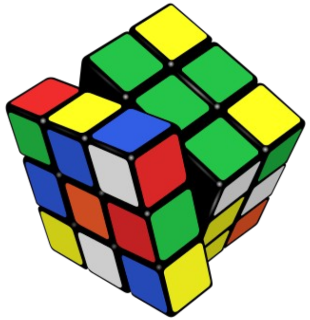
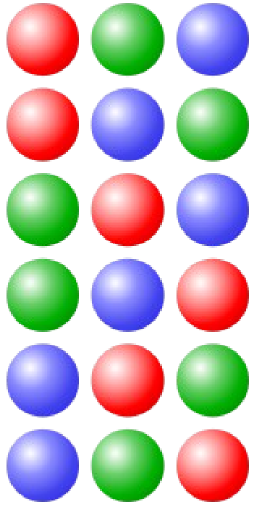
Combinatorics



Partial permutation (variace in czech)

Let have a set of **n** different elements. Then its k-th class partial permutation is a number of all ordered subsets of length **k**, denoted as $V_k(n)$. No repetition considered.

$$V_k(n) = n(n-1)(n-2)\ldots(n-k+1) = n!\ /\ (n-k)!$$



Ex.: Club has 30 members. In how many ways can be chosen chairman, vice-chairman and treasurer?

We choose 3 of the 30 elements, depending on the order - so these are variations:
$V_3(30) = 30 * 29 * 28 = 24360$.

Permutation (permutace in czech)

Let have a set of **n** different elements. Then its permutation is a number of ways how can be this set ordered, denoted as P(n). No repetition considered.

$$P(n) = n!$$

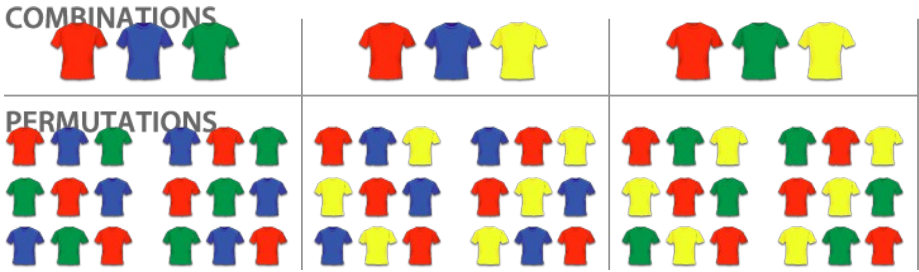
Ex.: How many numbers can be formed from the digits 1, 2, 3, 4, 5 if repetition is not allowed?

It is a permutation of set with 5 elements: P(5) = 5! = 120.

Combination (kombinace in czech)

Let have a set of **n** different elements. Then its k-th class combination of this set is a number in **how many ways k-length subsets can be chosen** if order does not matter, denoted as $C_k(n)$. No repetition considered.

$$C_k(n) = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1},$$



In other words: Let have a basket for k elements. The number of ways how to select its elements is given by k-combination from n possible elements.

Binomial coefficient is $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$. and it plays an important role in math.

It occurs as coefficients in the binomial theorem and it is a part of Pascal's triangle.
(kombinační číslo in czech)

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Ex.: How many ways can two card colors be selected?

We create two-element combinations of four possible colours,
i.e. C_2(4) = 6. (hearts, spades, diamonds, clubs)

1. ♥ ♦
2. ♥ ♠
3. ♥ ♣
4. ♦ ♠
5. ♦ ♣
6. ♠ ♣

With repetitions

Partial permutation with repetitions

Let have a set of **n** different elements. Then its k-th class partial permutation w repetition is a number of all ordered subsets of length **k**, denoted as $V'_k(n)$, where element repetition is allowed.

$$\mathbf{V'_k(n) = n^k}$$

Ex.: How many 4-digit numbers can be made of 1, 2, 3, 4, 5, 6?

This is a partial permutation w repetition: $V'_4(6) = 6^4 = 1296$.

