Cvičení 1 30.9.

Opakování SŠ matematiky

Př. 1 $x \in \mathbb{R}$: $\frac{4x-2}{2+x} \geq 0$

Podmínka $2+x \neq 0$ tj $x \neq -2$. Výraz na levé straně mění znaménko pouze pokud $(4x-2)$ nebo $(2+x)$ mění znaménko $\longrightarrow$ "rozdělíme na intervaly"

$x \in$	$(-\infty; -2)$	$(-2; \frac{1}{2})$	$\langle \frac{1}{2}; +\infty)$
$\frac{4x-2}{2+x}$	> 0	< 0	≥ 0

Tedy $\forall x \in (-\infty; -2) \cup \langle \frac{1}{2}; +\infty)$ platí $\frac{4x-2}{2+x} \geq 0$.

Př. 2 $|x+2| + 2 < 1$

$|x+2| < -1$ $\longrightarrow$ nemá řešení, jelikož výraz v absolutní hodnotě je vždy nezáporný!

Př. 3 $|x-1| < 3$

"nulové body"

$x-1=0 \implies x=1$

$$\begin{cases} x \leq 1 : & |x-1| = 1-x; \quad 1-x<3 \implies x>-2 \implies x \in (-2;1\rangle \\ x \geq 1 : & |x-1| = x-1; \quad x-1<3 \implies x<4 \implies x \in \langle 1;4) \end{cases}$$

Řešením je $x \in (-2;4)$

Př. 4 $\left|\frac{x+1}{x-1}\right| \leq 1$

nulové body: $x=-1$; $x=1$

- $x \leq -1$: $\frac{-x-1}{-x+1} \leq 1 \implies \left.\begin{array}{l}(-x-1) \leq 1\cdot(-x+1) \\ x+1 \geq x-1\end{array}\right\} \implies x \in (-\infty;-1\rangle$

- $x \in \langle -1;1)$: $\frac{x+1}{-x+1} \leq 1 \implies \left.\begin{array}{l}x+1 \leq -x+1 \\ 2x \leq 0\end{array}\right\} \implies x \in \langle -1;0\rangle$
  $x \leq 0$

- $x>1$: $\frac{x+1}{x-1} \leq 1 \implies \left.\begin{array}{l}x+1 \leq x-1 \\ 1 \leq -1\end{array}\right\}$ nemá řešení

Celkem tedy $x \in (-\infty;0\rangle$

Př. 5 $|x^2+2x-3| \geq |x^2+3x-4|$

→ hodnoty x v nichž výrazy v abs. hodnotách mění znaménko :

- $x^2+2x-3=0 \;\rightarrow\; x_{1,2} = \dfrac{-2 \pm \sqrt{4+12}}{2} = -1 \pm 2$

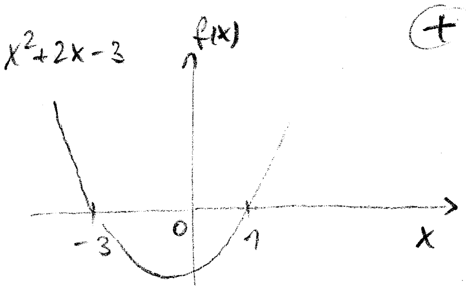
  $\rightarrow \boxed{x_1 = -3 \;;\; x_2 = 1}$

- $x^2+3x-4 = 0 \;\rightarrow\; x_{3,4} = \dfrac{-3 \pm \sqrt{9+16}}{2} \;\rightarrow\; \boxed{x_3 = -4} \;\; \boxed{x_4 = 1}$

- $x \in (-\infty; -4\rangle$: $\quad x^2+2x-3 \geq x^2+3x-4$

  $x \leq 1$ $\quad \Big\}\; x \in (-\infty; -4\rangle$

- $x \in \langle -4; -3\rangle$:



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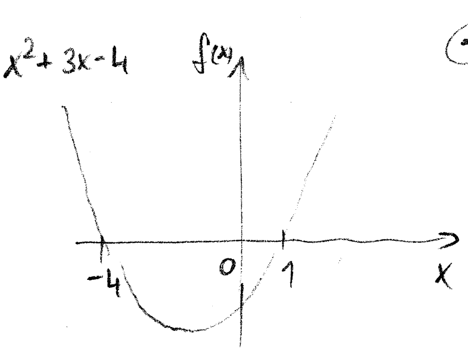

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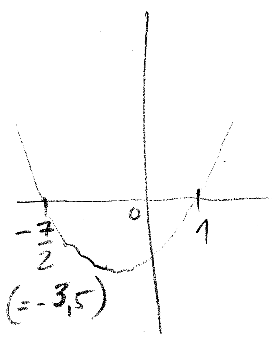
n.b.

$$x^2+2x-3 \geq -(x^2+3x-4)$$

$$2x^2+5x-7 \geq 0$$

$$x_{1,2} = \frac{-5 \pm \sqrt{\overbrace{25+56}^{9}}}{4} \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = -\frac{7}{2} \end{array}$$



$$x \in \left\langle -4; -\frac{7}{2} \right\rangle$$

• $x \in \langle 1; +\infty)$: $\left.\begin{aligned} x^2+2x-3 &\geq x^2+3x-4 \\ x &\leq 1 \end{aligned}\right\}$ $x = 1$

Celkem tedy: $x \in (-\infty; -\frac{7}{2}\rangle \cup \{1\}$.

Goniometrické funkce (rovnice)

- $\sin^2\alpha + \cos^2\alpha = 1$
- $\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

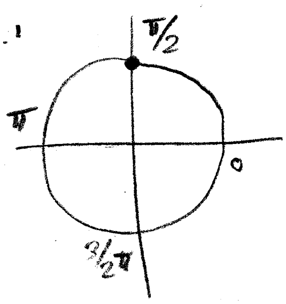
Př. 6 $x \in \mathbb{R}$: $tg(x) + cotg(x) = 2$

$$\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} = 2 \quad / \cdot \sin(x)\cos(x) \quad \text{kde} \quad \boxed{\begin{array}{c} ? \\ \sin(x) \neq 0 \\ \cos(x) \neq 0 \end{array}}$$

$$\sin^2(x) + \cos^2(x) = 2\sin(x)\cos(x)$$

$$1 = \sin(2x) \implies 2x = \frac{\pi}{2} + 2k\pi, \; k \in \mathbb{Z}$$

Pozn.!



