $\mathbf{C} = (1, 4, 2)$.

Required

Find vector, parametric, general and intercept equations of the plane given by points **A**, **B** and **C**. Using technical isometry, draw the plane by means of its intersections with coordinate planes.

Solution

The direction vectors of the plane are $\mathbf{u} = (2, 1, -2)$ and $\mathbf{v} = (-1, 3, -1)$. Then, the vector equation according eq. (4.18) is

$$\mathbf{P}(s, t) = (2 + 2s - t, 1 + s + 3t, 3 - 2s - t),\ (s, t) \in R^2$$

and parametric equations are

$$\begin{aligned} x(s, t) &= 2 + 2s - t, \\ y(s, t) &= 1 + s + 3t, \\ z(s, t) &= 3 - 2s - t,\ (s, t) \in R^2. \end{aligned}$$

To obtain the general equation of the plane, it is necessary to calculate the normal vector given by eq. (4.21), first

$$\mathbf{n} = \mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 3 & -1 \end{vmatrix} = (5, 4, 7).$$

After that, we can write

$$5x + 4y + 7z + d = 0. \tag{4.26}$$

By substituting the coordinates of any point located on the plane in eq. (4.26) we obtain equation with one unknown d. For example, after substituting $\mathbf{D} = (3, 5, 0)$ we obtain

$$15 + 20 + d = 0 \;\Rightarrow\; d = -35.$$

Finally, we can write general equation of the plane

$$5x + 4y + 7z - 35 = 0. \tag{4.27}$$

To obtain intercept equation of the plane, it is necessary to calculate intercepts p, q and r, first. Therefore, by substituting $y = 0$ and $z = 0$ in the eq. (4.27) we obtain $p = 7$, by substituting $x = 0$ and $z = 0$ we obtain $q = \frac{35}{4}$ and by substituting $x = 0$ and $y = 0$ we obtain $r = 5$. Thus, the intercept equation is

$$\frac{x}{7} + \frac{4}{35}y + \frac{z}{5} = 1.$$

The plane in technical isometry is drawn in fig. 4.9. □

■ Example 4.4 – Solid bounded by quadratic surfaces

Given

Quadratic surfaces

$$\rho : z = 4 - \sqrt{2x^2 + 2y^2}, \tag{4.28}$$

$$\sigma : 4x^2 + 4y^2 - z^2 - 4 = 0, \tag{4.29}$$

$$\omega : z = -2 - \sqrt{2 - x^2 - y^2}. \tag{4.30}$$

Required

Determine the type and characteristics of these surfaces. Draw top and front views of the solid bounded by surfaces ρ, σ and ω and sketch the solid in technical isometry. Find equations of intersection curves $k = \rho \cap \sigma$ and $l = \sigma \cap \omega$. How it is possible to generate the surface σ?

Solution

To recognize the type of the given quadratic surfaces and determine their characteristics, it is necessary to turn their equations into the canonical form. Then, the surfaces can be expressed by

$$\rho : \frac{x^2}{2} + \frac{y^2}{2} - \frac{(z-4)^2}{4} = 0,$$

$$\sigma : x^2 + y^2 - \frac{z^2}{4} = 1,$$

$$\omega : x^2 + y^2 + (z+2)^2 = 2.$$

It follows that the surface ρ is a cone of revolution with the vertex at point $\mathbf{V} = (0, 0, 4)$, semiaxes $a = b = \sqrt{2}$ and height $c = 2$. Due to the negative sign in front of square root in eq. (4.28), the part $z \leqslant 4$ has to be considered only.

The surface σ is a one-sheeted hyperboloid of revolution with centre at point $\mathbf{C} = (0, 0, 0)$, semiaxes $a' = b' = 1$, $c' = 2$ and its axis identical to z-axis. This surface can be generated by revolution of hyperbola

$$x^2 - \frac{z^2}{4} = 1 \tag{4.31}$$

or

$$y^2 - \frac{z^2}{4} = 1$$

about z-axis. Moreover, it is possible to create this surface by revolution of the straight line given by points $\mathbf{A} = (1, 1, -2)$ and $\mathbf{B} = (-1, 1, 2)$ about z-axis, for example.

Finally, the surface ω is a sphere with the centre at point $\mathbf{S} = (0, 0, -2)$ and radius $r = \sqrt{2}$. Due to the negative sign in front of square root in eq. (4.30), the lower hemisphere has to be considered only, i.e. $z \leqslant -2$.

To determine the intersection curve k, it is necessary to solve the set of equations

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{(z-4)^2}{4} = 0$$
$$x^2 + y^2 - \frac{z^2}{4} = 1,$$

to obtain $z_1 = 2$, $z_2 = \frac{34}{7}$. Due to the condition $z \leqslant 3$, only $z_1 = 2$ can be substituted into the equations of cone ρ and hyperboloid σ to get

$$k: \; x^2 + y^2 = 2, \; z = 2,$$

i.e. the circle in the plane $z = 2$ with the centre $\mathbf{S} = (0, 0, 2)$ and radius $r = \sqrt{2}$. Similarly, the solution of the set of equations

$$x^2 + y^2 - \frac{z^2}{4} = 1$$
$$x^2 + y^2 + (z-2)^2 = 2$$

is $z_1 = -2$, $z_2 = -1.2$. Due to the condition $z \leqslant -2$, only $z_1 = -2$ can be substituted into the equations of hyperboloid σ and sphere ω to get

$$x^2 + y^2 = 2, \; z = -2,$$

i.e. the circle in the plane $z = -2$ with the centre $S = (0, 0, -2)$ and radius $r = \sqrt{2}$.

The top, front and isometric views of the solid are drawn in fig. 4.23. Note that the top view of generating line $\mathbf{AB}$ of one-sheeted hyperboloid of revolution σ is tangent line to the throat parallel circle (see chapter 5) and the front view is the asymptote of hyperbola (4.31).

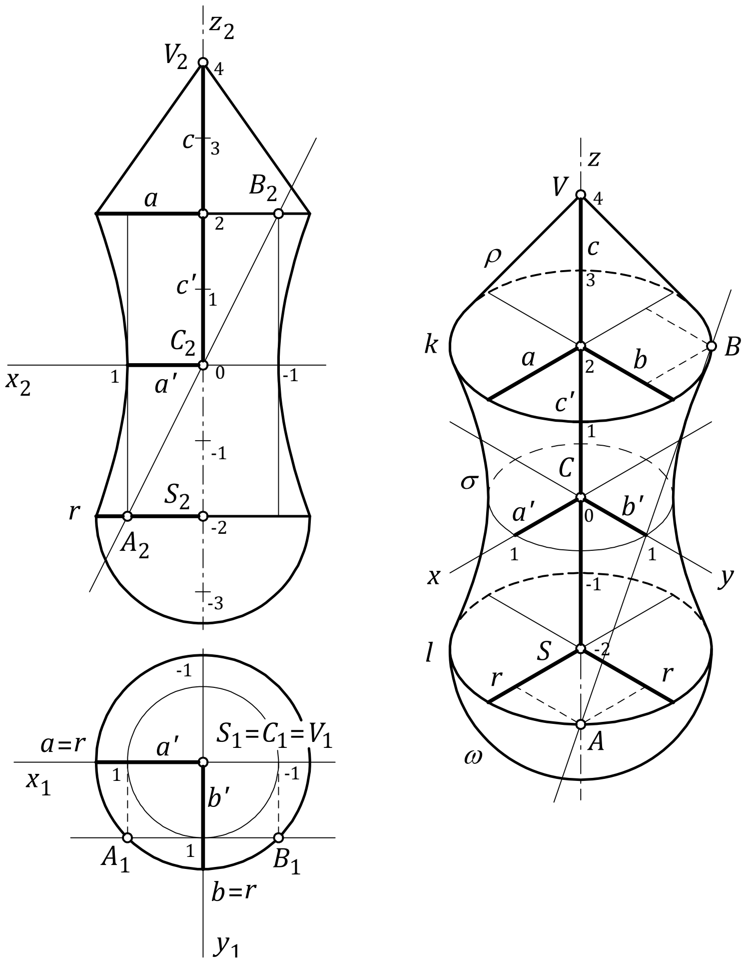


Figure 4.23: Solid bounded by quadratic surfaces

□

Chapter 5

Surfaces of revolution and their intersections

Surface of revolution $\sigma = (k, o)$ is a figure generated by revolution of a *generating curve* k about the given axis o, $k \neq o$, $k \not\subset \alpha$, $\alpha \perp o$, see example in fig. 5.1. Axis o is a straight line in three-dimensional space in general position with respect to the coordinate system and generating curve k is a spatial or planar curve. Without loss of generality, assume in this chapter that the axis o is z-axis (or parallel with z-axis, $o \perp \pi$), unless stated otherwise.

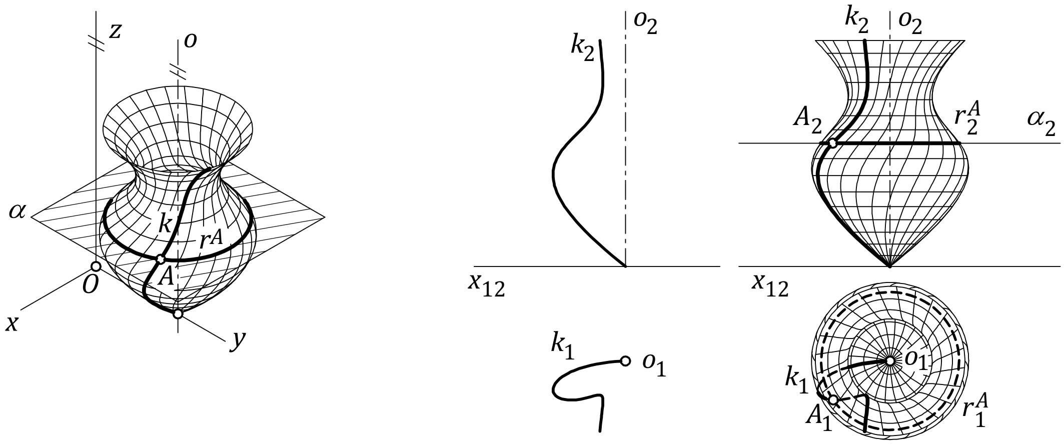


Figure 5.1: Surface of revolution generated by revolution of spatial generating curve in isometric view (left) and in Monge projection (right)

If the analytic representation of the generating curve k is given by vector equation

$$k: \ \mathbf{K}(v) = \big(x(v),\ y(v),\ z(v)\big),\ v \in [v_1, v_2],$$

the vector equation of the surface of revolution σ generated by revolution of curve k about z-axis is given by

$$\sigma: \ \mathbf{S}(u,v) = \big(x(v)\cos u - y(v)\sin u,\ x(v)\sin u + y(v)\cos(u),\ z(v)\big),\ u \in [0, 2\pi],\ v \in [v_1, v_2].$$

Trajectory of any point A on the generating curve is parametric u-curve of the surface called *parallel circle* or *parallel* r^A. Circle r^A lies in plane α perpendicular to the axis of revolution o

with the centre $S = \alpha \cap o$ and radius $||SA||$. Due to $o \perp \pi$, plane α is parallel with the horizontal plane of projection π and the parallel circle is projected as a circle without any distortion in the top view $r_1^A = (o_1, r = ||o_1A_1||)$. Front view r_2^A is straight line segment of length $2||o_1A_1||$, perpendicular to o_2 placed in true distance of plane α from the horizontal plane of projection π, see fig. 5.1 right. Parametric v-curves are congruent generating curves at individual revolved positions. Therefore, at any point A on the surface of revolution, there is located one parallel circle r^A and one revolved position of the generating curve k. Both systems of parametric curves create a mesh by means of which is possible to visualize the surface of revolution.

It is possible to create a surface of revolution of the same shape by revolving any generating curve suitably located on the surface of revolution about the axis of revolution. The easiest generating curve is a planar intersection m of a surface of revolution and a plane ρ passing through the axis of revolution $o \subset \rho$. The plane ρ is called *meridian plane* and the intersection curve m is called *meridian*. The meridian is a planar curve (or a pair of planar curves) symmetric with respect to the axis of revolution, see fig. 5.2 left. Meridian can be considered a generating curve, thus, at any point on the surface of revolution, one parallel circle and one meridian is located. If section plane ρ is parallel with the frontal plane of projection, the meridian is called *principal meridian* and plane ρ is called *principal meridian plane*. Intersection of surface of revolution and a half-plane beginning from the axis of revolution parallel with the frontal plane of projection is called *principal left half-meridian* or *principal right half-meridian*.

The top view m_1 of principal meridian is straight line segment on the straight line parallel with x_{12} passing through o_1. The front view m_2 of principal meridian is projected in true shape without any distortion, see fig. 5.2 right.

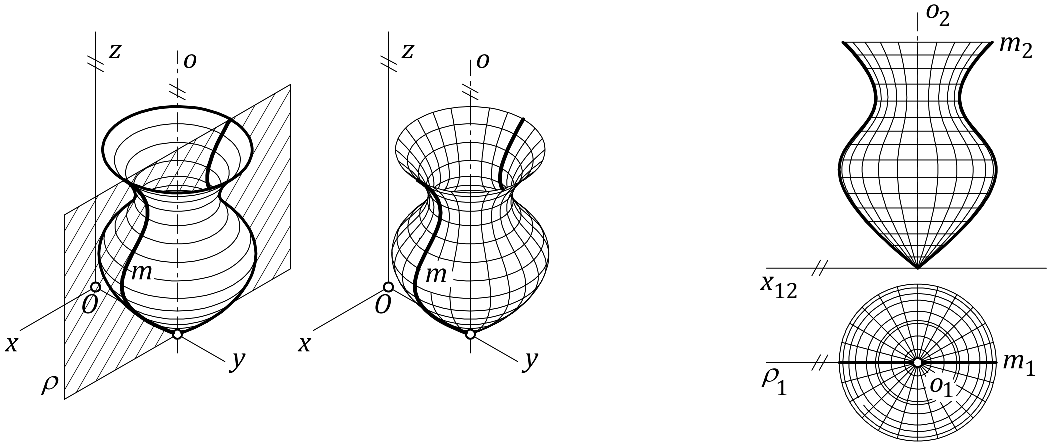
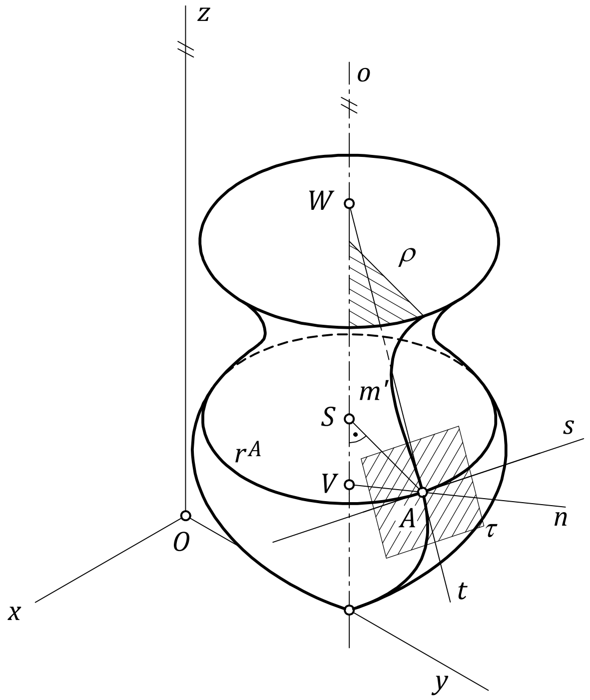


Figure 5.2: Surface of revolution generated by revolution of principal meridian in isometric view (left) and in Monge projection (right)

5.1 Properties of surfaces of revolution

Tangent plane τ at regular point A of a surface of revolution is determined by tangent lines to parametric curves passing through point A located on the surface of revolution, i.e. by the tangent line t to the generating curve and by the tangent line s to the parallel circle r^A, $\tau = (t, s)$. *Normal line* n at point A of a surface of revolution is perpendicular to the tangent plane τ, see fig. 5.3, where the tangent plane and the normal line to the surface of revolution at point A of half-meridian are drawn.



$s \perp SA$

$n \perp s$

$n \perp t$

Figure 5.3: Properties of surface of revolution

In the following list, the most important geometrical properties of surfaces of revolution are summarized.

- Surface of revolution is symmetric with respect to the axis of revolution o and with respect to any meridian plane ρ, $o \subset \rho$. It follows from the fact that each parallel circle is symmetric with respect to the centre and with respect to each diameter.

- Tangent plane $\tau = (t, s)$ of surface of revolution at point A of meridian is perpendicular to the meridian plane ρ. To show this property, it is necessary to find two intersecting straight lines located on plane ρ to which the plane τ is perpendicular. These straight lines are SA and o because $s \perp SA$ (s is a tangent line to parallel circle $r^A = (S, r = ||SA||)$) and $s \perp o$ because s lies in the plane of parallel circle r^A perpendicular to o.

- Normal line n at point A of a surface of revolution intersects axis o or it is parallel with the axis o. Consider meridian $A \in m'$ located in meridian plane $\rho = (o, A)$ and tangent plane $\tau = (s, t)$ at point A. Since $n \perp \tau$, the tangent s to the parallel circle r^A has to be perpendicular to the tangent line t to the meridian m'. Moreover, the tangent line $s \perp \rho$. It follows that normal line $n \subset \rho$. Both the axis o and the normal line n are located on one plane, therefore they are intersecting or parallel.

- Tangent line of meridian intersects the axis of revolution or it is parallel with the axis of revolution.

- Tangent lines of meridians at points along one parallel circle create a cone of revolution with vertex W on axis of revolution or a cylinder of revolution or a plane perpendicular to the axis of revolution.

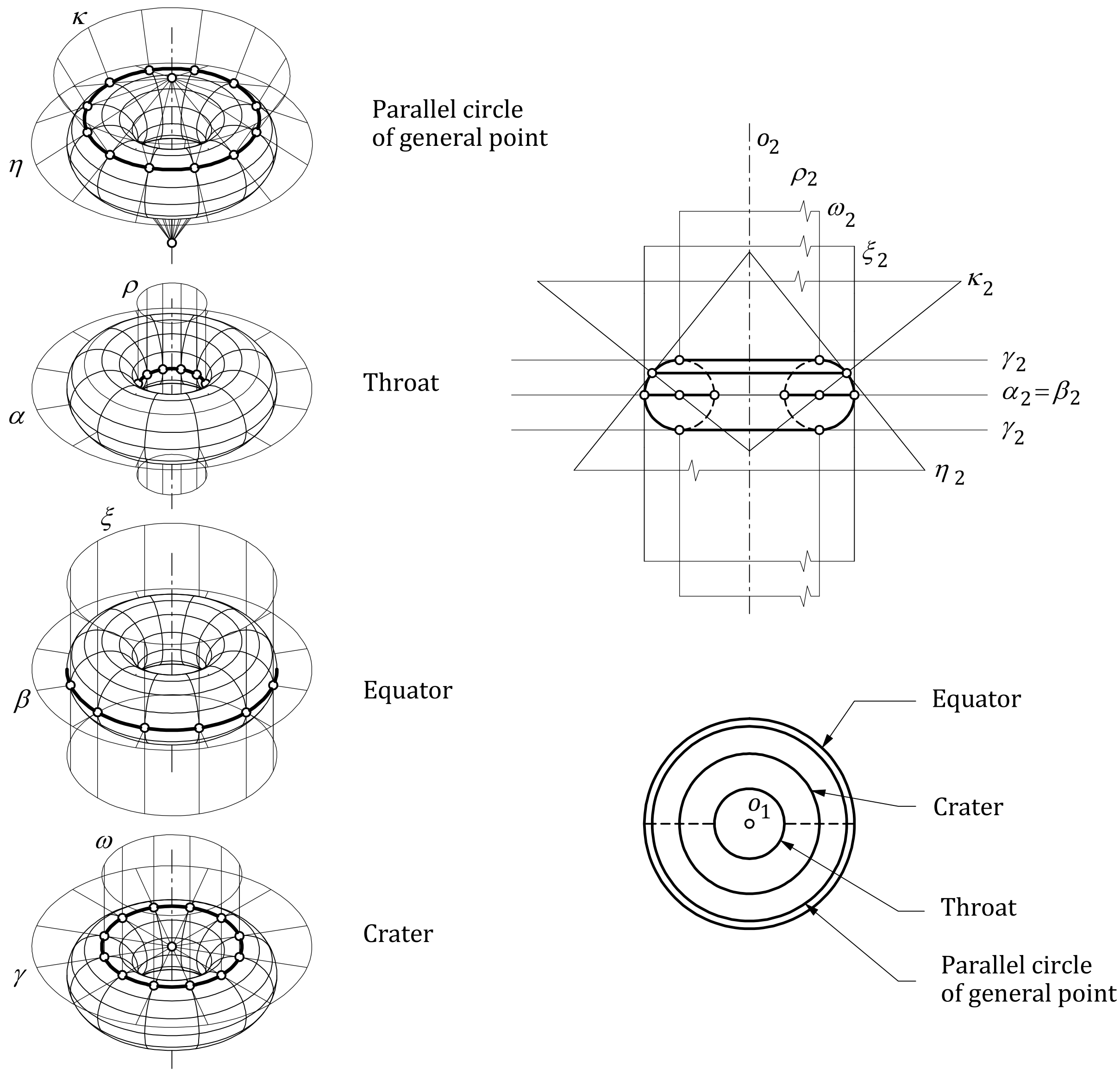


Figure 5.4: Properties of tangent and normal lines along parallel circles in isometric view (left) and in Monge projection (right)

- If the tangent lines of meridians at points along one parallel circle create a cone of revolution η called *tangent cone*, the normal lines of surface of revolution at points along the same parallel circle create a cone of revolution κ called *normal cone*, see fig. 5.4.
- If the tangent lines of meridians at points along one parallel circle create a cylinder of revolution ρ having internal contact with the surface of revolution, the parallel circle is called *throat*. The normal lines of the surface of revolution at points along the throat create a plane $\alpha \perp o$.
- If the tangent lines of meridians at points along one parallel circle create a cylinder of revolution ξ having external contact with the surface of revolution, the parallel circle is

called *equator*. The normal lines of the surface of revolution at points along the equator create a plane $\beta \perp o$.

- If the tangent lines at points along one parallel circle create a plane $\gamma \perp o$, this parallel circle is called *crater*. The normal lines of the surface of revolution along the crater create cylinder of revolution ω.

5.2 Example problems – surfaces of revolution

■ Example 5.1 – Missing front view of point on surface of revolution

Given

Generating curve k (a part of a circle with centre S) and axis o of surface of revolution $\sigma = (k, o)$, top view A_1 of point $A \in \sigma$ in Monge projection, see fig. 5.5 a).

Required

Using Monge projection, construct the missing front view A_2 of point $A \in \sigma$.

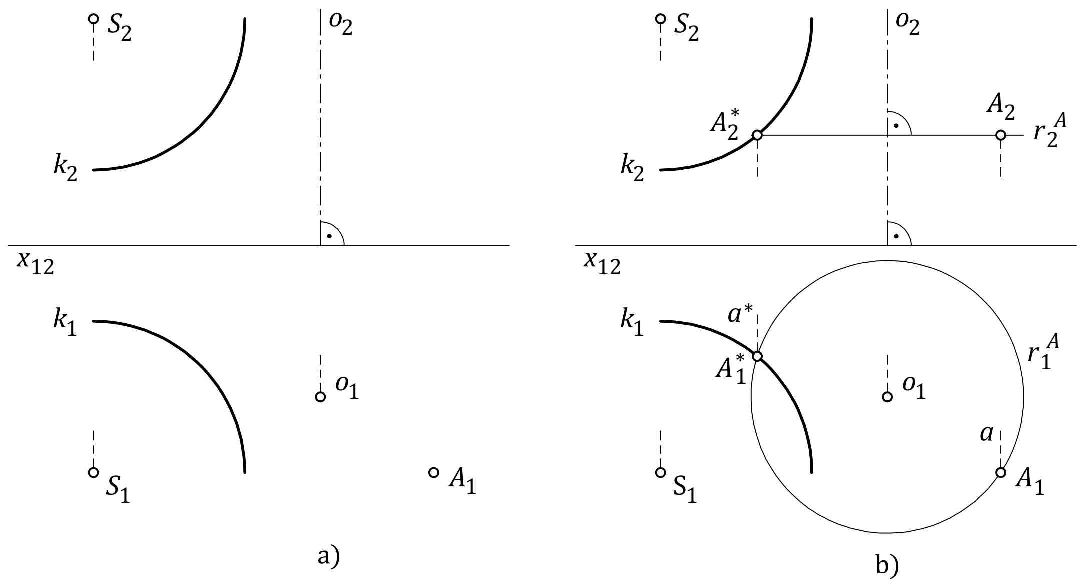


Figure 5.5: Construction of missing front view of point on surface of revolution

Analysis

Two lines (or curves) intersecting at the front view A_2 have to be determined. The first line is the ordinate a of point A. The second line is the front view of parallel circle r^A – the trajectory of point A. Parallel circle r^A can be constructed as common trajectory of points A and A^* on generating curve k with the same distance from horizontal plane of projection as is the distance of point A, i.e. $z_A = z_{A^*}$.

Graphical solution

1. Construct ordinate $a \perp x_{12}$, $A_1 \in a$, see fig. 5.5 b).
2. Draw top view $r_1^A = (o_1, r = ||o_1A_1||)$.
3. Top view $A_1^* = k_1 \cap r_1^A$.
4. Construct ordinate $a^* \perp x_{12}$, $A_1^* \in a^*$.
5. Front view $A_2^* = a^* \cap k_2$.
6. Construct front view $r_2^A \perp o_2$, $A_2^* \in r_2^A$.
7. Front view $A_2 = r_2^A \cap a$. □

■ Example 5.2 – Missing top view of point on surface of revolution

Given

Generating curve k (a part of a circle with centre S) and axis of revolution o of surface of revolution $\sigma = (k, o)$, front view A_2 of point $A \in \sigma$ in Monge projection, see fig. 5.6 a).

Required

Using Monge projection, construct the missing top view A_1 of point $A \in \sigma$.

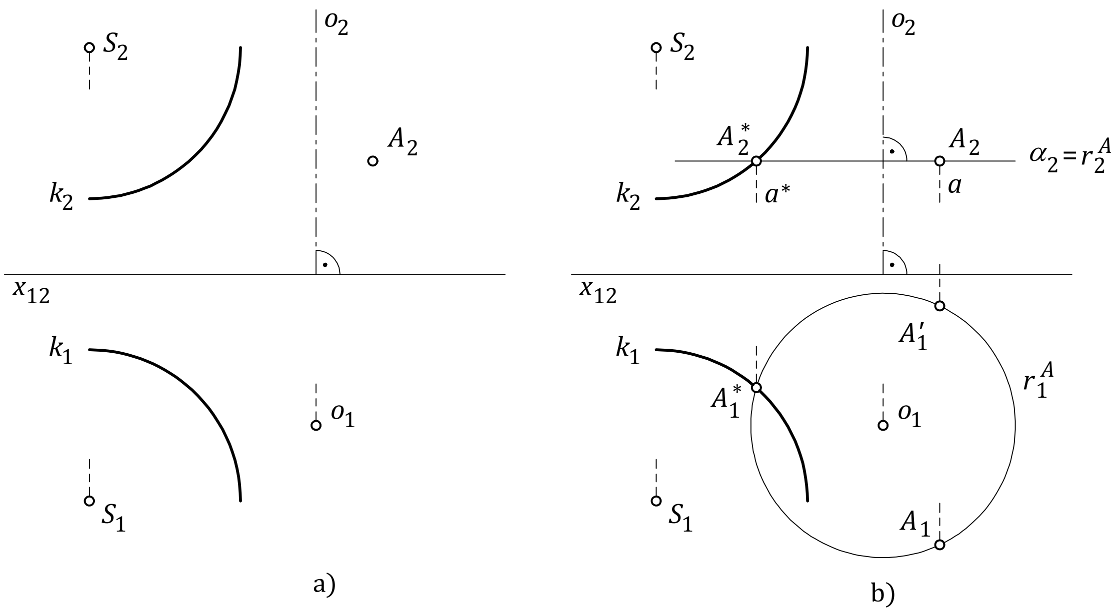


Figure 5.6: Construction of missing top view of point on surface of revolution

Analysis

Two lines intersecting at the top view A_1 have to be determined. The first line is the ordinate a of point A. The second line is the top view of parallel circle r^A – the trajectory of point A. The parallel circle r^A can be constructed as a common trajectory of point A and point A^* on generating curve k at the same distance from the axis of revolution as is the distance of point A, i.e. $||A_1 o_1|| = ||A_1^* o_1||$.

Graphical solution

1. Construct ordinate $a \perp x_{12}$, $A_2 \in a$, see fig. 5.6 b).
2. Construct front view α_2 of the plane in which the parallel circle r^A lies: $\alpha_2 \perp o_2$, $A_2 \in \alpha_2$.
3. Front view $A_2^* = k_2 \cap \alpha_2$.
4. Construct ordinate $a^* \perp x_{12}$, $A_2^* \in a^*$.
5. Top view $A_1^* = k_1 \cap a^*$.
6. Construct top view $r_1^A = (o_1, r = ||o_1 A_1^*||)$.
7. Two solutions are obtained: top views $A_1, A_1' = r_1^A \cap a$. □

■ Example 5.3 – Tangent plane and normal line at point on surface of revolution

Given

Principal left half-meridian m (a part of a circle with centre S) and axis of revolution o of surface of revolution $\sigma = (m, o)$ and top view A_1 of point $A \in \sigma$ in Monge projection, see fig. 5.7 a).

Required

Using Monge projection, construct tangent plane τ and normal line n at point A of the surface of revolution σ.

Analysis

Firstly, it is necessary to construct the front view A_2 of point A according to the procedure described in example 5.1. After that, the normal cone created by normal lines of surface σ at points along parallel circle r^A can be used to construct normal line n. Then, tangent plane $\tau = (s, t)$ is determined by tangent line t of meridian rotated into the position of point A and tangent line s of parallel circle r^A. To construct tangent line t at point A, the tangent cone created by tangent lines of meridians at points along r^A can be used.

Graphical solution

1. Construct front view A_2, see example 5.1 and fig. 5.7 b).
2. Draw top view $n_1 = o_1 A_1$.
3. Construct front view $n_2^* = S_2 A_2^*$.
4. Front view V_2 of the vertex of the normal cone $V_2 = o_2 \cap n_2^*$.

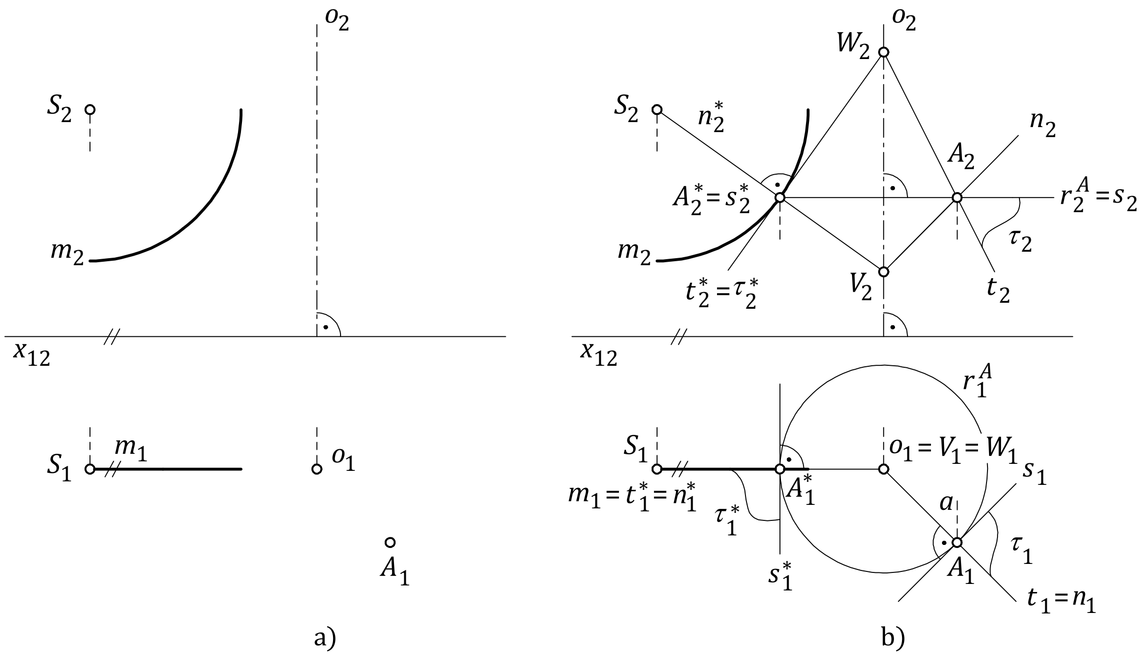


Figure 5.7: Tangent plane and normal line at point on surface of revolution

5. Draw front view $n_2 = V_2A_2$.
6. Top view $t_1 = o_1A_1$.
7. Construct top view $s_1 \perp t_1$, $A_1 \in s_1$.
8. Top view $\tau_1 = (t_1, s_1)$.
9. Construct front view $t_2^* \perp n_2^*$, $A_2^* \in t_2^*$.
10. Front view W_2 of the vertex of the tangent cone $W_2 = t_2^* \cap o_2$.
11. Draw front view $t_2 = W_2A_2$.
12. Front view $\tau = (t_2, s_2)$.

Note that for completeness of fig. 5.7 b), the top view τ_1^* of tangent plane τ^* and top view n_1^* of normal line n^* are drawn, even though these figures are not necessary for the construction of the required tangent plane τ and normal line n. □

■ Example 5.4 – Principal meridian of surface of revolution

Given

Generating curve k and axis of revolution o of surface of revolution $\sigma = (k, o)$ in Monge projection, see fig. 5.8 a).

Required

Using Monge projection, construct principal meridian m of the surface of revolution σ.

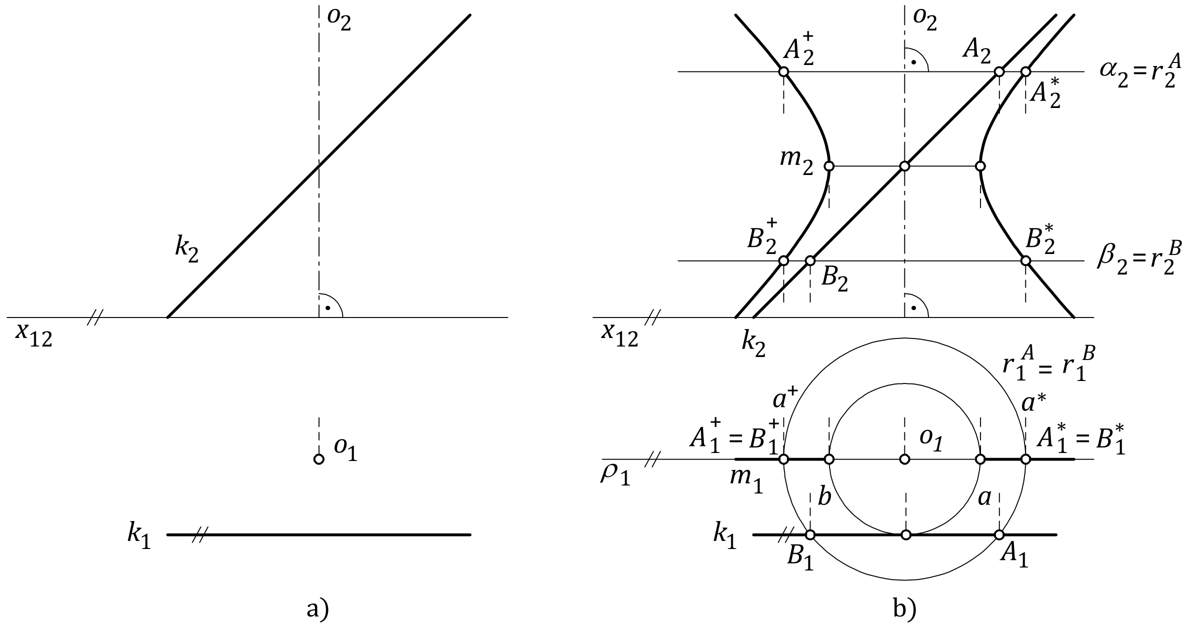


Figure 5.8: Principal meridian of surface of revolution

Analysis

Principal meridian m is the intersection of surface of revolution σ and principal meridian plane $\rho \perp \pi$, $o \in \rho$: $m = \sigma \cap \rho$. To construct the principal meridian, a pointwise approach is used, where the intersection points of the principal meridian plane and parallel circles of individual points suitably chosen on the surface of revolution are constructed. Finally, the principal meridian is drawn as a curve passing through all constructed points.

Graphical solution

1. Construct top view ρ_1 of principal meridian plane ρ: $\rho_1 \parallel x_{12}$, $o_1 \in \rho_1$, fig. 5.8 b).

2. Choose top view $A_1 \in k_1$ and find the adjacent front view $A_2 = a \cap k_2$, a is the ordinate of point A. Note that it is possible to choose front view $A_2 \in k_2$ and find the adjacent top view $A_1 \in k_1$, too.

3. Construct front view $\alpha_2 \perp o_2$, $A_2 \in \alpha_2$, α is the plane of parallel circle r^A.

4. Construct top view r_1^A of parallel circle r^A: $r_1^A = (o_1, r = ||o_1 A_1||)$.

5. Point of principal meridian is the intersection of the parallel circle r^A and principal meridian plane ρ. Here are two intersections A^* and A^+. Top views: $A_1^*, A_1^+ = r_1^A \cap \rho_1$.

6. Front views $A_2^* = \alpha_2 \cap a^*$, a^* is the ordinate of point A^* and $A_2^+ = \alpha_2 \cap a^+$, a^+ is the ordinate of point A^+.

7. Continue in a similar way to obtain a sufficient number of points on principal meridian. Do not forget points at special positions such as points at the minimal or maximal distance from the axis of revolution. Finally, draw the front view m_2 as a curve (hyperbola in this

case) passing through all constructed points. The top view $m_1 \subset \rho_1$ is a straight line segment.

Note that top view r_1^A intersects the top view k_1 of generating curve two times – at the chosen top view A_1 of point A and at the second top view B_1 of point B. The parallel circles $r^A \subset \alpha$ and $r^B \subset \beta$ of these two points have the same radius but different altitudes given by positions of front views α_2 and β_2. Thus, depending on the shape of the given generating curve, it is possible to proceed effectively and use the top view B_1 for construction of another point B^* and B^+ of principal meridian, see fig. 5.8 b). □

■ Example 5.5 – Intersection of surface of revolution and projecting plane $\rho \perp \nu$, $p = \sigma \cap \rho$

Given

Principal right half-meridian m, axis of revolution o of a surface of revolution $\sigma = (m, o)$ and section plane $\rho \perp \nu$ in Monge projection, see fig. 5.9 a).

Required

Using Monge projection, construct the intersection curve p of the surface σ and the given plane $\rho \perp \nu$.

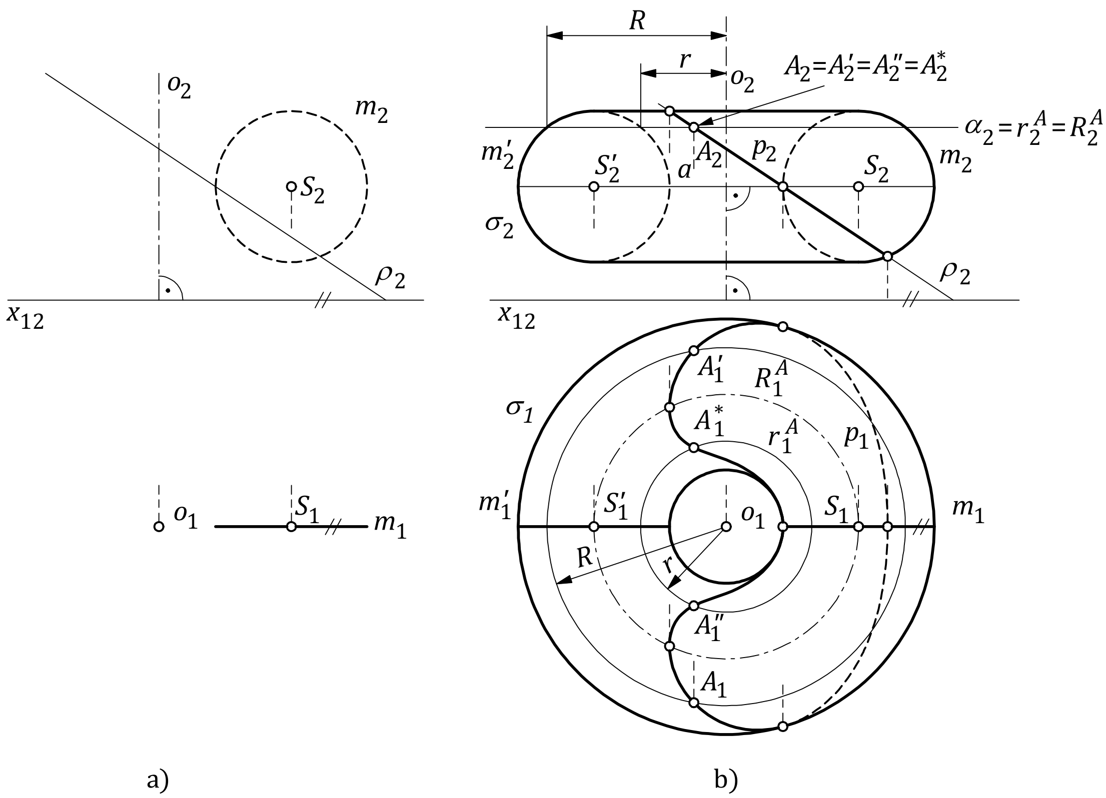


Figure 5.9: Intersection of surface of revolution and projecting plane $\rho \perp \nu$

Analysis

The surface of revolution σ generated by the given principal half-meridian m is a torus. Firstly, it is necessary to construct the front view σ_2 and the top view σ_1 of the torus, see fig. 5.9 b). To construct intersection curve p of the torus and the given plane ρ, a pointwise approach is used, where the intersection points of the section plane ρ and parallel circles of individual points suitably chosen on the surface of revolution are constructed. Finally, the curve of intersection p is drawn as a curve passing through all constructed points.

Graphical solution

1. Construct front view m'_2 of the principal left half-meridian symmetrically with respect to the front view o_2.

2. Construct front views of craters as straight line segments tangent to both half-meridians.

3. Construct top view of the throat and equator of the torus.

Since $\rho \perp \nu$, the front view p_2 of intersection curve p is projected as the straight line segment $p_2 \subset \rho_2$, see fig. 5.9 b). The top view p_1 can be obtained by the following pointwise construction.

1. Choose front view $A_2 \in p_2$.

2. Construct ordinate $a \perp x_{12}$, $A_2 \in a$.

3. Construct $\alpha_2 \perp o_2$, $A_2 \in \alpha_2$. Plane α is the plane of parallel circle of point A. In this case, two parallel circles r^A and R^A can be found on the torus.

