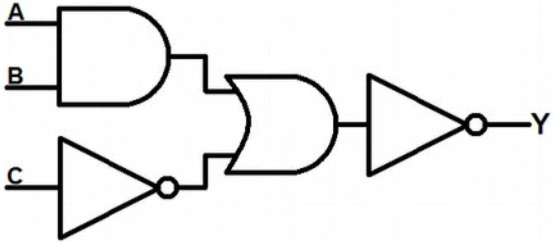
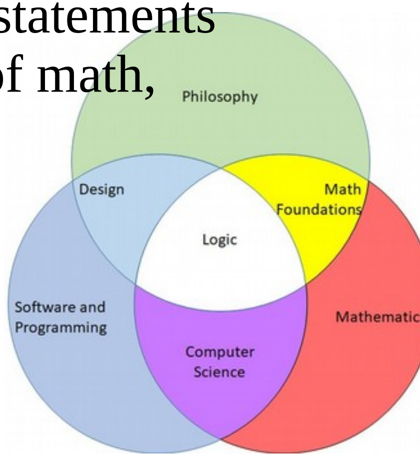
Mathematical logic

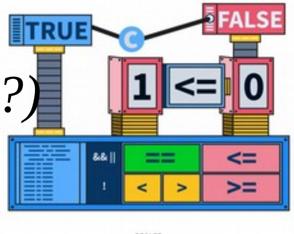


The most of mathematical results are formulated in the form of definitions, statements and theorems and their proofs. In order to understand the precise language of math, let see basics of mathematical logic.



<u>Statement</u>
= is a sentence which is either true or false.
= often contains assumptions (e.g.) and the implied claim
= typical is use of logical connectives
- e.g. *The moon is made of cheese.* *Number 42 is a perfect square.* *All dogs bite.*

<u>Operations with statements:</u>
- **negation of a proposition X** (non X or ¬ X)
- e.g. *The moon is not made of cheese.* (why not: *The moon is made of ice cream?*)
  There is a dog which does not bite.



Operations with statements

Conjunction of statements X and Y ($X \wedge Y$)
= read it as "both statements X and Y are valid / true" or just briefly "X and Y"
= in text as **AND**, in truth table similar to $X \cdot Y$
-e.g. I have a red car and Peter has a blue bike.
Tomorrow is Friday and it is the last day in the month.
My favourite number is 3 and it is not prime number.

Disjunction of statements X and Y ($X \vee Y$)
= read it as ,,X or Y "
= in text stands as **OR**, in truth table similar to $X+Y$
= it is the **inclusive or** (and not the sometimes used exclusive or => see XOR later)
-e.g. I have a red car or Peter has a blue bike.
Tomorrow is Friday or it is the last day in the month.
My favourite number is 13 or it is prime number.

Operations with statements

Implication of statements X and Y ($X \Rightarrow Y$) also logical consequence or conditional statement
= read it as "from X follows Y ", "if X holds, then Y also holds", ,,X implies Y "
,,X is a sufficient condition for Y ", ,,Y is a necessary condition for X"
= in text typically as **if..(premise) then..(conclusion)**
-e.g. If sun is shining then Peter has birthday.
If tomorrow is Friday then it is the last day in the month.
If number 22 is divisible by 4 then it is not prime number.

Equivalence of statements X and Y ($X \Longleftrightarrow Y$) also biconditional
- and read "X holds if and only if Y holds", "X is equivalent to Y "
= in text typically as **if and only if, is equivalent to**
-e.g. Sun is shining if and only if Peter has birthday.
Tomorrow is Friday if and only if it is the last day in the month.
The divisibility of number 22 by 4 is equivalent to number 22 not being a prime number.

Operations with statements

Statements are either:
true ... we mark +, 1 or ,,true"   or   **false** ... we mark as -, 0 or ,,false"

The truth values of all statements non X, X ∧ Y , … depending on the truth values of the statements X and Y can be seen from the following table.

X	Y	non X	$X \wedge Y$	$X \vee Y$	$X \Longrightarrow Y$	$X \Longleftrightarrow Y$
$+$	$+$	$-$	$+$	$+$	$+$	$+$
$+$	$-$	$-$	$-$	$+$	$-$	$-$
$-$	$+$	$+$	$-$	$+$	$+$	$-$
$-$	$-$	$+$	$-$	$-$	$+$	$+$

Exclusive OR

- denoted as XOR
- XOR is true if and only if the inputs differ (one is true, one is false)
- truth table:

A	B	$A \oplus B$
T	F	T
T	T	F
F	F	F
F	T	T

Mathematical quantifiers

Universal quantifier (symbol ∀)
= typically ∀ x ∈ I : V (x) - we read it: "for every x ∈ I the statement V (x) holds".
or ,,every x ∈ I has the property V (x)"
- e.g. "All the cars on this street are red."

Existential quantifier (symbol ∃)
= typically ∃ x ∈ I : V (x) - it reads as follows: ,,there exists x ∈ I
for which the statement V (x) holds", etc.
- e.g. "There is a car on this street that is red" or "At least one car on this street is red."