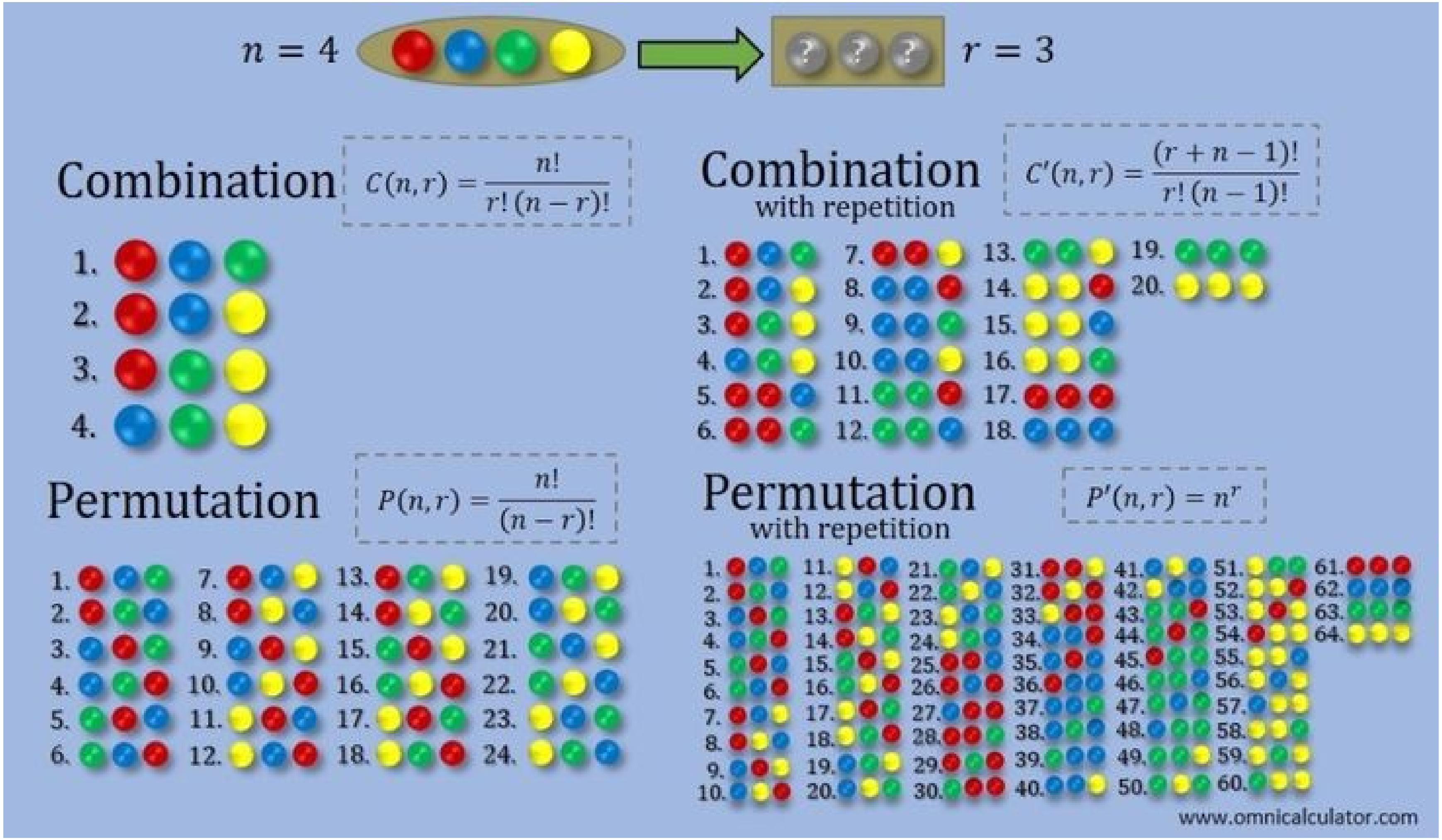
Permutation with repetitions

Let have a set of **n** different elements and we are forming sets of **k** elements of them, k>n. Let the first element be here k1-times, the second one k2-times… Then permutation w repetition is a number of ways how can be ordered set w possibly repeated elements formed.

P'(k1,k2,...) = k! / k1! k2! k3! ... , where k = k1+k2+..+kn.

Ex.: How many permutations can be formed from OKOLO?