4. Construct top view $r_1^A = (o_1, r)$ and $R_1^A = (o_1, R)$, radii r and R measure in the front view.

5. Top view of point on intersection curve p is the intersection of ordinate a and top views r_1^A and R_1^A of parallel circles: $A_1, A'_1 = R_1^A \cap a$, $A''_1, A^*_1 = r_1^A \cap a$ (depending on the position of point A on the torus, there are one, two, three or four intersections).

6. Continue in a similar way to obtain a sufficient number of points on intersection curve p. Do not forget points at special positions such as points located on the plane of craters, throat and equator of torus. Finally, draw the top view p_1 and indicate its visibility.

□

■ Example 5.6 – Intersection of surface of revolution and projecting plane $\rho \perp \pi$

Given

Principal right half-meridian m, axis of revolution o of a surface of revolution $\sigma = (m, o)$ and section plane $\rho \perp \pi$ in Monge projection, see fig. 5.10 a).

Required

Using Monge projection, construct the intersection curve p of the surface σ and the given plane $\rho \perp \pi$, $p = \sigma \cap \rho$.

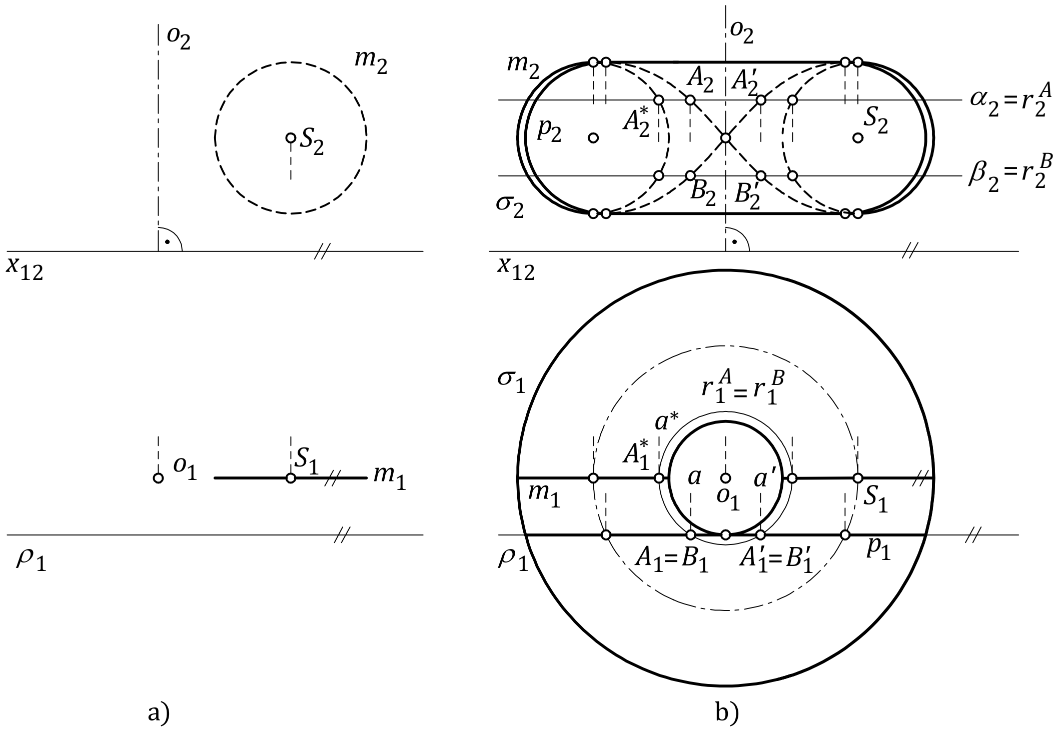


Figure 5.10: Intersection of surface of revolution and projecting plane $\rho \perp \pi$

Analysis

The surface of revolution σ generated by the given principal half-meridian m is a torus. The construction of its front view σ_2 and top view σ_1 is described in example 5.5. Since $\rho \perp \pi$, the top view p_1 of intersection curve p is projected as the straight line segment $p_1 \subset \rho_1$, see fig. 5.10 b). The front view can be obtained by pointwise construction, where the intersection points of the section plane and parallel circles of individual points suitably chosen on the surface of revolution are constructed. Finally, the curve of intersection is drawn as a curve passing through all constructed points.

Graphical solution

1. Choose top view $A_1 \in p_1$.
2. Construct ordinate $a \perp x_{12}$, $A_1 \in a$.
3. Construct top view $r_1^A = (o_1, r = ||o_1 A_1||)$.
4. Top view of point on principal meridian with parallel circle r^A: $A_1^* = m_1 \cap r_1^A$.
5. Construct ordinate $a^* \perp x_{12}$, $A_1^* \in a^*$.
6. Front view of point on principal meridian with the parallel circle r^A: $A_2^* = m_2 \cap a^*$.
7. Construct front view $\alpha_2 \perp o_2$, $A_2^* \in \alpha_2$. Plane α is the plane of parallel circle r^A.
8. Front view of point on intersection curve p: $A_2 = \alpha_2 \cap a$.

9. Continue in a similar way to obtain a sufficient number of points on intersection curve p. Do not forget points at special positions such as points located on craters, throat and equator of the torus. Finally, draw the front view p_2 of intersection curve p and indicate its visibility.

Note that the top view r_1^A intersects the top view m_1 two times. Thus, it is possible to use intersection of r_1^A and the right principal half-meridian to find the position of the front view α_2, too. Next, the top view r_1^A intersects the top view ρ_1 two times – at the chosen top view A_1 of point A and at the top view A'_1 of point A'. The front view A'_2 is the intersection of α_2 and ordinate a'. Finally, the ordinate a^* intersects the front view m_2 two times and position of front view β_2 can be determined, too. The plane β is the plane of parallel circle r^B with the same radius as r^A but different altitude. Ordinates a and a' intersect the front view β_2 at points B_2 and B'_2 on intersection curve p, see fig. 5.10 b). □

5.3 Intersection of surfaces of revolution

In general, intersection of two surfaces of revolution $\sigma = (m, o)$ and $\sigma' = (m', o')$ is a spatial curve $q = \sigma \cap \sigma'$ containing common points of both surfaces. A pointwise construction based on intersection of auxiliary surfaces with surfaces of revolution is used to find a sufficient number of common points of both surfaces. Each auxiliary surface intersects both surfaces at individual curve of intersection. These curves either intersect each other or not. Intersecting curves define a point on intersection curve q of both surfaces. Finally, the intersection curve q is drawn as a curve passing through all constructed points.

The auxiliary surface is chosen so that its intersection with the surfaces of revolution can be easily constructed. Depending on mutual position of surfaces of revolution, the following auxiliary surfaces are used.

- **Surfaces of revolution with identical axes** – no auxiliary surface is used. Depending on the shape of principal meridians m and m', the intersection curve q is a parallel circle (a set of parallel circles) common for both surfaces, see examples 5.7 and 5.8.

- **Surfaces of revolution with parallel axes** – auxiliary planes perpendicular to both axes are used. Each plane intersects both surfaces at a parallel circle. Intersecting circles define points on intersection curve of both surfaces of revolution, see example 5.9.

- **Surfaces of revolution with intersecting axes** – auxiliary spheres are used. The centre of all auxiliary spheres lies at the intersection of axes. Each sphere intersects both surfaces at two sets of parallel circles. Intersecting circles define points of the intersection curve of both surfaces of revolution, see example 5.10 and section 5.3.2.

- **Surfaces of revolution with skew axes** – auxiliary planes parallel with both axes are used. Each plane intersects both surfaces at intersection curve. Intersection curves which intersect each other define points on intersection curve of both surfaces of revolution. Graphical solution of this situation is beyond the scope of this textbook.

5.3.1 Example problems – intersection of surfaces of revolution

■ **Example 5.7 Surface of revolution and sphere with centre on axis of revolution**

Given

Surface of revolution $\sigma = (m, o)$ and sphere $\sigma' = (S, m')$ in Monge projection, see fig. 5.11 a).

Required

Using Monge projection, construct intersection curve $q = \sigma \cap \sigma'$.

Analysis

Axis of revolution of a sphere is any straight line passing through its centre. Since $S \in o$, the axis of the sphere can be considered identical to the axis o, thus no auxiliary surface is used. The intersection of both surfaces consists of two parallel circles r^A and r^B – trajectories of points A and B lying at intersections of principal meridians $A, B = m \cap m'$, see fig. 5.11 b).

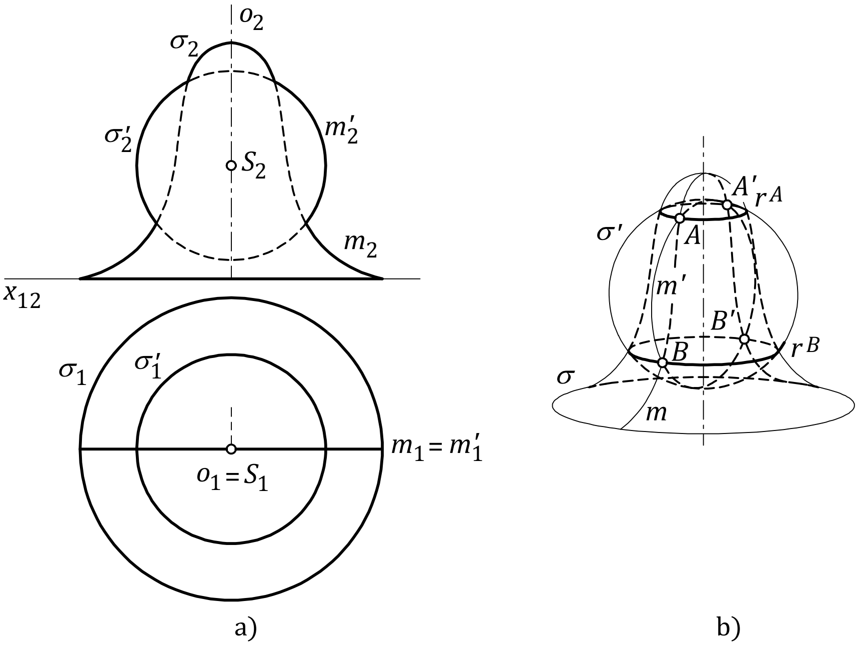


Figure 5.11: Intersection of surface of revolution and sphere

Graphical solution

1. Front views $A_2, A_2', B_2, B_2' = m_2 \cap m_2'$ of intersections of principal meridians, see fig. 5.12 b).
2. Draw front views $r_2^A = A_2A_2'$ and $r_2^B = B_2B_2'$ of parallel circles.
3. Draw top views $r_1^A = (o_1, r = \frac{1}{2}||A_2A_2'||)$ and $r_1^B = (o_1, r = \frac{1}{2}||B_2B_2'||)$ of parallel circles. □

■ Example 5.8 Surfaces of revolution with identical axes

Given

Surfaces of revolution $\sigma = (m, o)$ and $\sigma' = (m', o)$ in Monge projection, see fig. 5.13 a).

Required

Using Monge projection, construct intersection curve $q = \sigma \cap \sigma'$.

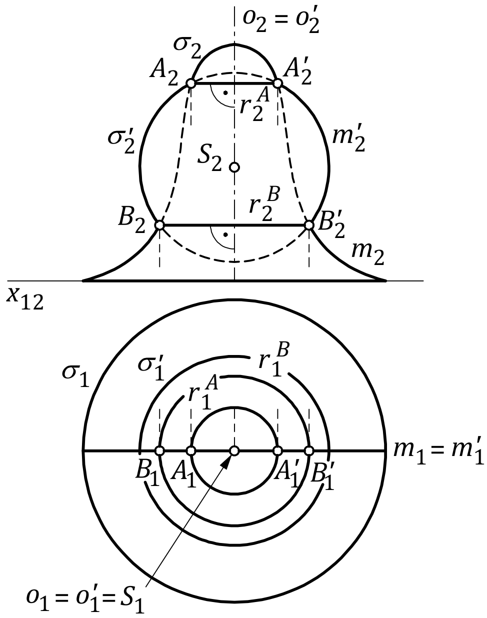


Figure 5.12: Intersection of surface of revolution and sphere – solution

Analysis

Since $o = o'$, no auxiliary surface is used. The intersection q of surfaces σ and σ' is parallel circle r^A – trajectory of point A lying at intersection of principal meridians $A = m \cap m'$, see fig. 5.13 b).

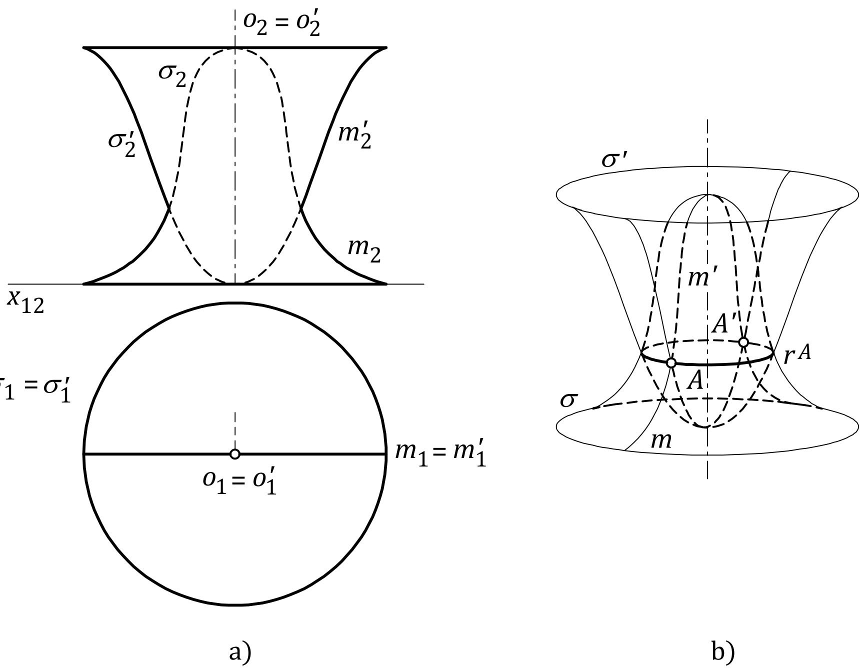


Figure 5.13: Intersection of surfaces of revolution with identical axes

Graphical solution

1. Front views $A_2, A_2' = m_2 \cap m_2'$ of intersections of principal meridians, see fig. 5.14.
2. Draw front view $r_2^A = A_2A_2'$ of parallel circle r^A.
3. Draw top view $r_1^A = (o_1, r = \frac{1}{2}||A_2A_2'||)$ of parallel circle r^A.

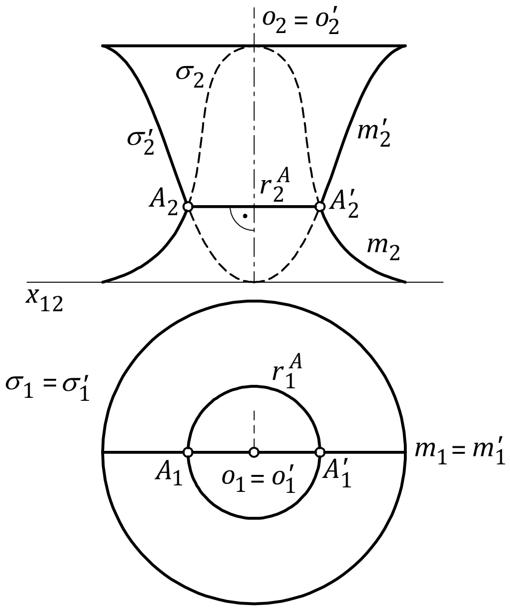


Figure 5.14: Intersection of surfaces of revolution with identical axes – solution

□

■ Example 5.9 Surfaces of revolution with parallel axes

Given

Cone of revolution $\sigma = (m, o)$ and sphere $\sigma' = (m', o')$ in Monge projection, see fig. 5.15 a).

Required

Using Monge projection, construct intersection curve $q = \sigma \cap \sigma'$.

Analysis

Since $o \parallel o'$, a set of auxiliary planes ρ perpendicular to axes o and o' is used. Each auxiliary plane ρ intersects the cone at parallel circle r^A and the sphere at parallel circle R^A. Point A of intersection curve q lies at the intersection of parallel circles $A = r^A \cap R^A$, see fig. 5.15 b).

Graphical solution

1. Draw front view $\rho_2 \perp o_2$ of auxiliary plane ρ, see fig. 5.16. The position of the plane ρ is chosen. It is delimited by front views $B_2, C_2 = m_2 \cap m_2'$ of common points B and C of

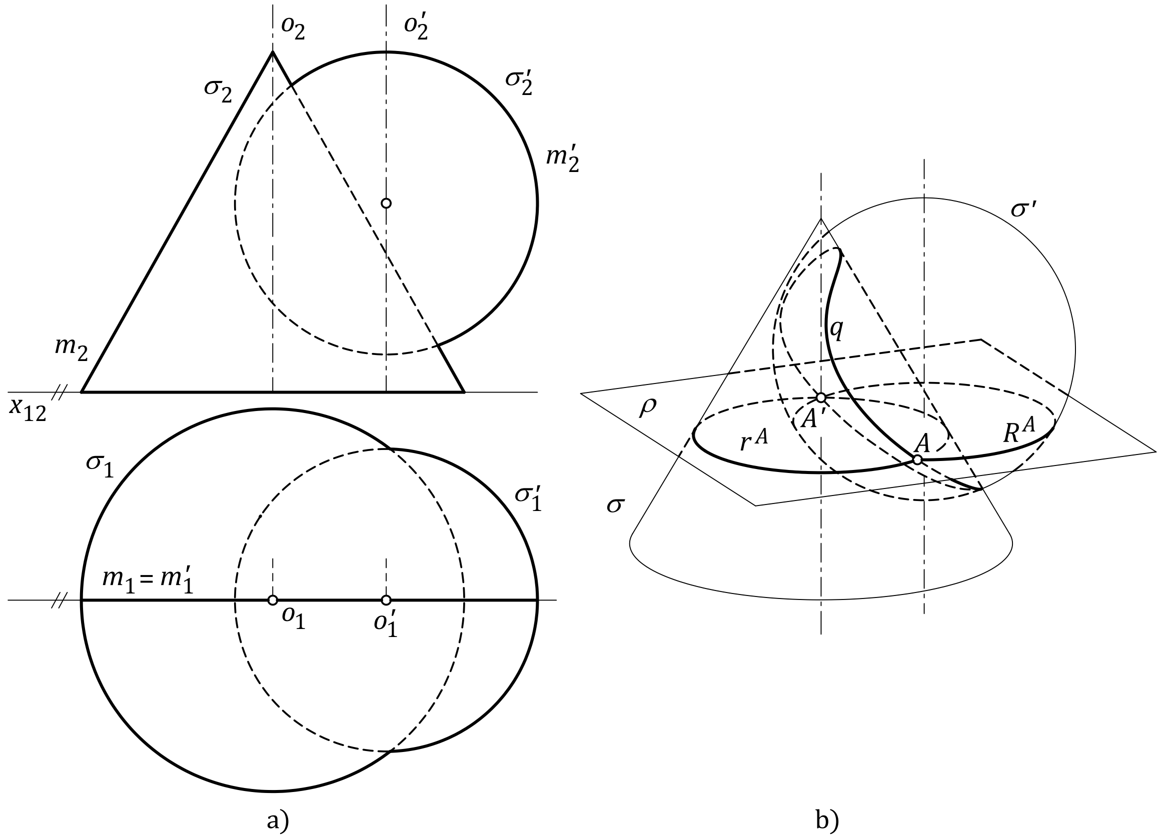


Figure 5.15: Intersection of surfaces of revolution with parallel axes

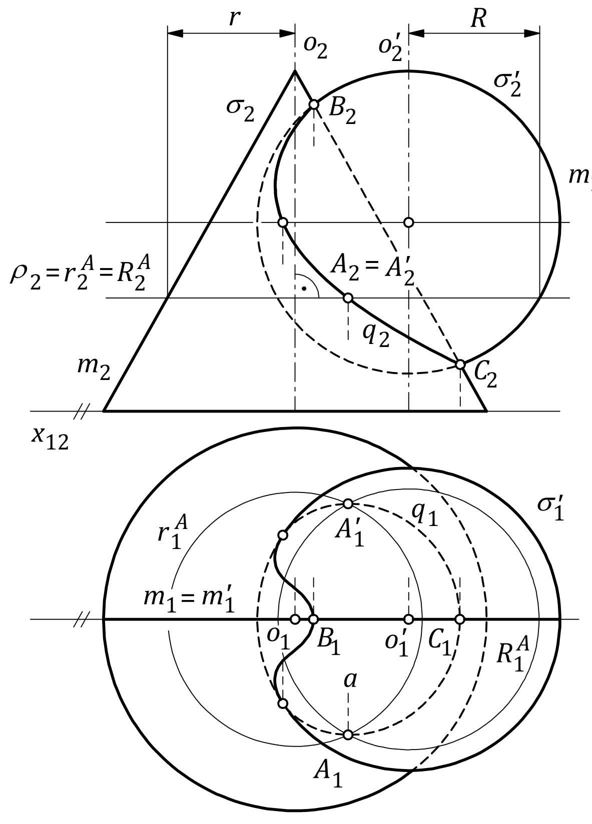


Figure 5.16: Intersection of surfaces of revolution with parallel axes – solution

principal meridians m and m'. The highest position of the auxiliary plane which intersects both surfaces is given by B_2, the lowest position by C_2.

2. Draw top views $r_1^A = (o_1, r)$ and $R_1^A = (o_1', R)$ of parallel circles $r^A = \sigma \cap \rho$ and $R^A = \sigma' \cap \rho$. Measure the radii r and R in the front view.

3. Top views $A_1, A_1' = r_1^A \cap R_1^A$ of points on intersection curve q.

4. Construct ordinate $a \perp x_{12}$, $A_1 \in a$. Since $o_1 o_1' \parallel x_{12}$, the top views A_1 and A_1' have the same ordinate a.

5. Front views $A_2, A_2' = \rho_2 \cap a$ of points on intersection curve p. Since $o_1 o_1' \parallel x_{12}$, the front views $A_2 = A_2'$.

6. Continue in a similar way to obtain a sufficient number of points on intersection curve q. Do not forget points at special positions such as points located on the equator of the sphere. Finally, draw the curve q and indicate its visibility. □

■ Example 5.10 Surfaces of revolution with intersecting axes

Given

A cone of revolution $\sigma = (m, o)$ and a cylinder of revolution $\sigma' = (m', o')$ in Monge projection, see fig. 5.17 a).

Required

Using Monge projection, construct intersection curve $q = \sigma \cap \sigma'$.

Analysis

Since the axes o and o' are intersecting, a set of auxiliary spheres with the centres at intersection of axes $S = o \cap o'$ is chosen. Each auxiliary sphere ρ intersects the cone at parallel circles r^A and s^A and the cylinder at parallel circles c^A and d^A. Points A, A', A'', A^* of intersection curve q lie at the intersections of these parallel circles, if they exist, see fig. 5.17 b).

Graphical solution

1. Draw front view $\rho_2 = (S_2, r_\rho)$ of auxiliary sphere ρ. The radius r_ρ is chosen in the range $r_\rho \in [r_{\min}, ||S_2 B_2||]$, where $r_{\min}$ is the radius of the sphere inscribed into the cylinder σ' (the smallest sphere which has the intersection with both surfaces) and B is common point of meridians m and m' at the maximum distance from the intersection S of axes of revolution.

2. Draw front views of parallel circles $r^A, s^A = \sigma \cap \rho$ – intersections of the auxiliary sphere and the cone, i.e. $r_2^A = 11''$, $s_2^A = 22'$, $1, 1', 2, 2' = m_2 \cap \rho_2$.

   Since $o, o' \parallel \nu$ the front views of parallel circles are straight line segments perpendicular to the corresponding axis.

3. Draw front views of parallel circles $c^A, d^A = \sigma' \cap \rho$ – intersections of the auxiliary sphere and the cylinder, i.e. $c_2^A = 33'$, $d_2^A = 44'$, $3, 3', 4, 4' = m_2' \cap \rho_2$. Since $o' \parallel \pi$, the top views of parallel circles c and d are straight line segments perpendicular to o_1'.

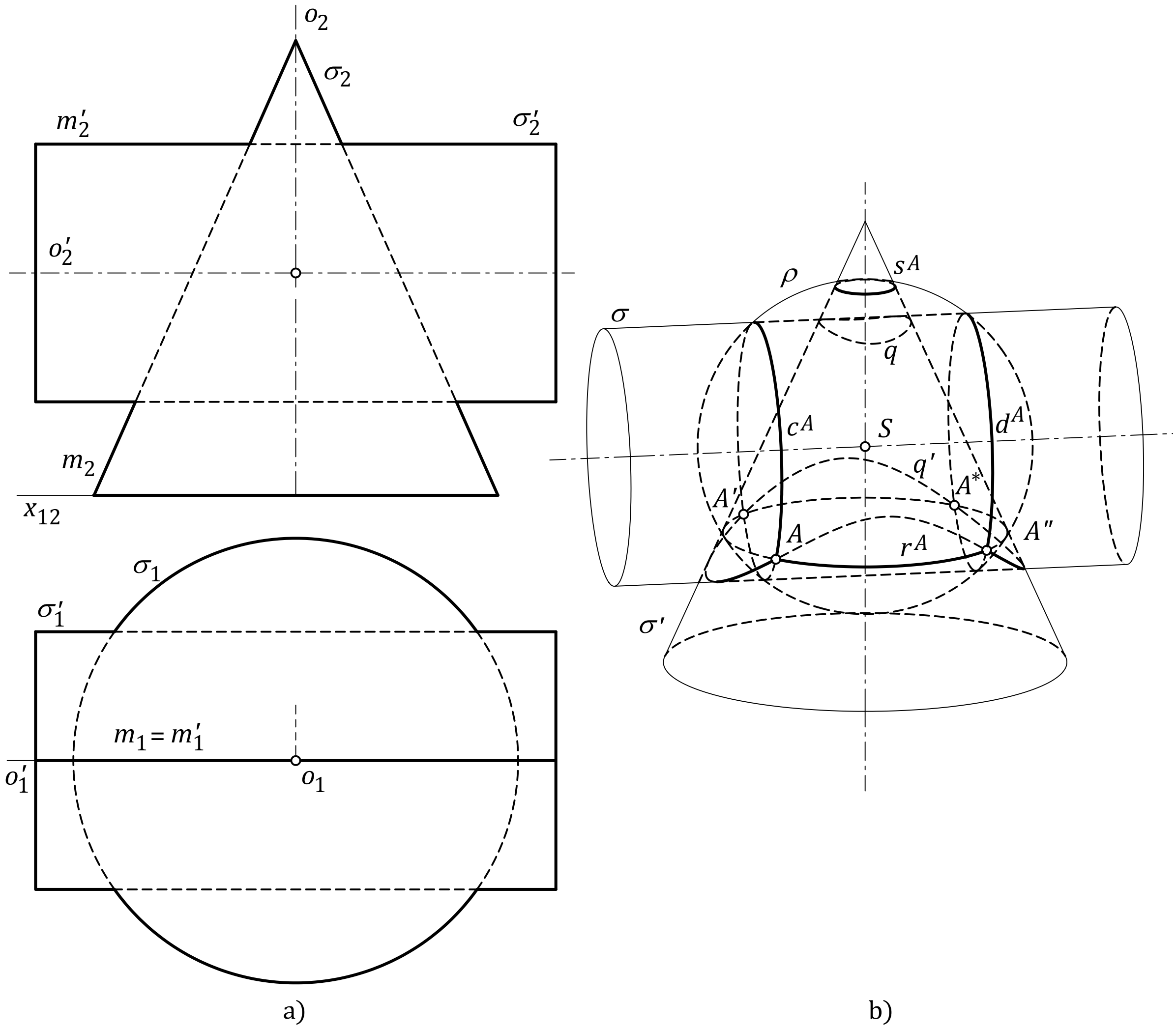


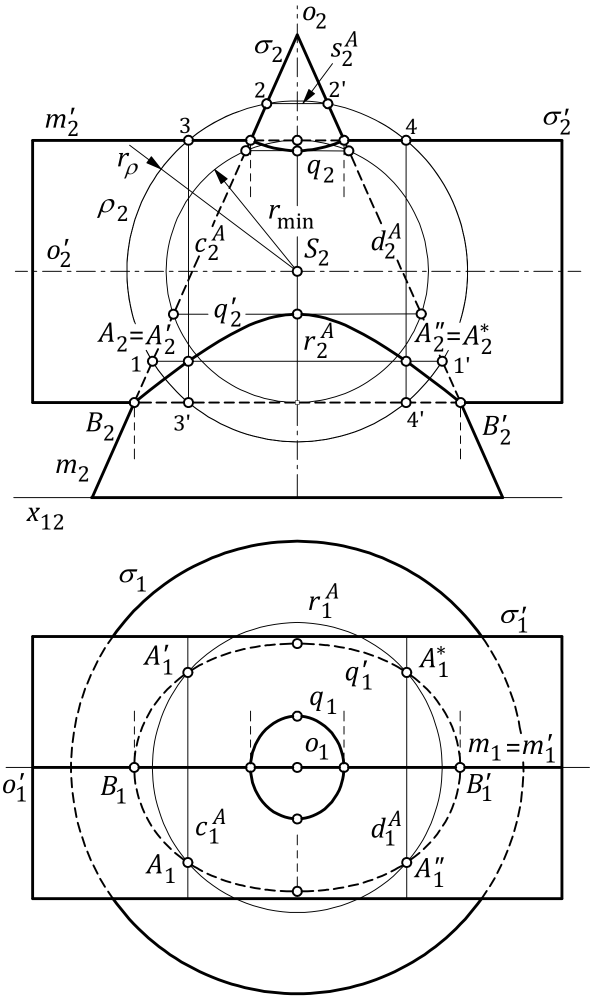
Figure 5.17: Intersection of surfaces of revolution with intersecting axes

4. Points on intersection curve q are possible intersections of parallel circles r^A, s^A, c^A and d^A. Front views $A_2 = A'_2 = r_2^A \cap c_2^A$ and $A''_2 = A^*_2 = r_2^A \cap d_2^A$. Parallel circle s^A has no intersection with any other parallel circle on the auxiliary sphere.

5. Draw top views $r_1^A = (o_1, r = \frac{1}{2}||11'||)$, $c_1^A \perp o'_1$ and $d_1^A \perp o'_1$.

6. Top views $A_1, A'_1 = r_1^A \cap c_1^A$ and $A''_1, A^*_1 = r_1^A \cap d_1^A$ of points on intersection curve p.

7. Continue in a similar way to obtain a sufficient number of points on intersection curve q. Do not forget points at special positions, i.e. points located on the sphere inscribed into the cylinder. Finally, draw the curve q and indicate its visibility.



b)

Figure 5.18: Intersection of surfaces of revolution with intersecting axes – solution

□

5.3.2 Degenerated intersection of two quadratic surfaces of revolution

Let us consider intersection of two quadratic surfaces of revolution (see section 4.5) with intersecting axes. A special case, when the spatial intersection curve degenerated into two conic sections, i.e. planar curves, is depicted in fig. 5.19.

To formulate the condition of degeneration, consider front views of the three situations depicted in fig. 5.20. Here, an intersection of a cone of revolution $\sigma = (m, o)$ and a cylinder of revolution $\sigma' = (m', o')$ with intersecting axes is drawn together with the smallest auxiliary sphere which intersects both surfaces. The cone and the axis of the cylinder are the same in all three cases. The radius of the cylinder is different. The plane given by axes o and o' is parallel with frontal plane of projection.

In situation depicted in fig. 5.20 a), the smallest auxiliary sphere is the sphere inscribed into the cylinder, i.e. the sphere contacts the cylinder along one parallel circle $11'$ and intersects the cone at two parallel circles $22'$ and $33'$. Therefore, only three parallel circles intersecting at four points are located on the auxiliary sphere. Two intersection Q_2 and Q_2' are visible in the front

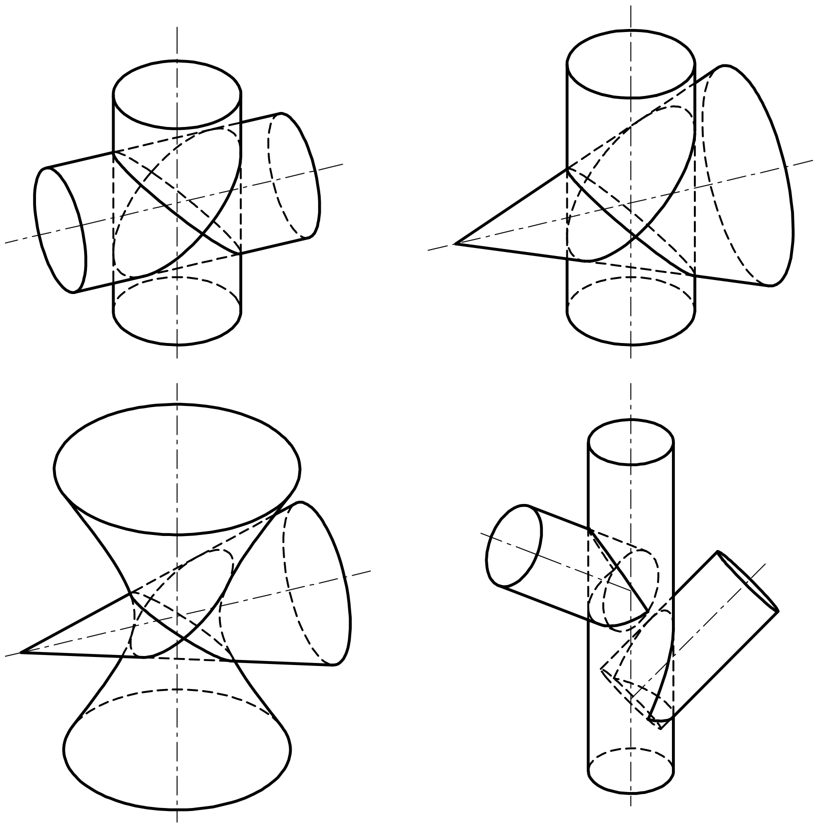


Figure 5.19: Degenerated intersection of quadratic surfaces of revolution with intersecting axes

view drawn in fig. 5.20 a). In the front view, points Q_2 and Q_2' are extreme points on both branches q_2 and q_2' of the intersection curve. Branches q and q' have no points of intersection.

Situation depicted in fig. 5.20 b) is similar. The smallest auxiliary sphere is the sphere inscribed into the cone, i.e. the sphere contacts the cone along one parallel circle $11'$ and intersects the cylinder at two parallel circles $22'$ and $33'$. Again, three parallel circles intersecting at four points are located on the auxiliary sphere. In the front view, points Q_2 and Q_2' are extreme points on both branches q and q' of the intersection curve. Branches q and q' have no points of intersection.

Finally, fig. 5.20 c) shows a special situation, where the smallest auxiliary sphere is inscribed into both surfaces. Consequently, there are only two parallel circles $11'$ and $22'$ intersecting at two points located on the auxiliary sphere. Front view Q_2 is the common extreme point on both branches q and q' of intersection curve. The branches q and q' are planar curves projected as straight line segments in the front view. Considering the position of the planes containing q and q' with respect to both surfaces of revolution, these planar curves are ellipses.

Thus, the following condition can be formulated: intersection of two quadratic surfaces of revolution with intersecting axes degenerates into two ellipses if there exists a sphere with the centre at intersection of the axes inscribed into both surfaces.

The degenerated cases of developable quadratic surfaces of revolution (cone and cylinder) are often applied in pipeline engineering due to the simplicity of their design and manufacturing. These situations are solved in chapter 8.

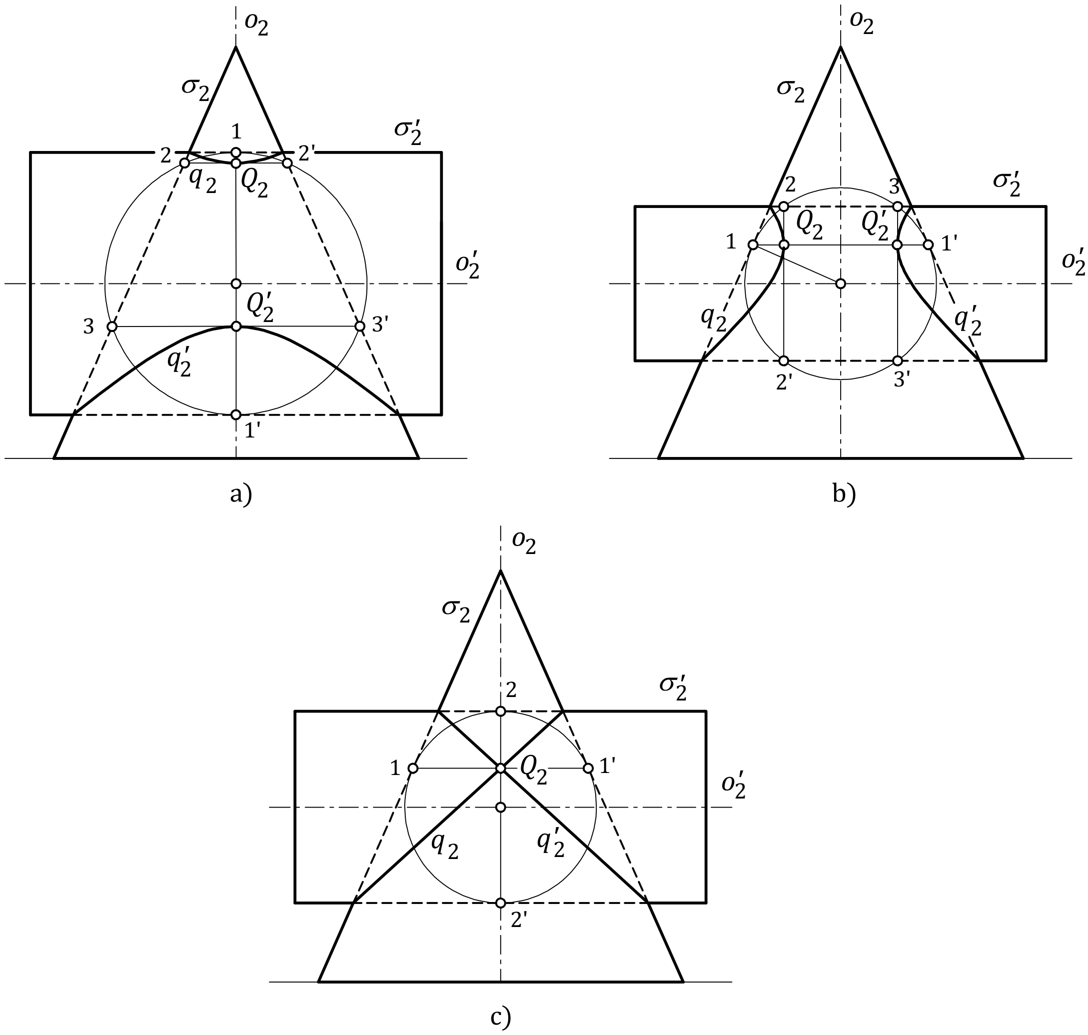


Figure 5.20: Intersection of quadratic surfaces of revolution with intersecting axes

Chapter 6

Helicoidal surfaces

Helicoidal surface σ is a figure generated by *screw motion* of *generating curve* k, see example in fig. 6.1. Screw motion is a composition of translation along axis o and revolution about axis o called *axis of screw motion*. Generating curve is a planar or spatial curve $k \neq o$. In this textbook, the length of translation directly proportional to the angle of revolution and constant distance of the generating curve from the axis of screw motion are considered.

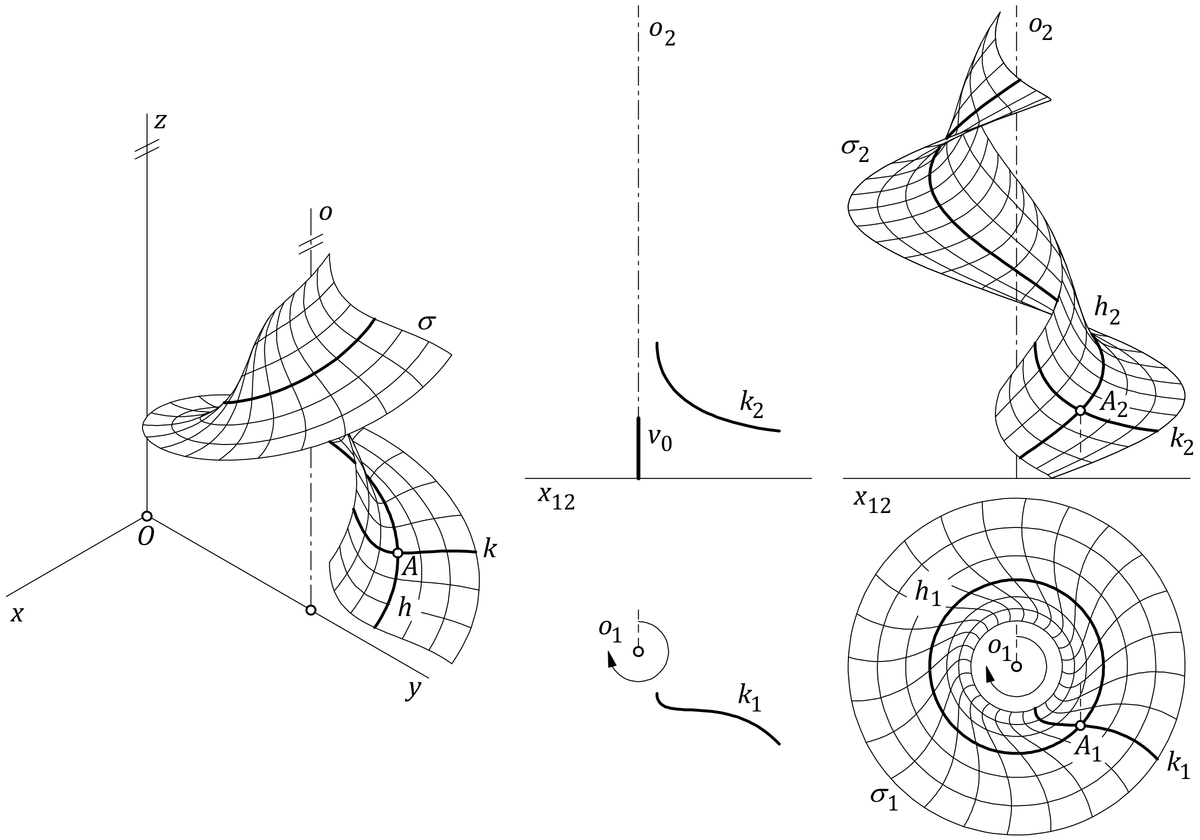


Figure 6.1: Helicoidal surface generated by right-handed screw motion of spatial generating curve in isometric view (left) and in Monge projection (right)

Screw motion is given by *axis* o, *parameter of screw motion* v_0 (i.e. translation directly proportional to the revolution about one radian), and right-handed or left-handed *orientation*. Axis o is a straight line in three-dimensional space in general position with respect to the coordinate system. Without loss of generality, assume the axis o is z-axis (or parallel with z-axis, i.e. $o \perp \pi$). The orientation of screw motion is designated by arrow around the top view o_1 in Monge projection: ↻ means right-handed (see fig. 6.1) and ↺ means left-handed (see fig. 6.2) orientation. The arrow denotes the descent direction of screw motion. Instead of parameter v_0, screw motion can be defined by *lead of screw motion* l, i.e. translation directly proportional to the revolution about 2π radians. One thread of screw motion is given by translation of one lead, i.e. by revolution about 2π radians.