$\sin\left(\frac{\pi}{2}\right) = 1$

$$\boxed{x \in \left\{\frac{\pi}{4} + k\pi; \; k \in \mathbb{Z}\right\}}$$

Př. 7 $\cos^2(x) - \sin(x) = 1 \quad ; x \in \mathbb{R}$

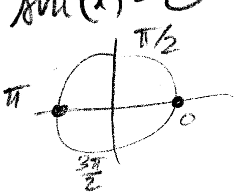
$$\left(1 - \sin^2(x)\right) - \sin(x) = 1$$

$$-\sin^2(x) - \sin(x) = 0 \quad \Big/ \ \text{sub.: } \sin(x) = t$$

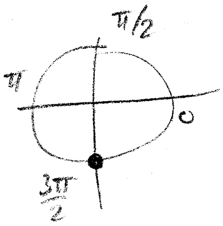
$$t^2 + t = 0$$

$$t \cdot (t+1) = 0 \Longrightarrow \left.\begin{array}{l} t = 0 \\ t = -1 \end{array}\right\} \begin{array}{l} \sin(x) = 0 \\ \sin(x) = -1 \end{array}$$

- $\sin(x) = 0$: $x \in \{k \cdot \pi \ / \ k \in \mathbb{Z}\}$



- $\sin(x) = -1$: $x \in \left\{\frac{3\pi}{2} + 2k\pi ; \ k \in \mathbb{Z}\right\}$



Celkem:

sjednocení

$$\boxed{x \in \{k\pi ; k \in \mathbb{Z}\} \cup \left\{\frac{3\pi}{2} + 2\pi k ; \ k \in \mathbb{Z}\right\}}$$

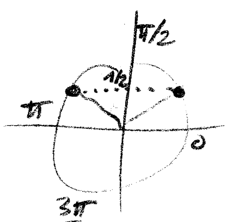
Př. 8 Řešte pro $x \in \mathbb{R}$: $\dfrac{1 - \sin(x)}{1 + \sin(x)} = \dfrac{1}{3}$

$\boxed{\sin(x) \neq -1}$

$$3\left(1 - \sin(x)\right) = 1 + \sin(x)$$

$$-4 \cdot \sin(x) = -2$$

$$\sin(x) = \frac{1}{2} \Longrightarrow \boxed{x \in \left\{\frac{\pi}{6} + 2k\pi ; \ k \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{6} + 2k\pi ; \ k \in \mathbb{Z}\right\}}$$



Př 9. Upravte (zjednodušte) dané výrazy. Určete pro jaká $x \in \mathbb{R}$ mají smysl.

a) $$\boxed{\frac{\cos^2(x)}{1+\sin(x)}} = \frac{1-\sin^2(x)}{1+\sin(x)} = \frac{(1-\sin(x))(1+\sin(x))}{1+\sin(x)} = \underline{\underline{1-\sin(x)}}$$

Podmínka: $1+\sin(x) \neq 0 \Rightarrow x \in \mathbb{R} \setminus \{\frac{3}{2}\pi + 2k\pi;\ k \in \mathbb{Z}\}$

Pozn.: ! Ale platí $\lim\limits_{x \to \frac{3}{2}\pi + 2k\pi} \frac{\cos^2(x)}{1+\sin(x)} = 2$

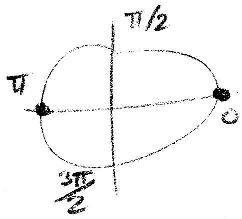
Tedy výraz má smysl pro $\forall x \in \mathbb{R}$.

b) $$\boxed{\cot g(x) + \frac{\sin(x)}{1+\cos(x)}} = \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{1+\cos(x)} = \frac{\cos(x) + \overbrace{\cos^2(x) + \sin^2(x)}^{1}}{\sin(x)\cdot(1+\cos(x))} =$$

$$= \frac{\cancel{\cos(x)+1}}{\sin(x)\cdot\cancel{(1+\cos(x))}} = \underline{\underline{\frac{1}{\sin(x)}}}$$

Podmínka: $\cos(x) \neq -1$
$\sin(x) \neq 0$

! Tato podmínka se tedy po úpravě už neuplatní



$x \in \mathbb{R} \setminus \{k\pi;\ k \in \mathbb{Z}\}$

c) $$\boxed{\frac{1}{1+tg^2(x)} + \frac{1}{1+\cot g^2(x)}} = \frac{1}{1+\frac{\sin^2(x)}{\cos^2(x)}} + \frac{1}{1+\frac{\cos^2(x)}{\sin^2(x)}} =$$

Podmínka: $\cos(x) \neq 0$
$\sin(x) \neq 0$

$$= \frac{\cos^2(x)}{\cos^2(x)+\sin^2(x)} + \frac{\sin^2(x)}{\sin^2(x)+\cos^2(x)} =$$

$$= \cos^2(x) + \sin^2(x) = \underline{\underline{1}}$$

! Tedy výraz má nakonec smysl i pro limitní případy $\cos(x) \to 0$; $\sin(x) \to 0$

$x \in \mathbb{R}$

Funkce

Př 10 Určete definiční obor dané funkce $y = f(x)$

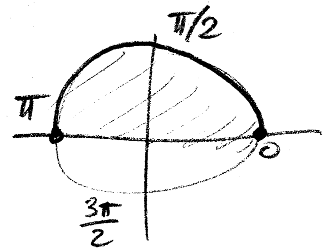
a) $y = 3x - 5 \implies \boxed{x \in \mathbb{R}}$ (zřejmé, lin. fce...)

b) $y = 4x^7 - 5x^3 + \frac{3}{2}x - 3 \implies \boxed{x \in \mathbb{R}}$ (polynom. fce)

c) $y = \frac{x^3 - 8}{x} \implies \boxed{x \neq 0}$ (nebo $x \in (-\infty; 0) \cup (0; +\infty)$ nebo $x \in \mathbb{R} \setminus \{0\}$)

d) $y = \frac{x-2}{\sqrt{x+5}} \implies x + 5 > 0 \implies \boxed{x \in (-5; +\infty)}$

e) $y = \sqrt{\sin(x)} \implies \sin(x) \geq 0 \implies$



$$x \in \bigcup_{k \in \mathbb{Z}} \langle 2k\pi; \pi + 2k\pi \rangle$$

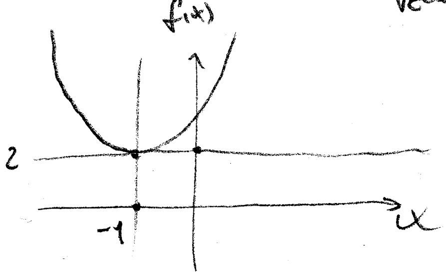
← sjednocení intervalů

nebo $x \in \mathbb{R} \setminus (\pi + 2k\pi; 2\pi + 2k\pi) \quad k \in \mathbb{Z}$