There is letter O 3-times. Therefore P(3,1,1) = 5! / 3! 1! 1! = 20.


n = 4
r = 3
Combination
C(n,r) = n! / r!(n − r)!
Combination
with repetition
C′(n,r) = (r + n − 1)! / r!(n − 1)!
Permutation
P(n,r) = n! / (n − r)!
Permutation
with repetition
P′(n,r) = n^r
www.omnicalculator.com

PS:

$$C_k^n = P_k^n / k!$$

Combination with repetitions

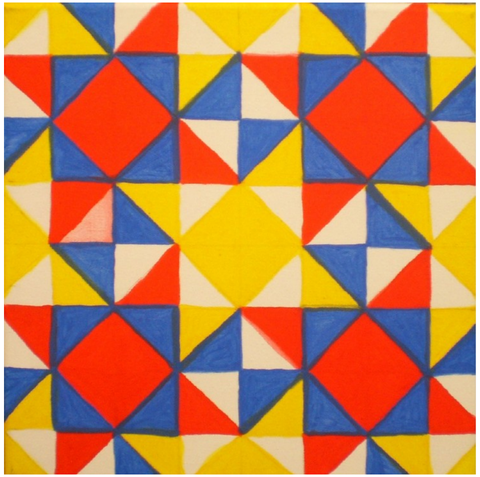
Let have a set of **n** different elements. Then its k-th class combination w repetitions is a number of subsets with length k, where elements can be repeated and order does not matter, i.e. $C'_k(n)$.

$$C'_k(n) = \left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

Ex.: They have 4 kinds of chocolates in the shop. How many ways can I buy 7 chocolates?

These are combinations of the seventh class of four elements with repetition, for which the number is valid:

$$C'_7(4) = \binom{4+7-1}{7} = \binom{10}{7} = 120$$



Jane has 5 T-shirts and 3 skirts. Between how many outfits can she decide?

[15]

There are 32 boys and 34 girls. How many pairs can they form? How long would be dance party, if every boy will dance with every girl for 1min?

[1088]

Mother bought 10 white rolls and 8 buns. Martin eats roll or bun. Then David takes one roll and one bun. In which case would have David more possibilities to choose between different bakery – if Martin eats roll or bun?

[roll: 72 > 70]

How many 4-digit numbers with different digits can be formed of 1,2,3,4,5?
How many of them is divisible by 5, odd?

37 120, 24, 72.

How many 4-digit numbers can be formed of 1,2,3,4,5 (allowing repetition)?
How many of them is divisible by 5, odd?

How many 5-digit numbers w dif. digits can be formed of 0,2,4,6,7,8,9?
How many of them is divisible by 4 and by 10?

38 2 160, 840, 360, 1 560.

There are 11 subjects in class 2.A. How many possible timetable of one day can be arranged, if teaching day consists of 6 subjects.

41 332 640.

There are 28 students and 30 seats. How many arrangements can be formed?

42 $\frac{30!}{2} \doteq 1,326 \cdot 10^{32}$.

There are 8 runners. How can be gold, silver and bronze medal decorated (no split result)?

43 336.

The partial permutation of 2-nd class gives 992 possibilities. How many elements has the set? No repetition.

44 32.

How can be 20 students ordered during inauguration?

47 $20! = 2\,432\,902\,008\,176\,640\,000.$

There are 10 czech and 5 english books. How many ways can they be ordered on the shelf, if the czech ones will be first and english ones after?

49 $10! \cdot 5! = 435\,456\,000.$

How many zeros does number 50! has at the end?

10 a) 12 nul;

There are 10 points in plane. How many lines are by them determined
a) if no three points lie on a line?
b) just 6 of them lie on one line?

52 a) 45; b) 31.

There are 15 points in space. How many planes are by them determined
a) if no three points lie on a line?
b) just 6 of them lie on one line?

How many diagonals have convex n-polygon?

57 $\frac{n(n-3)}{2}$.

In how many ways can be chosen school quadruplet for examination from 30 students?

58 27 405.

There are 8 sudo fighters. How many ways can be 4 fighters chosen for semifinal?

[70]

How many ways can be 12 players divided in two teams (of 6)?

60 924.

There are 4 girls and 8 boys. How can they be divided into two teams, where each team has 2 girls and 4 boys?

61 420.

There are 20 children, each with unique name. Two girls are called Jane and Helen.
How can be 8 children chosen, if the group should

a) contain Helen b) not contain Helen
c) contain Helen and Jane d) contain at least one of Jane or Helen
e) contain at most one of Jane or Helen f) not contain Helen neither Jane?

64 a) 50 388; b) 75 582; c) 18 564; d) 82 212; e) 107 406; f) 43 758.

How many ways can be 20 children divided in three groups if the first group has 10 children, the second one 6 and the third the rest?

66 38 798 760.

