Dynamic mode decomposition and its application to the flutter analysis

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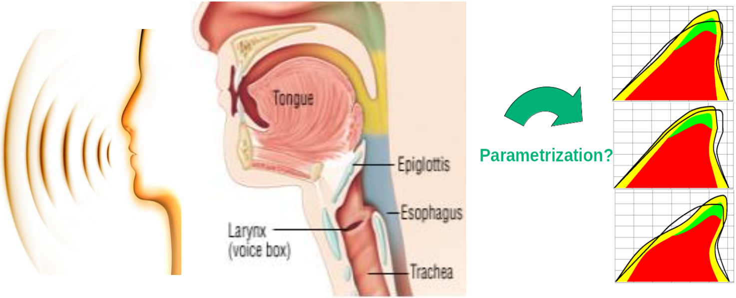
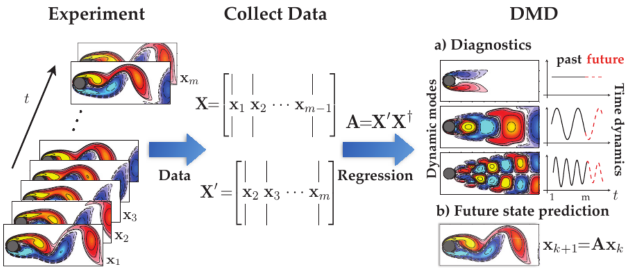
Motivation in background – parametrization of vocal folds vibration



⇒ introduction of **Dynamic mode decomposition**.

Dynamic mode decomposition

- Equation-free modeling and approximating dynamics from data
- Goal = find low-rank representation of high-dimensional system
- Local linearization, connection to the Koopman operator
- DMD modes have monofrequency content unlike POD
- Developed by Peter Schmid in 2009



DMD theory

Let's have a general system described by

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t, \mu). \qquad (1)$$

Its' solution gives us data $\mathbf{x}_i = \mathbf{x}(t_i)$.

Discrete-time representation of (1) or discretized PDE

$$\mathbf{x}_k = \mathbf{x}(k\Delta t) \quad \Rightarrow \quad \mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k).$$

The key of the DMD is to find matrix $\mathbb{A}$

$$\mathbf{x}_{k+1} = \mathbb{A}\mathbf{x}_k, \qquad k \in \{1, \dots, N-1\}$$

based on the given data set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$, e.g. state vectors. This matrix $\mathbb{A}$ approximates the original system by the linear one

$$\frac{d\mathbf{x}}{dt} = \mathcal{A}\mathbf{x}, \qquad \textit{where } \mathbb{A} = \exp(\mathcal{A}\Delta t).$$

Connection to Koopman operator

Koopman operator $\mathcal{K}$ is a linear, infinite-dimensional operator, which exactly represents a nonlinear dynamical system.

It is defined on the Hilbert space $\mathcal{H}$ of functions of $g : \mathbb{R}^n \mapsto \mathbb{R}$ by

$$\mathcal{K}g = g \circ \mathbb{F}, \qquad \textit{i.e. by} \quad \mathcal{K}g(\mathbf{x}_k) = g(\mathbb{F}(\mathbf{x}_k)) = g(\mathbf{x}_{k+1}).$$

DMD is finite-dimensional approximation of Koopman operator.

It is fundamentally different than linearizing the dynamics. The approximation quality depends on the chosen measurements $g(\mathbf{x})$. See also Carleman linearization.

DMD matrix

More construction of $\mathbb{A}$ possible, see e.g. [Tu etal., 2014].

One of the most favourable is following. Let's denote

$$\mathbb{X} = \begin{pmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_{N-1} \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{m,N-1} \qquad \textit{and} \qquad \mathbb{X}' = \begin{pmatrix} | & | & & | \\ \mathbf{x}_2 & \mathbf{x}_3 & \dots & \mathbf{x}_N \\ | & | & & | \end{pmatrix}$$

Then

$$\mathbb{X}' \approx \mathbb{A}\mathbb{X} \qquad \Rightarrow \qquad \mathbb{A} = \mathbb{X}'\mathbb{X}^{\dagger}, \quad i.e. \;\; \mathbb{A} \in \mathbb{R}^{m,m}!$$

which minimizes error

$$||\mathbb{X}' - \mathbb{A}\mathbb{X}||_F, \qquad \left(i.e. \quad \sum_{k=1}^{N-1} ||\mathbf{x}_{k+1} - \mathbb{A}\mathbf{x}_k||_2 \right).$$

In practice the DMD matrix $\mathbb{A}$ is approximated by a projection to the subspace defined by $r \ll m$ SVD L-vectors, i.e. by matrix $\tilde{\mathbb{A}} \in \mathbb{R}^{r,r}$.