f) $y = \ln(x^2 + 2x + 3) \implies x^2 + 2x + 3 > 0$

$$x_{1,2} = \frac{-2 \pm \sqrt{4 - 12}}{2} \implies D < 0$$

Pozn.! $x^2 + 2x + 3 = \underbrace{(x+1)^2 + 2}_{\text{vždy} > 0}$



$\implies$ neex. nulové body, tj průsečíky s osou x

$\to$ splněno $\forall x \in \mathbb{R}$

$\implies \boxed{x \in \mathbb{R}}$

Komplexní čísla

$$\boxed{i^2 = -1}$$

Př. 11 Upravte

a) $i^3 = i^2 \cdot i = \underline{\underline{-i}}$

b) $i^4 = i^2 \cdot i^2 = (-1)(-1) = \underline{\underline{1}}$

c) $(2+3i)(3-4i) = 6 - 8i + 9i - 12i^2 = \underline{\underline{18 + 1i}}$

d) $(2-3i)(1+4i) = 2 + 8i - 3i - 12i^2 = \underline{\underline{14 + 5i}}$

Př. 12 Určete absolutní hodnotu (velikost) komplexního čísla

a) $z = 3+4i \implies |z| = \sqrt{3^2 \cdot 4^2} = \underline{\underline{5}}$

b) $z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \implies |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \underline{\underline{1}}$

c) $z = \frac{\sqrt{6}-\sqrt{2}}{4} - \frac{\sqrt{6}+\sqrt{2}}{4} i \longrightarrow |z| = \sqrt{\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2} =$

$$= \sqrt{\frac{6 - 2\sqrt{6}\sqrt{2} + 2 + 6 + 2\sqrt{6}\sqrt{2} + 2}{16}} = \sqrt{\frac{16}{16}} = \underline{\underline{1}}$$

d) $z = \cos(x) + i \cdot \sin(x) \implies |z| = \sqrt{\cos^2(x) + \sin^2(x)} = \sqrt{1} = \underline{\underline{1}}$

Pozn.: $\boxed{e^{ix} = \cos(x) + i \cdot \sin(x)}$

Poloha přímek, soustavy rovnic

Př 13. Jsou dány přímky $p(P, \vec{u})$ a $q(Q, \vec{v})$
kde $P = [2; -1]$, $Q = [0; -2]$; $\vec{u} = (1; 2)$; $\vec{v} = (1; 1)$.
Určete vzájemnou polohu přímek.

$$p: \quad \begin{aligned} x &= 2 + 1 \cdot t \\ y &= -1 + 2 \cdot t \end{aligned} \qquad \text{resp.} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \overset{\overset{P}{\shortparallel}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} + t \cdot \overset{\overset{\vec{u}}{\shortparallel}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \; ; t \in \mathbb{R}$$

$$q: \quad \begin{aligned} x &= 0 + 1 \cdot s \\ y &= -2 + 1 \cdot s \end{aligned} \qquad \text{resp.} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} + s \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; ; s \in \mathbb{R}$$

Průsečík: $\left.\begin{pmatrix} x \\ y \end{pmatrix}\right|_p = \left.\begin{pmatrix} x \\ y \end{pmatrix}\right|_q$

$$\begin{aligned} (1) \quad & 2 + 1 \cdot t = 0 + 1 \cdot s \longrightarrow s = t + 2 \quad * \\ (2) \quad & -1 + 2 \cdot t = -2 + 1 \cdot s \end{aligned}$$

$$(2)^* \quad -1 + 2t = -2 + (t + 2)$$

$$\left.\begin{aligned} t &= 1 \\ s &= t + 2 = 3 \end{aligned}\right\}$$

Průsečík (nazveme opět X) je tedy $X = [3; 1]$

Pozn.: $\vec{u}, \vec{v}$ ... směrové vektory přímek p a q.

Př. 14 $p(P;\vec{u})$ a $q(Q,\vec{v})$ (poloha přímek)

$P=[1;0]$; $\vec{u}=(1;2)$

$Q=[3;5]$; $\vec{v}=(3;6)$

$p: \quad x = 1+t \qquad q: \quad x = 3+3s \qquad s,t \in \mathbb{R}$

$ y = 0+2t \qquad y = 5+6s$

Průsečík :

$$\begin{aligned} 1+t &= 3+3s \quad \cdot(-2) \\ 2t &= 5+6s \\ \hline -2 &= -1 \end{aligned}$$

$\rightarrow$ nemá řešení

$\Rightarrow$ přímky p, q jsou rovnoběžné, různé

Pozn.: To že jsou p, q rovnoběžné lze usoudit již z toho, že $\vec{u}$ a $\vec{v}$ jsou LZ.

!

$(3\vec{u}=\vec{v})$

Př. 15 Určete vzájemnou polohu přímek p a q.

$$p: \; x + 2y - 1 = 0$$
$$q: \; 3x + 6y = 2$$

$$\left.\begin{aligned}\vec{n}_p &= (1,2)\\ \vec{n}_q &= (3,6)\end{aligned}\right\}\; \vec{n}_q \text{ je násobkem } \vec{n}_p$$

$\Rightarrow$ rovnoběžné

ale p není násobkem q

$\left(x+2y-1 \neq k\cdot(3x+6y-2); \; k \in \mathbb{R}\setminus\{0\} \right)$

$\Rightarrow$ rovnoběžné, různé

Přímka v obecném tvaru:

Dvě přímky které mají rovnice

$$ax + by + c = 0$$
$$\bar{a}x + \bar{b}y + \bar{c} = 0$$

1. jsou rovnoběžné $\Leftrightarrow$ je-li vektor $\vec{n} = (a;b)$ nenulovým násobkem vektoru $\vec{n} = (\bar{a};\bar{b})$
2. jsou totožné, je-li jedna násobkem druhé
3. jsou různoběžné (mají průsečík), když má soustava jejich obecných rovnic právě 1 řešení.