There are 10 products in box, 3 of them defective. How can be 5 products chosen, if there is

a) no defective

b) exactly one defective

c) at most one defective

d) just two defective

e) at most two defective

f) at least two defective?

65 a) 21; b) 105; c) 126; d) 105; e) 231; f) 126.

If we increase number of element by 4 the number of all combinations of second class will be increased by 30. What is the original number of elements?

68 6.

How many symbols can be coded by Morse code if we allow words of length 1 up to 4 characters? Character is dot or dash.

71 30.

How many 5-digit numbers can be formed of digits 2,3,4, 6,7,9 if repetition is allowed?

72 7776.

There are 10 crayons in the box - 4 green, 3 red, 2 blue and pink.
How many ways can they be ordered?

73 12600.

There are 3 different types of notebooks in a shop. How many ways can be bought 5 notebooks?

74 21.

There are 5 different cakes in candy shop. How many ways can be bought 8 cake?

75 495.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7

Binomial theorem

It states how to compute and express all terms in n-power of bracket (x+y):

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

i.e. without sum notation:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n,$$

Human examples:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + nx^{n-1} + x^n.$$

Lets compute:

a) $(1+\sqrt{2})^5$

b) $(1-2\sqrt[3]{3})^6$

c) $(x+\sqrt{x})^4$

d) $\left(y-\frac{1}{2y}\right)^5$

e) $(a\sqrt[3]{a}-3)^3$

f) $\left(\sqrt{\frac{m}{n}}-\sqrt{\frac{n}{m}}\right)^5$

76 a) $41+29\sqrt{2}$; b) $97-708\sqrt[3]{3}-516\sqrt[3]{9}$; c) $x^4+4x^3\sqrt{x}+6x^3+4x^2\sqrt{x}+x^2$; d) $y^5-\frac{5}{2}y^3+\frac{5}{2}y-\frac{5}{4y}+\frac{5}{16y^3}-\frac{1}{32y^5}$; e) $a^4-9a^2\sqrt[3]{a^2}+27a\sqrt[3]{a}-27$; f) $\sqrt{mn}(\frac{m^2}{n^3}-\frac{5m}{n^2}+\frac{10}{n}-\frac{10}{m}+\frac{5n}{m^2}-\frac{n^2}{m^3})$.

Verify that x1= sqrt(2) – 2 is root of:

79 Ověřte, že číslo $x=\sqrt{2}-2$ je kořenem rovnice $x^5-10x^3-24x-16=0$.

Which member of the power does contain y^3?

Který člen binomického rozvoje $(y^2+y^{-1})^9$ obsahuje y^3?

86 6. člen.

Prove:

a) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n$

b) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^n\binom{n}{n}=0$

c) $\binom{n}{0}-2\binom{n}{1}+2^2\binom{n}{2}-\ldots+(-2)^n\binom{n}{n}=(-1)^n$

d) $4^n+\binom{n}{1}4^{n-1}+\binom{n}{2}4^{n-2}+\ldots+1\binom{n}{n}=5^n$

e) $1+4\binom{n}{1}+4^2\binom{n}{2}+\ldots+4^n\binom{n}{n}=5^n$

99 Návody: a) $2^n=(1+1)^n$; b) $0=(1-1)^n$; c) $(-1)^n=(1-2)^n$; d) $5^n=(4+1)^n$; e) $5^n=(1+4)^n$.

Lets compute and verify according to Moivre’s formula:

78 Umocněte podle binomické věty i podle Moivreovy věty:

a) $(1+\mathrm{i})^7$ b) $(\sqrt{2}-\mathrm{i}\sqrt{2})^6$ c) $(-2+2\mathrm{i}\sqrt{3})^5$

78 a) $8-8\mathrm{i}$; b) $64\mathrm{i}$; c) $-512-512\mathrm{i}\sqrt{3}$.

PS: de Moivre's theorem states $(\cos x + i \sin x)^n = \cos nx + i \sin nx,$

Derive formulas by using Moivre’s formula and binomial theorem:

98 Užitím binomické věty a věty Moivreovy odvoďte vzorce pro

a) $\sin 3x$, $\cos 3x$, b) $\sin 5x$, $\cos 5x$.

98 a) $\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = 4\cos^3 x - 3\cos x$; b) $\sin 5x = 5\sin x - 20\sin^3 x +$
$+ 16\sin^5 x$, $\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x$.