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DMD modes

System dynamics is determined by eigendecomposition of $\mathbb{A}$ on:

- (complex) eigenvalues λ_i and the related eigenvectors $\mathbf{\Phi}_i$

The given state-space trajectory is (approximately) reproduced by

$$\mathbf{x}(t_{k+1}) = \mathbf{x}_{k+1} \approx \mathbb{A}^k \mathbf{x}_1 = \sum_{i=1}^{M} \mathbf{\Phi}_i \exp(\omega_i t_k)\, b_i, \qquad (2)$$

where

- $M \leq N - 1$ DMD modes are selected (more options!).
- $\mathbf{b} = (b_i)$ is the initial amplitude of each mode ($\mathbf{b} = \mathbf{\Phi}^{\dagger} \mathbf{x}_1$).
- ω_i are approximate continuous-time eigenvalues, given as $\omega_i = \ln(\lambda_i)/\Delta t$.

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In the case $\mathbb{A} \approx \tilde{\mathbb{A}}$, $M \ll r$ DMD modes are typically selected.

Formula (2) can be used for future prediction (for $t_k > t_N$).

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DMD algorithm

1. Perform **truncated** singular value decomposition (SVD) of $\mathbb{X}$:

$$\mathbb{X} \approx \mathbb{U}\Sigma\mathbb{V}^\star, \qquad \mathbb{U} \in \mathbb{C}^{m,r} \dots$$

   Note: How to choose r? Choose e.g. $\sigma_{rr} > 10^{-3}$ or see [DMD_book].

2. Construct matrix $\tilde{\mathbb{A}}$ as:

$$\tilde{\mathbb{A}} = \mathbb{U}\mathbb{A}\mathbb{U}^\star = \mathbb{U}\mathbb{X}'\mathbb{V}\Sigma^{-1} \quad \in \mathbb{R}^{r,r}.$$

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3. Compute eigendecomposition of $\tilde{\mathbb{A}}$:

$$\tilde{\mathbb{A}}\mathbb{W} = \Lambda\mathbb{W}.$$

4. Reconstruct eigendecomposition of $\mathbb{A}$ from $\mathbb{W}$ and Λ:

$$\Lambda \checkmark \qquad \mathbf{\Phi} = \mathbb{X}'\mathbb{V}\Sigma^{-1}\mathbb{W}.$$

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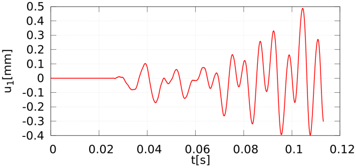
5. Select M DMD modes (e.g. based on criterion $\int \exp(\omega_i t) b_i \, dt$) and use formula (2) for dynamics reconstruction.

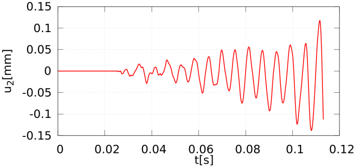
Flow-induced vibrations

- Vocal folds model [Valášek etal., Applications of Mathematics, 2019].
- Inlet velocity $\mathbf{v}_{\mathrm{Dir}}$ prescribed by penalization approach.
- Velocity $\mathbf{v}_{\mathrm{Dir}} = 1.95\,\mathrm{m/s}$ exceeds critical value $v_{\mathrm{crit}} \approx 1.9\,\mathrm{m/s}$.