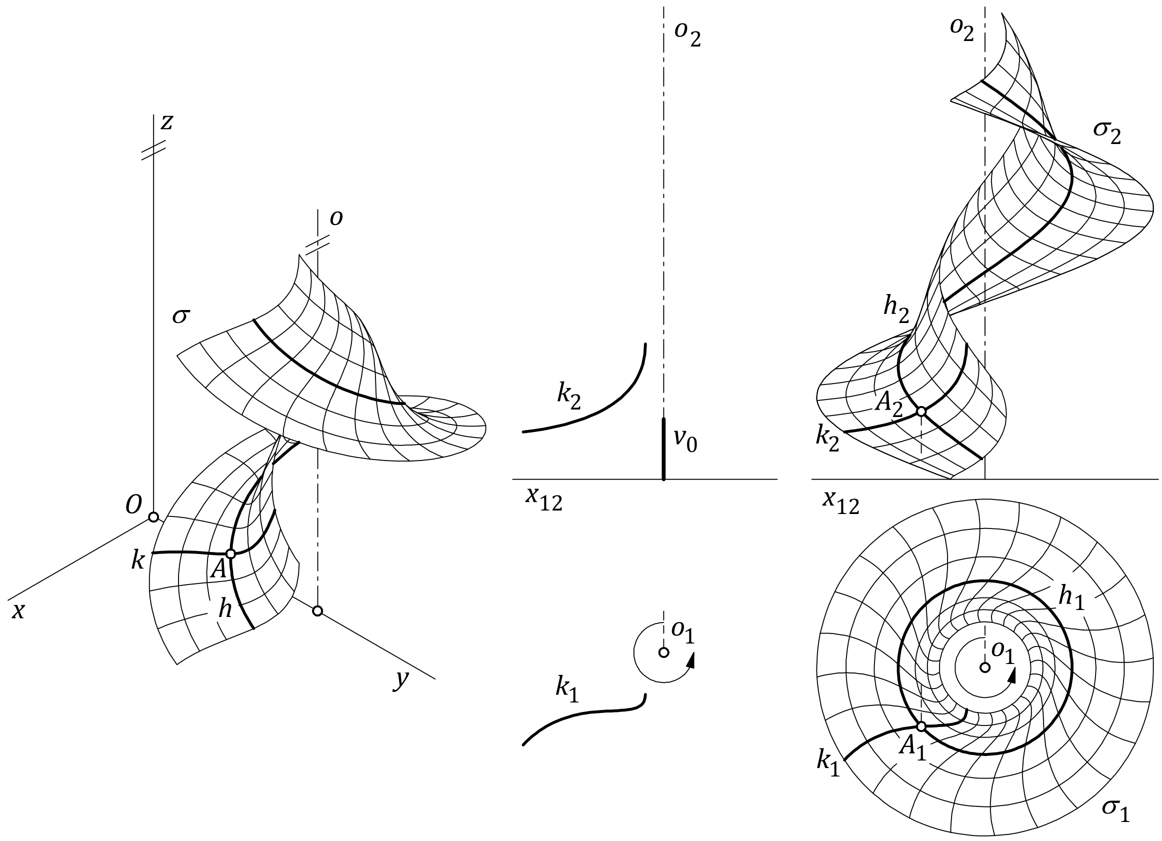


Figure 6.2: Helicoidal surface generated by left-handed screw motion of spatial generating curve in isometric view (left) and in Monge projection (right)

If the analytic representation of the generating curve $k \neq o$ is given by vector equation

$$k: \ \mathbf{C}(v) = \big(x(v), y(v), z(v)\big), \ v \in [v_1, v_2],$$

and $o = z$, the vector equation of one thread of the right-handed helicoidal surface σ generated by screw motion of curve k is given by

$$\begin{aligned}\sigma: \ \mathbf{S}(u,v) = &\big(x(v)\cos u - y(v)\sin u, \ x(v)\sin u + y(v)\cos u, v_0 u\big),\\ &u \in [0, 2\pi], \ v \in [v_1, v_2],\end{aligned}$$

and vector equation of the left-handed helicoidal surface κ is given by

$$\sigma: \ \mathbf{S}(u,v) = \big(x(v)\cos u - y(v)\sin u,\ -x(v)\sin u - y(v)\cos u, v_0 u\big),$$
$$u \in [0, 2\pi],\ v \in [v_1, v_2].$$

Parametric v-curves of one thread of the surface are congruent generating curves at individual screwed positions. Trajectory of any point A on generating curve is parametric u-curve of the surface called *helix*. Thus, at any point A on the helicoidal surface, there is located one helix h and one screwed position of the generating curve k. Both systems of parametric curves create a mesh useful for helicoidal surfaces visualization.

Tangent plane at regular point A of a helicoidal surface is determined by tangent lines to parametric curves passing through point A, i.e. by the tangent line to the generating curve and by the tangent line to the helix. Normal line at regular point A of a helicoidal surface is perpendicular to the tangent plane.

It is possible to create a helicoidal surface of the same shape by screw motion of any generating curve suitably located on the surface. There are two important planar curves on the helicoidal surface which are suitable as generating curves: *meridian* m and *normal section* c.

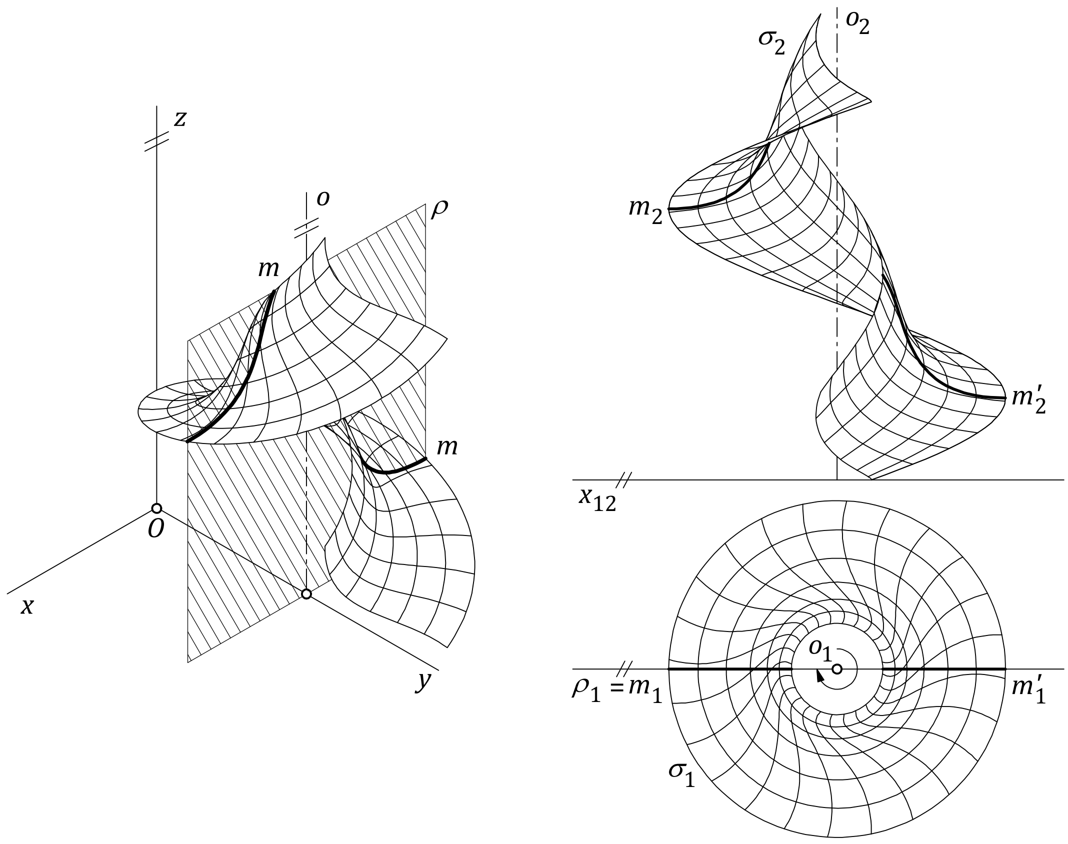


Figure 6.3: Principal meridian of helicoidal surface
in isometric view (left) and in Monge projection (right)

Meridian m of a helicoidal surface is a planar intersection of the helicoidal surface and a plane ρ passing through the axis o called *meridian plane*, see example in fig. 6.3. Considering one thread of helicoidal surface, the meridian is a pair of planar curves symmetric with respect to the axis of helicoidal surface translated in the axial direction of $\frac{l}{2}$. If the meridian plane is parallel with frontal plane of projection, the meridian is called *principal meridian* (*principal half-meridian*) and the plane is called *principal meridian plane.*

In Monge projection, the top view m_1 of principal meridian is a straight line segment (two straight line segments) on straight line parallel with x_{12} and passing through o_1. The front view m_2 of principal meridian is projected in true shape without any distortion, see fig. 6.3 right.

Normal section c of a helicoidal surface is a planar intersection of the helicoidal surface and a plane χ perpendicular to the axis o called *normal section plane*, see example in fig. 6.4. In Monge projection, the top view c_1 of normal section is projected in true size without any distortion. The front view c_2 of normal section is a straight line segment on the front view of normal section plane χ_2, see fig. 6.4 right.

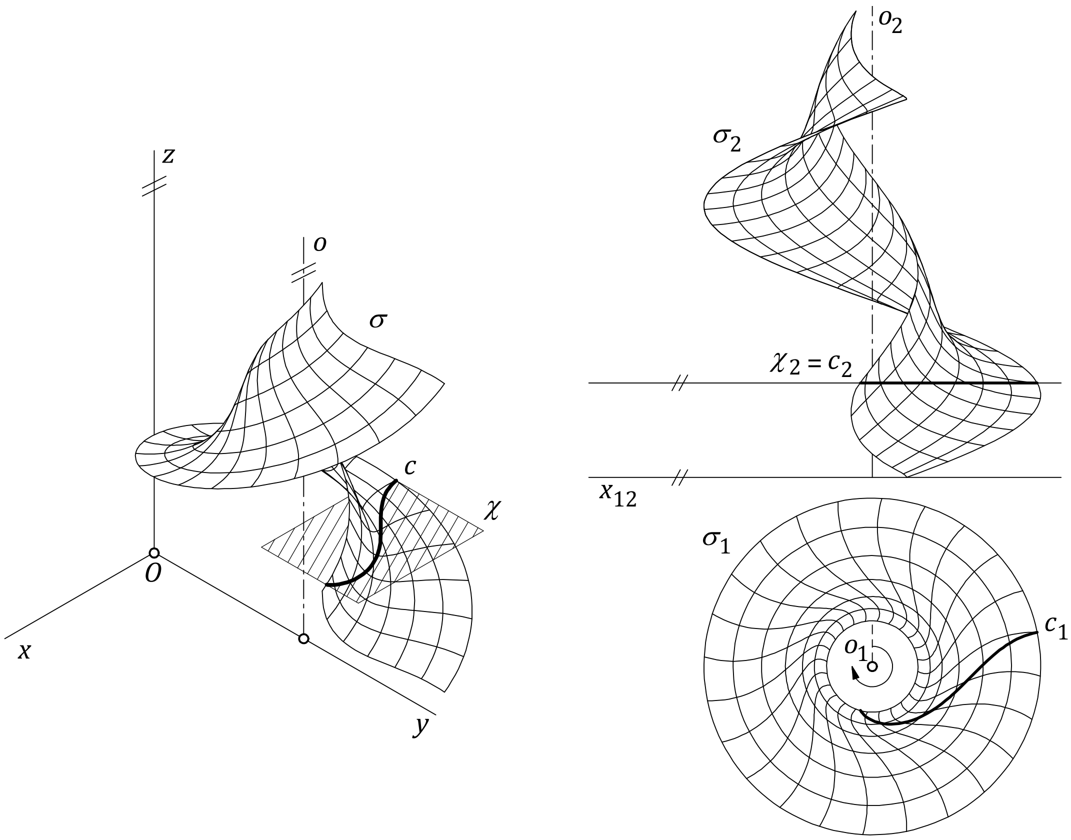


Figure 6.4: Normal section of helicoidal surface
in isometric view (left) and in Monge projection (right)

To obtain principal meridian or normal section of helicoidal surface, a pointwise construction based on solution of intersections between individual helices – parametric curves of the helicoidal surface and the section plane is used. Therefore, important properties of helix as a spatial curve and necessary basic constructions are mentioned in the following of this section first. Next, an overview of ruled helicoidal surfaces generated by screw motion of a straight line and cyclic helicoidal surfaces generated by screw motion of a circle is given. These surfaces are widely used in mechanical engineering. Finally, basic constructions on helicoidal surfaces such as construction of tangent plane, construction of principal meridian and construction of normal section are described in examples.

6.1 Properties of helix

The helix h is given by generating point $A \notin o$, axis o, parameter of screw motion v_0 (or lead of screw motion l) and orientation: $h = (A, o, v_0, \text{orientation})$ or $h = (A, o, l, \text{orientation})$. The vector equation of one thread of a right-handed helix with $o = z$ is

$$h : \mathbf{H}(u) = (r\cos u, r\sin u, v_0 u),\ \ u \in [0, 2\pi], \tag{6.1}$$

and vector equation of a left-handed helix with $o = z$ is

$$h : \mathbf{H}(u) = (r\cos u, -r\sin u, v_0 u),\ \ u \in [0, 2\pi],$$

where $r = ||A_1 o_1||$ is the distance of point A from axis o, see fig. 6.5.

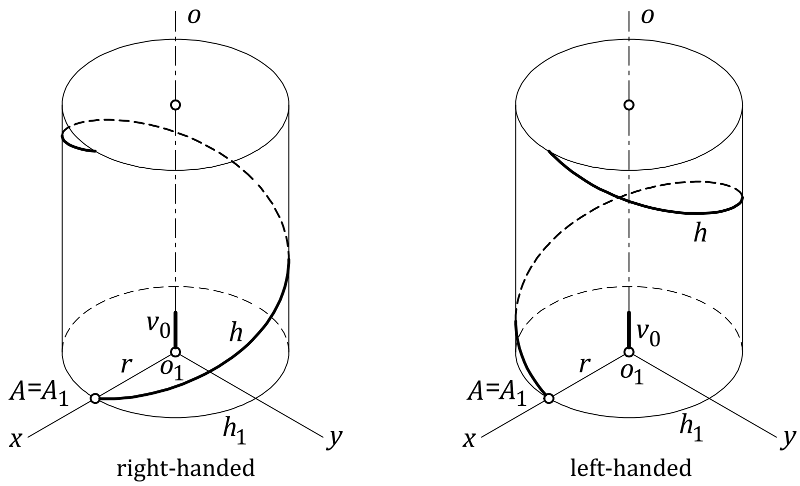


Figure 6.5: Helix

A right-handed helix given by eq. (6.1) is considered in the following list of important properties. The modifications necessary for a left-handed one are obvious.

- The top view $h_1 :\ \mathbf{H}_1(u) = (r\cos u, r\sin u, 0)$ of the helix h is a circle. It follows that helix lies on cylinder of revolution with axis o and radius r, see fig. 6.5. Construction of the top view of helix is described in example 6.1.

- The front view $h_2 :\ \mathbf{H}_2(u) = (r\cos u, 0, v_0 u)$ of the helix h is a cosine curve, see top views and front views of helices – parametric u-curves of helicoidal surfaces in fig. 6.1 and fig. 6.2. Construction of the front view of helix is described in example 6.1.

- The first curvature given by eq. (1.6) of helix is constant and equal to

$$^1k(u) = \frac{r}{r^2 + v_0^2}. \tag{6.2}$$

- The second curvature given by eq. (1.7) of helix is constant and equal to

$$^2k(u) = \frac{v_0}{r^2 + v_0^2}.$$

- Helix is a curve of *constant slope*, i.e. the angle α formed by tangent line to the helix and any plane perpendicular to the axis o is constant, see fig. 6.6. To prove this property, consider angle $\varphi = \frac{\pi}{2} - \alpha$ formed by unit tangent vector $\mathbf{H}'(u)$ of the helix (6.1) and coordinate vector $\mathbf{k} = (0,0,1)$ (direction vector of axis o). According to eq. (1.5), the unit tangent vector of helix is

$$\mathbf{H}'(u) = \left(\frac{-r \sin\ u}{\sqrt{r^2 + v_0^2}}, \frac{r \cos\ u}{\sqrt{r^2 + v_0^2}}, \frac{v_0}{\sqrt{r^2 + v_0^2}} \right)$$

and according to eq. (4.3), we get

$$\cos\varphi = \frac{v_0}{\sqrt{r^2 + v_0^2}}.$$

Since the parameter of screw motion v_0 and radius r are constant, the angle φ, and, consequently, α, has to be constant, too.

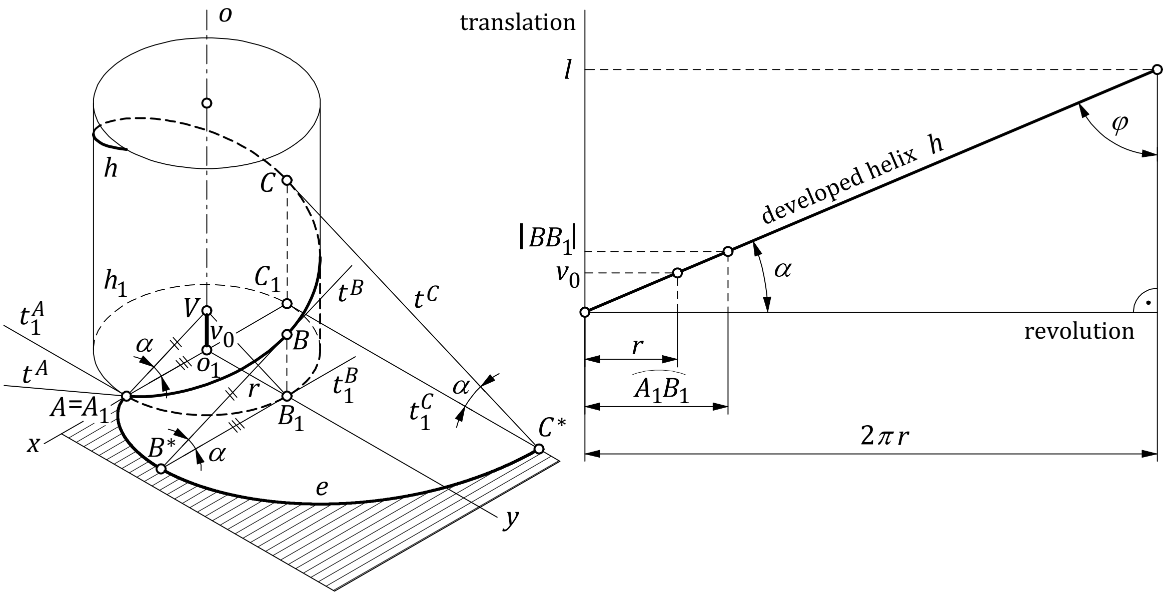


Figure 6.6: Properties of helix

- The development of a helix is a directly proportional graph with revolution (measured in the length of arc) on horizontal axis and translation on vertical axis, see fig. 6.6 right. The slope of this graph is equal to the angle α as follows from the previous properties. To draw this graph, it is necessary to determine one point of the developed helix. Depending on the given parameters, the coordinates of this point are (r, v_0) or $(2\pi r, l)$. The rectangle with the width equal to $2\pi r$ and height equal to l represents the development of the cylinder (see chapter 8) on which the helix is located.

  The graph of developed helix can be used to determine unknown translation of any point on helix from its known revolution or vice versa, see point $B \in h$ in fig. 6.6, for example.

- Intersection of tangent line to the helix and a plane perpendicular to the axis o lies on involute e obtained by rolling the top view of tangent line to the helix along the top view of the helix (see section 2.5).

  To understand this property, consider tangent line t^B to the helix h at point $B \in h$ in fig. 6.6 left. Tangent line t^B intersects (x, y) plane at point $B^* = t^B \cap t_1^B$. From the right angled triangle $\triangle BB_1B^*$ it follows that BB^* is the development of AB, i.e. $||BB^*|| = \widehat{AB}$, and B^*B_1 is the development of A_1B_1, i.e. $||B^*B_1|| = \widehat{A_1B_1}$. Therefore, the point B^* lies on involute e. Similarly, point $C^* = t^C \cap t_1^C$ lies on involute e, too.

- Tangent line to the helix is parallel with the generating line of a so called *directing cone of the helix* obtained by revolution of triangle $\triangle Ao_1V$ about axis o, see fig. 6.6 left. The boundary of circular base of the directing cone is the top view h_1 of helix h. It follows from equality of slope α of the directing cone and the helix. This property is used in construction of the tangent line to the helix, see example 6.2.

- The set of tangent lines to a helix forms a developable helicoidal surface σ called *tangent surface of a helix*. The intersection curve of tangent surface of a helix and a plane perpendicular to axis of helix is involute e, see fig. 6.7. To study properties of developable surfaces, see chapter 8.

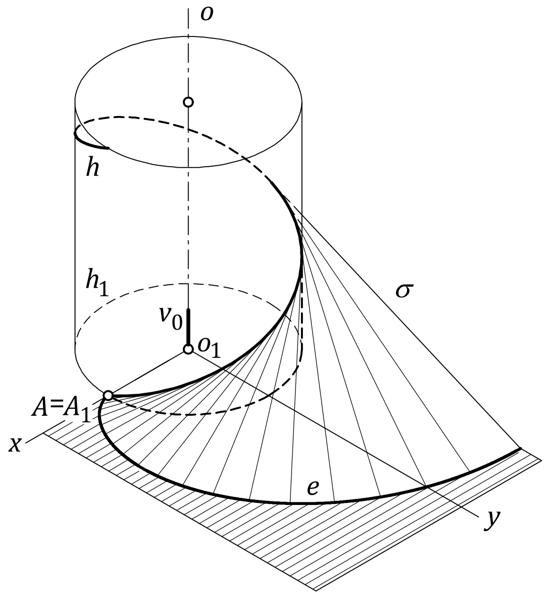


Figure 6.7: Tangent surface of a helix

6.1.1 Example problems – helix

■ Example 6.1 – Helix in Monge projection

Given

Helices $g = (A, o, l, \text{right-handed})$ and $h = (A, o, l, \text{left-handed})$ in Monge projection, see fig. 6.8 a) and fig. 6.9 a).

Required

Using Monge projection, construct top views and front views of both helices together with the cylinders of revolution on which the helices are located. Indicate the visibility of both helices.

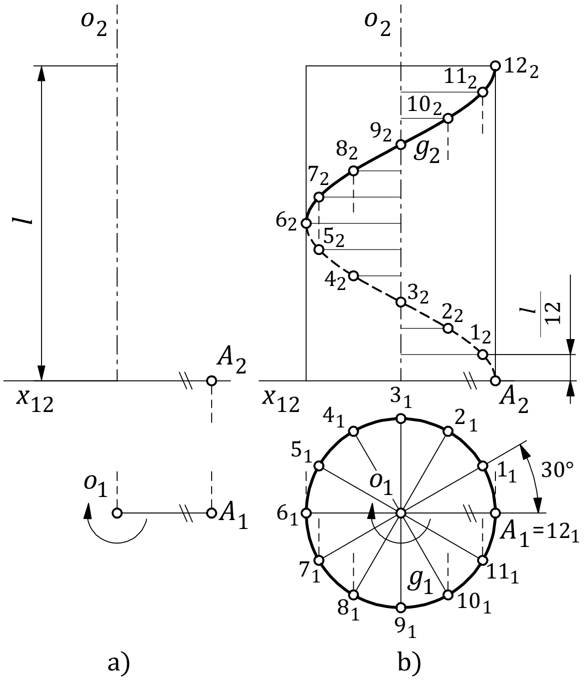


Figure 6.8: Right-handed helix in Monge projection

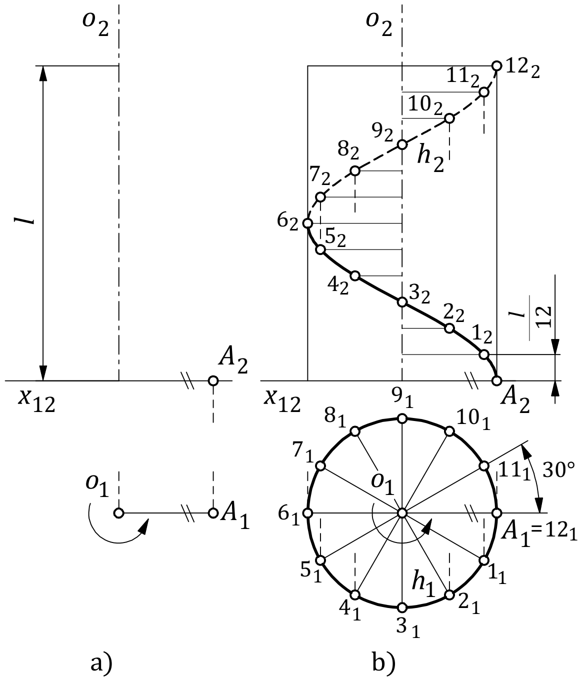


Figure 6.9: Left-handed helix in Monge projection

Analysis

The top view of helix as well as the top view of the corresponding cylinder of revolution is a circle $g_1 = (o_1, r = ||o_1A_1||)$, see fig. 6.8 b) and $h_1 = (o_1, r = ||o_1A_1||)$, see fig. 6.9 b). The front view of the cylinder of revolution is rectangle of width equal to $2||o_1A_1||$ and height equal to the lead l. The front view of the helix is a cosine curve.

Graphical solution

1. Starting from top view A_1, divide the top view g_1 (h_1) of the helix into a sufficient number n of equal parts. In fig. 6.8 b) and fig. 6.9 b) $n = 12$, thus the circle g_1 (h_1) is divided by $30°$. Top views of the dividing points are designated $A_1, 1_1, 2_1, \ldots, 12_1$.
2. Determine translation $\frac{l}{n}$ corresponding to the revolution by $\frac{360°}{n}$ and starting from z-level of point A (i.e. from x_{12} in this case), construct n equidistant lines parallel with x_{12} in the front view. The distance between individual equidistant lines is equal to $\frac{l}{n}$.
3. Construct ordinates passing through $1_1, 2_1, \ldots, 12_1$.
4. Front views $1_2, 2_2, \ldots, 12_2$ of dividing points lie at the corresponding intersections of ordinates and the set of equidistant lines.
5. Draw the front view g_1 (h_1) as a curve passing through the front views $A_2, 1_2, 2_2, \ldots, 12_2$ and indicate its visibility. □

■ **Example 6.2 – Tangent line to the helix**

Given

Helices $g = (A, o, v_0, \text{right-handed})$ and $h = (A, o, v_0, \text{left-handed})$ in Monge projection, see fig. 6.10 a) and fig. 6.11 a).

Required

Using Monge projection, construct tangent line to each helix at its generating point.

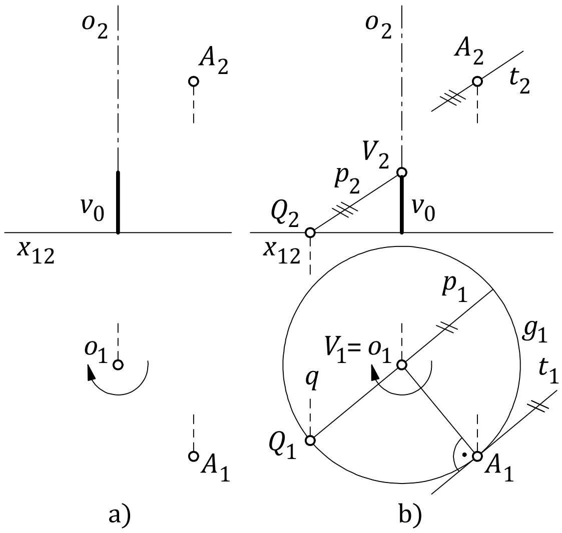


Figure 6.10: Tangent line to a right-handed helix

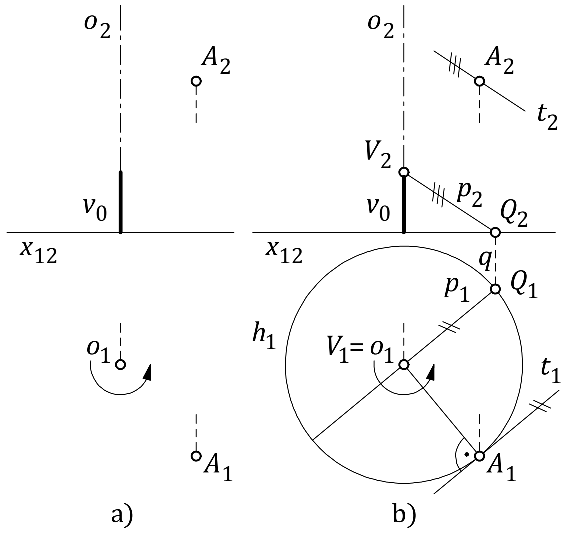


Figure 6.11: Tangent line to a left-handed helix

Analysis

Tangent line t to the helix is constructed by means of parallelism with generating line p of directing cone of the helix. Note that is not necessary to construct the front view of the helix.

Graphical solution

1. Construct top view $g_1 = (o_1, r = ||o_1A_1||)$, see fig. 6.10 b) (or $h_1 = (o_1, r = ||o_1A_1||)$, see fig. 6.11 b) in the case of left-handed helix h).
2. Draw o_1A_1.
3. Construct top view $t_1 \perp o_1A_1$, $A_1 \in t$ of tangent line to the helix.
4. Construct top view $p_1 \parallel t_1$, $o_1 \in p_1$ of generating line of the directing cone.
5. Top view $Q_1 = p_1 \cap g_1$ ($Q_1 = p_1 \cap h_1$). Point Q lies at the intersection of generating line p and the base g_1 (h_1) of the directing cone. Note that here are two possible top views of intersection $p_1 \cap g_1$ ($p_1 \cap h_1$). For the following constructions, it is necessary to use the correct intersection corresponding to the orientation of the helix.

6. Construct ordinate $q \perp x_{12}$, $Q_1 \in q$.

7. Front view $Q_2 = q \cap x_{12}$.

8. Draw front view $p_2 = Q_2V_2$ of generating line of the directing cone.

9. Construct front view $t_2 \parallel p_2$, $A_2 \in t_2$ of tangent line to the helix. □

■ Example 6.3 – Intersection of helix and plane perpendicular to axis of the helix

Given

Helices $g = (A, o, v_0, \text{right-handed})$ and $h = (A, o, v_0, \text{left-handed})$ in Monge projection, see fig. 6.12 a) and fig. 6.13 a).

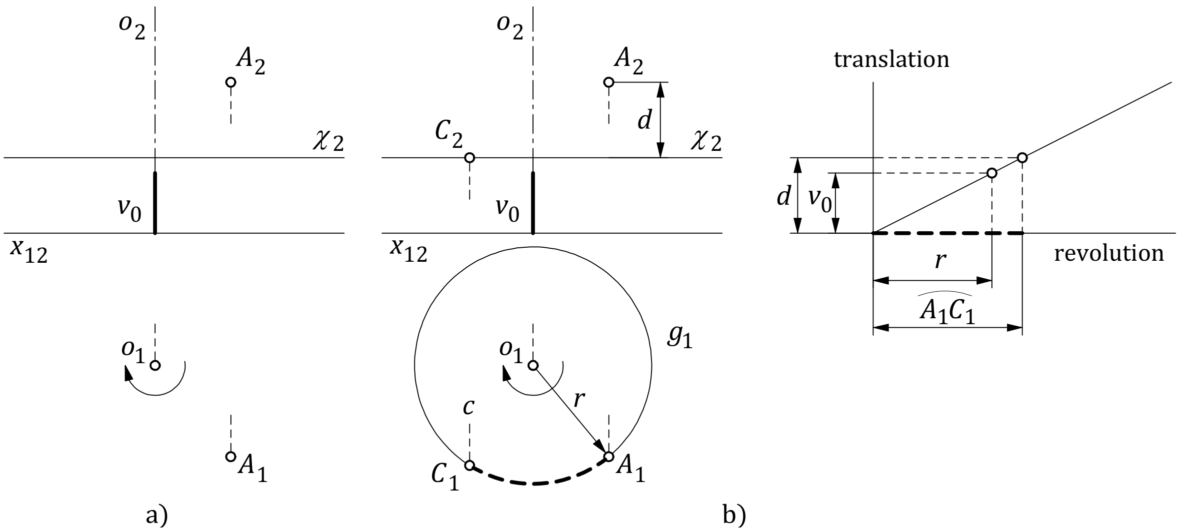


Figure 6.12: Intersection of right-handed helix and plane $\chi \perp o$

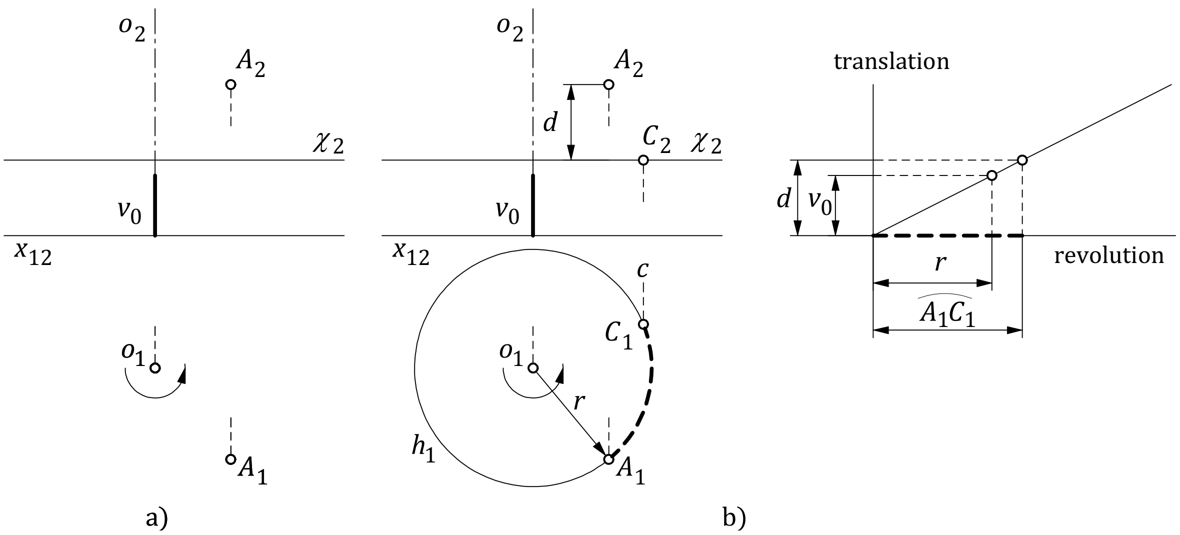


Figure 6.13: Intersection of left-handed helix and plane $\chi \perp o$

Required

Using Monge projection, construct intersection C of helices g and h with the given plane χ perpendicular to the axis of the helix.

Analysis

There are two points A and C on one helix. Point C lies in the plane $\chi \parallel \pi$, thus its z-coordinate is known and equal to the distance of χ_2 from x_{12}. To move a point from the A position to the C position along the helix, it is necessary to (1) translate downward ($z_A > z_C$) of the known distance d representing the difference of z-coordinates $d = |z_A - z_C|$, see fig. 6.12 b) and fig. 6.13 b), and (2) revolve of the unknown revolution corresponding to the translation d. To find the unknown revolution (measured in the length of arc), a graph of developed helix is used. Note that it is not necessary to construct the front view of the helix.

Graphical solution

1. Construct top view $g_1 = (o_1, r = ||o_1A_1||)$ (or $h_1 = (o_1, r = ||o_1A_1||)$ in the case of left-handed helix h).

2. Draw graph of developed helix g (h).

3. Measure the translation d in the front view and mark it on the vertical axis of the graph.

4. Determine the length of arc $\widehat{A_1C_1}$ corresponding to the translation d from the graph (designated by thick dashed line).

5. Measure the length of arc $\widehat{A_1C_1}$ along the top view g_1 (h_1) in the direction of arrow ↻ (↺) to find the top view C_1 (the corresponding arc is designated by thick dashed line). When measuring the length of the arc along a circle g_1 (h_1), it is possible to approximate it by the length of polygon, as is shown in fig. 2.33.

6. Construct ordinate $c \perp x_{12}$, $C_1 \in c$.

7. Front view $C_2 = \chi_2 \cap c$. □

■ Example 6.4 – Intersection of helix and axial plane

Given

Helices $g = (A, o, v_0, \text{right-handed})$ and $h = (A, o, v_0, \text{left-handed})$ in Monge projection, see fig. 6.14 a) and fig. 6.15 a).

Required

Using Monge projection, construct intersection C of helices g and h with the given plane ρ passing through the axis o.

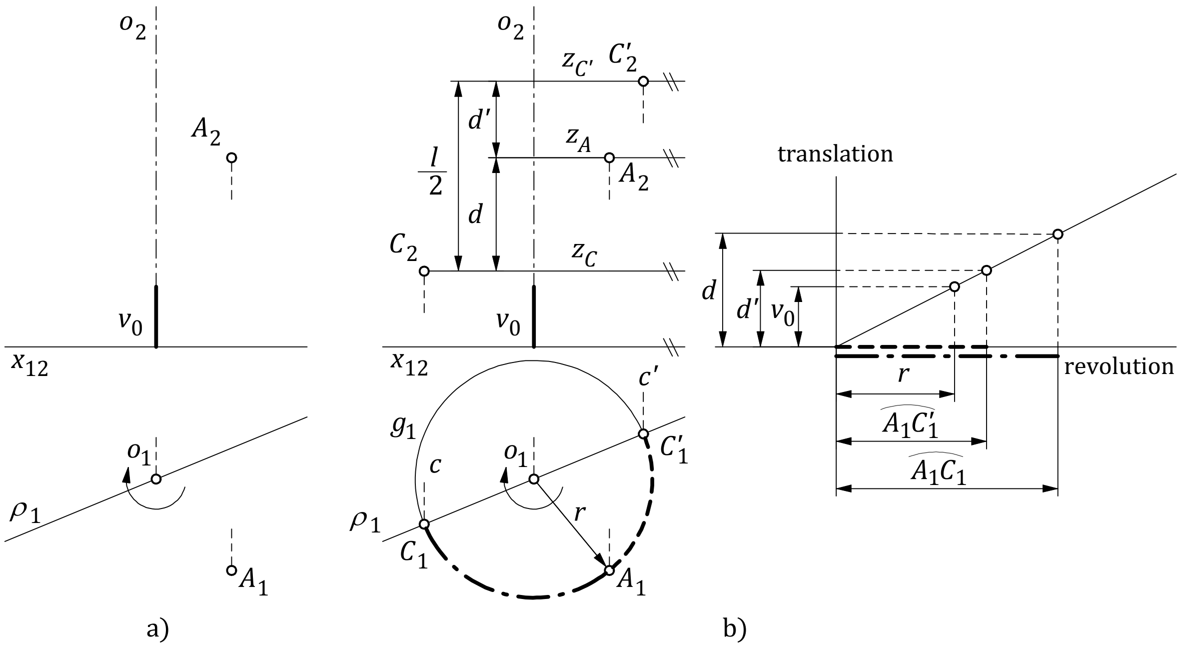


Figure 6.14: Intersection of right-handed helix and plane ρ passing through axis o

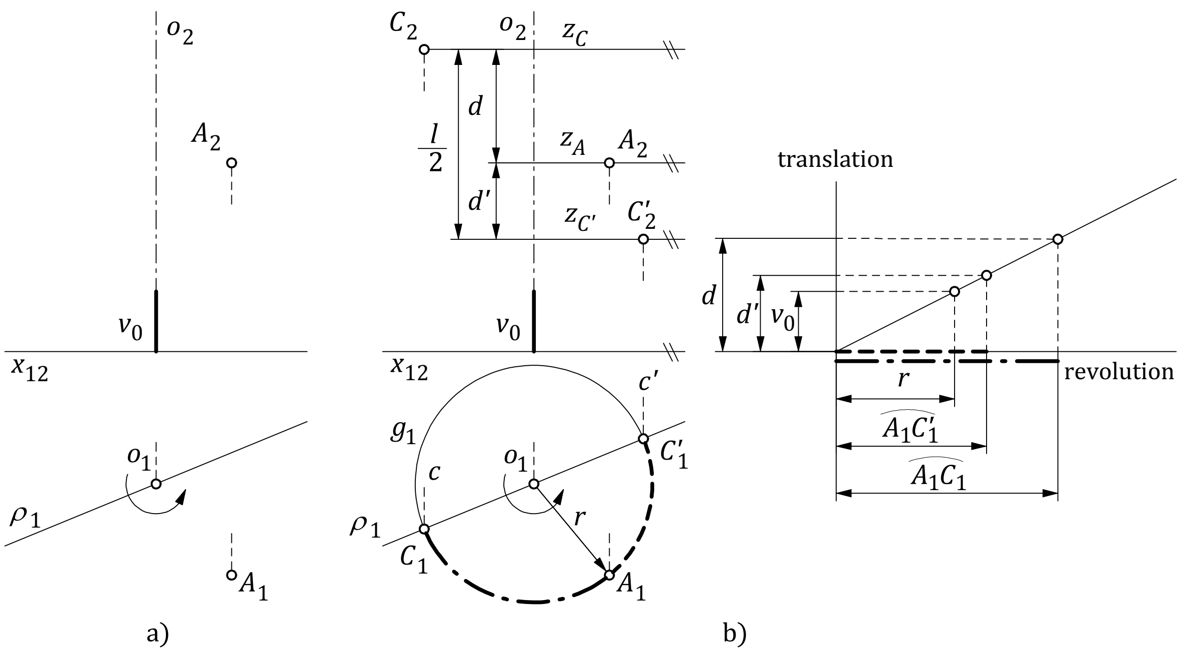


Figure 6.15: Intersection of left-handed helix and plane ρ passing through axis o

Analysis

Since the helix is an open cyclic curve, there are infinitely many intersection points of the helix and any plane passing through the axis of the helix. In this example, two intersections C and

C' of one thread of helix and the plane ρ are considered only. Note that is not necessary to construct the front view of the helix.