Soustavy lin. rovnic

Př. 16 (164)

$$\begin{aligned} (1)\quad & x - 2y + 2z = -9 \\ (2)\quad & 3x + 5y + 4z = 10 \\ (3)\quad & 5x + 12y + 6z = 29 \end{aligned}$$

$\cdot(-2)$ $+$ $\cdot(-3)$ $+$

1)

$$\begin{aligned} & x - 2y + 2z = -9 \quad (1) \\ & x + 9y = 28 \quad (2) \\ & \cancel{2x + 18y = 56} \quad (3) \end{aligned}$$

$\cdot(-1)$ $+$

2x násobek (2) rovnice

2)

$$\boxed{11y - 2z = 37} \quad (2)$$

Zvolíme parametr, např. $z = t$; $t \in \mathbb{R}$

3)

$$\Rightarrow \quad y = \frac{37 - 2t}{11} \; ; \; z = t$$

(1)

$$\Rightarrow \quad x = -9 + 2y - 2z = -9 + \frac{2\cdot(37-2t)}{11} - 2t$$

$$= \frac{-25 - 24t}{11}$$

Celkem máme:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{-25-24t}{11} \\ \frac{37-2t}{11} \\ t \end{pmatrix} \quad t \in \mathbb{R}$$

Vektory, vektorové prostory

① $\vec{u} = (2;4)^T$ $\vec{v} = (-1,6)^T$ $\vec{w} = (6,-10)^T$ $\vec{z} = (1,0)^T$

Určete $\vec{x} = \alpha \cdot \vec{u} + \beta \cdot \vec{v} + \gamma \cdot \vec{w} + \delta \cdot \vec{z}$

kde $\alpha = 2$, $\beta = 1$, $\gamma = -2$, $\delta = -8$

$$\vec{x} = 2 \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 6 \end{pmatrix} - 2 \cdot \begin{pmatrix} 6 \\ -10 \end{pmatrix} - 8 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -17 \\ -6 \end{pmatrix}$$

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② $\vec{u} = (4;1;2;1)^T$ $\vec{v} = (5;3;1;2)^T$ $\vec{w} = (0,0,0,50)$

$\vec{x} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = ?$ kde $\alpha = 2$, $\beta = -1$, $\gamma = 2$

$$\vec{x} = \begin{pmatrix} 3 \\ -1 \\ 3 \\ 100 \end{pmatrix}$$

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③ Vektorová rovnice

$\vec{a} - 2\vec{x} = \vec{b} - \vec{x}$ kde $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $\vec{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$x = ?$

$\vec{a} - \vec{b} = \vec{x}$

$$\vec{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

4) Skalární součin, rozhodněte zda jsou vektory kolmé!

a) $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

0, jsou kolmé

b) $\vec{u} = \begin{pmatrix} 5 \\ 4 \\ -2 \\ 3 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}$

−5, nejsou kolmé

c) $\vec{u} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$

0, kolmé

5) Pro jaké hodnoty $\alpha \in \mathbb{R}$ jsou dané vektory kolmé?

a)

$\vec{u} = (-2; \alpha + 3)$ $\vec{v} = (0; -1 + 2\alpha)$

$$0 = \vec{u} \cdot \vec{v} = (-2) \cdot 0 + (\alpha + 3) \cdot (-1 + 2\alpha) =$$

$$= -\alpha + 2\alpha^2 - 3 + 6\alpha = \boxed{2\alpha^2 + 5\alpha - 3 = 0}$$

$$\alpha_{1,2} = \frac{-5 \pm \sqrt{25 - 4 \cdot 2 \cdot (-3)}}{4} = \left\langle \begin{matrix} \frac{1}{2} \\ -3 \end{matrix} \right.$$

$$\boxed{\begin{matrix} \alpha_1 = \frac{1}{2} \\ \alpha_2 = -3 \end{matrix}}$$

b) $\vec{u} = (2, \alpha^2)$ $\vec{v} = (2\alpha, 1)$

$$\vec{u} \cdot \vec{v} = 4\alpha + \alpha^2 = 0$$

$$\alpha \cdot (4 + \alpha) = 0$$

$$\Rightarrow \boxed{\alpha_1 = 0} \quad \boxed{\alpha_2 = -4}$$

(6) Vypočítejte kosinus úhlu, který svírají zadané vektory

$$\left(\cos\alpha = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \right)$$

(A odhadněte velikost úhlu α)

a) $\vec{u} = (0;3;5)$

$\vec{v} = (4;-2;1)$

$$\cos\alpha = \frac{(0;3;5)\cdot(4;-2;1)}{\sqrt{0^2+3^2+5^2}\cdot\sqrt{4^2+(-2)^2+1^2}} =$$

$$= \frac{-1}{\sqrt{34}\,\sqrt{21}} = -\frac{1}{\sqrt{714}}$$

$\alpha \doteq 90^\circ$

b) $\vec{u} = (6;-2)$

$\vec{v} = (-3;1)$

$$\cos\alpha = \frac{(6;-2)\cdot(-3;1)}{\sqrt{6^2+(-2)^2}\cdot\sqrt{(-3)^2+1^2}} =$$

$$= \frac{-20}{\sqrt{40}\cdot\sqrt{10}} = -1$$

$\alpha = 180^\circ$

$\Rightarrow$ kolineární

(rovnoběžné)

(7) $(x+1, 3y) - (y-1, x) = (0, 8)$

určete x, y

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1) 2) $\begin{pmatrix} x+1 \\ 3y \end{pmatrix} - \begin{pmatrix} y-1 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

2) $x = 3y - 8 \Longrightarrow$ 1) $(3y-8)+1-(y-1) = 0$

$2y - 6 = 0$

$\boxed{\begin{matrix} y = 3 \\ x = 1 \end{matrix}}$

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(8) Pro dané vektory $\vec{a} = (1;1;1)$ určete $\alpha, \beta \in \mathbb{R}$

$\vec{b} = (1;1;2)$ tak že platí

$\vec{c} = (2;1;1)$

$\vec{a} \cdot (\alpha\vec{a} + \beta\vec{b} + \vec{c}) = 0$

$\vec{b} \cdot (\alpha\vec{a} + \beta\vec{b} + \vec{c}) = 0$

$3\alpha + 4\beta + 4 = 0$ $\quad / (-4)$

$4\alpha + 6\beta + 5 = 0$ $\quad / \cdot 3$

$\oslash \quad -16\beta + 18\beta - 16 + 15 = 0$

$\boxed{\beta = 1/2}$

$3\alpha + 4 \cdot \frac{1}{2} + 4 = 0$

$\boxed{\alpha = -2 \,; \beta = \frac{1}{2}}$

Lineární závislost / nezávislost (LZ / LN)

(9) Rozhodněte, zda jsou vektory $\vec{u} = (3;2;1;0)$ $\vec{v} = (4;0;2;1)$ a $\vec{w} = (-1;2;0;2)$ LZ/LN ?

Jaká je dimenze vektorového prostoru V nad těmito vektory generovaným?