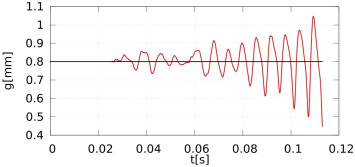
Flow-induced vibrations

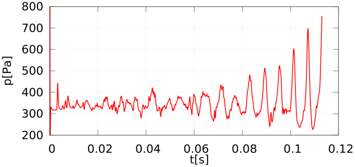
x- and y-component of VF displacement





Time behaviour of gap & pressure drop

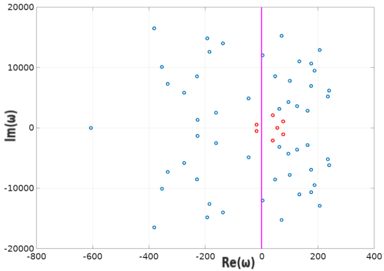


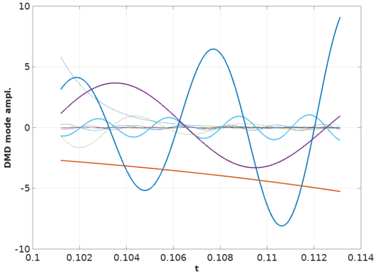


Application of DMD

- Last 300 time steps of structural displacements
- 7 DMD modes are chosen

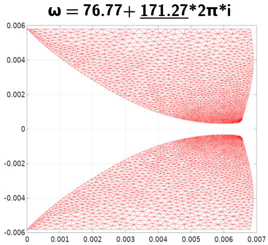
Spectrum of matrix $\tilde{\mathbb{A}}$ and DMD modes amplitudes.

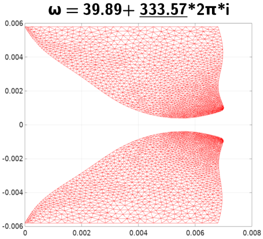




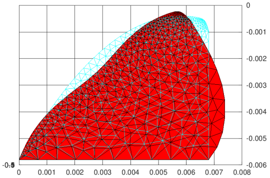
Application of DMD

DMD modes 1 & 3

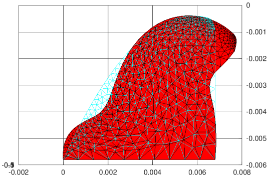




Similar to eigenmodes 3 (MAC = 84%) & 5 (MAC = 74%)



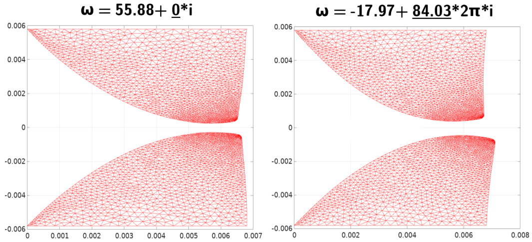
Frequency 181.3293 Hz



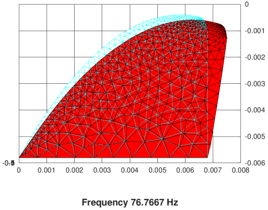
Frequency 317.2082 Hz

Application of DMD

DMD modes 2 (not oscillatory) & 4



Similar to eigenmodes ∅ & 1 (MAC = 85%)



Future state prediction

Future states can be predicted by formula

$$\mathbf{x}(t_{k+1}) = \mathbf{x}_{k+1} \approx \mathbb{A}^k \mathbf{x}_1 = \sum_{i=1}^{M} \mathbf{\Phi}_i \exp(\omega_i \, t_k) \, b_i, \quad k > N.$$

Prediction of two cycles (300 time steps)

Comparison

Aspect	POD	DMD
Principle	statistical	physical
Truncation error	optimal	high
Frequency content	mixed	pure
Noise sensitivity	low	high
Advantages	established	interpretation, prediction, control, system identification

Conclusion

- Introduction of DMD method
  - simple and efficient method of model reduction
  - good interpretation of results (decay/growth, frequency)
  - suitable also for measurements post-processing
  - computationally cheap method
  - applicable to systems with low-rank attractor (SVD spectrum)
- Many improvements of DMD – DMD modes orthogonalization, noise sensitivity reduction, ...
- Many possible extensions – control applications, system identification, ...
- Application to flutter analysis

References

basics & theory:

J. N Kutz, S. L. Brunton, B. W. Brunton, and J. L. Proctor.
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Hassan Arbabi, Milan Korda, and Igor Mezić.
A data-driven koopman model predictive control framework for nonlinear partial differential equations. *IEEE*, 2018.