Points C and C' lie in the plane $\rho \perp \pi$, thus their top views C_1 and C_1' are known, see fig. 6.14 b) and fig. 6.15 b). To move a point from the A position to the C or C' position along the helix, it is necessary to (1) revolve of the known angle $\angle A_1 o_1 C_1$ or $\angle A_1 o_1 C_1'$ (measured in the length of the corresponding arc), and, (2) translate of the unknown translation d or d' corresponding to the revolution of $\angle A_1 o_1 C_1$ or $\angle A_1 o_1 C_1'$. To find the unknown translation, a graph of developed helix is used.

Graphical solution

1. Construct top view $g_1 = (o_1, r = ||o_1 A_1||)$ (or $h_1 = (o_1, r = ||o_1 A_1||)$ in the case of left-handed helix h).
2. Draw graph of developed helix g (h).
3. Top views $C_1, C_1' = \rho_1 \cap g_1$ ($C_1, C_1' = \rho_1 \cap h_1$).
4. Construct ordinates $c, c' \perp x_{12}$, $C_1 \in c$, $C_1' \in c'$.
5. Measure the length of arcs $\widehat{A_1 C_1}$ (designated by thick dashed line) and $\widehat{A_1 C_1'}$ (designated by thick dot and dash line) along the top view of helix g_1 (h_1) and mark them on horizontal axis of the graph.
6. Construct auxiliary line $z_A \parallel x_{12}$, $A_2 \in z_A$.
7. Determine the translations d and d' corresponding to the revolution of angles $\angle A_1 o_1 C_1$ and $\angle A_1 o_1 C_1'$ from the graph and draw auxiliary lines $z_C \parallel z_A$ at the oriented distance d and $z_C' \parallel z_A$ at the oriented distance d' from z_A. The orientation depends on the orientation of screw motion: in the case of right-handed helix $z_C < z_A$ and $z_C' > z_A$, in the case of left-handed helix $z_C > z_A$, $z_C' < z_A$.
8. Front views $C_2 = c \cap z_C$ and $C_2' = c' \cap z_C'$. Note that the angle $\angle C_1 o_1 C_1' = 180°$, it follows, that $||C_2 C_2'|| = \frac{l}{2}$. □

6.2 Ruled helicoidal surfaces

Ruled helicoidal surfaces are generated by screw motion of a straight line k, $k \neq o$, $k \nparallel o$. According to the mutual position of the generating straight line k and the axis of screw motion o and the angle formed by these two straight lines, the following types of ruled helicoidal surfaces are distinguished.

- **Closed right ruled helicoidal surface** – the generating line k and the axis o are intersecting perpendicular lines, $k \perp o$, see fig. 6.16.
- **Open right ruled helicoidal surface** – the generating line k and the axis o are skew perpendicular lines, $k \perp o$, see fig. 6.17.
- **Closed oblique ruled helicoidal surface** – the generating line k and the axis o are oblique intersecting lines, $k \not\perp o$, see fig. 6.18.
- **Open oblique ruled helicoidal surface** – the generating line k and the axis o are oblique skew lines, $k \not\perp o$, see fig. 6.19.

Examples of ruled helicoidal surfaces are shown in figs. 6.16 to fig. 6.19. Each figure contains isometric view and Monge projection of the surface. In Monge projection, the definition figures, principal meridian $m \subset \rho$ and normal section $c \subset \chi$ of the surface are drawn, too.

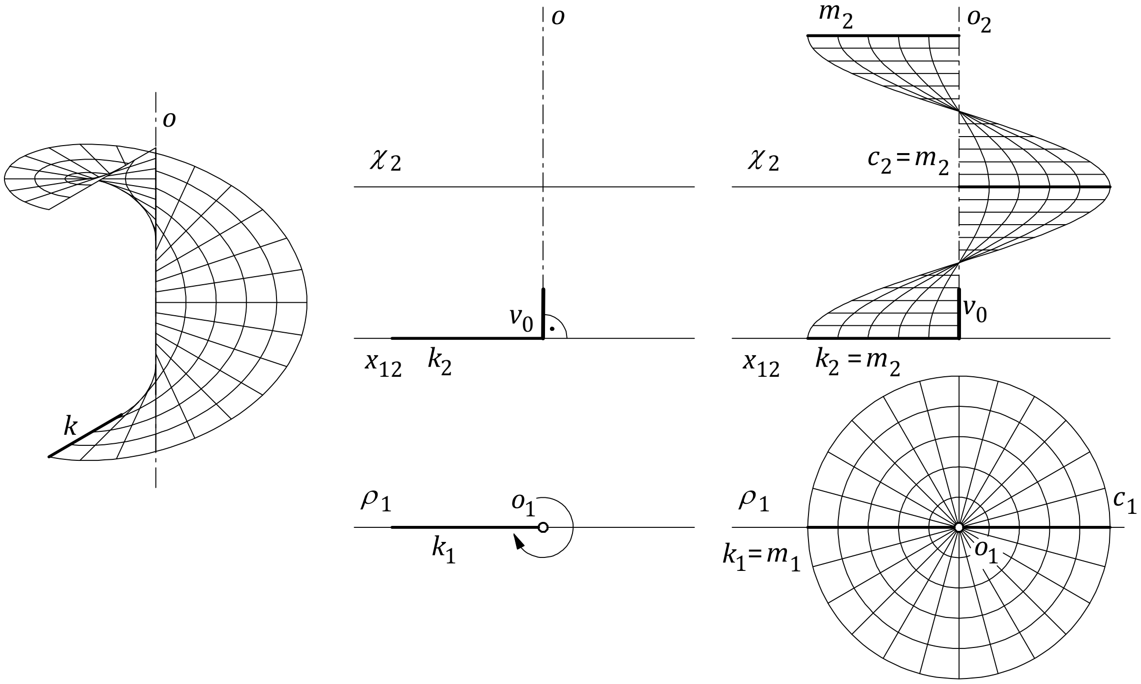


Figure 6.16: Closed right ruled helicoidal surface

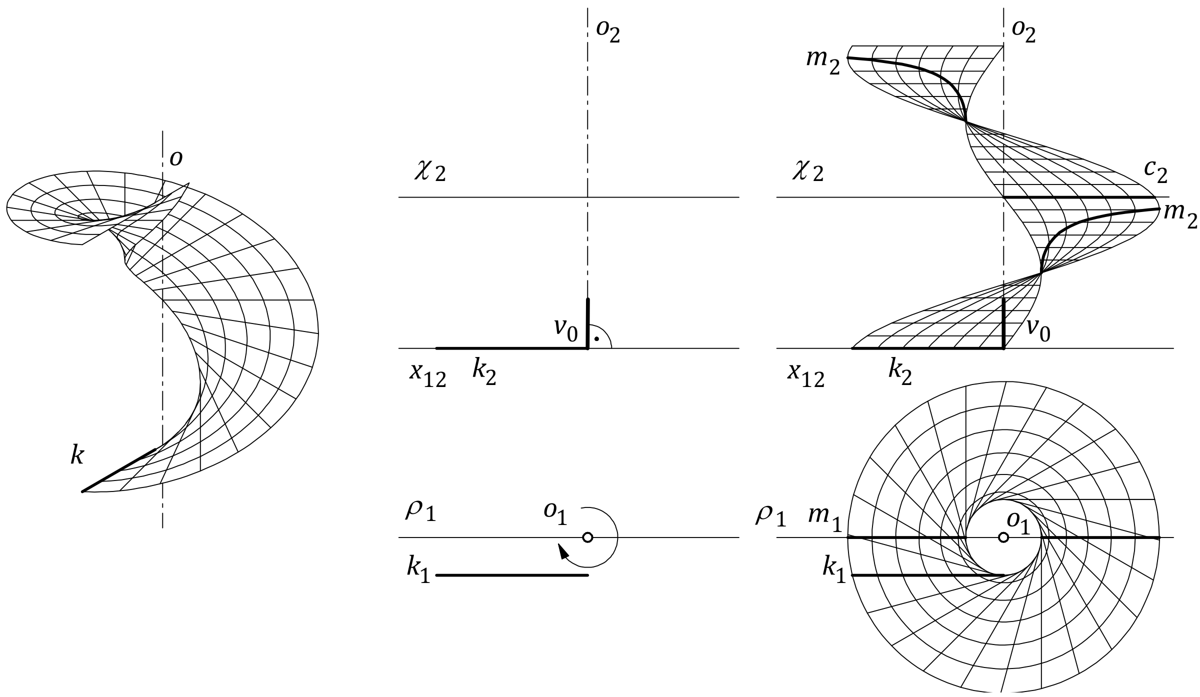


Figure 6.17: Open right ruled helicoidal surface

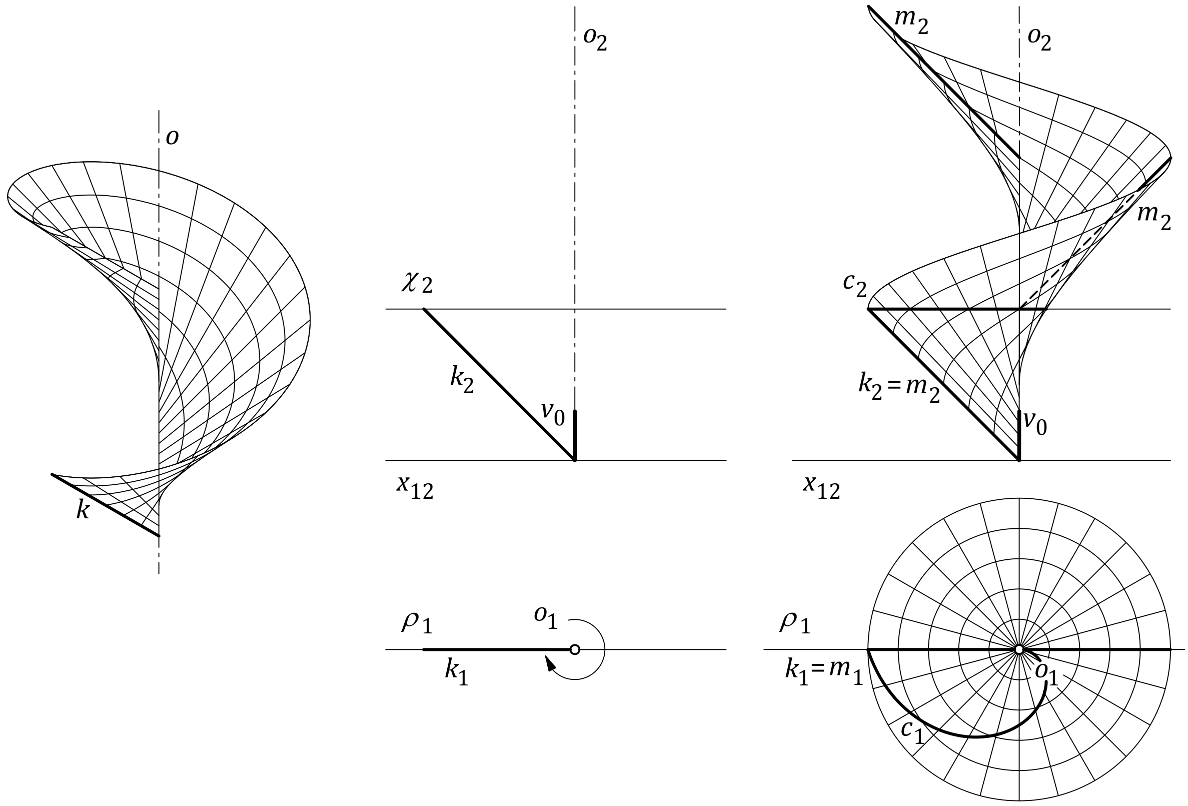


Figure 6.18: Closed oblique ruled helicoidal surface

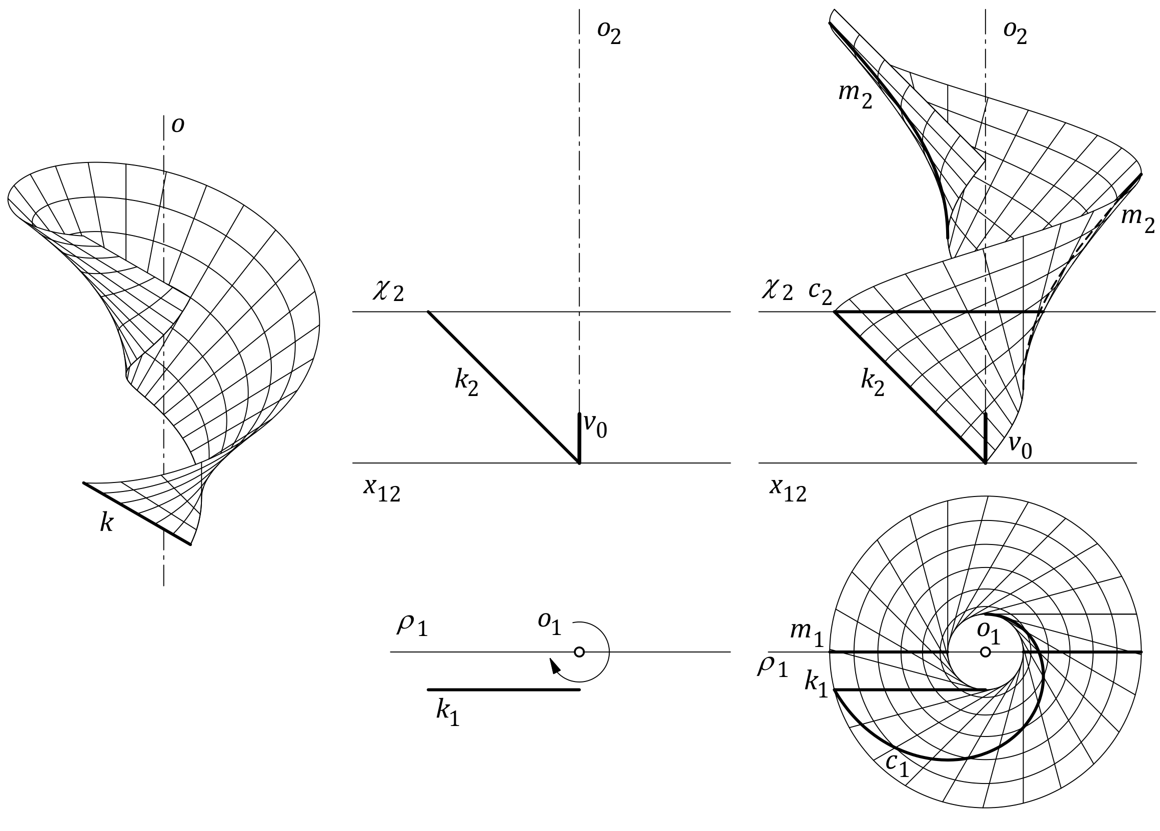


Figure 6.19: Open oblique ruled helicoidal surface

6.3 Cyclic helicoidal surfaces

Cyclic helicoidal surfaces are generated by screw motion of a circle $k = (S, r)$, $S \neq o$. According to the position of the *director plane*, i.e. the plane containing the generating circle, the following special types of cyclic helicoidal surfaces are distinguished.

- **Column surface** – the director plane of the circle k is perpendicular to the axis o, see fig. 6.20.
- **Axial cyclic helicoidal surface** – the director plane of the circle k is identical to the meridian plane of the surface, see fig. 6.21.
- **Pipe surface** – the director plane of the circle k is perpendicular to the helix generated by screw motion of the centre S of the circle, see fig. 6.22. This surface is called serpentine of Archimedes, too.

  Pipe surface belongs to the *canal* surfaces, i.e. envelope surfaces generated by motion of a sphere, see chapter 7.

Examples of cyclic helicoidal surfaces are shown in figs. 6.20 to fig. 6.22. Each figure contains isometric view and Monge projection of the surface. In Monge projection, the definition figures, principal meridian $m \subset \rho$ and normal section $c \subset \chi$ of the surface are drawn, too.

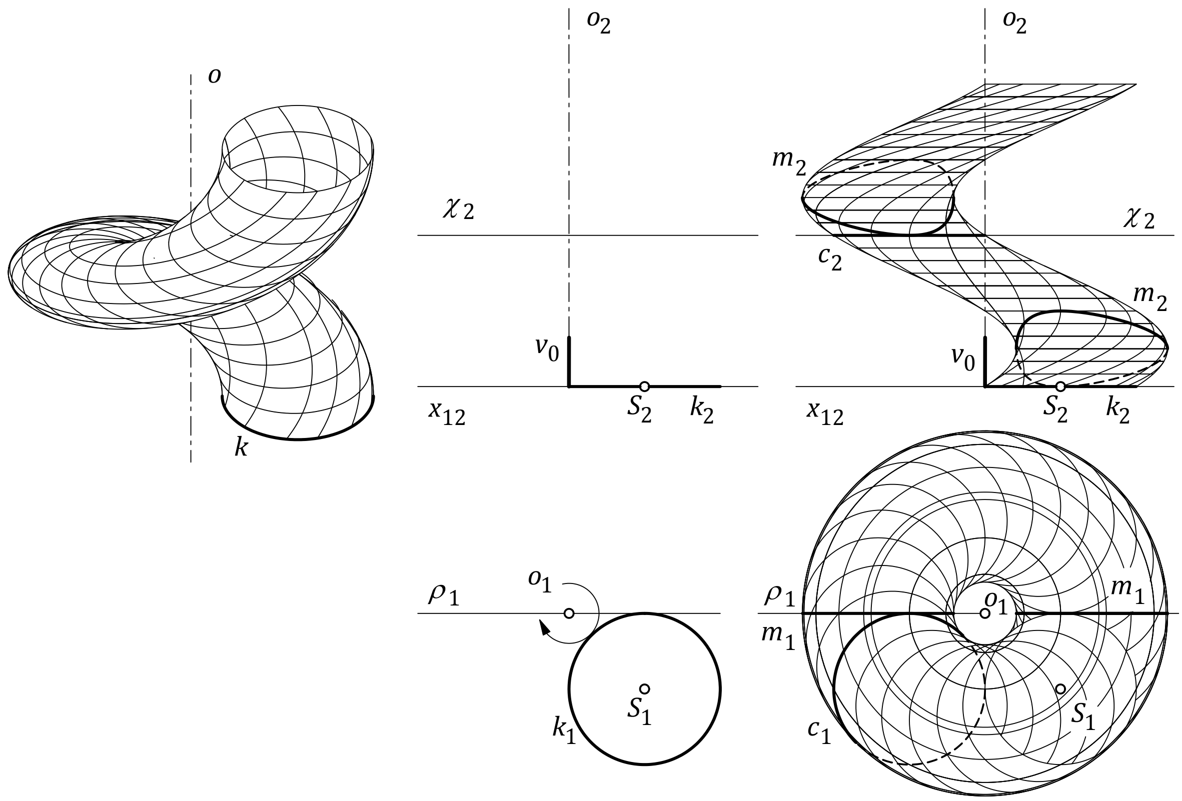


Figure 6.20: Column surface

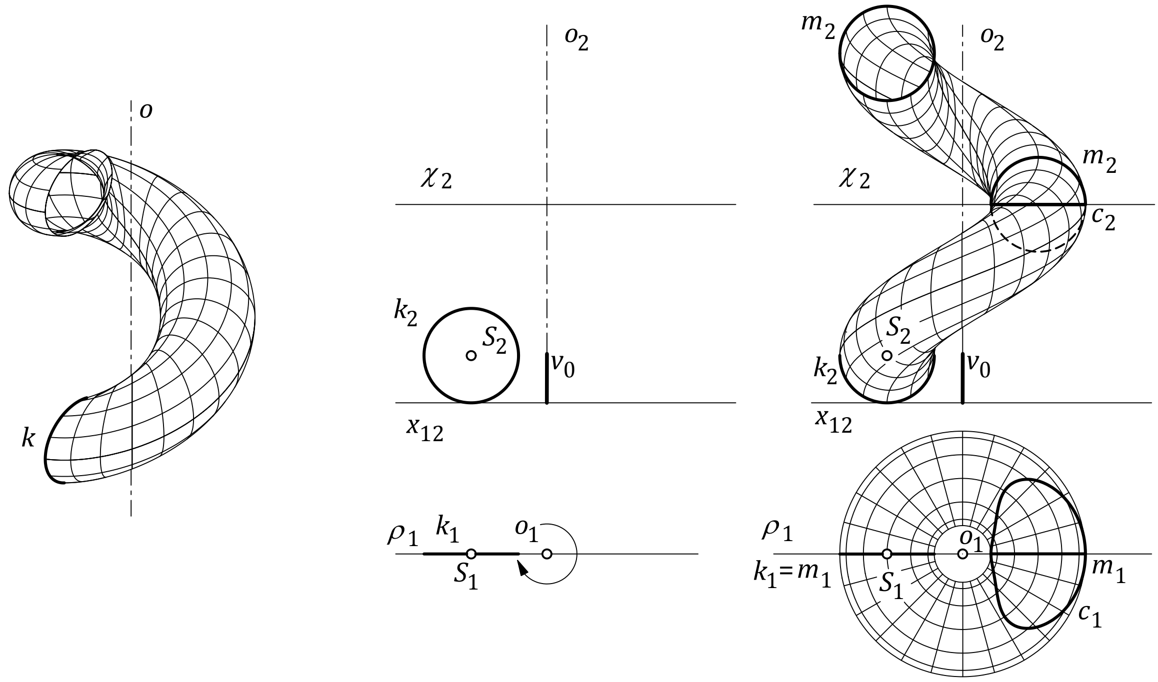


Figure 6.21: Axial cyclic helicoidal surface

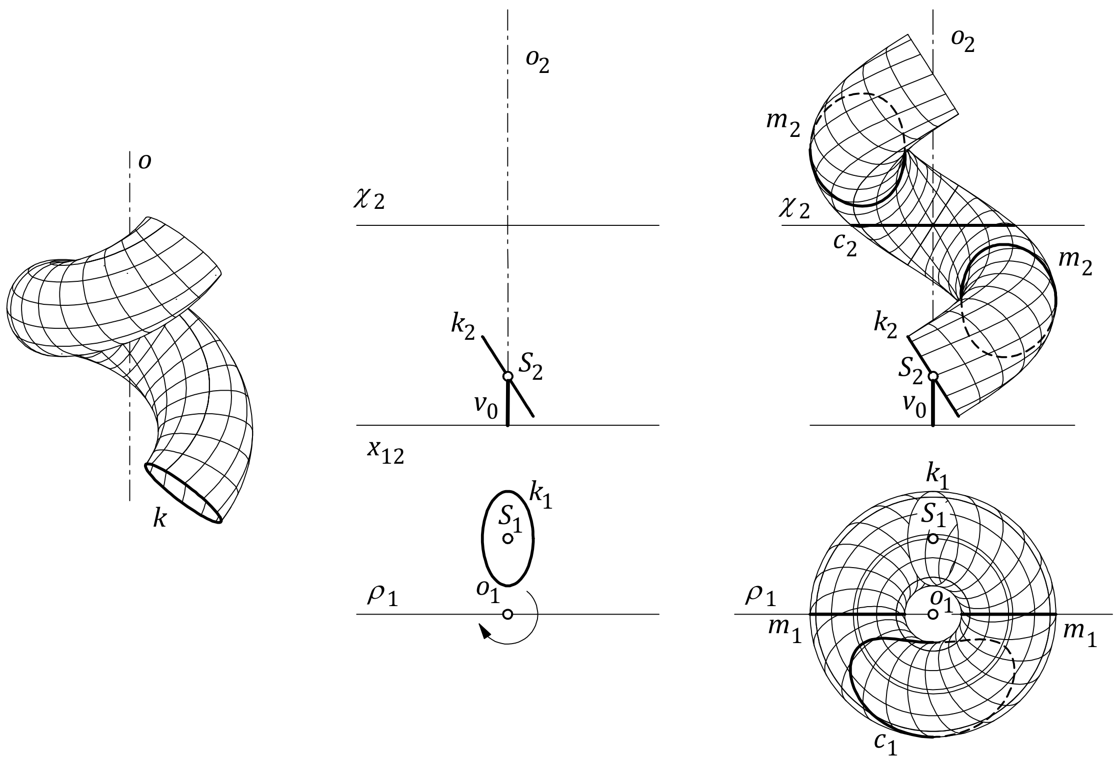


Figure 6.22: Pipe surface (serpentine of Archimedes)

6.4 Example problems – helicoidal surfaces

■ Example 6.5 – Tangent plane at point on helicoidal surface

Given

Helicoidal surface $\sigma = (k, o, v_0, \text{left-handed})$, point $A \in k$ in Monge projection, see fig. 6.23 a).

Required

Using Monge projection, construct tangent plane τ at point A on helicoidal surface σ.

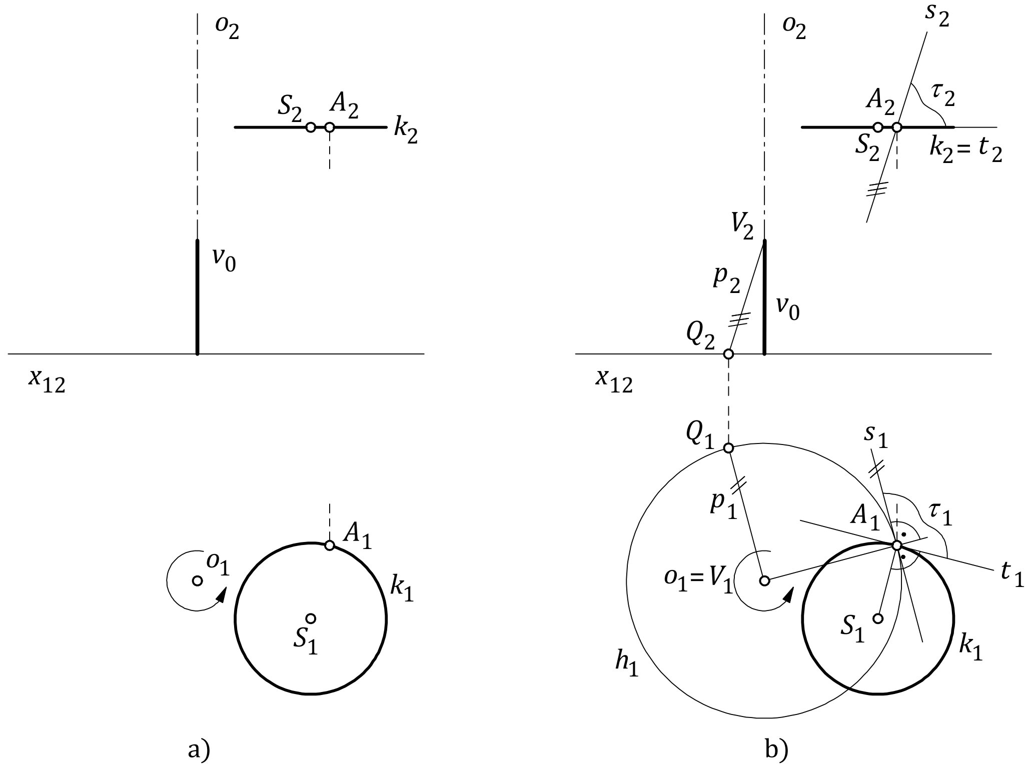


Figure 6.23: Tangent plane of helicoidal surface

Analysis

The surface σ is a column surface. The tangent plane at point A on helicoidal surface is given by tangent line t to the generating curve k and tangent line s to the helix – trajectory of point A. The top view t_1 is tangent line to the top view k_1, i.e. tangent line to the circle. The front view t_2 is tangent line to the front view k_2, i.e. tangent line to the straight line segment. Tangent line s to the helix of point A is constructed according to the procedure described in example 6.2.

Note that neither the front view of the helix of point A nor the front view of helicoidal surface has to be constructed.

Graphical solution

1. Draw top view S_1A_1.

2. Construct top view $t_1 \perp S_1A_1$, $A_1 \in t_1$.

3. Front view $t_2 = k_2$.

4. Construct tangent line s, see example 6.2.

5. Top view $\tau_1 = (t_1, s_1)$, front view $\tau_2 = (t_2, s_2)$. □

■ Example 6.6 – Principal meridian of helicoidal surface

Given

Helicoidal surface $\sigma = (k, o, v_0, \text{right-handed})$ in Monge projection, see fig. 6.24.

Required

Using Monge projection, construct principal meridian m of one thread of helicoidal surface σ.

Analysis

The surface σ is open oblique ruled helicoidal surface. To construct principal meridian, a pointwise approach is used, i.e. for a sufficient number of suitable chosen points on the generating curve k, intersection points of their helices and the principal meridian plane ρ are constructed. Finally, the principal meridian is drawn as a curve passing through all the constructed points.
Note that neither the front view of the helices nor the front view of the helicoidal surface have to be constructed.
To construct the intersection of each individual helix with the principal meridian plane, the procedure described in example 6.4, where the intersection of helix and axial surface is solved, can be applied.

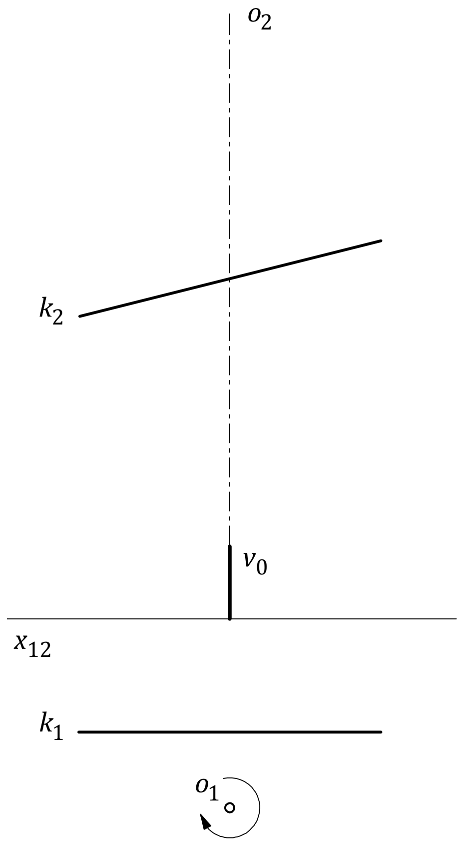


Figure 6.24: Principal meridian of helicoidal surface – task setting

In graphical solution, either graph of developed helix of each considered point or a graph of developed helix of only one suitably chosen point is used to find the unknown translation corresponding to the known revolution.

The former approach consists of a construction of many graphs, as is shown in fig. 6.25. Points 1, 2, 3, 4 and 5 represent the chosen points on the generating curve. In the top view, their helices are projected as circles with centre at o_1 and radii r^1, r^2, r^3, r^4 and r^5. These circles intersect the top view ρ_1 of principal meridian plane at points $1^+, 2^+, 3^+, 4^+$ and 5^+ on the top view m_1^+ of the right half-meridian and at points $1^*, 2^*, 3^*4^*$ and 5^* on the top view m_1^* of the left half-meridian. Graphs of all developed helices are drawn and the unknown descent $d_1^+, d_2^+, d_3^+, d_4^+$ and d_5^+ corresponding to the revolution $\widehat{11^+}, \widehat{22^+}, \widehat{33^+}, \widehat{44^+}$ and $\widehat{55^+}$ as well as the unknown ascent $d_1^*, d_2^*, d_3^*, d_4^*$ and d_5^* corresponding to the revolution $\widehat{11^*}, \widehat{22^*}, \widehat{33^*}, \widehat{44^*}$ and $\widehat{55^*}$ is read from the individual graphs.

Obviously, this approach is highly time consuming and provides unprecise results. The smaller radius of the circle (top view of the developed helix) is, the shorter length of approximating polygon edge has to be chosen to obtain comparably accurate length of all the approximated arcs according to fig. 2.33.

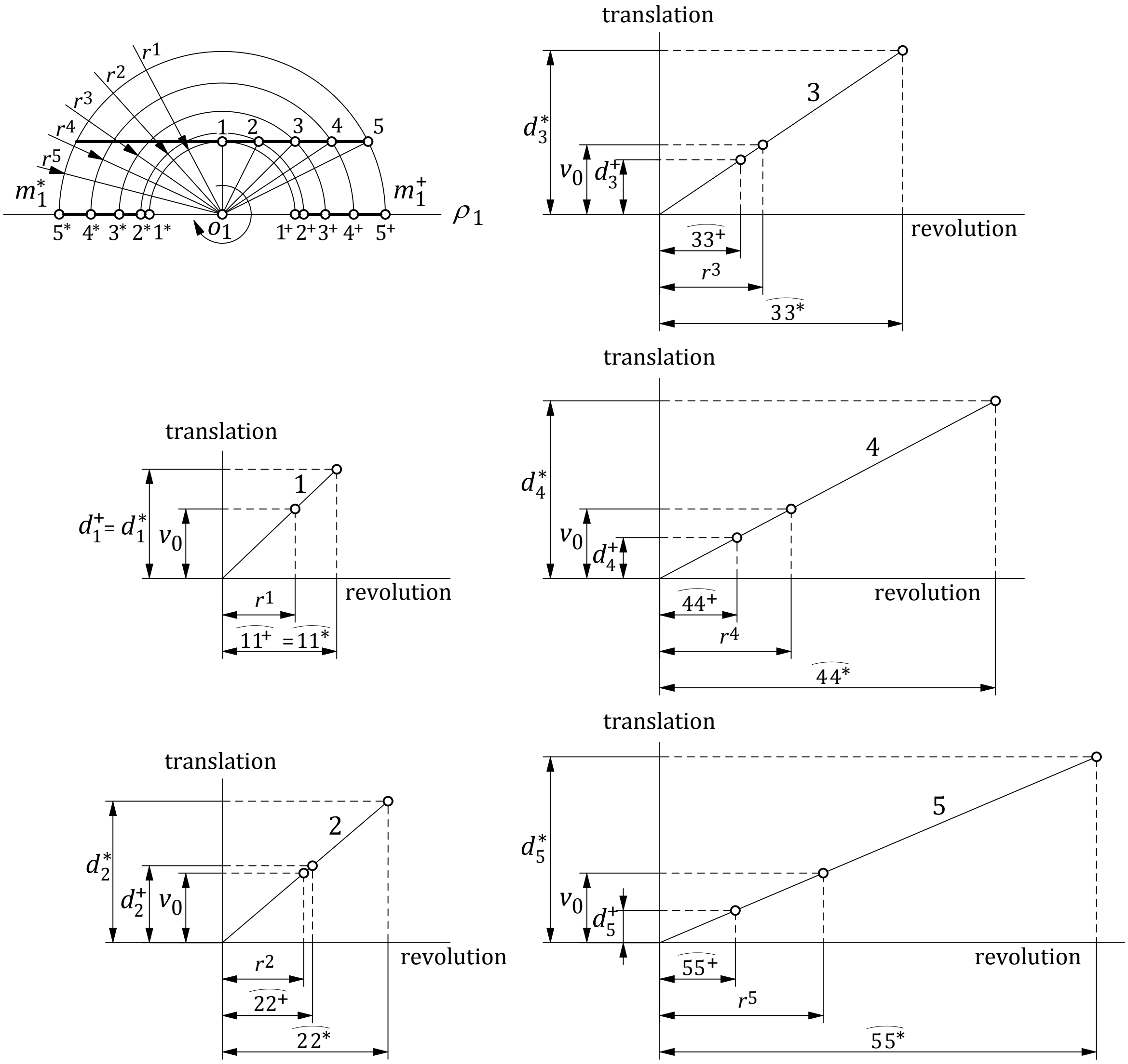


Figure 6.25: Determination of unknown translation
by means of graphs of all developed helices

A more effective approach is shown in fig. 6.26. The graph of helix h of point 5 (with the biggest radius r^5) is developed here. Since the angle $\angle 1^+ o_1 1 = \angle 5^+ o_1 1'$, $\angle 2^+ o_1 2 = \angle 5^+ o_1 2', \ldots,$ the known revolution of all the other points can be expressed in the length of arc on the top view h_1. Then the graph of developed helix of only one point has to be drawn and the unknown translation of all the other points can be read from this graph, see fig. 6.26. This approach is applied in graphical solution of this example described below.

Note that the graph of developed helix of arbitrary point can be used. To obtain the most precise results by hand drawn procedure, the graph of the helix with the biggest radius in the top view is recommended. However, the most effective way is to choose the helix, radius of which top view is equal to v_0. In this case, the known revolution measured in the length of arc on the top view of this helix is equal to the unknown translation. Therefore, it is not necessary to construct any graph of developed helix.

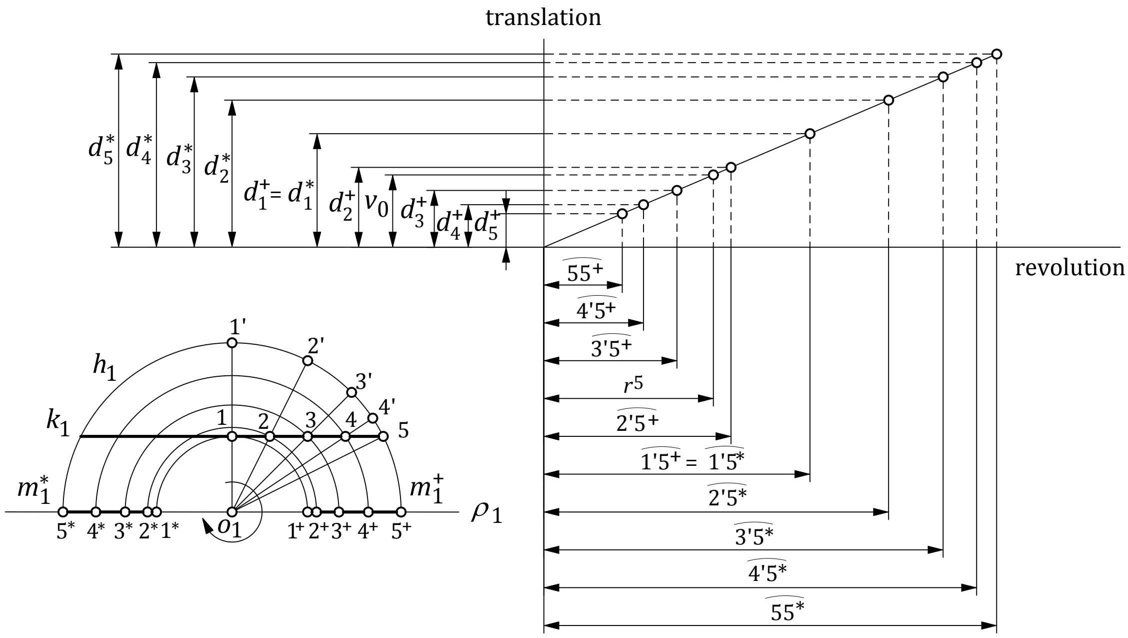


Figure 6.26: Determination of unknown translation
by means of graph of one developed helix

Graphical solution

1. Construct top view $\rho_1 \parallel x_{12}$, $o_1 \in \rho_1$ of principal meridian plane, see fig. 6.27.
2. Determine point $K \in k$ at the maximal distance from axis o.
3. Construct top view $h_1 = (o_1, r = ||o_1K_1||)$ of helix of point K.
4. Construct graph of developed helix h of point K.
5. Choose point $A \in k$, draw o_1A_1, construct top view $g_1 = (o_1, r = ||o_1A_1||)$ of helix of point A and construct auxiliary line $z_A \parallel x_{12}$, $A_2 \in z_A$.
6. Top views $A_1^+, A_1^* = \rho_1 \cap g_1$ of points on principal meridian.
7. Construct ordinates $a^+ \perp x_{12}$, $A_1^+ \in a^+$ and $a^* \perp x_{12}$, $A_1^* \in a^*$.
8. Measure length of arcs $\widehat{K_1^+A_1'}$ and $\widehat{A_1'K_1^*}$, $A_1' = o_1A_1 \cap h_1$, $K_1^+, K_1^* = h_1 \cap \rho_1$.
9. Using the graph of developed helix h, determine the corresponding translation d^+.
10. Construct auxiliary lines $z_A^+ \parallel z_A$ and $z_A^* \parallel z_A$ at oriented distances d^+ and d^*.

11. Front views $A_2^+ = a^+ \cap z_A^+$, $A_2^* = a^* \cap z_A^*$ of points on principal meridian.

12. Continue in a similar way to obtain a sufficient number of points on principal meridian. Do not forget points at special positions such as a point at the minimal or maximal distance from axis o. Finally, draw the front view m_2^+ and m_2^* as curves passing through all constructed points. The top view $m_1^+ \subset \rho_1$ and $m_1^* \subset \rho_1$ are straight line segments.

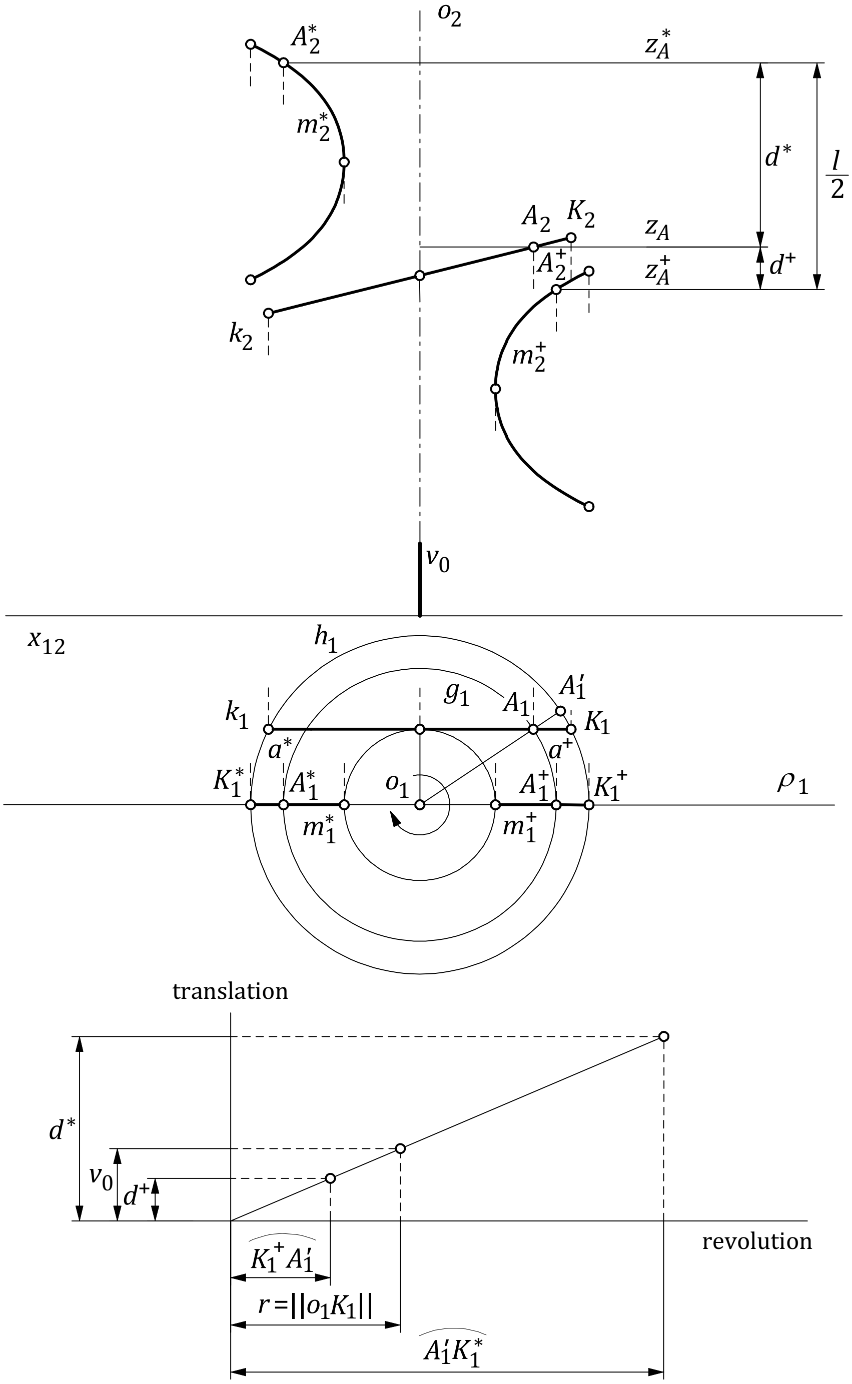


Figure 6.27: Principal meridian of helicoidal surface – solution

□

■ **Example 6.7 – Normal section of helicoidal surface**

Given

Helicoidal surface $\sigma = (m, o, v_0, \text{left-handed})$ in Monge projection, see fig. 6.28.