($) $\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{o}$

$$3\alpha + 4\beta + (-1)\gamma = 0$$
$$2\alpha + 0\beta + 2\gamma = 0 \longrightarrow \alpha = -\gamma$$
$$1\alpha + 2\beta + 0\cdot\gamma = 0 \longrightarrow \alpha = -2\beta$$
$$\left.\begin{array}{l}\alpha = -\gamma \\ \alpha = -2\beta\end{array}\right\} \gamma = 2\beta \; *$$
$$0\alpha + 1\beta + 2\gamma = 0 \longrightarrow \beta = -2\gamma \; *$$

$$* \quad \begin{array}{rl} \gamma = 2\beta & \\ -2\gamma = \beta & \big| \cdot (-2) \\ \hline 5\gamma = 0 & \end{array} \Bigg\} \; \boxed{\gamma = 0} \Longrightarrow \boxed{\begin{array}{l}\beta = 0 \\ \alpha = 0\end{array}}$$

$\Rightarrow$ Soustava ($) má jediné, triviální řešení

$\alpha = \beta = \gamma = 0 \Rightarrow \vec{u}, \vec{v}, \vec{w}$ jsou LN !

$\Rightarrow \dim V = 3$; V je podprostor $\mathbb{E}_4$ (4 složky)

(10) LZ/LN? dimenze vekt. prostoru V, který je nad danými vektory generován?
Které vektory tvoří jeho bázi?

a) $\vec{u}=(2;1)$ $\vec{v}=(-1;3)$

LN, 2, $(\vec{u},\vec{v})$ $(\vec{u}\neq k\vec{v}$ $k\in\mathbb{R})$
→ neleží na jedné přímce...

b) $\vec{u}=(1;4;2)$ $\vec{v}=(3;2;2)$

LN, 2, $(\vec{u},\vec{v})$

c) $\vec{u}=(2;0;3)$ $\vec{v}=(1;1;0)$, $\vec{w}=(0;-2;1)$

$$2\alpha+\beta=0 \longrightarrow \beta=-2\alpha$$
$$\beta-2\gamma=0 \qquad \beta=2\gamma$$
$$\left.\begin{array}{l}\beta=-2\alpha\\ \beta=2\gamma\end{array}\right\}\ \gamma=-\alpha$$
$$3\alpha+\gamma=0 \longrightarrow \gamma=-3\alpha$$

$$-\alpha=-3\alpha$$

$$\Rightarrow \alpha=0 \Rightarrow \gamma=\beta=0 \Rightarrow LN,\ 3,\ (\vec{u},\vec{v},\vec{w})$$

d) $\vec{u} = (-1,2)$ $\vec{v} = (0,3)$ $\vec{w} = (5,7)$

LZ, 2, kterákoli dvojice z $\vec{u}, \vec{v}, \vec{w}$

e) $\vec{u} = (2;3;-2)$ $\vec{v} = (3;0;1)$ $\vec{w} = (0;9;-8)$

$$2\alpha + 3\beta + 0\gamma = 0 \longrightarrow \beta = \frac{-2}{3}\alpha$$
$$3\alpha + 0\beta + 9\gamma = 0 \longrightarrow \gamma = \frac{-3}{9}\alpha = \frac{-1}{3}\alpha$$
$$-2\alpha + 1\beta - 8\gamma = 0$$

$$-2\alpha + 1\cdot\left(\frac{-2}{3}\alpha\right) - 8\left(-\frac{1}{3}\alpha\right) = 0$$

$$\frac{-6}{3}\alpha \; \frac{-2}{3}\alpha + \frac{8}{3}\alpha = 0$$

$$0 = 0$$

$\Longrightarrow$ splněno $\forall \alpha \in \mathbb{R}$

$\Longrightarrow$ označme α jako parametr, např. $\alpha = t \in \mathbb{R}$

Potom: $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} t \\ -2/3\,t \\ -1/3\,t \end{pmatrix} = \begin{pmatrix} 1 \\ -2/3 \\ -1/3 \end{pmatrix} t, \; t \in \mathbb{R}$

LZ, 2, kterákoli 2-jice ...

f) $\vec{u} = (2;4;3;0)$ $\vec{v} = (1;1;0;0)$

$\vec{w} = (3;1;3;0)$ $\vec{r} = (1;0;2;0)$

$$2\alpha + 1\beta + 3\gamma + 1\delta = 0 \quad \cdot(-2) \quad \cdot(-3)$$
$$4\alpha + 1\beta + 1\gamma + 0\delta = 0 \quad +$$
$$3\alpha + 0\beta + 3\gamma + 2\delta = 0 \quad \cdot(2) \; +$$
$$\cancel{0\alpha + 0\beta + 0\gamma + 0\delta = 0}$$ (splněna automaticky)

$$2\alpha + 1\beta + 3\gamma + 1\delta = 0$$
$$0\alpha - 1\beta - 5\gamma - 2\delta = 0 \quad \cdot(-3)$$
$$0\alpha - 3\beta - 3\gamma + 1\delta = 0 \quad +$$

---

1) $2\alpha + 1\beta + 3\gamma + 1\delta = 0$

2) $0\alpha - 1\beta - 5\gamma - 2\delta = 0$

3) $0\alpha + 0\beta + 12\gamma + 7\delta = 0$

horní Δ tvar (později budeme vycházet z maticového zápisu)

Volíme $\delta = t;\ t \in \mathbb{R}$

3) $\rightarrow \gamma = \frac{-7}{12} t$

2) $\rightarrow \beta = -2\delta - 5\gamma = -2t + \frac{35}{12} t = \frac{11}{12} t$

1) $\rightarrow \alpha = \dfrac{-\delta - 3\gamma - \beta}{2} = \dfrac{-t + \frac{21}{12} t - \frac{11}{12} t}{2} = \frac{-1}{12} t$

Celkem: $\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \frac{t}{12} \begin{pmatrix} -1 \\ 11 \\ -7 \\ 12 \end{pmatrix}$

$\boxed{\text{LZ, 3, } \vec{u}, \vec{v}, \vec{w} \text{ (neopr.)}}$

(11) Určete pro jaké hodnoty parametru $\lambda \in \mathbb{R}$

jsou vektory $\vec{u} = (\lambda - 3; 2; 1)$

$\vec{v} = (2; 5+\lambda; 0)$ lineárně závislé?

$\vec{w} = (2, 0, 1)$

Jaká je v těchto případech dimenze vek. prostoru V který je nad těmito vektory generován?

$$\alpha \vec{u} + \beta \vec{v} + \gamma \vec{w} = \vec{o}$$

$$\beta = \frac{2}{5+\lambda}$$

1) $(\lambda-3)\cdot\alpha + 2\cdot\beta + 2\gamma = 0$

2) $2\cdot\alpha + (5+\lambda)\cdot\beta + 0\gamma = 0 \longrightarrow \beta = \frac{-2}{5+\lambda}\alpha$ $\quad 5+\lambda \neq 0$

3) $1\cdot\alpha + 0\cdot\beta + 1\gamma = 0 \longrightarrow \alpha = -\gamma$

1) $(\lambda-3)\cdot(-\gamma) + 2\cdot\frac{+2}{5+\lambda}\gamma + 2\gamma = 0 \quad / \cdot(-1)\cdot(5+\lambda)$

$$(\lambda-3)(\lambda+5)\cdot\gamma - 4\gamma - 2(5+\lambda)\gamma = 0$$

$$(\lambda^2 + 2\lambda - 15 - 4 - 10 - 2\lambda)\cdot\gamma = 0$$

$$\boxed{(\lambda^2 - 29)\gamma = 0}$$

Vypočtem případ $(\lambda^2 - 29) = 0 \Rightarrow$ LZ

$\rightarrow$ Pro $(\lambda^2 - 29) \neq 0$ musí být nutně $\boxed{\beta = 0}$ LN, 3, $\vec{u}, \vec{v}, \vec{w}$

$$\Rightarrow \boxed{\alpha = \gamma = 0}$$

Cvičení 3.