Negation of statements with quantifiers

- in order to negate the statement

  => let use the opposite quantifier and the negation of the original statement

  non (∀ x ∈ I : V (x)) == ∃ x ∈ I : (non V (x)) and vice versa

- e.g. All days are raining. => At least one day is not raining.

  There is a a flying horse. => No horse does fly.

AND

Non (X and Y) = non X OR non Y

- e.g. Today is Friday and Jane has a blue car. => Today is not Fr. or Jane doesn’t have blue car.

OR

Non (X or Y) = non X AND non Y

- e.g. Today is Friday or Jane has a blue car. => Today is not Fr. and Jane doesn’t have blue car.

Negation of composed statements

IMPLICATION
Non (X => Y) == X AND NON Y
- e.g. If today is Friday then Jane has a blue car. => Today is Fr. and Jane does not have a blue car.
PS. Counterpositive is: non Y => non X. It has the same truth values as X=>Y.

Logical equivalences involving conditional statements

1. $p \implies q \equiv \neg p \vee q$
2. $p \implies q \equiv \neg q \implies \neg p$
3. $p \vee q \equiv \neg p \implies q$
4. $p \wedge q \equiv \neg(p \implies \neg q)$
5. $\neg(p \implies q) \equiv p \wedge \neg q$

EQUIVALENCE
Non (X <=> Y) ==   non X <=> Y
- e.g. Sun is shining if and only if Peter has birthday. => Sun is not shining if and only if Peter has birthday.

Logical equivalences involving biconditionals

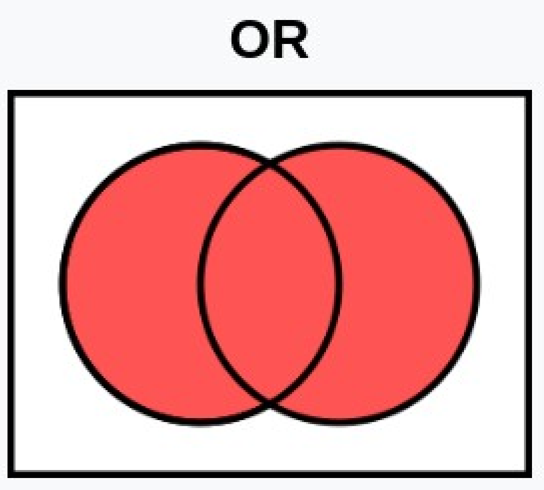
1. $p \iff q \equiv (p \implies q) \wedge (q \implies p)$
2. $p \iff q \equiv \neg p \iff \neg q$
3. $p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
4. $\neg(p \iff q) \equiv p \iff \neg q \equiv p \oplus q$

Where $\oplus$ represents XOR.

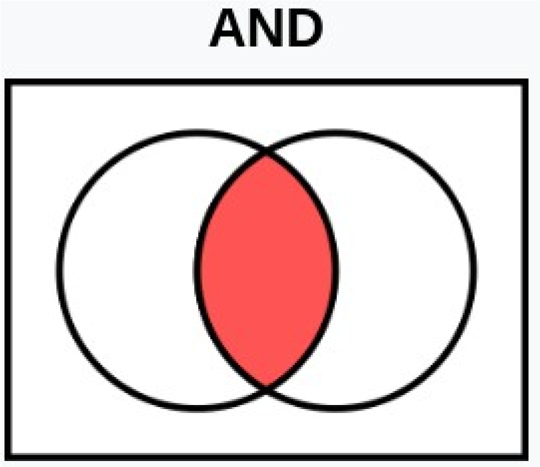
PS. Implication and equivalence can be fully expressed by AND, OR:
X=>Y == nonX OR Y,   X<=>Y == (X AND Y) OR (nonX AND nonY)
https://discrete.openmathbooks.org/dmoi3/sec_intro-statements.html

Sets

- the same operations as with statements can be performed with sets
  as operations union and intersection imitate logical conjunction and disjunction
- see also De Morgan's laws



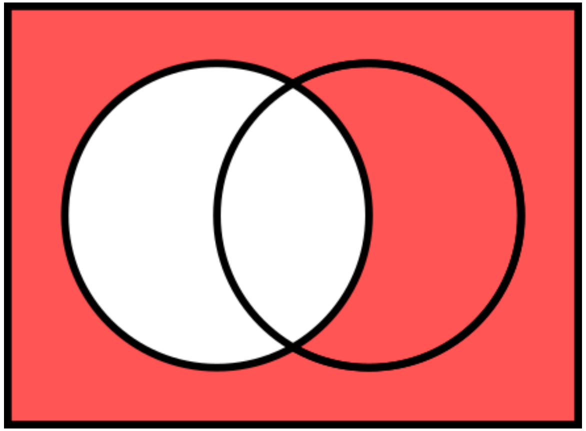
Similar to union of sets A and B



Similar to intersection of sets A and B

Further operations with sets

Complement of A in U



Similar to negation of A

Set difference A \ B