Required

Using Monge projection, construct normal section c of helicoidal surface σ by the given plane χ.

Analysis

The surface σ is an axial cyclic helicoidal surface. To construct normal section, a pointwise approach is used, i.e. for a sufficient number of suitable chosen points on the principal meridian m, intersection points of their helices and the section plane χ are constructed. Finally, the normal section is drawn as a curve passing through all the constructed points.
Note that neither the front view of the helices nor the front view of the helicoidal surface have to be constructed.
To construct the intersection of each individual helix with the principal meridian plane, the procedure described in example 6.3, where the intersection of helix and surface perpendicular to the axis of the helix is solved, can be applied. To determine unknown revolution corresponding to the known translation, graph of developed helix of point on principal meridian at the maximum distance from the axis is used.

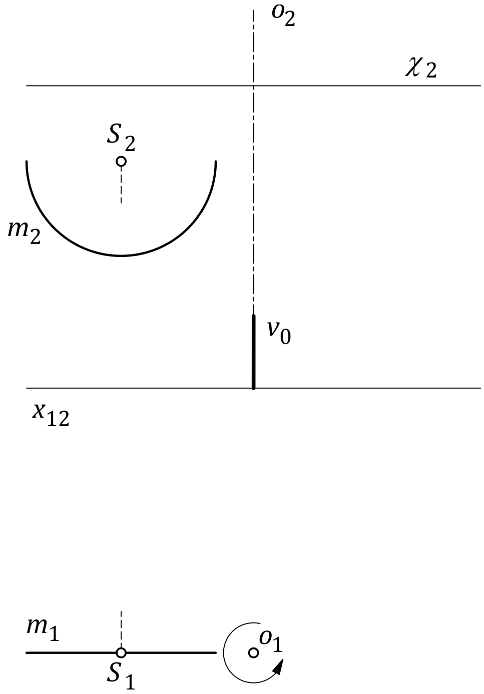


Figure 6.28: Normal section of helicoidal surface – task setting

Graphical solution

1. Determine point $K \in m$ at the maximal distance from axis o, see fig. 6.29.
2. Construct top view $h_1 = (o_1, r = ||o_1K_1||)$ of helix of point K.
3. Construct graph of developed helix h of point K.
4. Choose point $A \in m$.
5. Measure translation $d = ||A_2\chi_2||$.
6. Using the graph of developed helix h, determine the corresponding revolution $\widehat{K_1A_1'}$.
7. Measure the length of arc $\widehat{K_1A_1'}$ along the top view h_1 in the opposite direction of arrow ↻ (section plane is higher than point A) to find the top view A_1'.
8. Draw o_1A_1'.
9. Construct top view $g_1 = (o_1, r = ||o_1A_1||)$ of helix of point A.

10. Top view $A_1^* = o_1A_1' \cap g_1$ of point on normal section c.
11. Front view $A_2^* = \chi_2 \cap a$, a is the ordinate of point A^*.
12. Continue in a similar way to obtain a sufficient number of points on normal section. Do not forget points at special positions such as a point at the minimal distance from axis o. Finally, draw the top view c_1 as a curve passing through all the constructed points. The front view $c_2 \subset \chi_2$ is a straight line segment.

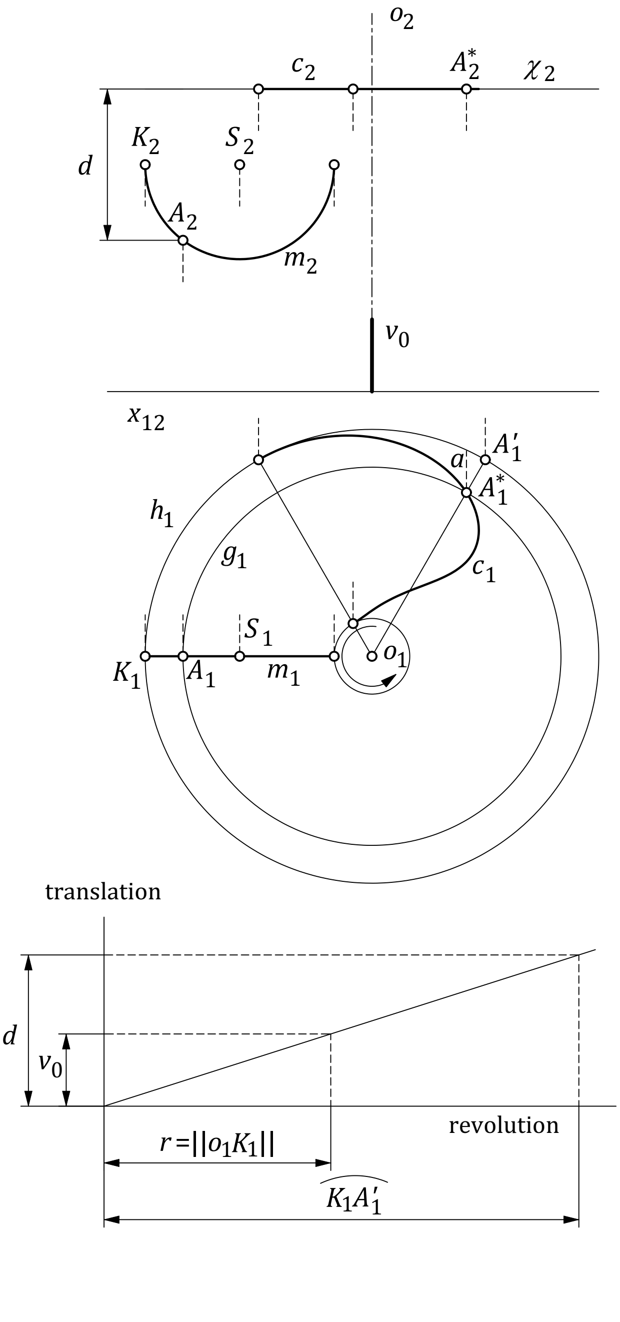


Figure 6.29: Normal section of helicoidal surface – solution

□

Chapter 7

Envelope surfaces

Let us consider a solid

$$\kappa: \ \mathbf{B}(u,v,t) = (x(u,v,t), y(u,v,t), z(u,v,t)), \ u \in [u_1,u_2], \ v \in [v_1,v_2], \ t \in [t_1,t_2]$$

generated by continuous motion of a *generating surface*

$$\sigma: \ \mathbf{S}(u,v) = (x(u,v), y(u,v), z(u,v)), \ u \in [u_1,u_2], \ v \in [v_1,v_2]$$

along a trajectory

$$\tau: \ \mathbf{T}(t) = (x(t), y(t), z(t)), \ t \in [t_1,t_2].$$

Then, a part

$$(\sigma): \ \mathbf{E}(s,t) = (x(s,t), y(s,t), z(s,t)), \ s \in [s_1,s_2], \ t \in [t_1,t_2]$$

of superficies of this solid is called *envelope surface* if the following conditions are satisfied.

- Surface $\mathbf{E}(s,t)$ and any uv-parametric surface of the solid $\mathbf{B}(u,v,t)$ are tangent along s-parametric curve of surface $\mathbf{E}(s,t)$ called *characteristic curve of envelope surface*

$$c: \ \mathbf{C}(s) = (x(s), y(s), z(s)), \ s \in [s_1,s_2].$$

- At each point on the surface $\mathbf{E}(s,t)$, there exists a common tangent plane and normal line of the surface $\mathbf{E}(s,t)$ and one uv-parametric surface of the solid $\mathbf{B}(u,v,t)$.
- There does not exist any surface which is simultaneously a part of surface $\mathbf{E}(s,t)$ and any uv-parametric surface of the solid $\mathbf{B}(u,v,t)$.

In general, the shape of the generating surface as well as the trajectory can be arbitrary. However, the complexity of envelope surface determination is strongly dependent on the shape of the generating surface, the shape of trajectory and the type of motion.

7.1 Types of motion

If the trajectory is a spatial freeform curve, the motion is called *curvilinear*. If the position of moving figure with respect to the trajectory of curvilinear motion is preserved, the motion is called *general*. Note that the position of moving figure with respect to the trajectory can be defined by position of the figure with respect to the Frenet moving trihedron, see chapter 1.

If the position of moving figure with respect to the world coordinate system is preserved during the general motion, the motion is called *translational*. *Screw motion* (see chapter 6) is a special case of general motion when the trajectory of motion is a helix. *Rotation* (see chapter 5) is a special case of general motion when the trajectory of motion is a circle. *Linear motion* is a special case of translational motion when the trajectory is a straight line. Linear motion is called *translation*, too.

Generally, the characteristic curve of envelope surface generated by general or translational motion consistently changes its shape during the motion. In the case of screw motion, rotation and translation, the shape of characteristic curve is constant.

Example of the whole procedure of envelope surface generation is given in fig. 7.1. The generating surface is a surface of revolution $\sigma = (m, o)$ given by axis o and meridian m, see fig. 7.1 a) the trajectory is planar freeform curve τ. Several positions of the generating surface along the trajectory are drawn in fig. 7.1 b). The generating surface moves by general motion, because its position with respect to the trajectory is preserved. In particular, the meridian plane of the generating surface and the normal plane of the trajectory are identical. The generated solid κ is drawn in fig. 7.1 c) and the resulting envelope surface (σ) together with characteristic curve c in fig. 7.1 d). In the case of envelope surface generated by general motion along a planar trajectory, the characteristic curve is the meridian of the generating surface located in normal plane of the trajectory.

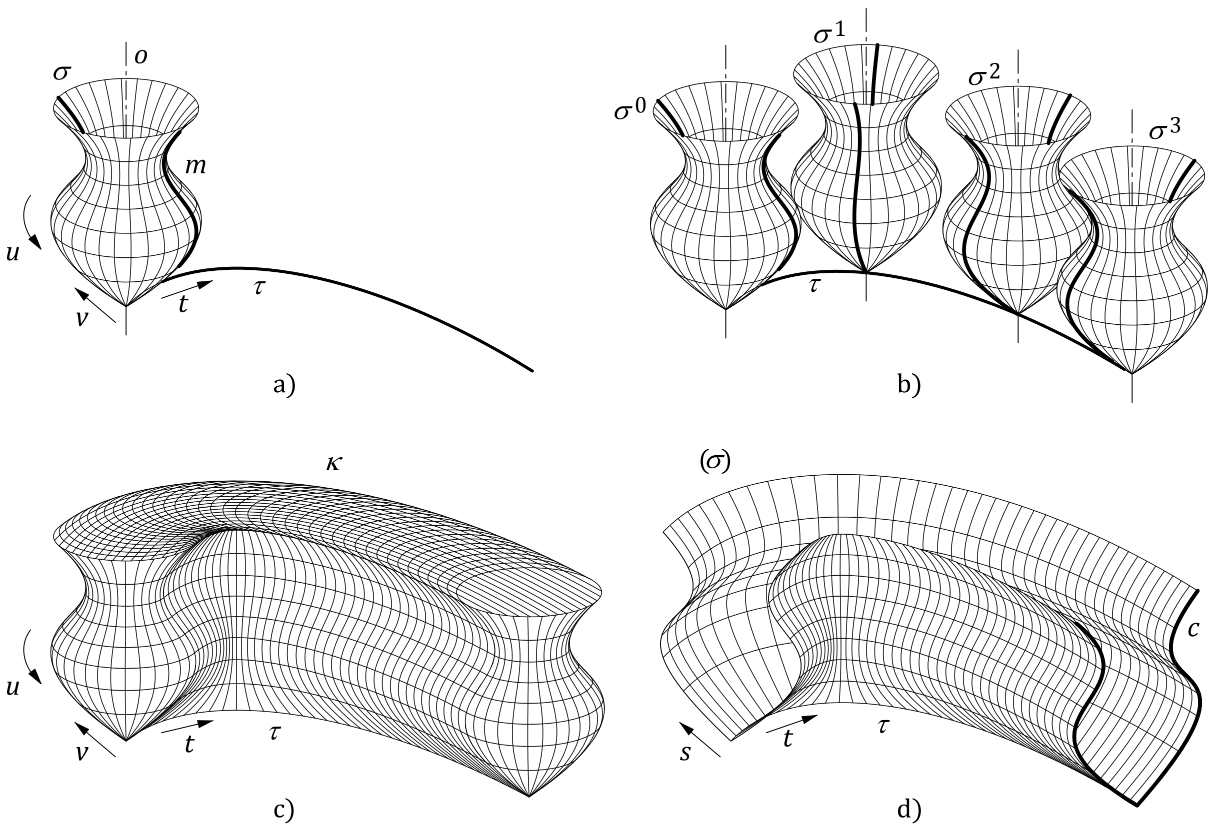


Figure 7.1: Envelope surface

7.2 Properties of envelope surfaces

There are several important properties of envelope surfaces that result from the above given definition.

1. Generating surface $\mathbf{S}(u,v)$ is a uv-parametric surface of the solid $\mathbf{B}(u,v,t)$ at each instant.
2. Characteristic curve is a curve of contact between the envelope surface and any position of the generating surface. It follows that the envelope surface $\mathbf{E}(s,t)$ can be considered a set of characteristic curves. In other words – it can be generated by the motion of characteristic curve only, instead of the whole generating surface. This property is very useful, because the whole process of solution is simplified to the determination of characteristic curve.
3. At each point on a characteristic curve, there is a common tangent plane of both generating and envelope surfaces. It follows that the common tangent plane contains the tangent line to the trajectory of the considered point on the characteristic curve.
4. Characteristic curve can be considered the intersection curve of two infinitely close positions of the generating surface.

In the following sections, determination of characteristic curve of envelope surfaces generated by translation, rotation and screw motion of a plane, sphere and surface of revolution are described.

7.3 Envelope surfaces generated by motion of a plane

Envelope surface generated by motion of a plane is a ruled surface, i.e. surface generated by motion of a straight line (see chapter 8). Characteristic curve is common generating line of the generating plane and envelope surface. It follows from the following considerations: tangent plane of any plane is the same plane. Thus, the generating plane is simultaneously the common tangent plane of the generating and envelope surfaces at each point on the characteristic curve. Moreover, characteristic curve can be considered the intersection curve of two infinitely close positions of generating plane, i.e. a straight line.

- **Envelope surface generated by translation of a plane** – if the direction of translation is parallel with the generating plane σ, the envelope surface (σ) is identical to the plane σ, characteristic curve c is a straight line perpendicular to the direction of translation, see fig. 7.2.

  If the direction of translation is not parallel with the generating plane, the envelope surface does not exist.

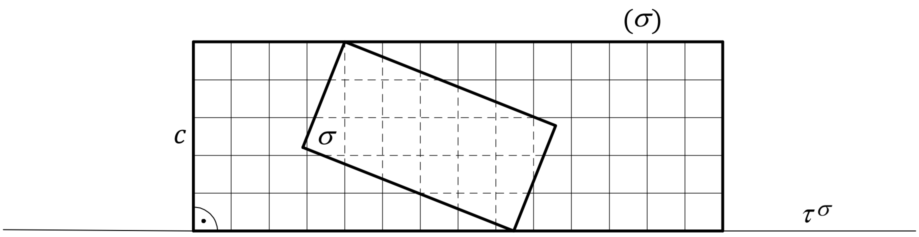


Figure 7.2: Envelope surface generated by translation of a plane

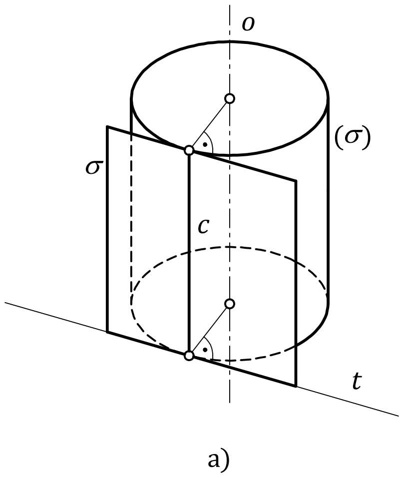


Figure 7.3: Envelope surfaces generated by rotation of a plane

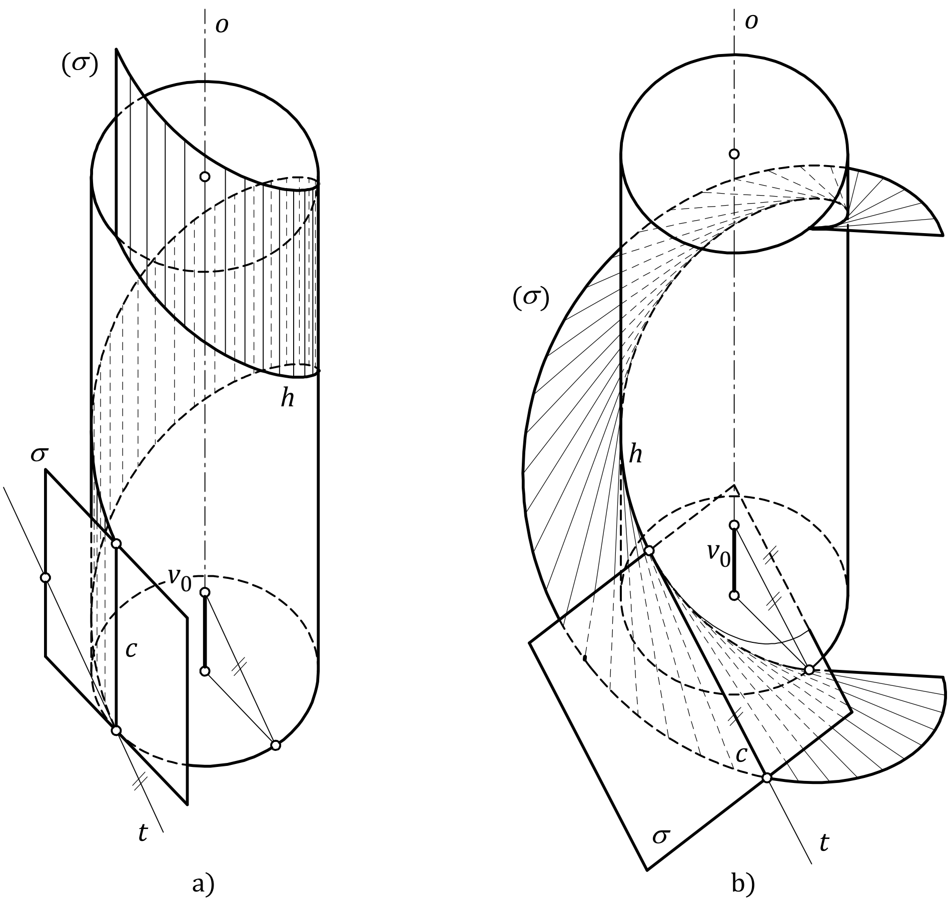


Figure 7.4: Envelope surfaces generated by screw motion of a plane

- **Envelope surface generated by rotation of a plane** – if the axis o of revolution lies on the generating plane σ, the envelope surface does not exist.

  If the axis o is parallel with the generating plane, the envelope surface is a cylinder of revolution with axis o. The characteristic curve is the generating line of the cylinder parallel with axis o, see fig. 7.3 a) and example 7.1.

  If the axis o intersects the generating plane and $o \not\perp \sigma$, the envelope surface is a cone of revolution. The characteristic curve is the generating line of the cone passing through axis o, see fig. 7.3 b) and example 7.2.

  If $o \perp \sigma$, the envelope surface is identical to the plane σ, the characteristic curve is a straight line perpendicular to axis o and intersecting axis o.

- **Envelope surface generated by screw motion of a plane** – if the axis o of screw motion lies on the generating plane σ, the envelope surface does not exist.

  If the axis o is parallel with the generating plane, the envelope surface is a cylinder of revolution. The characteristic curve is the generating line of the cylinder parallel with axis o, see fig. 7.4 a) and example 7.3.

  If the axis o intersects the generating plane and $o \not\perp \sigma$, the envelope surface is a tangent surface of the helix – trajectory of screw motion. The characteristic curve is the tangent line of the helix, see fig. 7.4 b) and example 7.4.

  If $o \perp \sigma$, the envelope surface does not exists.

7.4 Envelope surfaces generated by motion of a sphere

In general, envelope surface generated by motion of a sphere is called *canal surface*. The characteristic curve of a canal surface is the principal circle of the generating sphere located in normal plane to the trajectory. To show it, consider sphere σ moving by its centre S along a freeform trajectory τ^σ, as is depicted in fig. 7.5 a).

Tangent plane τ of the sphere at any point A on the sphere is perpendicular to normal line $n = SA$. If point A is a point on the characteristic curve c of envelope surface (σ) generated by motion of the sphere, then the tangent line s which passes through point A and is parallel with the tangent line t to the trajectory τ^σ passing through the centre S lies on tangent plane τ. This occurs if the normal line n is perpendicular to the tangent line t. A normal line of a sphere is any line passing through its centre. All normal lines of the sphere perpendicular to the tangent line t create normal plane ν. Thus, the characteristic curve c of the envelope surface is the intersection curve of the normal plane ν and the sphere σ. Obviously, this intersection is the principal circle of the sphere σ located on normal plane ν of trajectory τ^σ.

This approach can be used to construct characteristic curve at arbitrary position of the moving sphere. It follows that the canal surface, see fig. 7.5 b), can be generated by motion of characteristic curve, too.

- **Envelope surface generated by translation of a sphere** – the envelope surface is a cylinder or revolution with axis parallel with the direction of translation. The characteristic curve is parallel circle of the cylinder, see fig. 7.6 a) and example 7.5.

- **Envelope surface generated by rotation of a sphere** – the envelope surface is a torus. The characteristic curve is meridian of the torus, see fig. 7.6 b) and example 7.6.

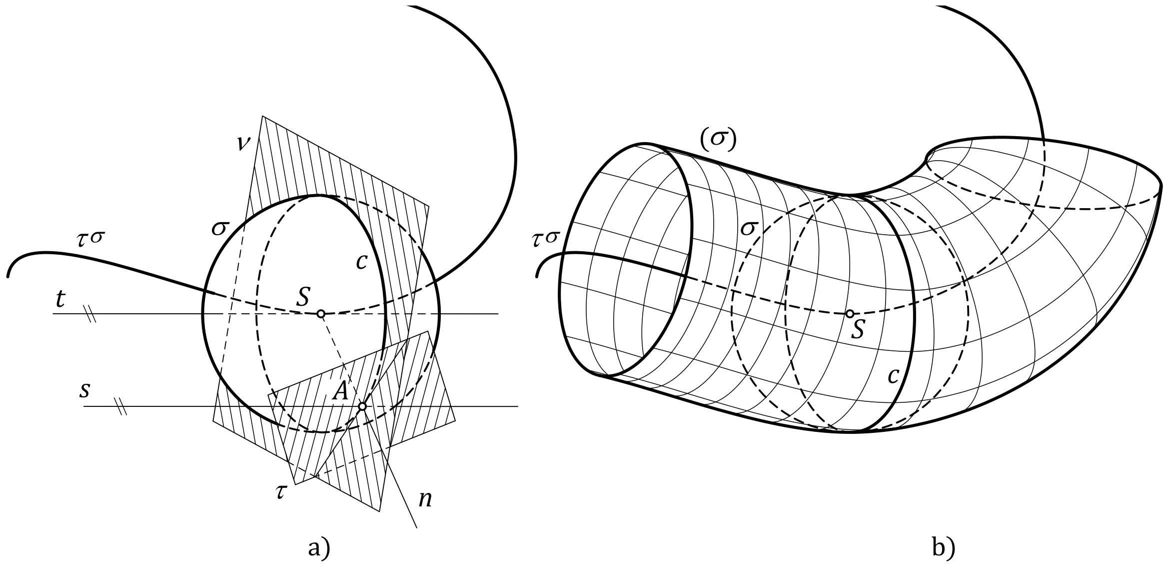


Figure 7.5: Canal surface

- **Envelope surface generated by screw motion of a sphere** – the envelope surface is a pipe surface (serpentine of Archimedes). The characteristic curve is principal circle of the sphere located in the normal plane of the helix – trajectory of the motion, see fig. 7.6 c) and example 7.7.

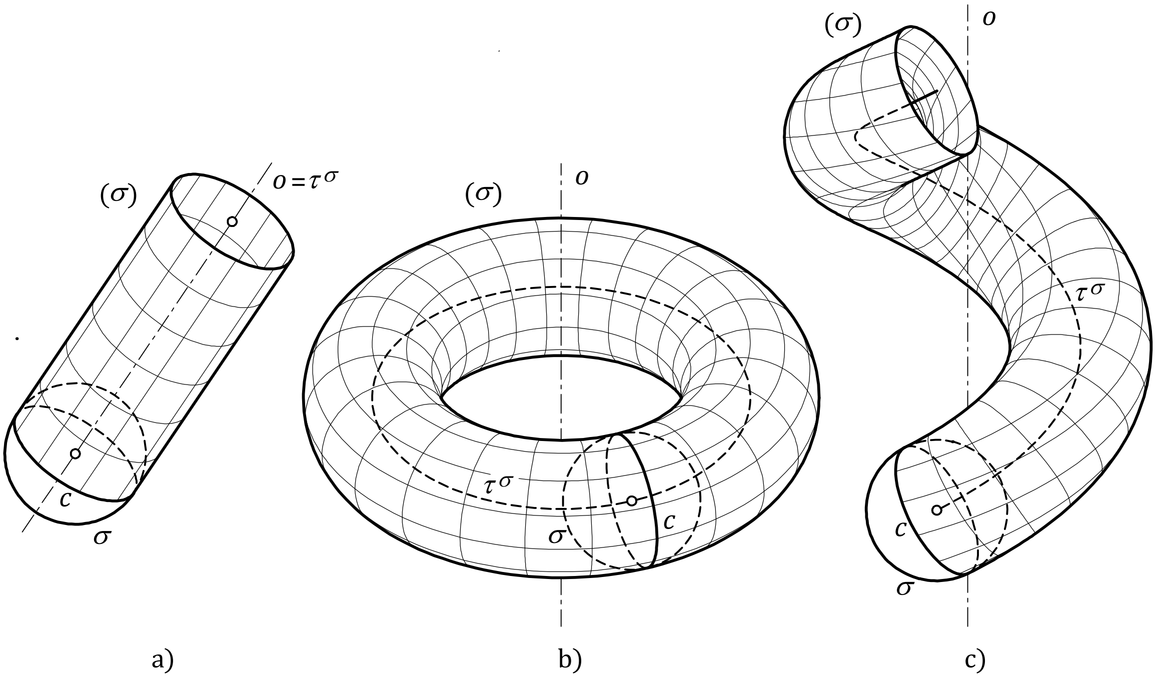


Figure 7.6: Envelope surface generated by motion of a sphere

7.5 Envelope surfaces generated by motion of a surface of revolution

Procedure of determination of envelope surface generated by motion of a sphere is applied in pointwise construction of characteristic curve of envelope surface generated by motion of a surface of revolution. The generating surface of revolution is replaced by a sufficient number of inscribed spheres with centers on the axis of revolution moving in the same way as the original generating surface. Possible points of characteristic curve of envelope surface generated by motion of the surface of revolution are located on characteristic curves of partial canal surfaces generated by motion of individual inscribed spheres.

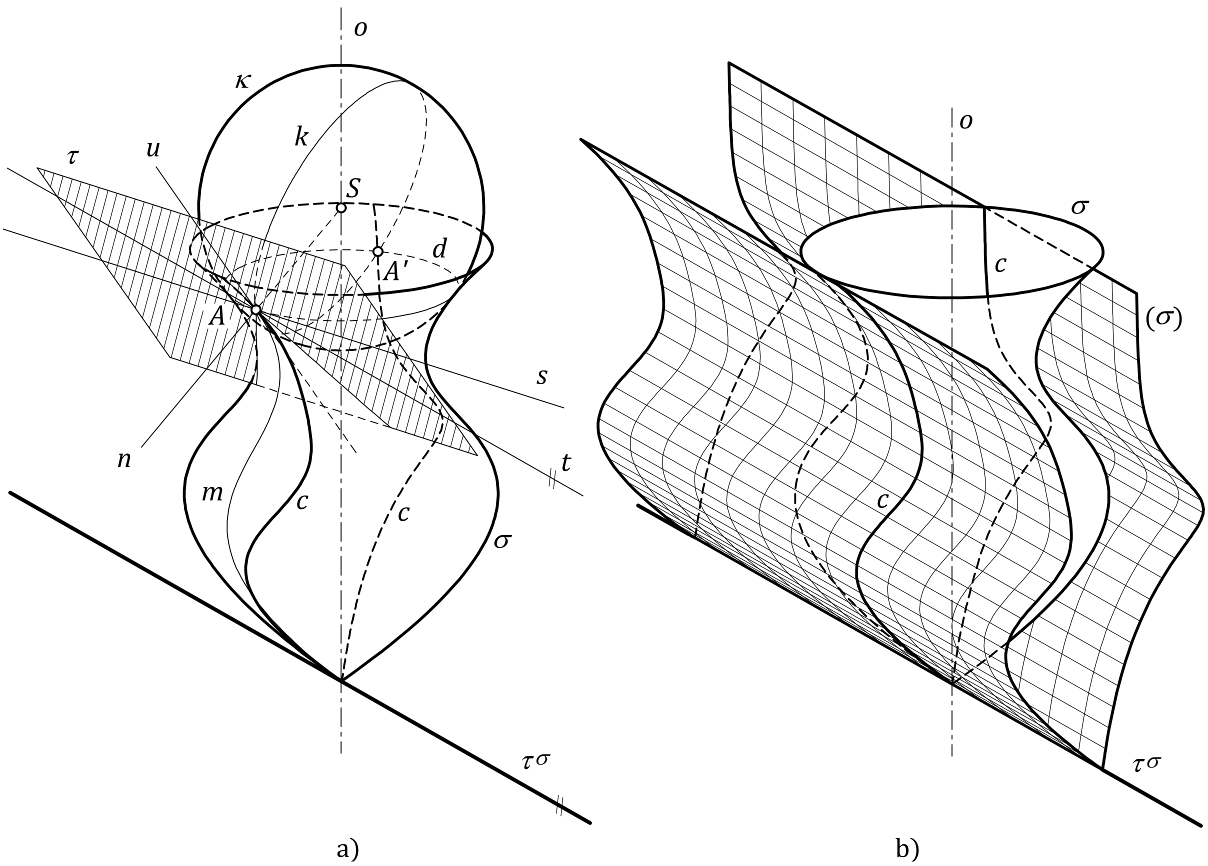


Figure 7.7: Envelope surface generated by translation of surface of revolution

To understand the method of inscribed spheres applied in construction of a point on the characteristic curve, see fig. 7.7 a). Consider a surface of revolution $\sigma = (m, o)$ moving along the trajectory τ^σ (to keep the readability of the picture and without loss of generality, the trajectory is a straight line intersecting axis o, $\tau^\sigma \not\perp o$). There are two parametric curves at each point A of the surface of revolution – meridian m and parallel circle d. The tangent plane τ is determined by tangent line u to the meridian and tangent line s to the parallel circle. The sphere κ is inscribed into the surface of revolution σ so that the parallel circle d is the common curve of the sphere κ and the surface σ. Thus, there is the common tangent plane τ and the normal line n at each point of the parallel circle to both sphere κ and surface σ. The characteristic curve of partial canal surface generated by motion of the sphere is the principal circle k located in normal plane of the trajectory.

If there exists intersection of parallel circle d and characteristic curve k, then this point is a point of characteristic curve c. The tangent line t passing through this point is parallel with the tangent line to the trajectory and located on the tangent plane τ, see two intersections A and A' in fig. 7.7 a).

To obtain a sufficient number of points on the characteristic curve c, it is necessary to consider a sufficient number of spheres inscribed into the generating surface of revolution. The resulted characteristic curve c is drawn as a curve passing through the points constructed by the above described procedure. The envelope surface depicted in fig. 7.7 b) is generated by the motion of characteristic curve.

7.6 Example problems – envelope surfaces

■ Example 7.1 – Rotation of a plane parallel with axis o

Given

Generating plane $\sigma \perp \pi$ represented by rectangle $ABCD$ and axis $o \perp \pi$ of revolution in Monge projection, see fig. 7.8 a).

Required

Using Monge projection, construct characteristic curve c of envelope surface (σ) and a part of envelope surface (σ) corresponding to the rectangle $ABCD$. Indicate the visibility.

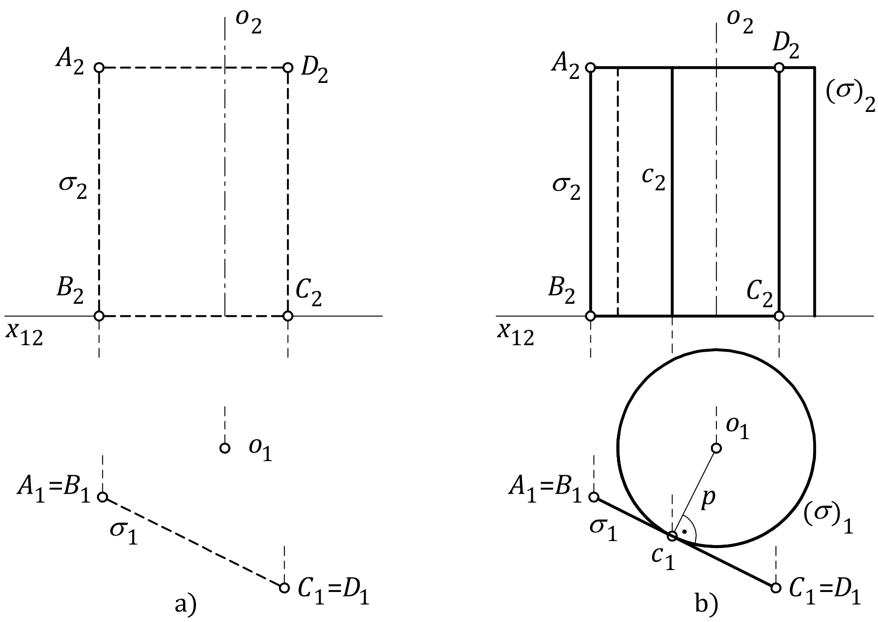


Figure 7.8: Envelope surface generated by rotation of plane parallel with axis o

Analysis

Envelope surface (σ) is a cylinder of revolution with axis o, see fig. 7.3 a). Characteristic curve c is a common generating line of the plane σ and cylinder (σ), it follows, that $c \parallel o$. Since $o \perp \pi$

and $\sigma \parallel o$, the top view $(\sigma)_1$ of the cylinder $(\sigma)_1$ is a circle with centre at o_1 and radius equal to the distance $d(o, \sigma)$. This distance is projected in true size in the top view.

Graphical solution

- Construct straight line $p \perp A_1C_1$, $o_1 \in p$, see fig. 7.8 b).
- Top view $c_1 = p \cap A_1C_1$ of characteristic curve c.
- Construct top view $(\sigma)_1 = (o_1, r = ||c_1o_1||)$ of clinder (σ).
- Construct front view $c_2 \parallel o_2$, the position of c_2 is given by ordinate of top view c_1.
- Construct front view $(\sigma)_2$. □

■ Example 7.2 Rotation of a plane intersecting axis o

Given

Generating plane σ represented by rectangle $ABCD$ and axis of revolution o in Monge projection, see fig. 7.9 a).

Required

Using Monge projection, construct characteristic curve c of envelope surface (σ) and a part of envelope surface (σ) corresponding to the rectangle $ABCD$. Indicate the visibility.

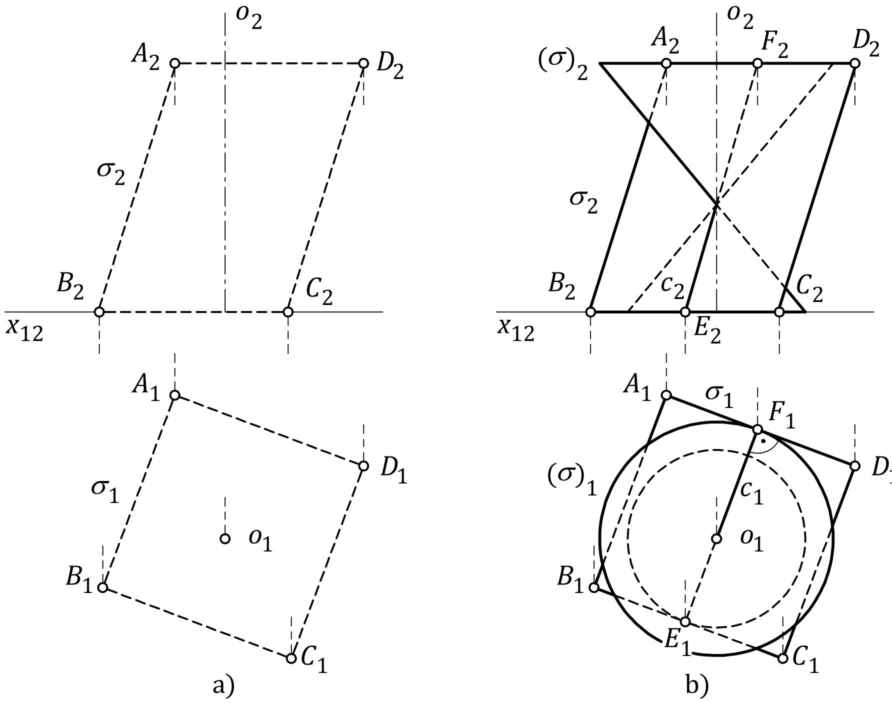


Figure 7.9: Envelope surface generated by rotation of plane intersecting axis o

Analysis

Envelope surface (σ) is a cone of revolution with axis o, see fig. 7.3 b). Characteristic curve c is a common generating line of plane σ and cone (σ), it follows, that c passes through axis of the cone. Since $o \perp \pi$, the top view $(\sigma)_1$ of the cone is projected as two circles with centers at o_1 and radii equal to the distance $d(o, AD)$ and $d(o, BC)$. Both these distances are projected in true size in the top view.

Graphical solution

1. Construct top view c_1 of characteristic curve c: $c_1 \perp A_1D_1$, $o_1 \in c_1$, see fig. 7.9 b).
2. Top views $E_1 = c_1 \cap B_1C_1$ and $F_1 = c_1 \cap A_1D_1$.
3. Construct circles $(o_1, r = ||o_1E_1||)$ and $(o_1, ||o_1F_1||)$ – the top view $(\sigma)_1$ of cone (σ).
4. Construct front views $E_2 \in B_2C_2$ and $F_2 \in A_2D_2$.
5. Front view $c_2 = E_2F_2$.
6. Construct front view $(\sigma)_2$. □

■ Example 7.3 – Screw motion of a plane parallel with axis o

Given

Generating plane $\sigma \perp \pi$ and screw motion $(\sigma, o, v_0, \text{left-handed})$ in Monge projection, see fig. 7.10 a).

Required

Using Monge projection, construct characteristic curve c of envelope surface (σ) generated by screw motion of the generating plane σ between the horizontal plane of projection π and the given plane ρ.

Analysis

Envelope surface (σ) is cylinder of revolution with axis o, see fig. 7.4 a). Characteristic curve c is a common generating line of the cylinder and the generating plane σ. Since $\sigma \parallel o$, the top view c_1 is point of contact between the top view $h_1 = (o_1, r = d(o, \sigma))$ of helix h – trajectory of screw motion and the top view σ_1 of the generating plane, see fig. 7.10 b).

Graphical solution

1. Construct straight line $p \perp \sigma_1$, $o_1 \in p$, see fig. 7.10 b).
2. Top view $c_1 = p \cap \sigma_1$ of characteristic curve.
3. Construct front view $c_2 \parallel o_2$, the position of c_2 is given by ordinate of top view c_1.

Note that the top view h_1 of helix does not have to be drawn. □

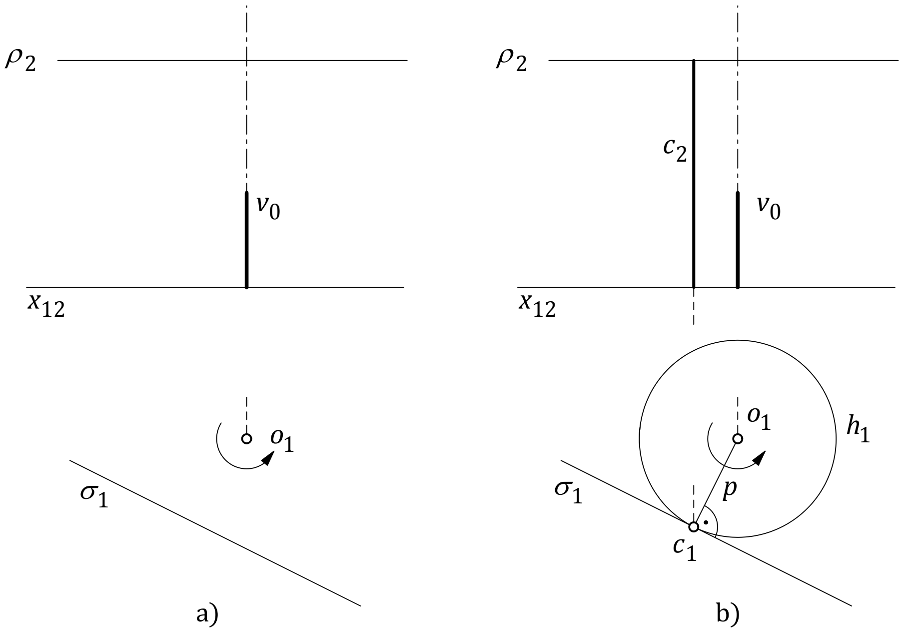


Figure 7.10: Characteristic curve of envelope surface generated by screw motion of plane parallel with axis o

■ Example 7.4 – Screw motion of a plane intersecting axis o

Given

Generating plane $\sigma \perp \nu$ and screw motion $(\sigma, o, v_0, \text{left-handed})$ in Monge projection, see fig. 7.11 a).