Podprostory vektorového prostoru

Př. 1 Rozhodněte zda $\mathbb{V}'$ je podprostorem vektorového prostoru $\mathbb{V}$.

(Uzavřenost vůči operacím "sčítání prvků" a "násobení prvků reálnými čísly")

a) $\mathbb{V} = V(E_3)$, $\mathbb{V}' = \{(a;b;c);\ a;b;c \in \mathbb{R},\ a+b=0\}$

Mějme 2 prvky z $\mathbb{V}'$: $\underset{\nearrow (a+b)=0}{(a;b;c)}$ a $\underset{\nearrow (\bar{a}+\bar{b})=0}{(\bar{a};\bar{b};\bar{c})}$ a reálné číslo $\alpha \in \mathbb{R}$.

Chceme ověřit zda: $\boxed{\begin{array}{l}(a;b;c)+(\bar{a};\bar{b};\bar{c}) \in \mathbb{V}' \\ \alpha\cdot(a;b;c) \in \mathbb{V}'\end{array}}$ I), a II)

$$\Rightarrow\ (a;b;c)+(\bar{a};\bar{b};\bar{c}) = (a+\bar{a};\ b+\bar{b};\ c+\bar{c})$$

platí $(a+\bar{a})+(b+\bar{b})=0$? I)

ANO: $(a+\bar{a})+(b+\bar{b}) = \underbrace{\underbrace{(a+b)}_{0}+\underbrace{(\bar{a}+\bar{b})}_{0}}_{\text{z předpokladu}} = 0$

(asociativní zákon)

$\Rightarrow \alpha \cdot (a; b; c) = (\alpha a; \alpha b; \alpha c)$

Platí $\alpha a + \alpha b = 0$? II)

ANO: $\alpha a + \alpha b \overset{\text{distributivní zákon}}{=} \alpha \cdot \underbrace{(a+b)}_{0 \text{ (z předpokladu)}} = 0$

Závěr: Součet 2 libovolných prvků z $\mathbb{V}'$ je opět prvek z $\mathbb{V}'$
Součin libovolného reálného čísla ($\alpha \in \mathbb{R}$) a libovolného prvku z $\mathbb{V}'$ ((a, b, c)) je opět prvek z $\mathbb{V}'$!

$\Rightarrow \mathbb{V}'$ je podprostor prostoru $\mathbb{V}$

---

b) $\mathbb{V} = \mathbb{V}(\mathbb{E}_3)$; $\mathbb{V}' = \{(x, y, z), x, y, z \in \mathbb{R}; x+y=1\}$

I) $(x, y, z) + (\bar{x}, \bar{y}, \bar{z}) = (x+\bar{x}, y+\bar{y}, z+\bar{z})$

Musí platit: $(x+\bar{x}) + (y+\bar{y}) = 1$

$\Rightarrow (x+\bar{x}) + (y+\bar{y}) = \underbrace{(x+y)}_{1} + \underbrace{(\bar{x}+\bar{y})}_{1} = 2$

$\Rightarrow \mathbb{V}'$ není podprostorem $\mathbb{V}$ (selhala hned vlastnost I)

---

c) $\mathbb{V} = \mathbb{V}(\mathbb{E}_2)$; $\mathbb{V}' = \{(a,b);\ a,b \in \mathbb{R},\ b = 0\}$

I) $(a,b) + (\bar{a},\bar{b}) = (a+\bar{a},\ b+\bar{b})$

$b + \bar{b} \overset{?}{=} 0$, platí ($b = 0$; $\bar{b} = 0$) ✓

II) $\alpha \cdot (a;b) = (\alpha a,\ \alpha b)$

$\alpha b \overset{?}{=} 0$, platí ($b = 0$, $\alpha \in \mathbb{R}$) ✓

$\Longrightarrow$ $\mathbb{V}'$ je podprostorem $\mathbb{V}$

---

d) $\mathbb{V} = \mathbb{V}(\mathbb{E}_2)$; $\mathbb{V}' = \{(x,y),\ x,y \in \mathbb{R},\ y \geq 0\}$

I) $(x,y) + (\bar{x},\bar{y}) = (x+\bar{x},\ y+\bar{y})$

Zřejmě, pokud $y \geq 0$ a $\bar{y} \geq 0$ tak $y + \bar{y} \geq 0$

$\Rightarrow$ platí ✓

II) $\alpha \cdot (x,y) = (\alpha x,\ \alpha y)$

? $\alpha y \geq 0$ $\Rightarrow$ ne vždy ! (pokud $\alpha < 0$

tak $\alpha y < 0$ pro $y \geq 0$)

$\Rightarrow$ neplatí

$\mathbb{V}'$ není podprostorem $\mathbb{V}$

e) $\mathbb{V} = \mathbb{V}(\mathbb{E}_3)$, $\mathbb{V}' = \{(a+2; 2a-1; a+3); a \in \mathbb{R}\}$

I) $(a+2; 2a-1; a+3) + (\bar{a}+2; 2\bar{a}-1; \bar{a}+3) =$

$= ((a+\bar{a})+4; 2(a+\bar{a})-2; (a+\bar{a})+6)$

$\Rightarrow$ neplatí

$\Rightarrow$ $\mathbb{V}'$ není podprostorem $\mathbb{V}$

Matice

Př. 2 $A = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix}$; $B = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 2 & -1 \end{pmatrix}$

$\alpha = 2$ $\beta = 3$

$\alpha \cdot A + \beta \cdot B = ?$

$$2 \cdot \begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} + 3 \begin{pmatrix} -2 & 4 & 3 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 16 & 19 \\ 0 & 6 & 9 \end{pmatrix}$$