References

system identification:

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Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *PNAS*, 2016.

extensions:

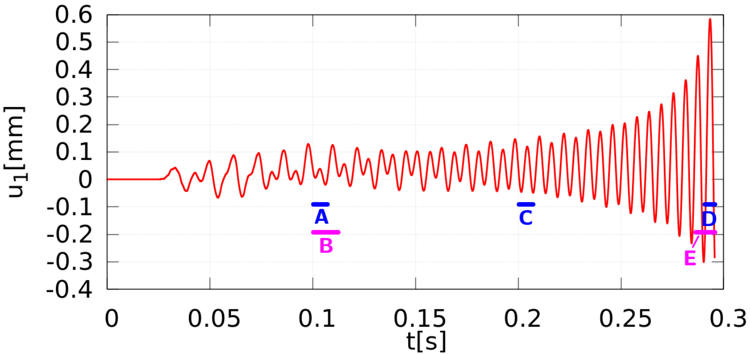
Bernd R. Noack, W. Stankiewicz, M. Morzyski, and P. J. Schmid.
Recursive dynamic mode decomposition of transient and post-transient wake flows.
Journal of Fluid Mechanics, 2016.

S. Le Clainche, R. Moreno-Ramos, P. Taylor, and J. M. Vega.
New robust method to study flight flutter testing.
Journal of Aircraft, 2019.

Thank for your attention :)

Results in CM paper

VF displacement of slightly different flutter simulation
⇒ 5 different intervals analyzed by DMD

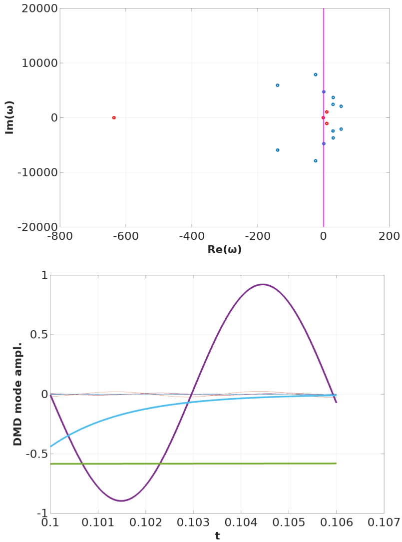


Comparison of DMD decompositions

case	time steps	DMD modes	ω of dominant mode	$e_{\text{pred}} \cdot 10^{-3}$
A	150	6	$5.81 + 168.5 \cdot 2\pi \cdot i$	1.3096
B	300	6	$53.2 + 5.8 \cdot 2\pi \cdot i$	1.2364
C	150	4	$9.8 + 168.5 \cdot 2\pi \cdot i$	0.47357
D	150	5	$65.69 + 167.3 \cdot 2\pi \cdot i$	'small'
E	300	5	$1000.5 + 101.5 \cdot 2\pi \cdot i$	'big'

Results in CM paper

CASE **C**



Results in CM paper

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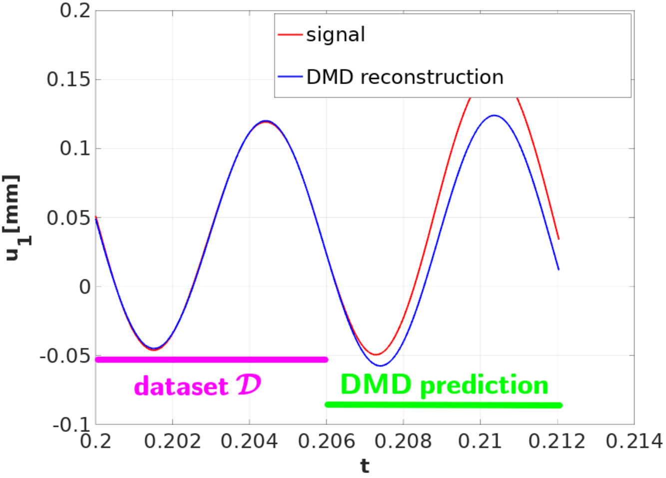

DMD modes

DMD mode with a) big growth rate and $Im(\omega) = 168.45\,\text{Hz}$
b) rapidly decaying and $Im(\omega) = 0$