Required

Using Monge projection, construct characteristic curve c of envelope surface (σ) generated by screw motion of the generating plane σ.

Analysis

Envelope surface σ is tangent surface of helix h with the same slope as is the slope of the generating plane, see fig. 7.4 b). Characteristic curve c is tangent line t to the helix h located in the generating plane σ. Since $\sigma \perp \nu$, the front view $t_2 = c_2 = \sigma_2$. Reverse procedure of construction of tangent line to the helix described in example 6.2 can be applied to find the top view $t_1 = c_1$ of tangent line $t = c$.

Graphical solution

1. Front view $A_2 = \sigma_2 \cap o_2$ of generating point of the helix h.

2. Construct front view p_2 of generating line p of directing cone of the helix h: $p_2 \parallel \sigma_2$, $V_2 \in p_2$. Note that V_2Q_2 is principal meridian of the directing cone of the helix h due to $\sigma \perp \nu$.

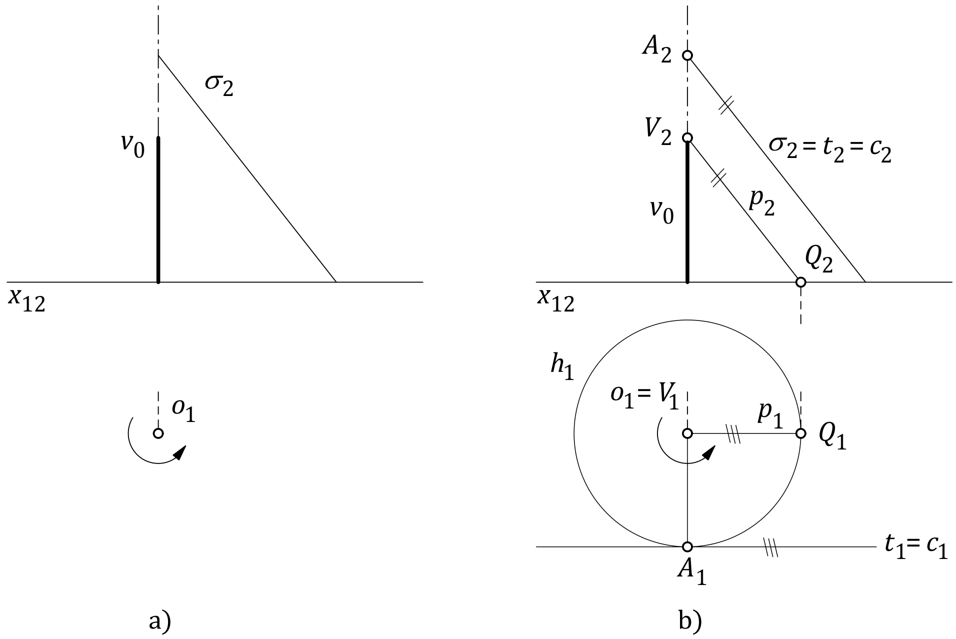


Figure 7.11: Characteristic curve of envelope surface generated by screw motion of plane intersecting axis o

3. Front view $Q_2 = x_{12} \cap p_2$.
4. Construct top view Q_1 in the principal meridian plane of the directing cone, $Q_1Q_2 \perp x_{12}$.
5. Construct top view $h_1 = (o_1, r = ||o_1Q_1||)$ of the helix h.
6. Determine top view $A_1 \in h_1$, $A_2A_1 \perp x_{12}$ corresponding to the orientation of screw motion.
7. Construct top view t_1 of tangent line to the top view h_1, $A_1 \in t_1$. Top view c_1 of characteristic curve $c_1 = t_1$. □

■ Example 7.5 – Translation of a sphere

Given

Generating sphere $\sigma = (S, r)$ and trajectory τ, $S \in \tau$ in Monge projection, see fig. 7.12 a).

Required

Using Monge projection, construct characteristic curve c of envelope surface generated by translation of the sphere σ along the trajectory τ. Construct part of envelope surface (σ) corresponding to the drawn part of the trajectory τ.

Analysis

Envelope surface (σ) is a cylinder of revolution with axis τ, see fig. 7.6 a). Characteristic curve c is principal circle of the sphere located in the projecting plane perpendicular to π. Therefore, the characteristic curve is projected as a straight line segment in the top view and as an ellipse in the front view, see fig. 3.27 a).

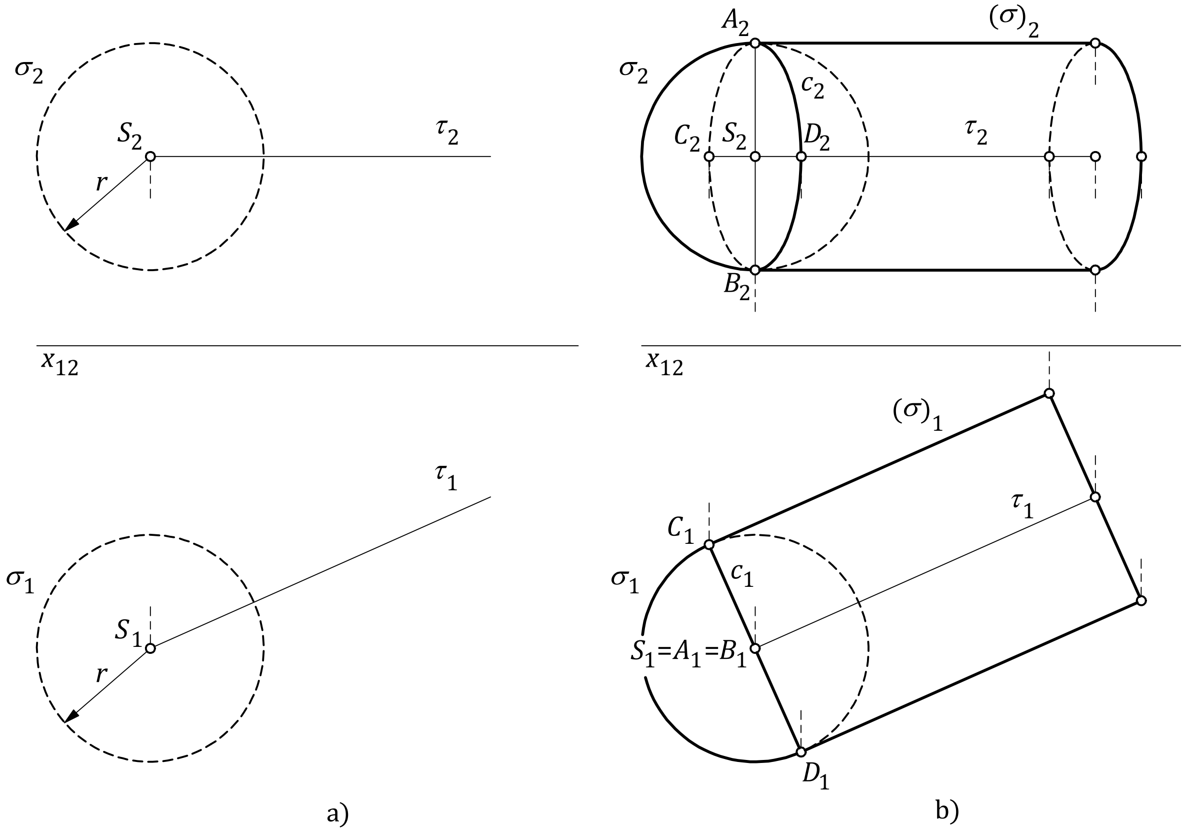


Figure 7.12: Cylinder of revolution generated by translation of sphere

Graphical solution

1. Construct top view c_1 of characteristic curve c: $c_1 = C_1D_1$, $C_1D_1 \perp \tau_1$, $S_1 \in C_1D_1$, $C_1, D_1 \in \sigma_1$, see fig. 7.13 b).
2. Construct front view of major axis A_2B_22 of the ellipse into which the characteristic curve is projected, $A_2B_2 \perp \tau_2$, $S_2 \in A_2B_2$, $A_2, B_2 \in \sigma_2$.
3. Draw the front view c_2 of characteristic curve as the ellipse given by major axis A_2B_2 and minor axis C_2D_2. Use approximation by osculation circles described in section 3.2.4.
4. Construct top and front view of cylinder (σ). The left base of the cylinder and characteristic curve are identical. The right base of the cylinder and characteristic curve are congruent. □

■ Example 7.6 – Rotation of a sphere

Given

Generating sphere $\sigma = (S, r)$ and axis of revolution $o \perp \pi$ in Monge projection, see fig. 7.13 a).

Required

Using Monge projection, construct characteristic curve c of envelope surface (σ) generated by rotation of the sphere σ about the axis o, principal meridian m of envelope surface and envelope surface (σ).

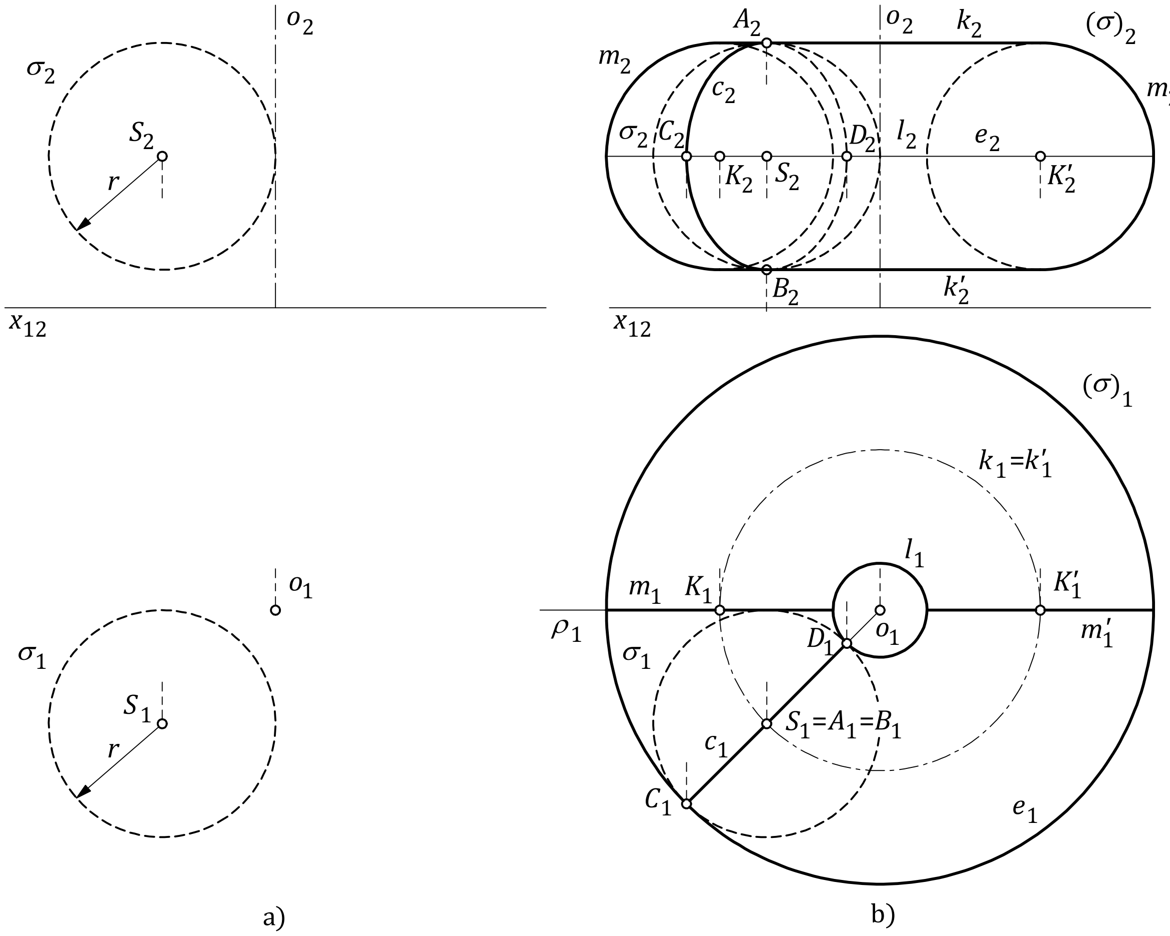


Figure 7.13: Torus generated by rotation of sphere

Analysis

Envelope surface (σ) is a torus with axis o, see fig. 7.6 b). Characteristic curve c is principal circle of the sphere located in the projecting plane perpendicular to π. Therefore, the characteristic curve is projected as a straight line in the top view and as an ellipse in the front view, see fig. 3.27 a). Principal meridian is a circle located in principal meridian plane of the torus. Thus, its direct construction can be used instead of pointwise construction of principal meridian of surface of revolution described in example 5.4.

Graphical solution

1. Construct top view c_1 of characteristic curve: $c_1 = C_1D_1$, $C_1D_1 = o_1S_1 \cap \sigma_1$, see fig. 7.13 b).
2. Construct front view of major axis $A_2B_2 \perp x_{12}$, $S_2 \in A_2B_2$, $A_2, B_2 \in \sigma_2$.
3. Draw front view c_2 of characteristic curve as the ellipse given by major axis A_2B_2 and minor axis C_2D_2. Use approximation by osculation circles described in section 3.2.4.

4. Construct top view $(\sigma)_1$ of torus (σ), i.e. top views of throat $l_1 = (o_1, r = ||o_1 D_1||)$ and equator $e_1 = (o_1, r = ||o_1 C_1||)$.

5. Construct top view ρ_1 of principal meridian plane of torus: $\rho_1 \parallel x_{12}$, $o_1 \in \rho_1$.

6. Top view of principal meridian $m_1, m'_1 = \rho_1 \cap (\sigma)_1$.

7. Construct front view of principal meridian, i.e. circles $m_2 = (K_2, r)$, $m'_2 = (K'_2, r)$, $K_2, K'_2 \in e_2$, e_2 is the front view of equator.

8. Construct front view $(\sigma)_2$ of torus (σ), i.e. front views of craters $k_2 \perp o_2$, $A_2 \in k_2$ and $k'_2 \perp o_2$, $B_2 \in k'_2$ and corresponding parts of the left and right principal half-meridians. □

■ Example 7.7 – Screw motion of a sphere

Given

Generating sphere σ and screw motion $(\sigma, o, v_0, \text{left-handed})$ in Monge projection, see fig. 7.14.

Required

Using Monge projection, construct characteristic curve c and right principal half-meridian m of envelope surface (σ) generated by the given screw motion of the sphere σ.

Analysis

Envelope surface (σ) is a pipe surface (serpentine of Archimedes), see fig. 7.6 c). Characteristic curve c is principal circle of the sphere located in the normal plane of helix h – trajectory of the centre of the sphere. Since $S_2 \in o_2$, the tangent line to the helix h at centre S is parallel with the frontal plane of projection ν. Consequently, the normal line on the helix at centre S lies in projecting plane perpendicular to the frontal plane of projection ν. The circle c located in this plane is projected as an ellipse in the top view and as a straight line segment in the front view, see fig. 3.27 b). To construct the principal right half-meridian, the pointwise construction described in example 6.6 is applied.

Note that to draw the ellipse c, approximation by osculation circles (see section 3.2.4) can be used. However, the points on ellipse c used in pointwise construction of principal meridian have to be constructed as precisely as possible. Therefore, the precise construction of sufficient number of additional points on the ellipse by means of parallelogram method (see section 3.2.4) is recommended. After revolving these points into the principal meridian plane, more precise graphical solution of principal meridian is obtained comparing with estimation of points on ellipse approximated by osculation circles.

Graphical solution

1. Construct top view $h_1 = (o_1, r = ||o_1 S_1||)$ of helix h, see fig. 7.15.

2. Construct tangent line t to the helix h according to the procedure described in example 6.2.

3. Construct front view $\chi_2 \perp t_2$, $S_2 \in \chi_2$ of plane of characteristic curve.

4. Construct front view c_2 of characteristic curve c: $c_2 = C_2 D_2$, $C_2, D_2 = \chi_2 \cap \sigma_2$.

5. Construct top view of major axis $A_1 B_1 \perp x_{12}$, $S_1 \in A_1 B_1$, $A_1, B_1 \in \sigma_1$.

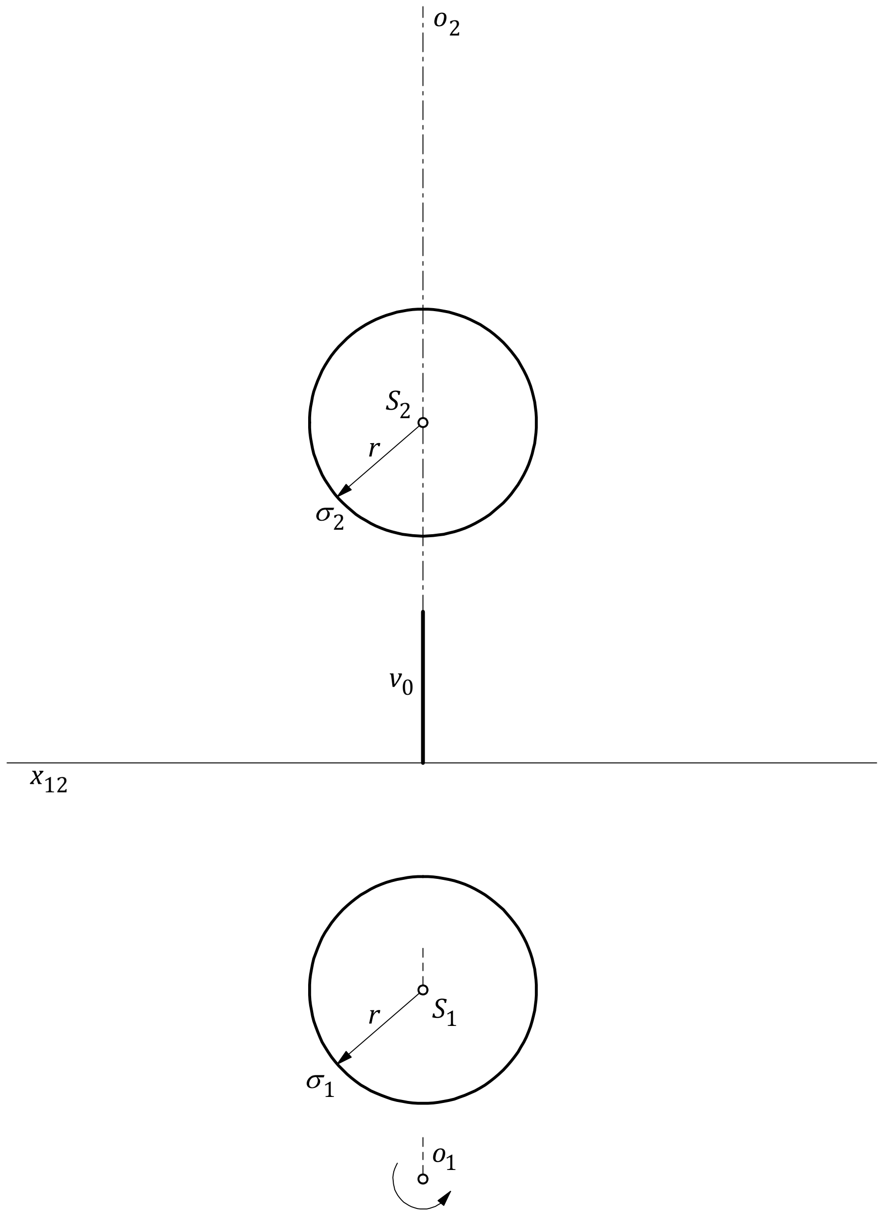


Figure 7.14: Pipe surface (serpentine of Archimedes) generated by screw motion of sphere (task setting)

6. Draw top view c_1 of characteristic curve c as the ellipse given by major axis A_1B_1 and minor axis C_1D_1. Use approximation by osculation circles described in section 3.2.4 and construct sufficient number of points on ellipse by means of parallelogram method described in section 3.2.4.

7. Use construction described in example 6.6 to construct principal right half-meridian m. The graphical solution for vertices of the ellipse c is drawn in fig. 7.15, the graph of developed helix h is drawn in fig. 7.16.

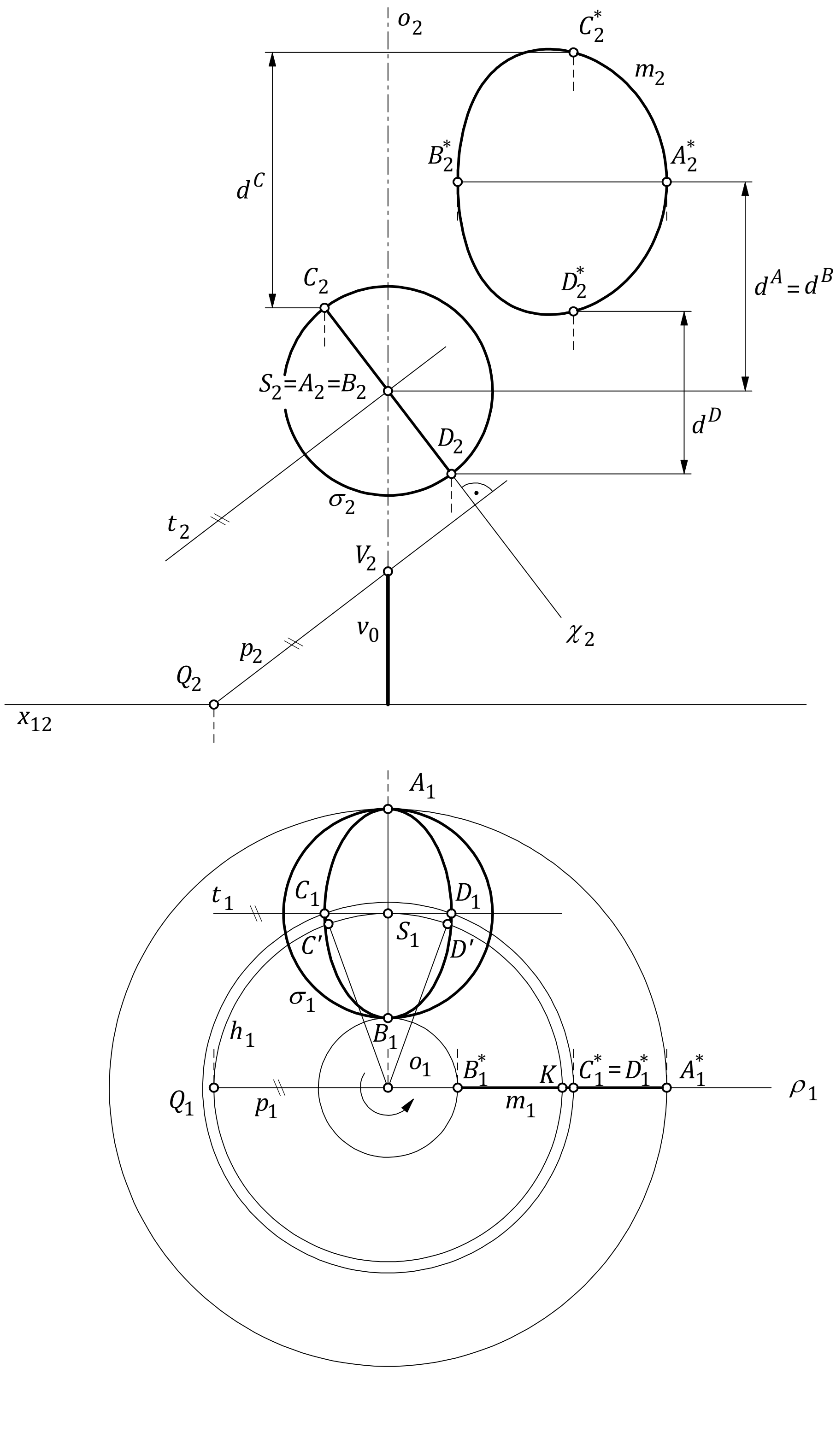


Figure 7.15: Pipe surface (serpentine of Archimedes) generated by screw motion of sphere (solution)

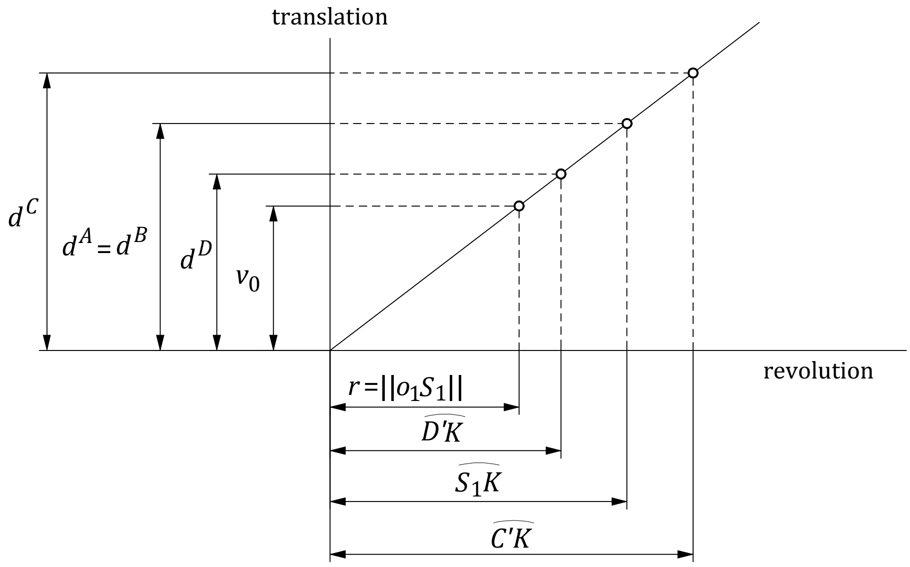


Figure 7.16: Graph of developed helix h – trajectory of the center of generating sphere

□

■ Example 7.8 – Rotation of a cylinder of revolution

Given

Generating cylinder of revolution σ and axis of revolution o in Monge projection, see fig. 7.17.

Required

Using Monge projection, construct characteristic curve c and left principal half-meridian of envelope surface (σ) generated by rotation of the cylinder σ about the given axis o.

Analysis

Pointwise construction based on method of inscribed spheres (see fig. 7.7) into the cylinder of revolution σ is used to construct characteristic curve c. Point on characteristic curve is constructed as intersection of two circles located on the inscribed sphere κ, if the intersection exists. The first circle is the common parallel circle d of the sphere κ and the cylinder of revolution σ. The second one is characteristic curve k of partial torus generated by revolution of the sphere κ.

Since the axis a of cylinder σ is parallel with the frontal plane of projection ν, the parallel circle d lies in projecting plane perpendicular to the frontal plane of projection ν. Therefore, it is projected as a straight line segment in the front view and as an ellipse in the top view, see fig. 3.27 b). The characteristic curve k of torus is a circle located in the projecting plane χ perpendicular to the horizontal plane of projection π. Thus, it is projected as a straight line segment in the top view and as an ellipse in the front view, see fig. 3.27 a) and example 7.6, where the characteristic curve of envelope surface generated by revolution of a sphere is solved.

Pointwise construction described in example 5.4 is used to construct the left principal half-meridian of the envelope surface generated by revolution of the cylinder σ about axis o.

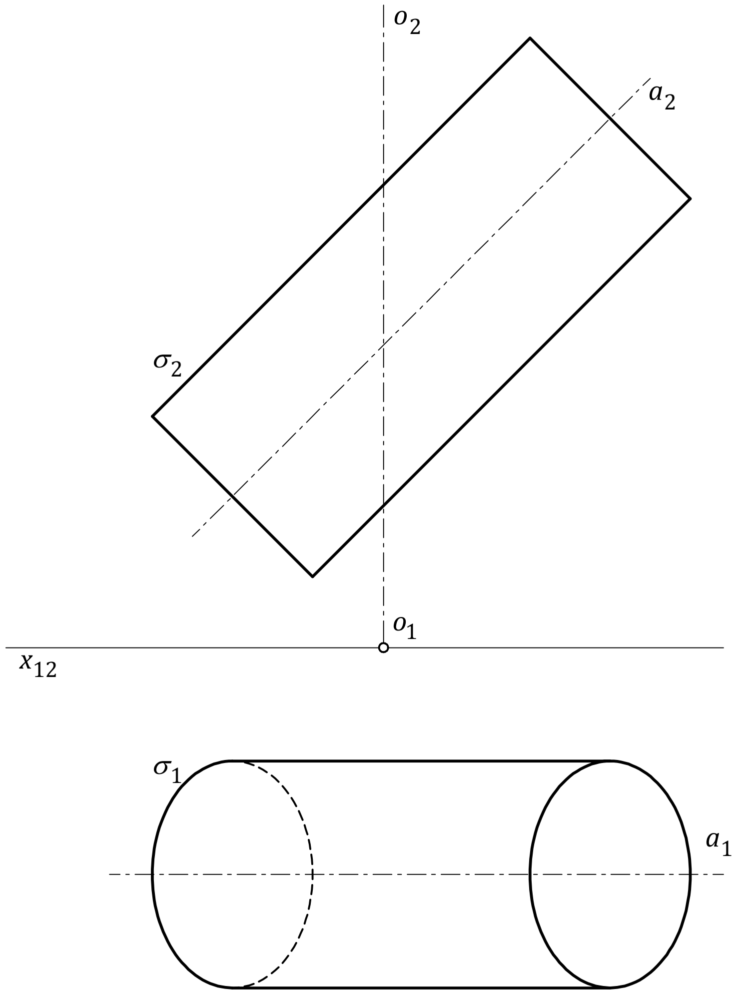


Figure 7.17: Envelope surface generated by rotation of cylinder of revolution

Graphical solution

1. Choose front view $S_2 \in a_2$ of center S of the sphere κ inscribed into the cylinder σ and determine top view $S_1 \in a_1$, $S_1S_2 \perp x_{12}$ (or choose top view $S_1 \in a_1$ and determine the front view $S_2 \in a_2$, $S_1S_2 \perp x_{12}$).

2. Construct the common parallel circle d of the cylinder σ and the sphere κ as the straight line segment $d_2 \perp a_2$, $S_2 \in d_2$ and the corresponding ellipse d_1 (congruent with both bases of the cylinder) according to fig. 3.27 b) approximated by osculation circles. Note that neither the front view κ_2 nor the top view κ_1 have to be constructed.

3. Construct top view $\chi_1 = o_1S_1$ of plane of characteristic curve k of partial torus generated by revolution of the sphere κ.

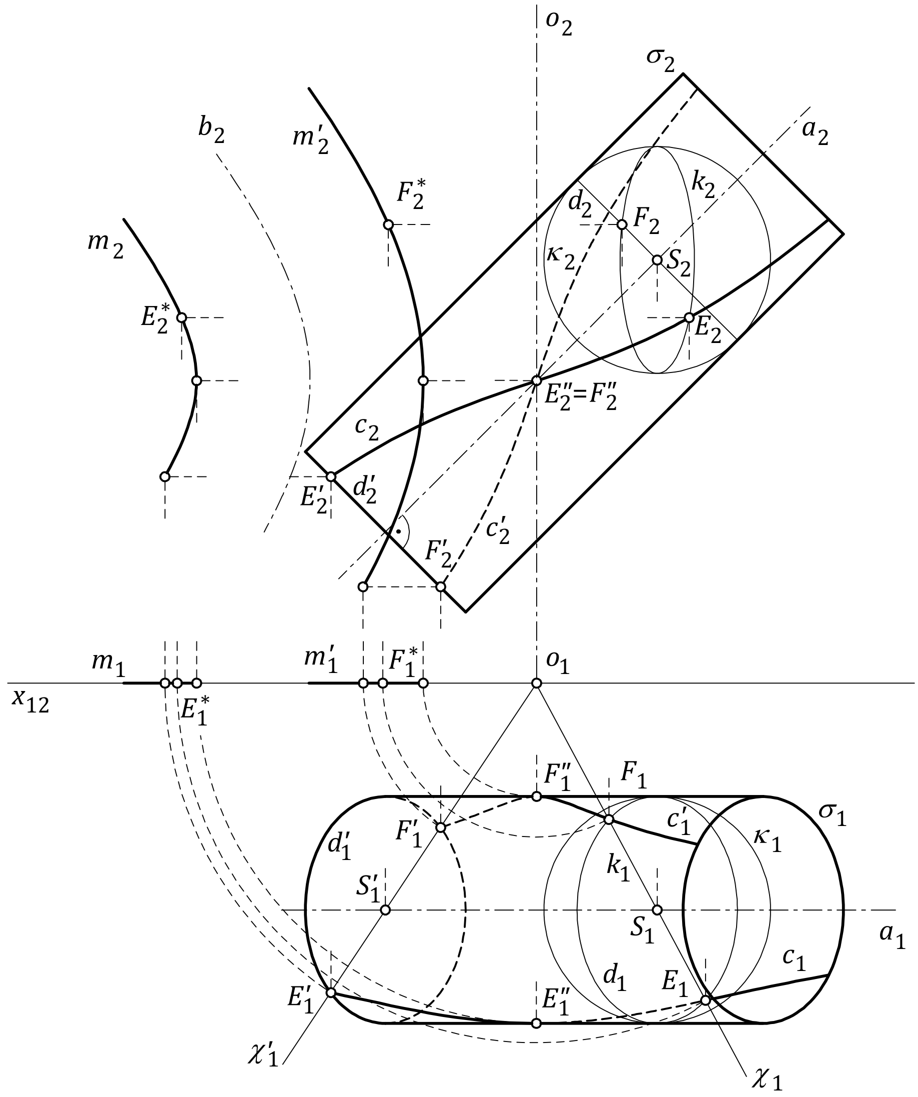


Figure 7.18: Envelope surface generated by rotation of cylinder of revolution – solution

4. Top views $E_1, F_1 = \chi_1 \cap d_1$ of points on characteristic curve c. Note that neither the top view (straight line segment k_1) nor the front view (ellipse k_2) have to be constructed.

5. Determine front views $E_2, F_2 \in d_2$, $E_1E_2 \perp x_{12}$, $F_1F_2 \perp x_{12}$.

6. Continue in the similar way to obtain sufficient number of points on characteristic curve c. Do not forget points at special positions, such as end points of characteristic curve denoted by E' and F' on the left base of the cylinder and points on the right base of the cylinder (without denotation) and points E'' and F''. Top views E_1'' and F_1'' determine the points at which the visibility of characteristic curve is changed in the top view.

7. Use pointwise construction described in example 5.4 to construct left principal half-meridian. Left principal half-meridian has two branches m and m'. Note that points on principal meridian corresponding to points E'' and F'' determine the points at the minimal distance from axis o.

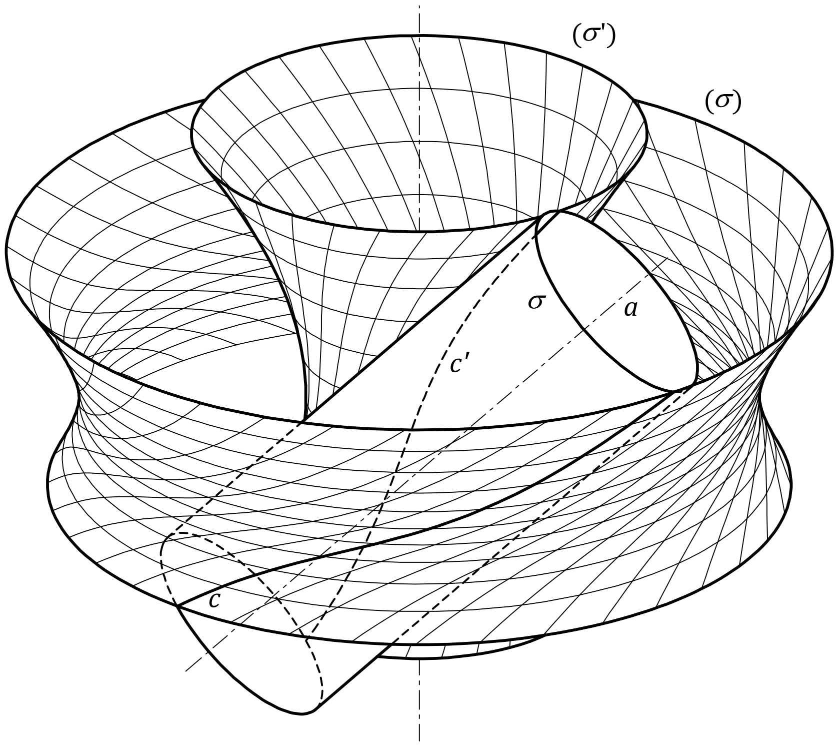


Figure 7.19: Envelope surface generated by rotation of cylinder of revolution (isometric view)

Isometric view of the given and solved figures is drawn in fig. 7.19.

Note that both branches of the envelope surface (σ) generated by rotation of cylinder of revolution (axis of cylinder a and the given axis of revolution o are skew lines) are offset surfaces of the one-sheeted hyperboloid of revolution generated by rotation of axis a about axis o. Thus, the curves m_2 and m_2' in fig. 7.20 are offset curves to the hyperbola b_2 – principal meridian of the one-sheeted hyperboloid of revolution. □

■ Example 7.9 – Rotation of a surface of revolution

Given

Generating surface of revolution σ and axis of revolution o in Monge projection, see fig. 7.20.

Required

Using Monge projection, construct characteristic curve c and right principal half-meridian of envelope surface (σ) generated by rotation of the surface σ about the given axis o.

Analysis

Pointwise construction based on method of inscribed spheres (see fig. 7.7) into the surface of revolution σ is used to construct characteristic curve c. Point on characteristic curve is constructed as intersection of two circles located on the inscribed sphere κ, if the intersection exists. The

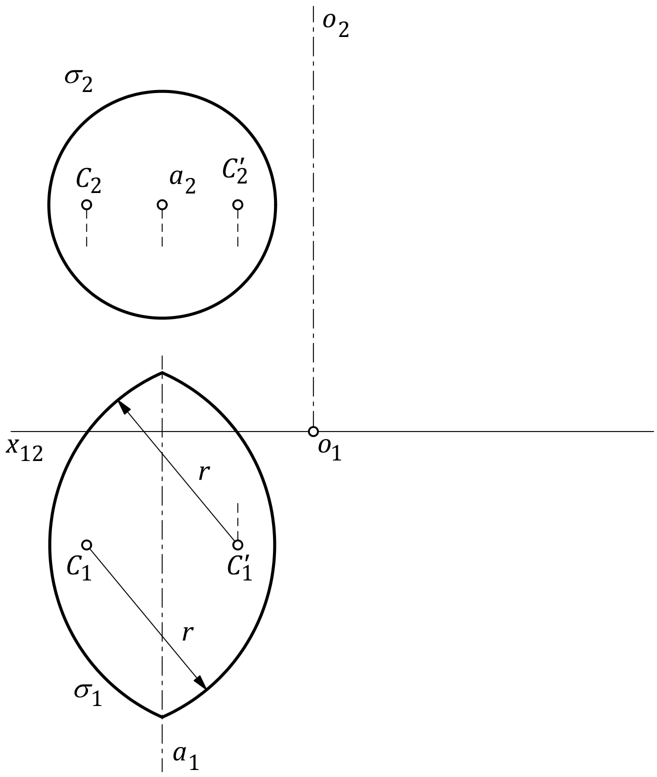


Figure 7.20: Envelope surface generated by rotation of surface of revolution (task setting)

first circle is the common parallel circle d of the sphere κ and the surface of revolution σ. The second one is characteristic curve k of partial torus generated by revolution of the sphere κ.

Since the axis a of surface σ is perpendicular to the frontal plane of projection ν, the parallel circle d lies in principal plane parallel with the frontal plane of projection ν. Therefore, it is projected as a straight line segment in the top view and as a circle in the front view, see fig. 3.26 b). The characteristic curve k of torus is a circle located in the projecting plane χ perpendicular to the horizontal plane of projection π. Thus, it is projected as a straight line segment in the top view and as an ellipse in the front view, see fig. 3.27 a) and example 7.6, where the characteristic curve of envelope surface generated by rotation of a sphere is solved.

Pointwise construction described in example 5.4 is used to construct the right principal half-meridian of the envelope surface generated by rotation of the surface σ about axis o.

Graphical solution

1. Draw top view $d_1 \perp a_1$ of common parallel circle d of the surface σ and the inscribed sphere κ. The position of d_1 is suitably chosen, see fig. 7.21.

2. Top view $E_1 = \sigma_1 \cap d_1$ of point E.

3. Construct front view $d_2 = (a_2, r = d(E_1, a_1))$ of parallel circle d.

4. Top view $S_1 = E_1C_1' \cap a_2$ of the centre S of inscribed sphere κ. Note that neither the top view κ_1 nor the front view κ_2 have to be constructed.

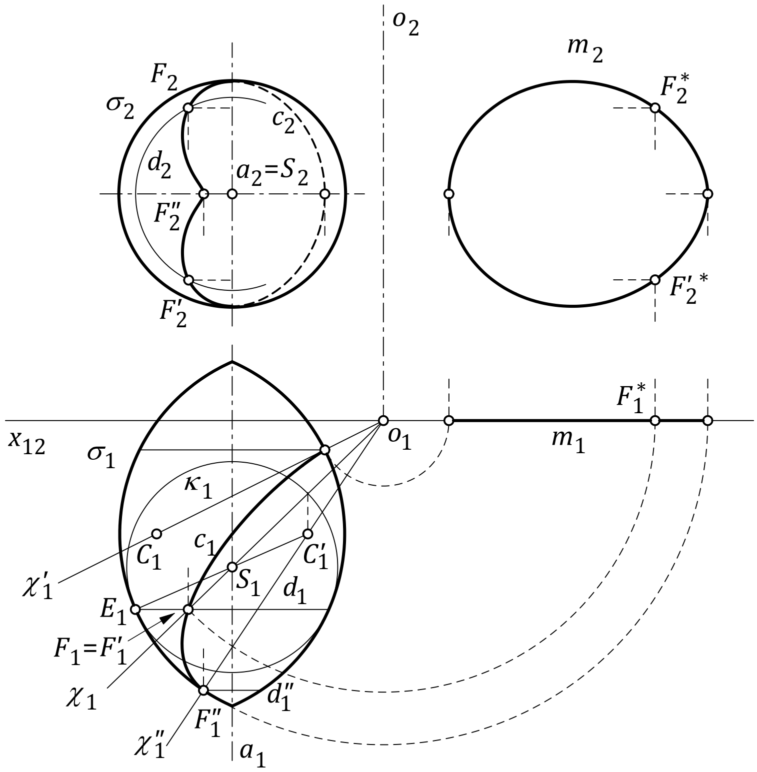


Figure 7.21: Envelope surface generated by rotation of surface of revolution – solution

5. Top view $\chi_1 = o_1S_1$ of plane of characteristic curve of partial torus generated by rotation of the inscribed sphere.

