Derivative of $\arctan(x)$

Let's use our formula for the derivative of an inverse function to find the derivative of the inverse of the tangent function: $y = \tan^{-1} x = \arctan x$.

We simplify the equation by taking the tangent of both sides:

$$\begin{aligned} y &= \tan^{-1} x \\ \tan y &= \tan(\tan^{-1} x) \\ \tan y &= x \end{aligned}$$

To get an idea what to expect, we start by graphing the tangent function (see Figure 1). The function $\tan(x)$ is defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. It's graph extends from negative infinity to positive infinity.

If we reflect the graph of $\tan x$ across the line $y = x$ we get the graph of $y = \arctan x$ (Figure 2). Note that the function $\arctan x$ is defined for all values of x from minus infinity to infinity, and $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.

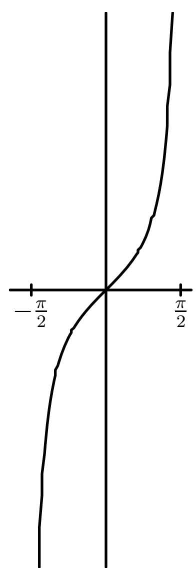

Figure 1: Graph of the tangent function.

You may know that:

$$\begin{aligned} \frac{d}{dy} \tan y &= \frac{d}{dy} \frac{\sin y}{\cos y} \\ &\vdots \\ &= \frac{1}{\cos^2 y} \\ &= \sec^2 y \end{aligned}$$

tedy: (tg y)' = 1/ cos^2 y

též viz tabulka derivací

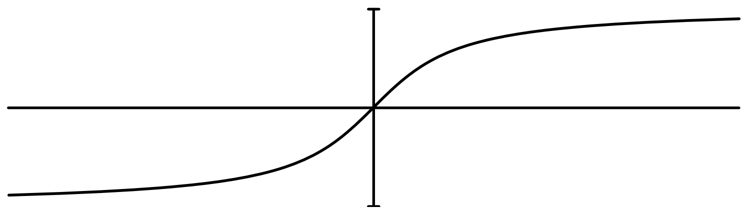

Figure 2: Graph of $\tan^{-1} x$.

(If you haven't seen this before, it's good exercise to use the quotient rule to verify it!)

We can now use implicit differentiation to take the derivative of both sides of our original equation to get:

y = arctg x => proto potřeba derivovat dle x

$$\begin{aligned} \tan y &= x \\ \frac{d}{dx}(\tan(y)) &= \frac{d}{dx}x \\ \text{(Chain Rule)} \quad \frac{d}{dy}(\tan(y))\frac{dy}{dx} &= 1 \\ \left(\frac{1}{\cos^2(y)}\right)\frac{dy}{dx} &= 1 \\ y' = \frac{dy}{dx} &= \cos^2(y) \end{aligned}$$

jak ale vyjádřit y(x) pomocí x?

Důležitá úvaha, co platí v pravoúhlém trojúhelníku

Or equivalently, $y' = \cos^2 y$. Unfortunately, we want the derivative as a function of x, not of y. We must now plug in the original formula for y, which was $y = \tan^{-1} x$, to get $\underline{y' = \cos^2(\arctan(x))}$. This is a correct answer but it can be simplified tremendously. We'll use some geometry to simplify it.

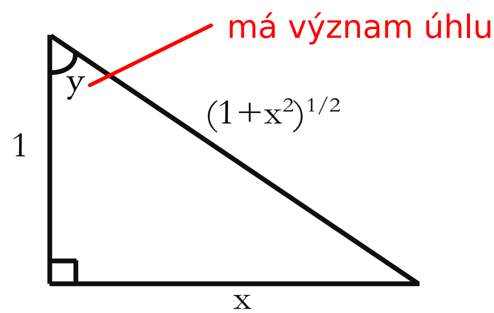

Figure 3: Triangle with angles and lengths corresponding to those in the example.

In this triangle, $\tan(y) = x$ so $y = \arctan(x)$. The Pythagorean theorem tells us the length of the hypotenuse: přepona

$$h = \sqrt{1+x^2}$$

and we can now compute: a proto

$$\cos(y) = \frac{1}{\sqrt{1+x^2}}.$$

From this, we get:

$$\cos^2(y) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

so:

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

Nakonec dostáváme

In other words,

$$\boxed{\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.}$$