Pr. 3 $\quad A \cdot B = ?$

a) $A = \begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}$; $B = \begin{pmatrix} 3 & 4 \\ 7 & 11 \end{pmatrix}$

$$\begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 7 & 11 \end{pmatrix} = \begin{pmatrix} 4\cdot 3 - 2\cdot 7 & 4\cdot 4 - 2\cdot 11 \\ 5\cdot 3 - 3\cdot 7 & 5\cdot 4 - 3\cdot 11 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 & -6 \\ -6 & -13 \end{pmatrix}}}$$

b) $\underset{2\times 3}{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \end{pmatrix}$; $\underset{3\times 3}{B} = \begin{pmatrix} 3 & 1 & 2 & 4 \\ 0 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 21 & 5 & 3 & 11 \\ 10 & 19 & 5 & 2 \end{pmatrix}$$

c) $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 3 \end{pmatrix}$$

Př. 4 Vypočítejte :

a) $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^3$; b) $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \end{pmatrix}^T$; c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$

---

a) $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix}$

$\begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 15 & 20 \\ 20 & 35 \end{pmatrix}$

---

b) $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$

---

c) I) $\boxed{A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$

A^2

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

A^3

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

A^4

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$\vdots$

$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ (odhadem) + důkaz indukcí

Indukce :

$A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

předpoklad

$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

$\Rightarrow A^{n+1} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$

ověření :

$A^{n+1} = A \cdot A^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix} \Rightarrow$ platí ✓

II) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = I + B$

kde $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

dohledat !

Potom $A^n = (I+B)^n = \left(\text{binomická věta}\right) =$

$$= \binom{n}{0} I^n + \binom{n}{1} I^{n-1} \cdot B + \binom{n}{2} I^{n-2} \cdot B^2 +$$

$$\dots \binom{n}{n-1} I^1 B^{n-1} + \binom{n}{n} B^n$$

$\boxed{I^n \overset{?}{=} I}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Ale ! $B^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$B^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$

$B^3 = B^4 = \dots B^k \, (k \geq 0) = O$

Tedy: $(I+B)^n = \binom{n}{0} I^n + \binom{n}{1} \cdot I^{n-1} \cdot B =$

$$= I + n \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + n \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$$

$$= \underline{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}}$$

Př. 5 $A = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}$; $A \cdot A^T = ?$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Př. 6 Určete hodnost matice s řádky tvořenými vektory: $(1;2;3)$; $(2;-1;1)$ a $(1;7;8)$

$$(A=) \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & 7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \end{pmatrix}$$

$\rightarrow$ Horní Δ matice, hodnost: $h(A) = 2$

singulární $(A_{3\times 3}$; $h(A) < 3)$

Př. 7 Nalezněte hodnost matice. Pokud je čtvercová, rozhodněte zda je singulární / regulární.

a)
$$A = \begin{pmatrix} 1 & -2 & 3 \\ -3 & -6 & -9 \\ 4 & 8 & 12 \end{pmatrix}$$

b)
$$B = \begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix}$$

c)
$$C = \begin{pmatrix} 2k-2 & 3 & 0 & 0 \\ 0 & k-1 & 3 & 0 \\ 0 & 2 & k & 0 \\ 0 & 0 & 0 & k-3 \end{pmatrix}$$

---

a)
$$\begin{pmatrix} 1 & -2 & 3 \\ -3 & -6 & -9 \\ 4 & 8 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & -12 & 0 \\ \cancel{0} & \cancel{16} & \cancel{0} \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & -12 & 0 \end{pmatrix}$$

$h(A) = 2$, singulární

b)
$$\begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -2 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & 3 & -1 \\ 4 & -3 & 1 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 3 & 4 & -2 \\ 0 & -10 & -10 & 6 \\ \cancel{0} & \cancel{-5} & \cancel{-5} & \cancel{3} \\ \cancel{0} & \cancel{-15} & \cancel{-15} & \cancel{9} \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -2 \\ 0 & -10 & -10 & 6 \end{pmatrix}$$

$h(B) = 2$, singulární

c)
$$\begin{pmatrix} 2k-2 & 3 & 0 & 0 \\ 0 & k-1 & 3 & 0 \\ 0 & 2 & k & 0 \\ 0 & 0 & 0 & k-3 \end{pmatrix} \begin{array}{c} \\ (-2) \\ +\downarrow \\ (k-1) \end{array} \sim \begin{pmatrix} 2k-2 & 3 & 0 & 0 \\ 0 & k-1 & 3 & 0 \\ 0 & 0 & k^2-k-6 & 0 \\ 0 & 0 & 0 & k-3 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} (2k-2) & 3 & 0 & 0 \\ 0 & (k-1) & 3 & 0 \\ 0 & 0 & (k-3)(k+2) & 0 \\ 0 & 0 & 0 & (k-3) \end{pmatrix}$$

$k=3$:
$$\begin{pmatrix} 4 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{pmatrix}$$
$h(C)=2$
singulární

$k=-2$:
$$\begin{pmatrix} -6 & 3 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} -6 & 3 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$
$h(C)=3$
singulární

$k=1$:
$$\begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ \cancel{0} & \cancel{0} & \cancel{-6} & \cancel{0} \\ 0 & 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$
$h(C)=3$
singulární

Tedy pro $k \neq 3; -2; 1$ je matice C regulární

Př. 8 Vypočítejte determinanty zadaných matic

a) $\begin{vmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{vmatrix}$ b) $\begin{vmatrix} -1 & -3 \\ -2 & 5 \end{vmatrix}$ c) $\begin{vmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{vmatrix}$

d) $\begin{vmatrix} 0 & 5 & -2 & 3 \\ 1 & 2 & 0 & 0 \\ 5 & 2 & 3 & 2 \\ 2 & -1 & 2 & 3 \end{vmatrix}$

---

a) $$\begin{vmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{vmatrix} = \cos(x)\cdot\cos(x) - (-\sin(x))\cdot\sin(x) = \cos^2(x) + \sin^2(x) = \underline{\underline{1}}$$

b) $$\begin{vmatrix} -1 & -3 \\ -2 & 5 \end{vmatrix} = (-1)\cdot 5 - (-3)\cdot(-2) = \underline{\underline{-11}}$$

c) $$\begin{array}{|ccc|} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{array}$$
$$\begin{array}{ccc} 1 & i & 1+i \\ -i & 1 & 0 \end{array}$$

$$= 1 + 0 + 0 - (1+i)(1-i) - 0 - i\cdot(-i) =$$
$$= 1 - (1 - i^2) + i^2 = 2i^2 = \underline{\underline{-2}}$$

d)

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} \\ 2 \\ \\ \\ \end{array} & \begin{vmatrix} 0 & 5 & -2 & 3 \\ 1 & 2 & 0 & 0 \\ 5 & 2 & 3 & 2 \\ 2 & -1 & 2 & 3 \end{vmatrix} \end{array} = (-1)^{2+1} \cdot 1 \cdot \begin{vmatrix} 5 & -2 & 3 \\ 2 & 3 & 2 \\ -1 & 2 & 3 \end{vmatrix} + (-1)^{2+2} \cdot 2 \cdot \begin{vmatrix} 0 & -2 & 3 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{vmatrix} + 0 + 0 =$$

$$= -1 \cdot \left(45 + 12 + 4 - (-9) - 20 - (-12)\right)$$

$$+ 2 \cdot \left(0 + 30 + (-8) - 18 - 0 - (-30)\right) =$$

$$= -62 + 2 \cdot 34 = -62 + 68 = \underline{\underline{6}}$$

Cvičení 4.