6. Top view $F_1 = \chi_1 \cap d_1$ of point F on characteristic curve c.

7. Construct front view $F_2 \in d_2$, $F_1F_2 \perp x_{12}$ of point on characteristic curve c. Here are two intersections F_2 and F_2' of ordinate F_1F_2 and d_2.

8. Continue in a similar way to obtain sufficient number of points on characteristic curve c. Finally, draw the characteristic curve as a curve passing through all the constructed points. Do not forget points at special positions given by position $\chi_1' = o_1C_1$ and $\chi_1'' = o_1C_1'$.

9. Use pointwise construction described in example 5.4 to construct right principal half-meridian m.

Isometric view of the given and solved figures is drawn in fig. 7.22.

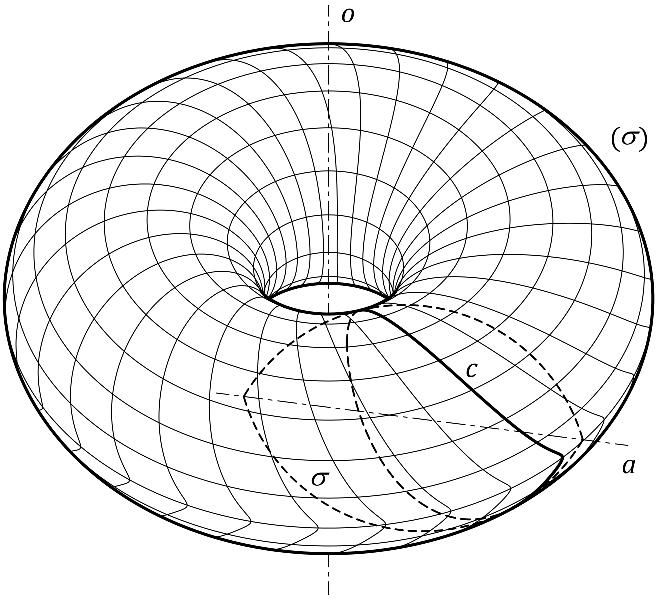


Figure 7.22: Envelope surface generated by rotation of surface of revolution (isometric view)

□

Chapter 8

Developable surfaces

The *development* of a surface is an isometric mapping of the surface onto a plane, i.e. the lengths and angles of curves located on the surface are preserved. The surface is called *developable* if it possible to unfold or unroll it into a planar figure without any distortion such as stretching or tearing. The developed planar figure gives the true size of each area of the original curved surface.

Developable surface is a special *ruled* surface with zero Gaussian curvature everywhere (see chapter 1). Ruled surface is generated by continuous motion of a straight line called *generating line* along a spatial curve called *directing curve* (*directrix*). From the geometrical point of view, the ruled surface has the same tangent plane at all points along a generating line while the surface is located in the neighbourhood of this generating line in one halfspace determined by the tangent plane. A ruled surface that is not developable is called *warped*.

Example of ruled developable surfaces is given in fig. 8.1. A cylinder of revolution and a cone of revolution are drawn here together with the tangent plane at points on the generating line g. Tangent plane at a point on a surface of revolution is given by tangent line to the generating curve (generating line g itself) and tangent line t to the parallel circle, see chapter 5. All tangent lines to the parallel circles at points on the generating line in fig. 8.1 are parallel, therefore only one tangent plane along the generating line exists.

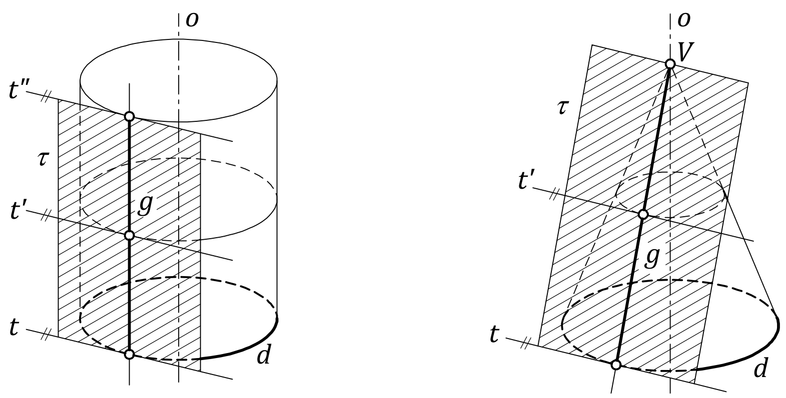


Figure 8.1: One tangent plane along the generating line of developable surfaces

Example of warped surfaces is given in fig. 8.2. A one-sheeted hyperboloid of revolution and a closed right ruled helicoidal surface are drawn here together with several tangent planes at points on the generating line g. In the case of one-sheeted hyperboloid of revolution, the tangent lines to the parallel circles of points on the generating line are not parallel, therefore, infinite number of tangent planes (called a *sheaf* of planes) along the generating line exists. Similarly, tangent plane at a point on helicoidal surface is given by tangent line to the generating curve (generating line g itself) and tangent line t to the helix, see chapter 6. The tangent lines to the helices of points on the generating line are not parallel, therefore a sheaf of tangent planes along the generating line exists, too.

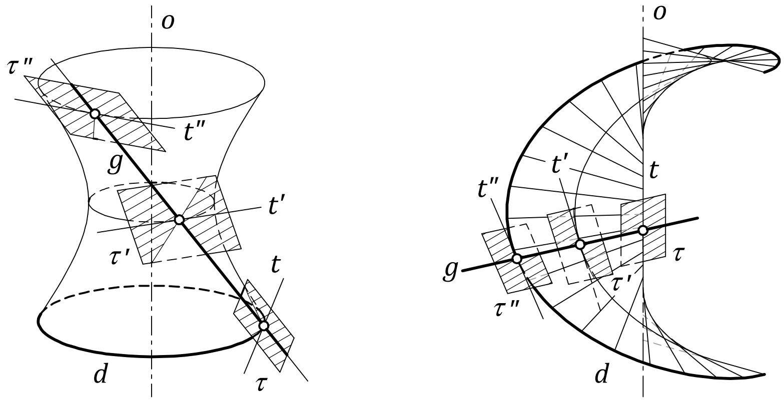


Figure 8.2: Sheaf of tangent planes along the generating line of warped surfaces

8.1 Types of developable surfaces

Developable surfaces include the following types.

- **Plane** – is given by translation of the generating line g along a directing line d (directrix), see example in fig. 8.3 a). Plane is the basic developable surface.

- **Cylinder** – *general cylinder* (*general cylindrical surface*) is given by continuous translation of generating line g along a spatial directing curve d, see example in fig. 8.3 b). If the directing curve is a circle and the generating line is not perpendicular to the plane of the directing circle, the cylinder is called *oblique*, see fig. 8.3 c). If the directing curve is a circle and the generating line is perpendicular to the plane of the directing circle, the cylinder is called *right*, see fig. 8.3 d). Right cylinder is a surface of revolution.

- **Cone** – *general cone* (*general conical surface*) is given by a fixed point (vertex) V and spatial directing curve d, see example in fig. 8.3 e). The union of all straight lines passing through the vertex and any point of the directing curve creates a cone. If the directing curve is a circle and straight line given by vertex and the centre of the directing circle is not perpendicular to the plane of the directing circle, the cone is called *oblique*, see fig. 8.3 f). If the directing curve is a circle and straight line given by vertex and the centre of the directing circle is perpendicular to the plane of the directing circle, the cone is called *right*, see fig. 8.3 g). Right cone is a surface of revolution.

- **Tangent surface of a spatial curve** – the surface is given by motion of generating line g in the direction of tangent vector to the spatial directing curve g. Tangent surface of a spatial curve can be considered an envelope surface generated by motion of osculation plane along a spatial directing curve. If the directing curve is a helix, tangent surface of the helix is created, see section 6.1 and example 8.7.

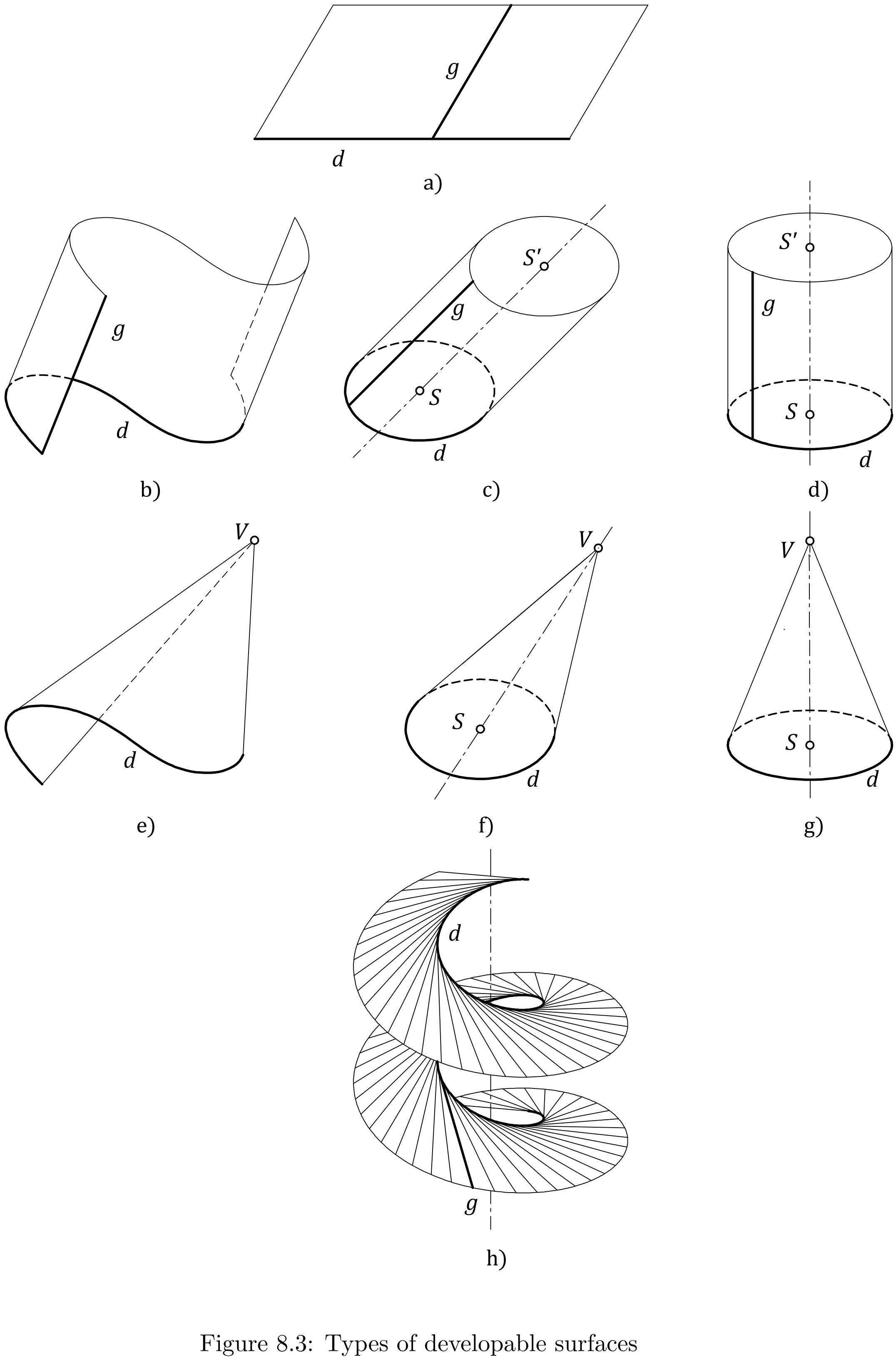


Figure 8.3: Types of developable surfaces

8.2 Methods of development

In general, the construction of development consists of selecting suitable lines (individual positions of generating lines) and important curves located on a curved surface and determination of true or approximate mutual relationship of these lines and curves. The developed figure is obtained by reproduction of the relationship on a plane. There are the following basic methods of development.

- **Parallel-line development** – a set of parallel lines is selected on the surface. This method is useful in the case of cylinder development, where lines parallel with the generating line of the cylinder are chosen and the cylinder is approximated by inscribed prism, see examples 8.1, 8.2 and 8.3.

- **Radial-line development** – lines passing through one point are selected on the surface. This method is suitable for cones, where the lines pass through the vertex and the cone is approximated by inscribed pyramid, see examples 8.4 and 8.6.

- **Tangent-line development** – tangent lines to the directing curve are selected on the surface. This method is suitable for tangent surface of spatial curves, where the surface is approximated by a set of quadrilateral faces, see example 8.7.

- **Triangulation** – the surface is split into a set of triangular areas. This method can be used in the case of any surface development even if the surface is theoretically not developable. In such a case, the surface is approximated by irregular polyhedra with triangular faces. The development consists in construction of true shapes of individual triangular faces. Graphical solution is beyond the scope of this textbook, example is given in fig. 8.18.

8.2.1 Example problems – developable surfaces

■ **Example 8.1 – Development of right cylinder**

Given

Right cylinder (cylinder of revolution) $\sigma = (d, g)$ and section plane $\chi \perp \nu$ in Monge projection, see fig. 8.4.

Required

Construct development σ_0 of a part of the cylinder between the horizontal plane of projection π and the section plane χ.

Analysis

The directing circle d lies in the horizontal plane of projection. The generating line $g \perp d$ and $g \parallel \nu$. The intersection $e = \sigma \cap \chi$ is an ellipse. Parallel-line development method is used, where the cylinder is approximated by n-sided right prism inscribed into the cylinder. Edges of the prism parallel with generating line g are projected in true length in the front view. Edges of the prism base are projected in true size in the top view. Length of base edges can be approximated by the chordal length of the polygon inscribed into the directing circle. For a more accurate development, these lengths can be approximated according to fig. 2.33 or calculated as semicircumference $2\pi r/n$, where r is the radius of the cylinder. Then, the directing circle d is developed into a straight line d_0 of length equal to $2\pi r$. Edges parallel with generating line g are developed in straight lines perpendicular to the d_0.

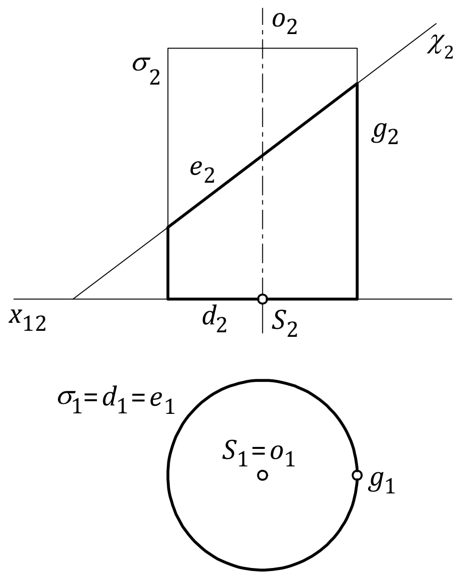


Figure 8.4: Development of right cylinder – task setting

Graphical solution

1. Divide the top view d_1 into sufficient number of n ($n = 12$ at least) equal parts, see fig. 8.5, where $n = 12$. Top views $0_1, 1_1, \ldots, 12_1$ of points along the top view d_1 of the directing circle d are obtained.

2. Construct front views $0_2, 1_2, \ldots, 12_2$ of dividing points along the front view d_2 of the directing circle d.

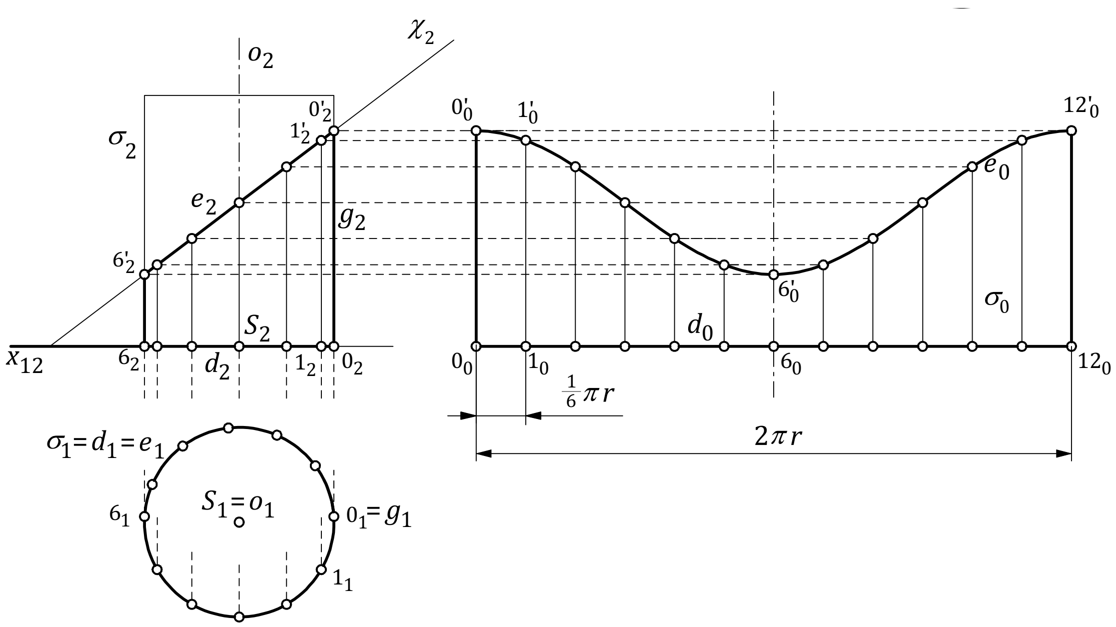


Figure 8.5: Right cylinder development – solution

3. Construct front views $0'_2, 1'_2, \ldots, 12'_2$ of dividing points along the front view e_2 of ellipse e.

4. Draw front views $0_2 0'_2, 1_2 1'_2, \ldots, 12_2 12'_2$ of prism edges.

5. Draw development d_0 of the directing circle as a straight line. Mark equidistant dividing points $0_0, 1_0, \ldots, 12_0 \in d_0$ so that $||0_0 1_0||, ||1_0 2_0||, \ldots, ||11_0 12_0|| = \widehat{0_1 1_1}$. Note that d_0 can be drawn in arbitrary position anywhere. However, optimal position is shown in fig. 8.5, where d_0 is drawn in extension of d_2 (i.e. x_{12} in this case).

6. Construct lines at points $0_0, 1_0, \ldots, 12_0$ perpendicular to d_0.

7. Measure the true length of prism edges and mark them on the corresponding lines so that $||0_2 0'_2|| = ||0_0 0'_0||, ||1_2 1'_2|| = ||1_0 1'_0||, \ldots, ||12_2 12'_2|| = ||12_0 12'_0||$.

8. Draw the developed ellipse e_0 as a curve passing through points $0'_0, 1'_0, \ldots, 12'_0$. Edge $6_0 6'_0$ is the axis of symmetry of the developed shape. □

■ Example 8.2 – Development of degenerated intersection of two right cylinders

Given

Right cylinder $\sigma = (o, m)$ and axis o' of right cylinder $\sigma' = (o', m')$ in Monge projection, see fig. 8.6.

Required

Using Monge projection, construct the front view σ_2 of the cylinder σ and the front view σ'_2 of cylinder σ' so that the intersection of both cylinders is degenerated. Construct development σ_0 and σ'_0 of parts of both cylinders corresponding to the drawn parts of their axes. Do not draw the top view of this situation.

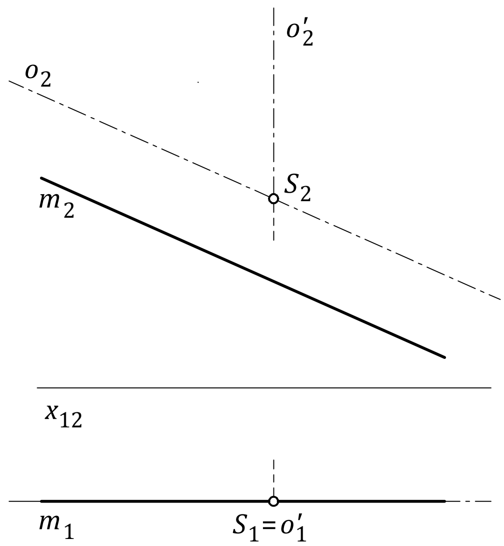


Figure 8.6: Development of degenerated intersection of two right cylinders – task setting

Analysis

Situation drawn in fig. 8.6 represents the intersection of two cylinders of revolution with intersecting axes, see chapter 5. If the intersection is degenerated, then the front view of auxiliary sphere with centre S_2 inscribed into both cylinders (see section 5.3.2) has to be constructed first. Secondly, the front views of both cylinders can be constructed and parallel-line development described in example 8.1 can be individually applied on each cylinder.

The prism edges parallel with axes of cylinders are projected in true length in the front view. Base edges appear in true length in auxiliary views obtained by revolving the directing circles into the principal plane parallel with the frontal plane of projection.

Due to symmetry, only half-circles are necessary to construct in auxiliary views. To construct the developed shape, the top view of both cylinders does not have to be drawn.

Graphical solution

Graphical solution is given in fig. 8.7.

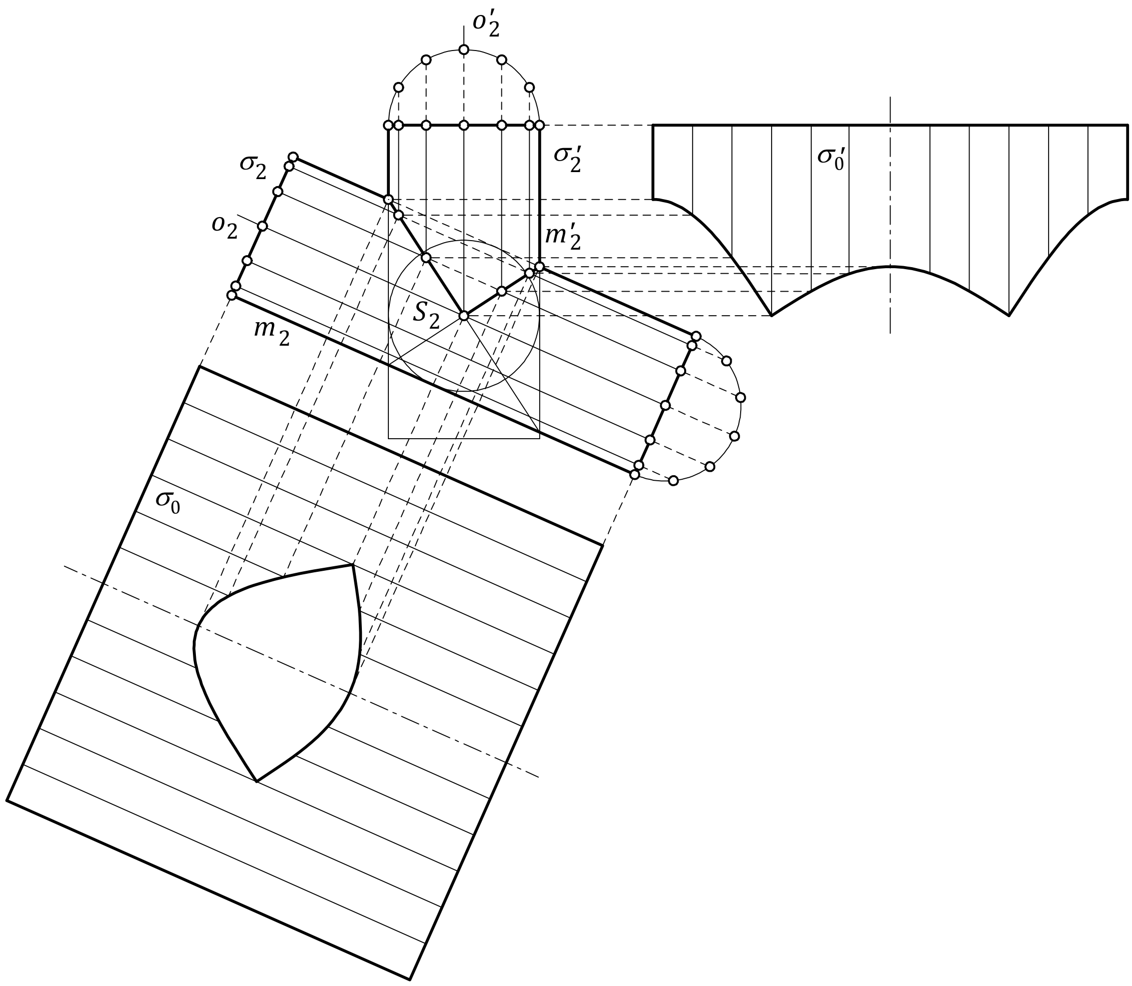


Figure 8.7: Development of degenerated intersection of two right cylinders – solution

□

■ **Example 8.3 – Development of oblique cylinder**

Given

Oblique cylinder $\sigma = (d, g)$ and section plane $\chi \perp g$ in Monge projection, see fig. 8.8.

Required

Construct development σ_0 of the cylinder σ.

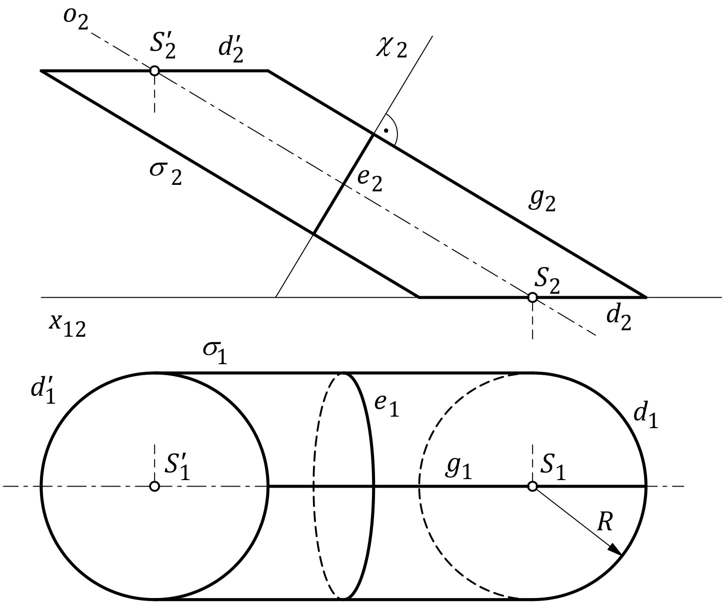


Figure 8.8: Development of oblique cylinder
task setting

Analysis

The directing circle d lies in the horizontal plane of projection. The generating line g is parallel with the frontal plane of projection. The normal section $e = \sigma \cap \chi$ of the cylinder is an ellipse lying in the plane perpendicular to the generating line g.

Parallel-line development method is used, where the cylinder is approximated by an n-sided oblique prism inscribed into the cylinder, see fig. 8.9. Edges of the prism parallel with generating line g are projected in true length in the front view and their perpendicularity with normal section e is preserved in the development.

Normal section e appears in true size in auxiliary view obtained by revolving the normal section into the principal plane of projection parallel with the frontal plane of projection. The length of base edges of the prism with vertices on normal section e can be approximated by chordal length of polygon inscribed into the revolved ellipse (lengths r, r' and r'' in fig. 8.9). For a more accurate development, these lengths can be approximated according to fig. 2.33.

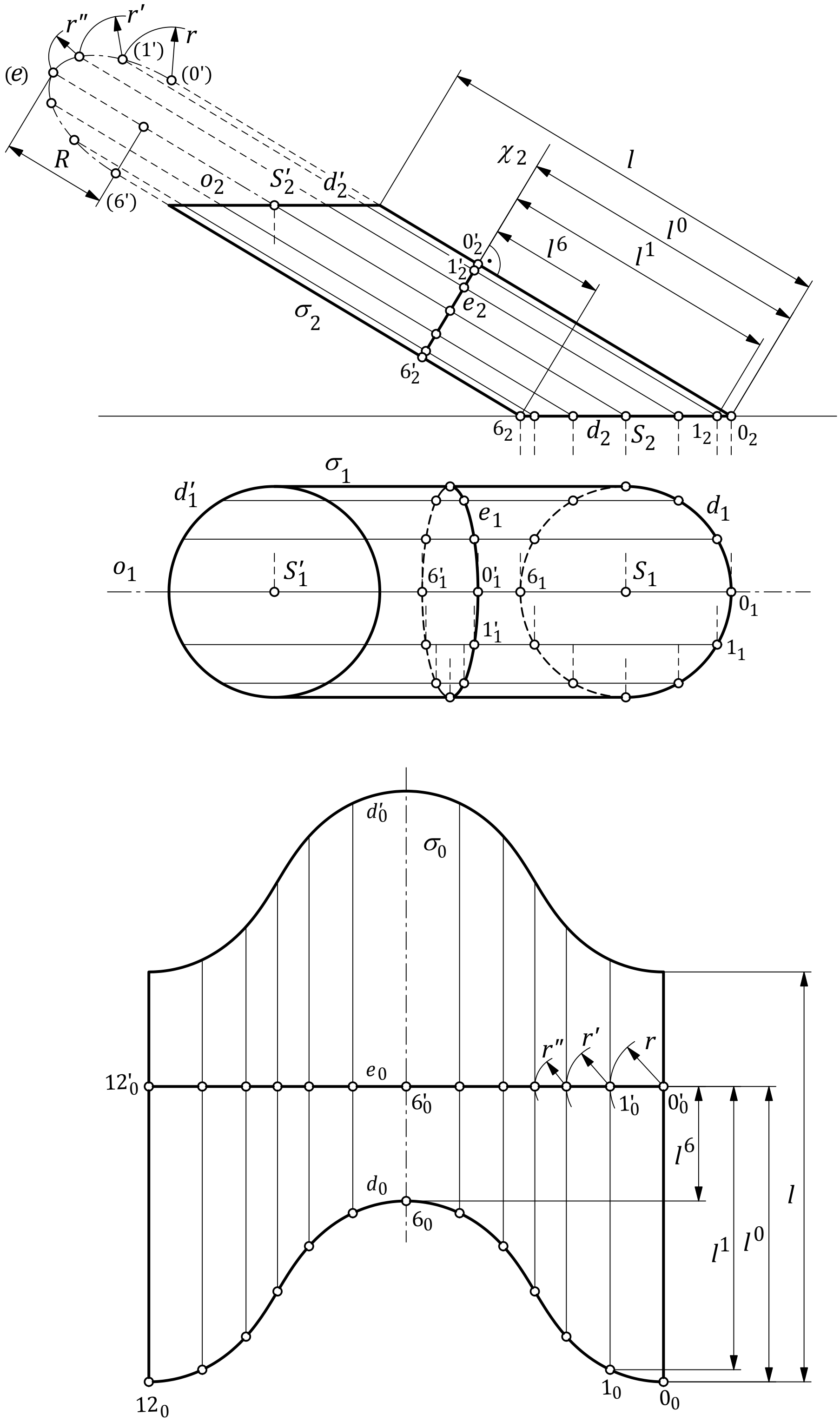


Figure 8.9: Development of oblique cylinder – solution

Graphical solution

1. Divide the top view d_1 into a sufficient number of n ($n = 12$ at least) equal parts, see fig. 8.9, where $n = 12$. Top views $0_1, 1_1, \ldots, 12_1 \in d_1$ of points along the directing circle d are obtained.

2. Construct front views $0_2, 1_2, \ldots, 12_2 \in d_2$ of dividing points.

3. Construct front views $0'_2, 1'_2, \ldots, 12'_2 \in e_2$ of dividing points along the normal section.

4. Draw front views $0_2 0'_2, 1_2 1'_2, \ldots, 12_2 12'_2$ of prism edges. Note that the top views of prism edges do not have to be drawn.

5. Revolve ellipse e into the principal plane parallel with the frontal plane of projection and construct auxiliary view (e) of the ellipse. The length of the major axis is equal to $2R$, where R is the radius of the directing circle. The length of the minor axis is equal to $||0'_2 6'_2||$. Note that due to symmetry, only a half of the revolved ellipse is necessary to construct.

6. Construct dividing points $(0'), (1'), \ldots, (12') \in (e)$ in auxiliary view.

7. Draw development e_0 of the normal section as a straight line. Mark dividing points $0_0, 1_0, \ldots, 12_0 \in e_0$ so that $||0_0 1_0|| = \overset{\frown}{(0')(1')}$, $||1_0 2_0|| = \overset{\frown}{(1')(2')}$ and $||2_0 3_0|| = \overset{\frown}{(2')(3')}$. Use symmetry to mark all other points.

8. Construct lines at points $0_0, 1_0, \ldots, 12_0$ perpendicular to e_0.

9. Measure the true lengths of prism edges between the horizontal plane of projection and the normal section plane and mark them on the corresponding lines in the development so that $||0'_2 0_2|| = ||0'_0 0_0||, ||1'_2 1_2|| = ||1'_0 1_0||, \ldots, ||12'_2 12_2|| = ||12'_0 12_0||$.

10. Draw the developed directing circle d_0 as a curve passing through points $0_0, 1_0, \ldots, 12_0$.

11. Complete the development by d'_0 of the same shape as d_0 at the distance l. The edge passing through point 6_0 is the axis of symmetry of the developed shape. □

■ Example 8.4 – Development of right cone

Given

Right cone (cone of revolution) $\sigma = (d, V)$ and section plane $\chi \perp \nu$ in Monge projection, see fig. 8.10.

Required

Construct development σ_0 of a part of the cone σ between the horizontal plane of projection π and the section plane χ.

Analysis

The directing circle d lies in the horizontal plane of projection. The intersection $e = \sigma \cap \chi$ is an ellipse. The right cone development is a sector of a circle with radius l equal to the length of generating line of the cone, see fig. 8.11. The sector angle φ is given by $\varphi = 360r/l$, where r is the radius of the directing circle.

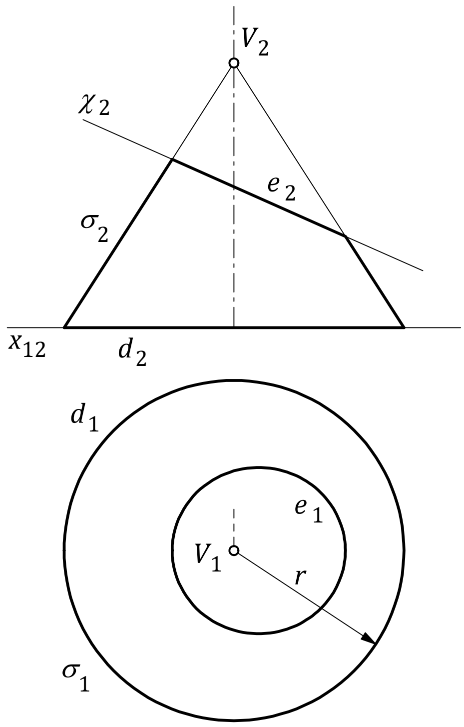


Figure 8.10: Development of right cone – task setting

Radial-line development method is used, where the cone is approximated by an n-sided right pyramid inscribed into the cone. Edges of the pyramid passing through the vertex V are not generally parallel with the frontal plane of projection. Thus, it is necessary to construct their true lengths. Edges of the pyramid base are projected in true size in the top view. The length of base edges can be approximated by chordal length of the polygon inscribed into the directing circle. For a more accurate development, this length can be approximated according to fig. 2.33 or it is possible to use a protractor and measure the corresponding angle φ/n.

Graphical solution

1. Divide the top view d_1 into a sufficient number of n ($n = 12$ at least) equal parts, see fig. 8.11, where $n = 12$. Top views $0_1, 1_1, \ldots, 12_1 \in d_1$ of points along the directing circle d are obtained.

2. Construct front views $0_2, 1_2, \ldots, 12_2 \in d_2$ of dividing points.

3. Construct front views $0_2V_2, 1_2V_2, \ldots, 12_2V_2$ of pyramid edges. Note that top views of pyramid edges passing through the vertex do not have to be drawn.

4. Front views $0'_2 = e_2 \cap 0_2V_2$, $1'_2 = e_2 \cap 1_2V_2, \ldots$, $12'_2 = e_2 \cap 12_2V_2$ of dividing points along the ellipse.

5. Determine the true length $l^{0'}, l^{1'}, \ldots, l^{12'}$ the pyramid edges by revolving of pyramid edges into the principal meridian plane. Edges $0V = 12V$ and $6V$ are projected in true length in the front view, because they are the principal meridians of the cone.

6. Construct the development d_0 of the directing circle as a sector of a circle with radius l. Mark dividing points $0_0, 1_0, \ldots, 12_0 \in d_0$ so that $\widehat{0_0 1_0}, \widehat{1_0 2_0}, \ldots, \widehat{11_0 12_0} = \widehat{0_1 1_1}$.

7. Draw lines $0_0V_0, 1_0V_0, \ldots, 12_0V_0$.

8. Measure the constructed true length of pyramid edges between the horizontal plane of projection and the section plane χ by compass and mark them on the corresponding lines so that $||0_0 0_0'|| = l^{0'}$, $||1_0 1_0' = l^{1'}||, \ldots$, $||12_0 12_0'|| = l^{12'}$.

9. Draw the developed ellipse e_0 as a curve passing through points $0_0', 1_0', \ldots, 12_0'$. Edge 6_0V_0 is the axis of symmetry of the developed shape.

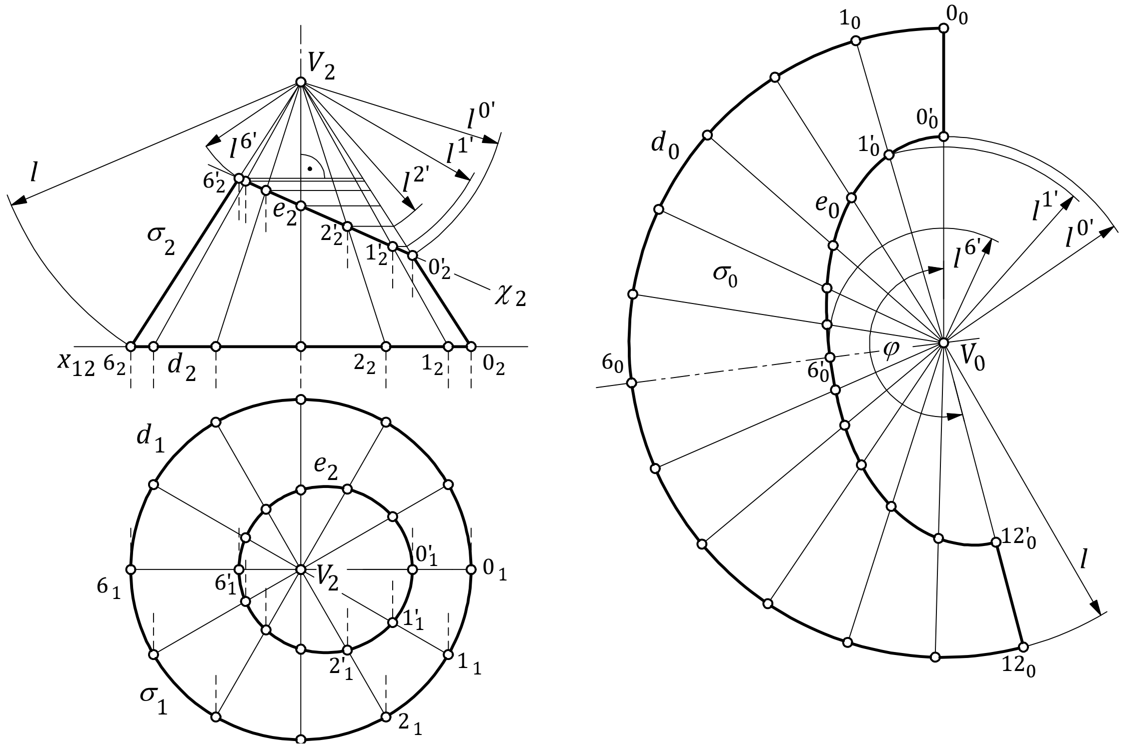


Figure 8.11: Development of right cone – solution

□

■ Example 8.5 – Development of degenerated intersection of right cylinder and right cone

Given

Right cylinder $\sigma = (o, m)$, axis o' and vertex V of right cone $\sigma' = (m', V)$ in Monge projection, see fig. 8.12.

Required

Using Monge projection, construct the front view σ_2 of the cylinder σ and the front view σ_2' of a right cone so that the intersection of these quadratic surfaces is degenerated. Construct development σ_0 and σ_0' of parts of both surfaces corresponding to the drawn parts of their axes. Do not draw the top view of this situation.

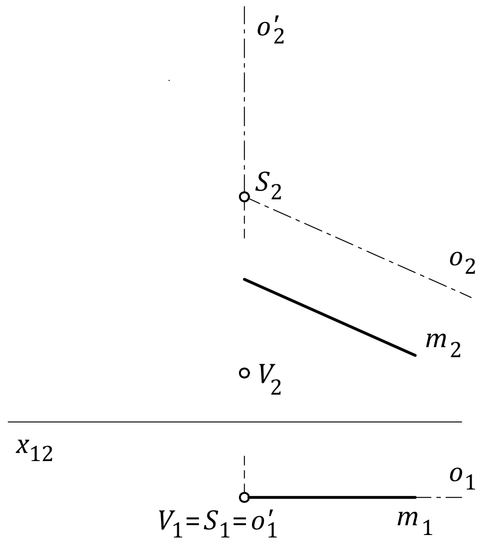


Figure 8.12: Development of degenerated intersection of right cylinder and cone – task setting

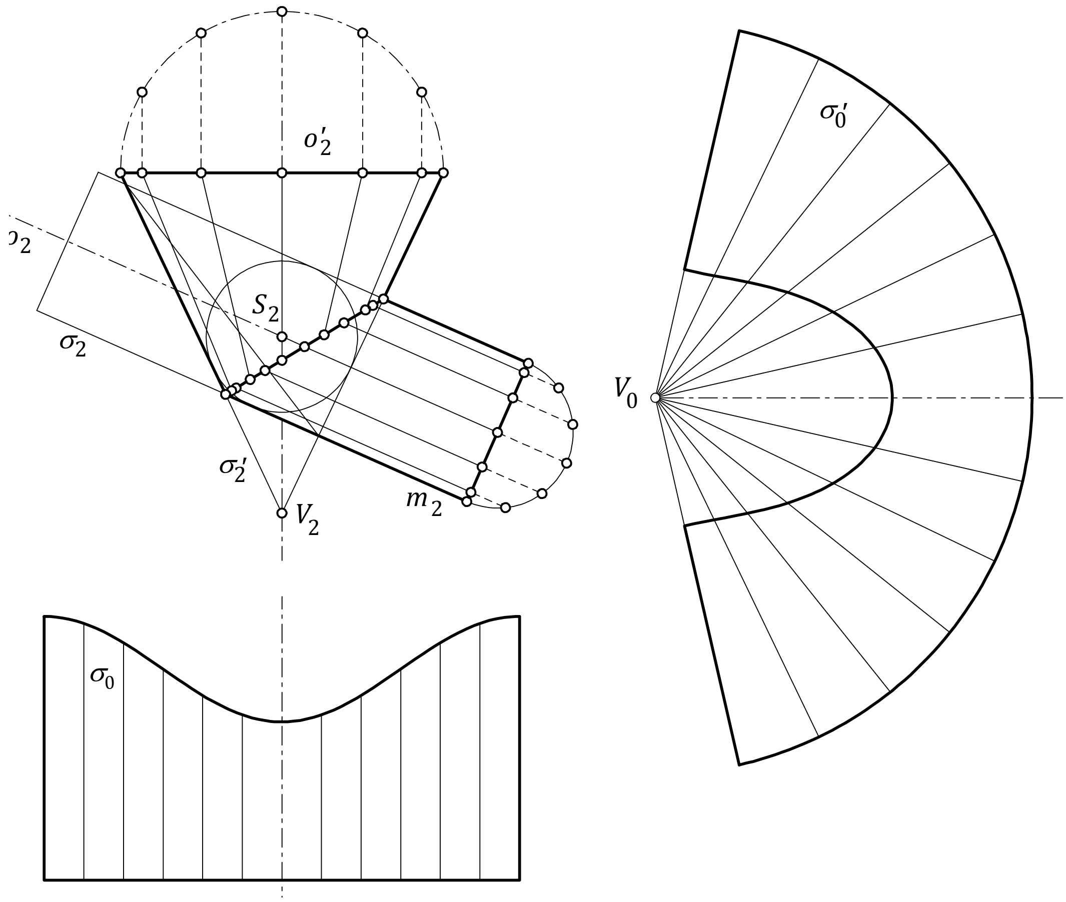


Figure 8.13: Development of degenerated intersection of right cylinder and cone – solution

Analysis

Situation drawn in fig. 8.12 represents an intersection of two quadratic surfaces of revolution with intersecting axes, see chapter 5. If the intersection is degenerated, the front view of auxiliary sphere with centre at S_2 inscribed into both surfaces (see section 5.3.2) has to be constructed first. After that, the front view of both surfaces can be constructed. Next, parallel-line development described in example 8.1 can be applied on the cylinder and radial-line development described in example 8.4 can be applied on the cone. To construct the developed shape, the top view of both surfaces does not have to be drawn.

Graphical solution

Graphical solution is given in fig. 8.13. □

■ Example 8.6 – Development of oblique cone

Given

Oblique cone $\sigma = (d, V)$ and section plane $\chi \perp \nu$ in Monge projection, see fig. 8.14.

Required

Construct development σ_0 of a part of the cone σ between the horizontal plane of projection π and the section plane χ.

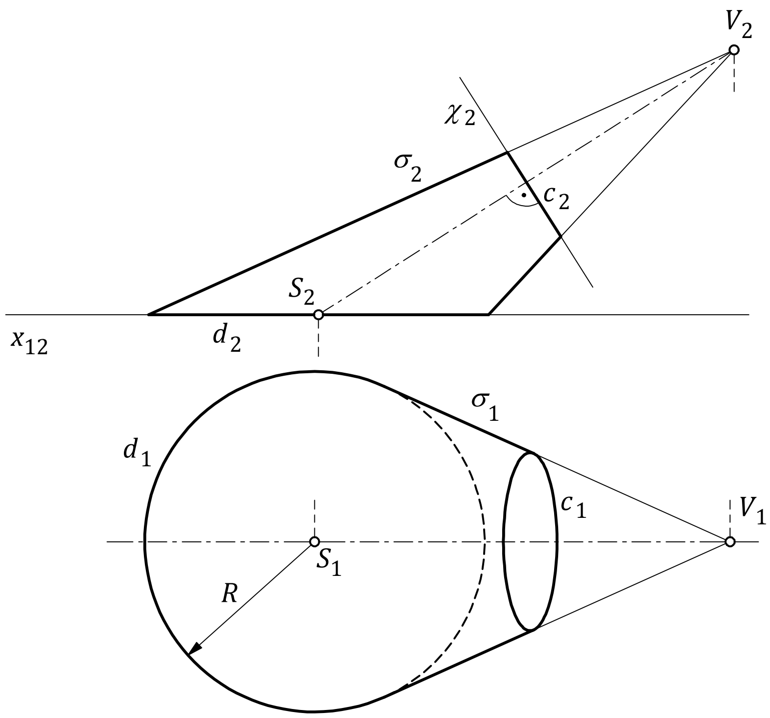


Figure 8.14: Development of oblique cone – task setting

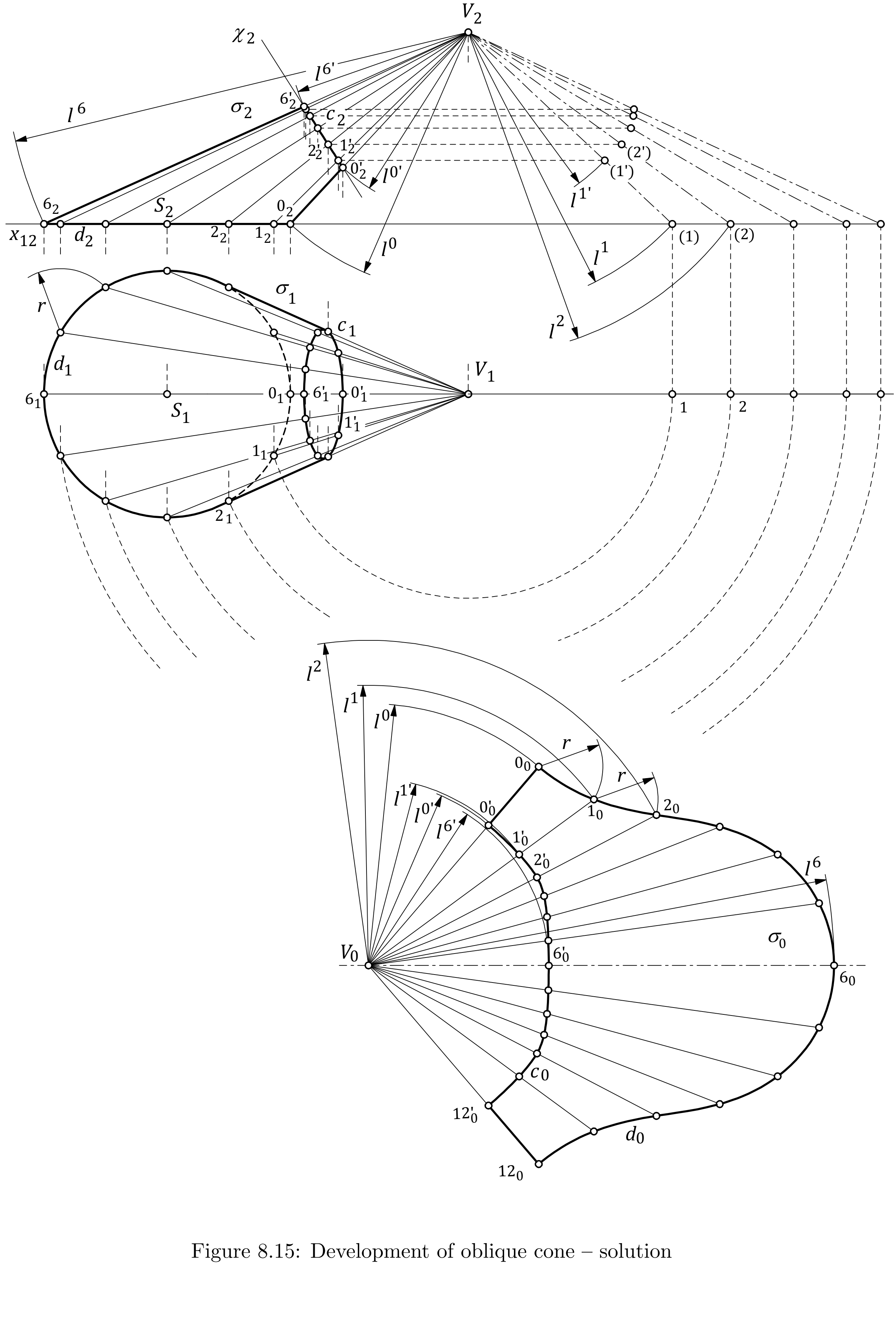


Figure 8.15: Development of oblique cone – solution

Analysis

The directing circle d lies in the horizontal plane of projection. Radial-line development method is used, where the cone is approximated by an n-sided pyramid. Edges of the pyramid passing through the vertex V are not generally parallel with the frontal plane of projection. Thus, it is necessary to construct their true lengths. Edges of the pyramid base are projected in their true size in the top view. The length of the base edges can be approximated by the chordal length r of the edge of polygon inscribed into the directing circle. Note that the normal section $c = \sigma \cap \chi$ is not an ellipse in this case.

Graphical solution

1. Divide the top view d_1 into a sufficient number of n ($n = 12$ at least) equal parts, see fig. 8.15, where $n = 12$. Top views $0_1, 1_1, \ldots, 12_1 \in d_1$ of points along the directing circle d are obtained.

2. Construct front views $0_2, 1_2, \ldots, 12_2 \in d_2$ of dividing points.

3. Construct front views $0_2V_2, 1_2V_2, \ldots, 12_2V_2$ of pyramid edges. Note that top views of pyramid edges passing through the vertex do not have to be drawn.

4. Front views $0'_2 = c_2 \cap 0_2V_2$, $1'_2 = c_2 \cap 1_2V_2, \ldots$, $12'_2 = c_2 \cap 12_2V_2$ of dividing points along the normal section.

5. Determine true length $l^0, l^1, \ldots, l^{12}$ and $l^{0'}, l^{1'}, \ldots, l^{12'}$ of pyramid edges by revolving pyramid edges into the principal plane parallel with the frontal plane of projection. Edges $0V = 12V$, $6V$, $0'V = 12'V$ and $6'V$ are projected in their true length in the front view, because they are parallel with the frontal plane of projection.

6. Construct individual triangular faces of pyramid, i.e. $\triangle 0_0 1_0 V_0$ so that $||0_0 1_0|| = r$, $||1_0 V_0|| = l^1$ and $||V_0 0_0|| = l^0$; $\triangle 1_0 2_0 V_0$ so that $||1_0 2_0|| = r$, $||2_0 V_0|| = l^2$ and $||V_0 1_0|| = l^1$; etc.

7. Draw the development d_0 of the directing circle as a curve passing through points $0_0, 1_0, \ldots, 12_0$.

8. Construct points $0'_0, 1'_0, \ldots, 12'_0$ at the distances $l^{0'}, l^{1'}, \ldots, l^{12'}$ on the corresponding pyramid edges in the development.

9. Draw the development c_0 of the normal section as a curve passing through points $0'_0, 1'_0, \ldots, 12'_0$. The pyramid edge $6_0 6'_0$ is the axis of symmetry of the developed shape. □

■ Example 8.7 – Development of tangent surface of helix

Given

Helix $h = (A, o, v_0, \text{right-handed})$ in Monge projection, see fig. 8.16 a).

Required

Construct the development σ_0 of a half thread of tangent surface σ of helix h between the helix and the horizontal plane of projection.

Analysis

Since the first curvature of the helix is constant, see eq. (6.2), the development h_0 of helix is a circle with radius

$$R = \frac{r^2 + v_0^2}{r}. \tag{8.1}$$

The radius R can be determined constructionally according to the Euclidean theorem, see fig. 8.16 b). The intersection of the helix and the horizontal plane of projection is an involute e of top view h_1 of the helix. The development e_0 of this involute is an involute of the developed helix h_0. The tangent surface σ of the helix is a set of tangent lines to the helix.

The tangent-line development method is used and the surface is approximated by planar quadrilaterals joined along edges – tangent lines to the helix.

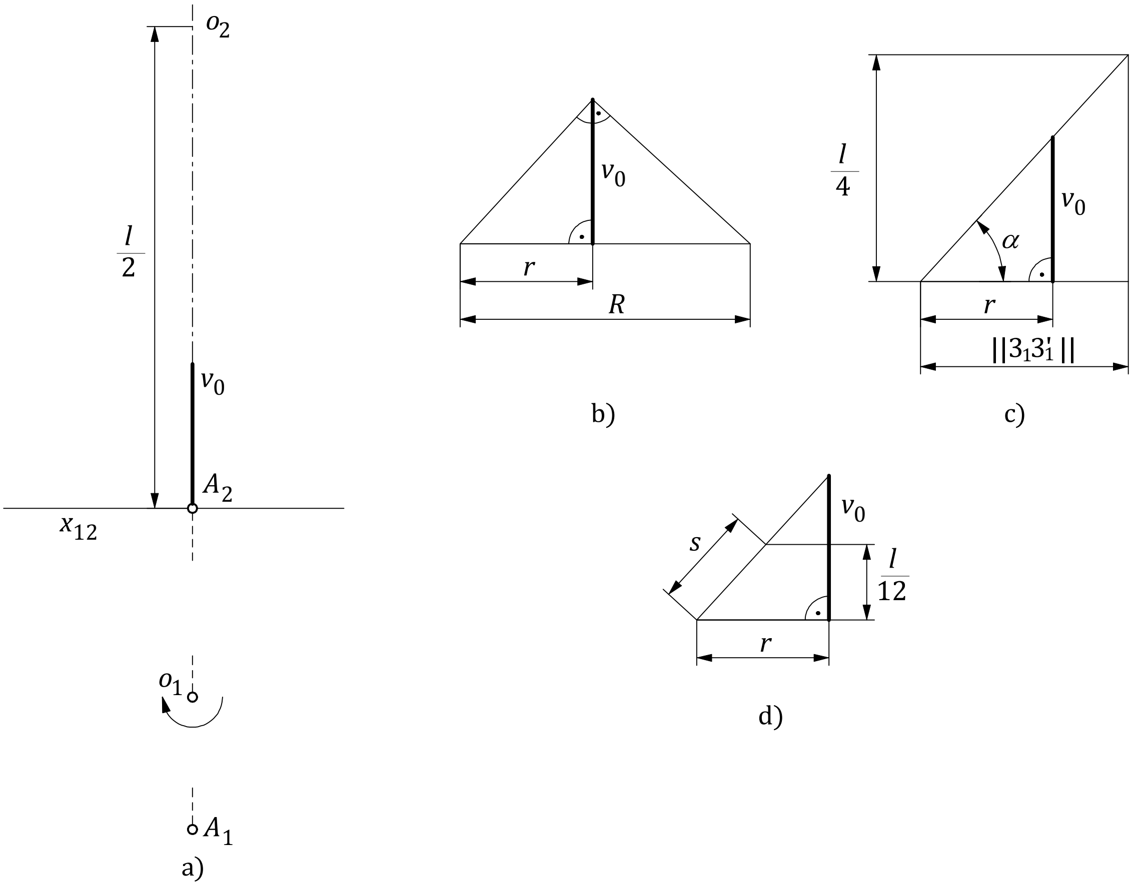


Figure 8.16: Development of tangent surface of helix – task setting and analysis

Graphical solution

1. Construct the top view $h_1 = (o_1, r = ||o_1A_1||)$ of helix. Divide a semicircle belonging to the half of thread in a sufficient number of n equal parts ($n = 6$ at least), see fig. 8.17, where $n = 6$. Top views $0_1, 1_1, \ldots, 6_1$ of points along the helix are obtained. Construct their front views $0_2, 1_2, \ldots, 6_2$, see example 6.1. Construct tangent lines $t^0, t^1, \ldots, t^6$ at each dividing point, see example 6.2.

2. Front views $0'_2, 1'_2, \ldots, 6'_2 \in e_2$ of points on involute e lie at the intersections of tangent lines and folding line x_{12}.

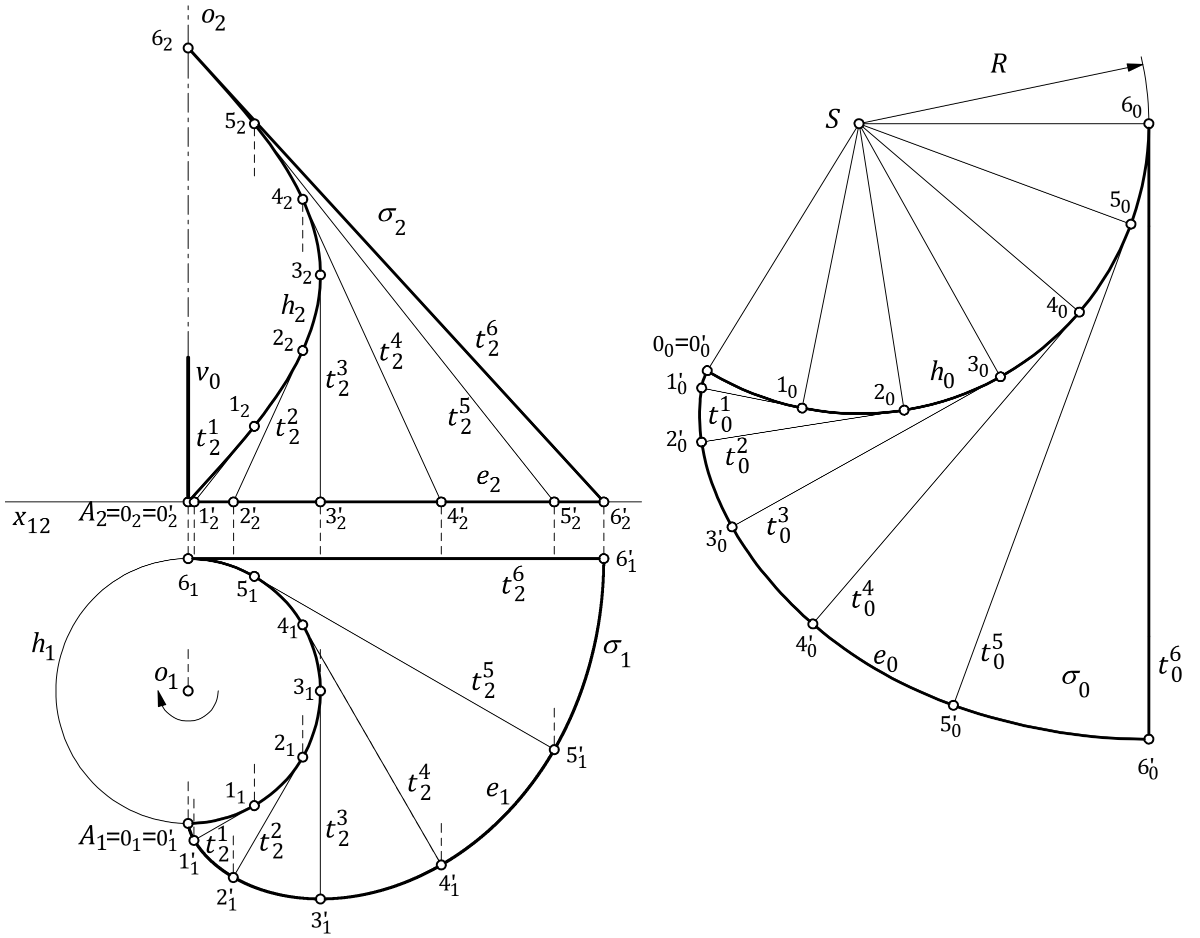


Figure 8.17: Development of tangent surface of helix – solution

3. Construct top views $0_2', 1_2', \ldots, 6_2'$ as intersections of the ordinates of points and the corresponding top views of tangent lines. Tangent line t^3 is perpendicular to the folding line x_{12}, i.e. coincides with the ordinate constructed from front view $3_2'$. Therefore, it is necessary to determine the distance $||3_1 3_1'||$ as is drawn in fig. 8.16 c), for example.

4. Draw the development h_0 as a sector of a circle with radius R determined by construction according to fig. 8.16 b).

5. Determine the length of arc $s = \widehat{01}$ from the graph of developed helix according to fig. 8.16 d).

6. Mark points $0_0, 1_0 \ldots, 6_0 \in h_0$ along the development h_0 so that $\widehat{0_0 1_0}, \widehat{1_0 2_0}, \ldots, \widehat{5_0 6_0} = s$, the approximation of the length of the arc by the length of the polygon given in fig. 2.33 can be used.

7. Construct tangent lines $t_0^0, t_0^1, \ldots, t_0^6$ to the development h_0.

8. Determine points $1_0, 2_0, \ldots, 6_0 \in e_0$ so that $||1_0 1_0'|| = s$, $||2_0 2_0'|| = 2s, \ldots, ||6_0 6_0'|| = 6s$.

9. Draw the development e_0 as a curve passing through points $0_0', 1_0', \ldots, 6_0'$. □

8.3 Developable transition surfaces

Transition surface (*transition piece*) is a surface connecting two differently shaped or sized ducts represented by their planar profiles. If vector equations of both profiles are known, the transition surface σ can be expressed as a ruled surface given by two directing curves $d: \mathbf{D}(u)$ and $e: \mathbf{E}(u)$ with the same domain of parametrization $u \in [u_1, u_2]$ as follows

$$\sigma: \mathbf{S}(u,v) = (1-v)\mathbf{D}(u) + v\mathbf{E}(u), \ u \in [u_1, u_2], \ v \in [v_1, v_2], \tag{8.2}$$

see example in fig. 8.18. However, the surface given by eq. (8.2) is not generally developable. Usually, the directing curves d and e are divided into the same number of equally spaced parts and approximated by polygons $\bar{d}$ and $\bar{e}$. The warped ruled surface is broken into a series of small triangles with edges and vertices at the dividing points. The set of triangles $\bar{\sigma}$ approximates the original form. The development consists in construction of true size of all triangles arranged in an appropriate way. Finally, the polygons $\bar{d}$ and $\bar{e}$ are interpolated by smooth curves $\bar{d_0}$ and $\bar{e_0}$ and the development $\bar{\sigma_0}$ is accomplished.

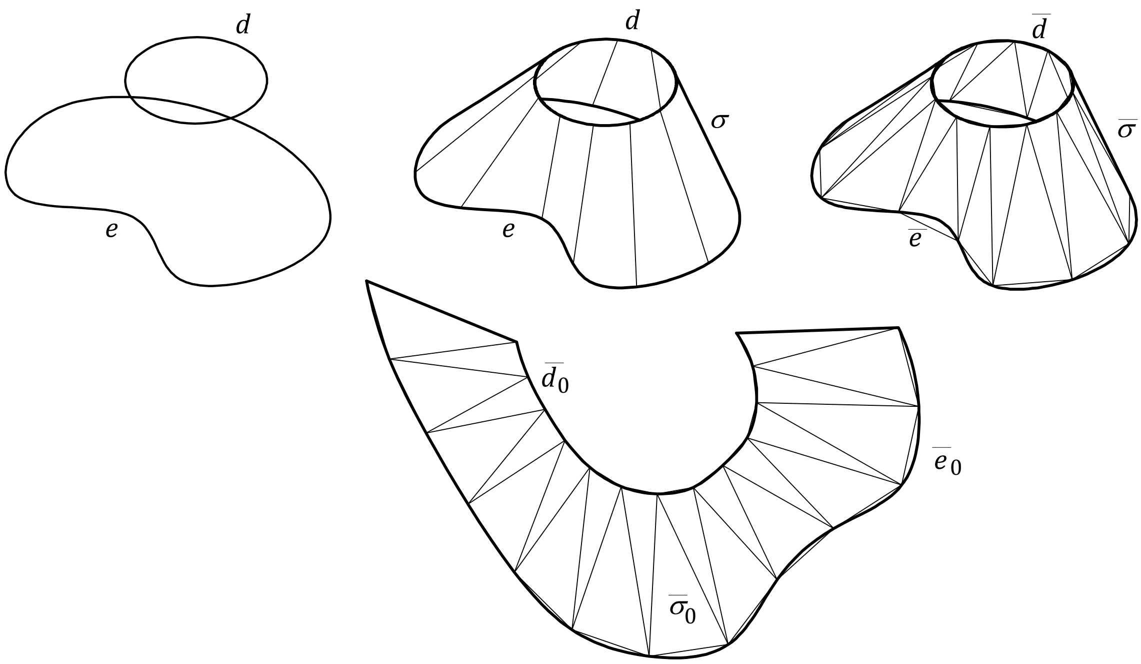


Figure 8.18: Triangulation of warped ruled surface

Here, the graphical solution of a smooth developable transition surface between circular and polygonal profiles located in parallel or intersecting planes is considered only. In such a case, the smooth developable transition surface is composed of triangles and portions of oblique cones so that each triangle lies on the tangent plane of the attached oblique cone. The following rules can be formulated to determine the triangles and oblique cones.

- The number of triangles is equal to the number of polygon edges.
- The number of oblique cones is equal to the number of polygon vertices.

- Each edge of the polygon belongs to one triangle. The third vertex of the triangle is located on the circle.
- Vertices of all triangles located on the circle break the circle in circular segments. Each circular segment defines the directing circle of the corresponding oblique cone.
- Vertices of individual oblique cones coincide with vertices of polygon.

8.3.1 Problem examples – developable transition surfaces

■ Example 8.8 – Transition surface between two profiles in parallel planes

Given

Polygon $ABCD \subset \pi$ and circle $k = (S, r)$ in Monge projection, see fig. 8.19 a).

Required

Construct smooth developable transition surface between the polygon $ABCD$ and circle k.

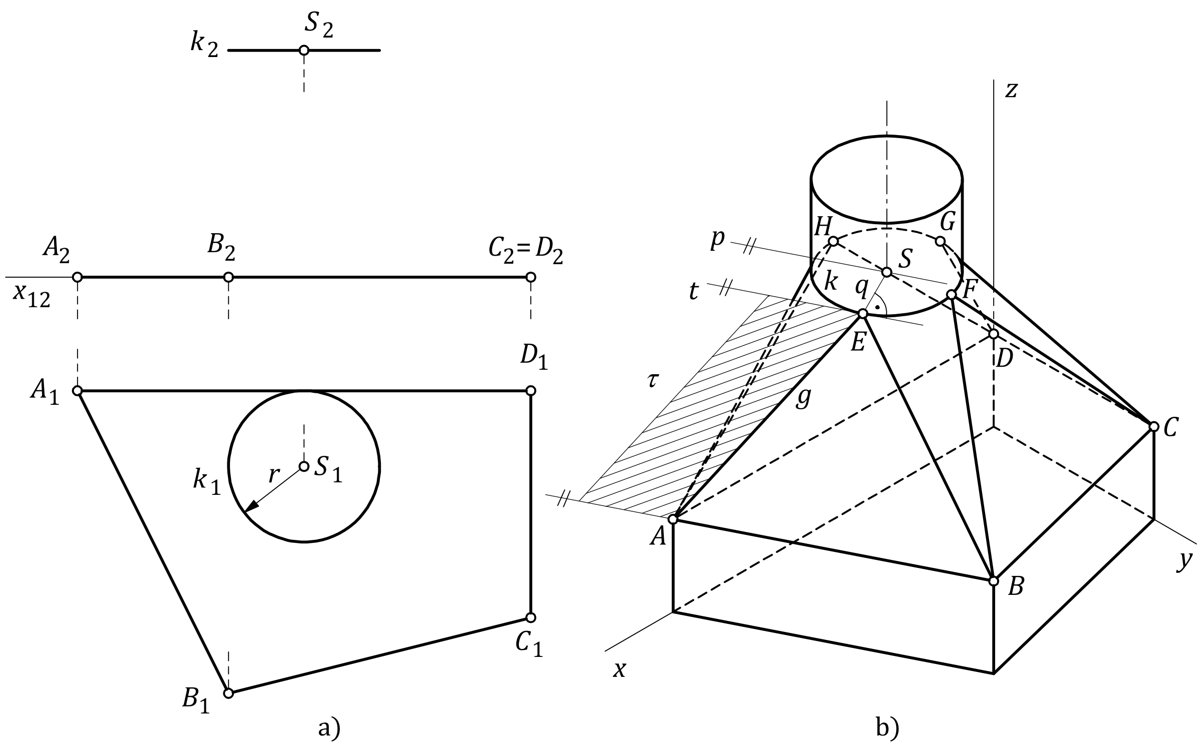


Figure 8.19: Transition surface between two profiles in parallel planes

Analysis

Transition surface is created by four triangles $\triangle ABE$, $\triangle BCF$, $\triangle CDG$ and $\triangle DAH$ and four parts of oblique cones given by vertices A, B, C and D and circular segments $\widehat{HE}$, $\widehat{EF}$, $\widehat{FG}$ and $\widehat{GH}$ in the given order, see fig. 8.20 b).

Each edge of the polygon determines one edge of one triangle and each vertex of the polygon determines the vertex of one oblique cone. For example, let us consider edge AB of triangle $\triangle ABE$ and the attached cones with vertices at points A and B. The position of the third vertex E on the circle k is determined so that the triangle $\triangle ABE$ lies in tangent plane τ of both attached cones. Tangent plane τ along the generating line $g = AE$ of oblique cone is given by the generating line itself and tangent line t to the directing circle.

Obviously, the tangent line t has to be parallel with the edge AB. Since both profiles lie in parallel planes, it is possible to find radius of the circle k parallel with the edge AB, see line $p \parallel AB$ containing such radius in fig. 8.19 b). Then, the intersection of straight line $q \perp p$ and circle k determines point E – point of contact between the circle k and tangent line $t \parallel AB$.

The development of transition surface consists in construction of the four triangles in true size and construction of all small triangles in true size produced by development of oblique cones described in example 8.6.

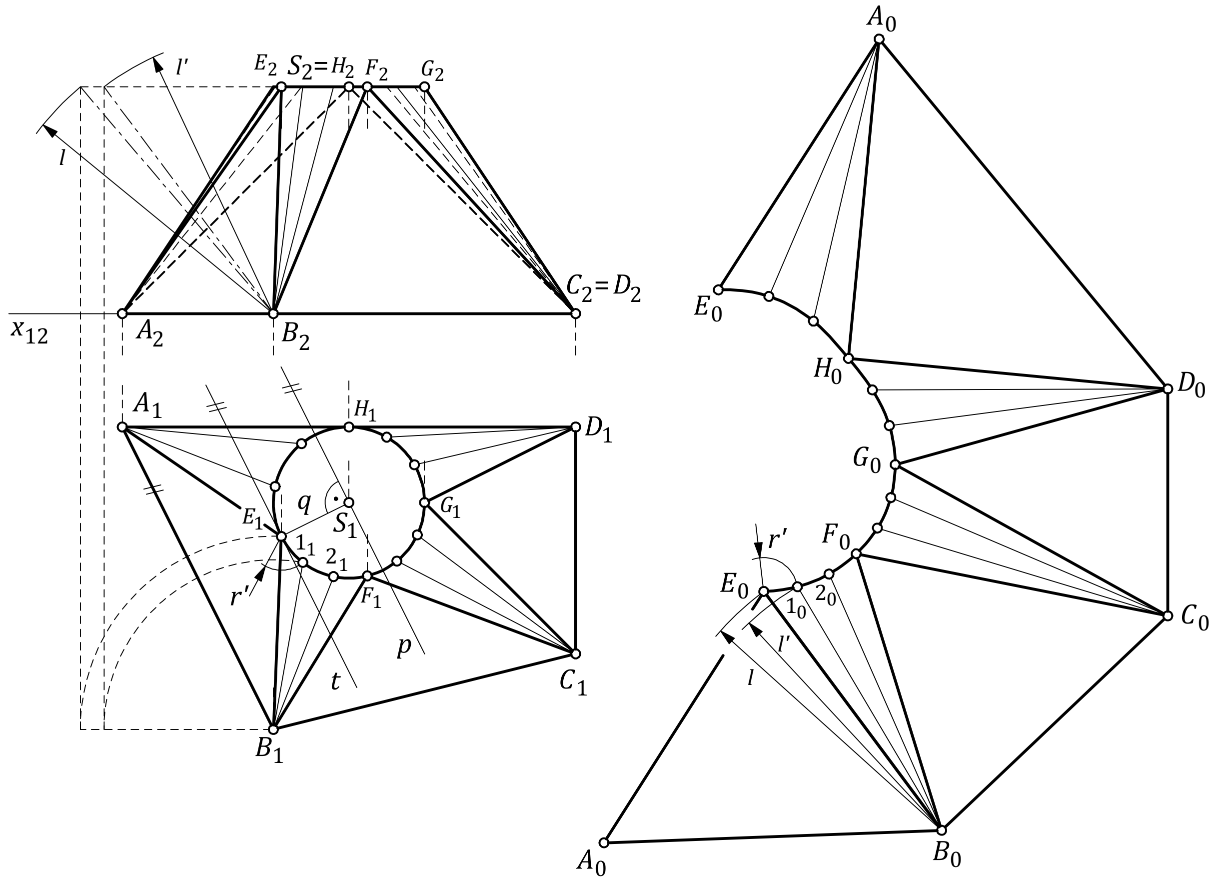


Figure 8.20: Transition surface between two profiles in parallel planes – solution

Graphical solution

1. Construct straight line $p \parallel A_1B_1$, $S_1 \in p$, see fig. 8.20.

2. Construct straight line $q \perp p$, $S_1 \in q$.

3. Top view $E_1 = k_1 \cap q$.

4. Construct front view $E_2 \in k_2$, $E_1E_2 \perp x_{12}$.
5. Construct true size of triangle $\triangle ABE$, i.e. the development $\triangle A_0B_0E_0$.
6. Construct the development of oblique cone given by vertex at point B and directing circular sector $\widehat{EF}$ according to the procedure described in example 8.6.
7. Continue in a similar way to obtain points $F_1, G_1H_1 \in k_1$ and, finally, the whole developed shape. □

■ Example 8.9 – Transition surface between two profiles in intersecting planes

Given

Rectangle $ABCD \subset \nu$ and circle $k = (S, r) \subset \pi$ in Monge projection, see fig. 8.21 a).

Required

Construct smooth developable transition surface between the rectangle $ABCS$ and circle k.

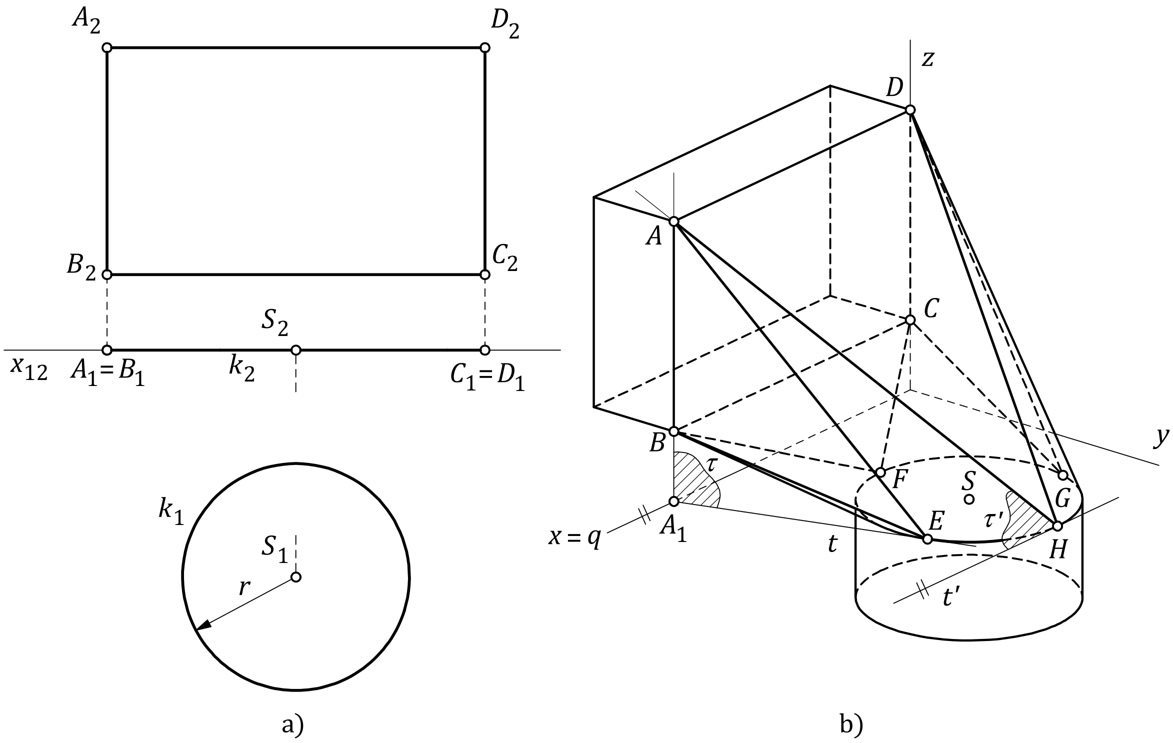


Figure 8.21: Transition surface between two profiles on intersecting planes – task setting

Analysis

Transition surface is created by four triangles $\triangle ABE$, $\triangle BCF$, $\triangle CDG$ and $\triangle DAH$ and four parts of oblique cones given by vertices A, B, C and D and circular segments $\widehat{HE}$, $\widehat{EF}$, $\widehat{FG}$ and $\widehat{GH}$ in the given order, see fig. 8.22 b). Each edge of the rectangle determines one edge of one triangle and each vertex of the rectangle determines the vertex of one oblique cone. Since the rectangle $ABCD$ and the circle k lies on intersecting planes, it is necessary to determine intersection $q = \nu \cap \pi$ of these planes, first (in this case, the intersection q is identical to x-axis). After that, intersections of all edges of rectangle (generally of polygon) and straight line q is necessary to find and construct tangent lines to the circle k from these intersections. Points of contact between the circle and tangent lines determine the vertices of oblique cones.

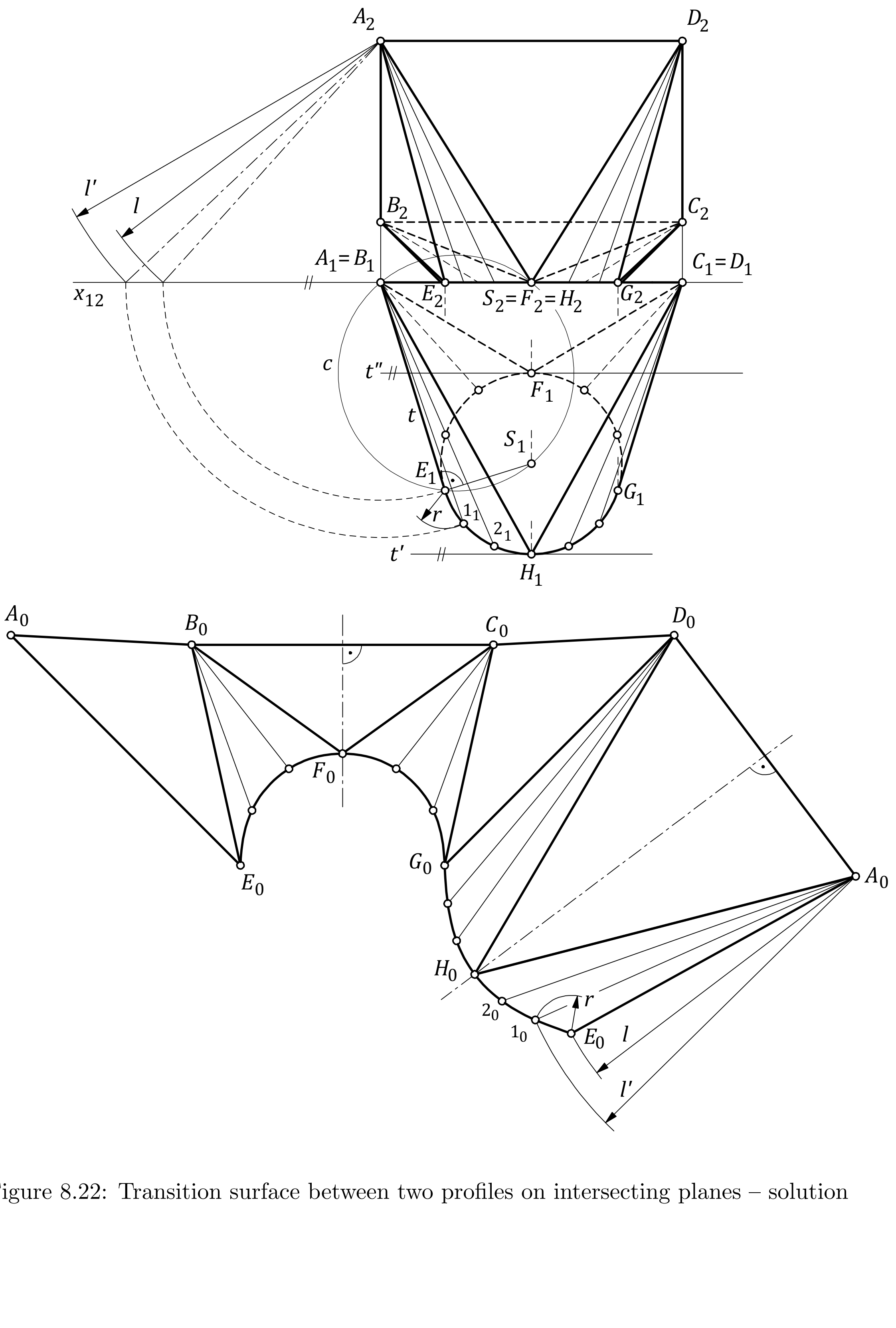


Figure 8.22: Transition surface between two profiles on intersecting planes – solution

Let us consider edge AB of triangle $\triangle ABE$ and the attached cones with vertices at points A and B in fig. 8.21 b), for example. The position of the third vertex E on the circle k is determined so that the triangle $\triangle ABE$ lies in tangent plane τ of both attached cones. Tangent plane τ along the generating line $g = AE$ of the oblique cone is given by the generating line itself and tangent line t to the directing circle. Tangent line t passes through the intersection $A_1 = q \cap AB$ of x-axis and the edge AB. Point E is the point of contact between the tangent line t and the circle k.

Special situations occur in the case of edges BC and DA. Since they are parallel with x-axis, the intersection lies at infinity. Consequently, the tangent line to the circle k is parallel with x-axis, see tangent line t', point of contact H and tangent plane $\tau' = (HA, t')$ in fig. 8.21 b).

The development of transition surface consists in construction of the four triangles in true size and construction of all small triangles in true size produced by development of oblique cones described in example 8.6.

Graphical solution

The procedure of construction is obvious, see fig. 8.22. Point E_1 of contact between the tangent line t and the circle k is constructed according to Thales' theorem (circle c with diameter A_1S_1). Bisectors of angles $\angle B_0F_0C_0$ and $\angle D_0H_0A_0$ are possible axes of symmetry of the developed shape. □

Bibliography

[1] J. Černý: "Geometry", textbook of FCE CTU in Prague, 1996

[2] E.G. Pare, R.O. Loving, I.L. Hill and R.C. Pare, "Descriptive Geometry (9th Edition)", Peachpit Press (1996)

[3] D.L. Ryan, "CAD/CAE Descriptive Geometry", CRC Press (1991)

[4] M. Kargerová, "Geometry and graphics", textbook of FS CTU in Prague, 2005

[5] D. Velichová, "Constructive geometry", e-book, http://www.evlm.stuba.sk/ velichova/Geometria/KOGE/book.html

[6] "Review of analytic geometry", http://www.stewartcalculus.com/data/ESSENTIAL

[7] "Review of conic sections", http://www.stewartcalculus.com/data/ESSENTIAL