Př. 1

$$\begin{vmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{vmatrix} = ?$$

$$\det A_{n\times n} = a_{i1} \cdot A_{i1} + a_{i2} \cdot A_{i2} + \dots + a_{in} A_{in}$$

kde $A_{ij} = (-1)^{i+j} A^*_{ij}$

kde A^*_{ij} je determinant čtvercové matice $(n\times 1) \times (n-1)$, která vznikne z matice A vynecháním i-tého řádku a j-tého sloupce.

$$\begin{vmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{vmatrix} = (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 2 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} + (-1)^{1+3} \cdot 2 \cdot \begin{vmatrix} 2 & 0 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{vmatrix}$$

$$+ (-1)^{1+4} \cdot 2 \cdot \begin{vmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = -2 \cdot 8 + 2 \cdot (-8)$$

$$-2 \cdot 8 = \underline{\underline{-48}}$$

Pozn.: Vyměníme-li v matici A dva řádky / sloupce $\Rightarrow$ determinant je potom roven $-\det A$.

Př. 2

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & 0 & 0 \\ -1 & -1 & 1 & 0 \end{pmatrix}; \quad \det A = ?$$

$$\begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & 0 & 0 \\ -1 & -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 & 0 \\ 0 & -1 & -1 & 1 \\ a & b & 0 & 0 \\ -1 & -1 & 1 & 0 \end{vmatrix} = (-1)^{2+4} \cdot 1 \cdot \begin{vmatrix} 1 & -1 & -2 \\ a & b & 0 \\ -1 & -1 & 1 \end{vmatrix} =$$

$$= b + 2a + 0 - 2b - 0 + a = \underline{3a - b}$$

Př. 3 $\quad A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

a) vypočítejte matice A^{-1} a B^{-1}

b) vypočítejte matici X, která vyhovuje rovnici: $A \cdot X \cdot B = C$

---

a) $$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 15 & 5 & 5 & 0 \\ 15 & 6 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 15 & 5 & 5 & 0 \\ 0 & 1 & -5 & 3 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|cc} 15 & 0 & 30 & -15 \\ 0 & 1 & -5 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$B^{-1} = \frac{1}{\det B}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}^T ;$$ B_{ij} ... doplňky matice B (k prvku b_{ij})

$$= \frac{1}{\det B}\begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11}\end{pmatrix}$$

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 2 & 0\end{pmatrix}$$

b) $$X = A^{-1} \cdot C \cdot B^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3\end{pmatrix}\cdot\begin{pmatrix} 2 & 2 \\ 2 & -1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 3 & -1 \\ 2 & 0\end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 5 \\ -4 & -13\end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 16 & -2 \\ -38 & 4\end{pmatrix} = \begin{pmatrix} 8 & -1 \\ -19 & 2\end{pmatrix}$$

Př 4 $$A = \begin{pmatrix} 2\alpha-1, & \alpha+1, & 1 \\ 1, & -1, & 1 \\ 1, & 1 & 3\end{pmatrix}$$

a) Pro jaké hodnoty α existuje A^{-1}?

b) Najděte A^{-1} pro $\alpha = 2$

---

a) $$\begin{vmatrix} 2\alpha-1 & \alpha+1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 3\end{vmatrix} = (2\alpha-1)(-1)(3) + 1\cdot1\cdot1 + 1\cdot(\alpha+1)\cdot1$$
$$- 1\cdot(-1)\cdot1 - 1\cdot1\cdot(2\alpha-1)$$
$$- 3\cdot(\alpha+1)\cdot1 =$$

$$= -6\alpha+3+1+\alpha+1+1-2\alpha+1-3\alpha-3 =$$
$$= \boxed{-10\alpha+4}$$

Tedy pro $\alpha = \frac{2}{5}$ je $\det A = 0$ a tedy A^{-1} neexistuje (A je singulární).

Pro $\alpha \neq \frac{2}{5}$ je A regulární a A^{-1} existuje.

b) $\alpha = 2$

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right)^{(\cdot 1/3)} \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1/3 & 1/3 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1/3 & 1/3 & 0 & 0 \\ 0 & -2 & 2/3 & -1/3 & 1 & 0 \\ 0 & 0 & 8/3 & -1/3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & -1/3 & 1/6 & -1/2 & 0 \\ 0 & 0 & 1 & -1/8 & 0 & 3/8 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3/8 & 0 & -1/8 \\ 0 & 1 & 0 & 1/8 & -1/2 & 1/8 \\ 0 & 0 & 1 & -1/8 & 0 & 3/8 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/8 & 1/2 & -2/8 \\ 0 & 1 & 0 & 1/8 & -1/2 & 1/8 \\ 0 & 0 & 1 & -1/8 & 0 & 3/8 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 2/8 & 1/2 & -2/8 \\ 1/8 & -1/2 & 1/8 \\ -1/8 & 0 & 3/8 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 2 & 4 & -2 \\ 1 & -4 & -1 \\ -1 & 0 & 3 \end{pmatrix}$$

Pozn.:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \cdot \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}}_{A} = \begin{pmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \to \text{prohození řádků 1 a 2}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_2} \cdot \underbrace{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}}_{A} = \begin{pmatrix} a & b & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \to \text{sečtení 1 a 2 řádku (na 1 řádku...)}$$

$$\Rightarrow \boxed{\underbrace{E_m \cdots E_2 \cdot E_1}_{A^{-1}} \cdot A = I} \Rightarrow E_m \cdots E_2 \cdot E_1 (A|I) \Rightarrow (I \mid \underbrace{E_m \cdots E_2 \cdot E_1}_{A^{-1}})$$

Př. 5 $A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ Pro jaké hodnoty α_1 a α_2 existuje A^{-1}? Vypočítejte A^{-1}!

$\det A = \alpha_1 \cdot \alpha_2 \Rightarrow \det A \neq 0 \Leftrightarrow \alpha_1 \cdot \alpha_2 \neq 0$

$\Rightarrow A^{-1}$ existuje pro $\alpha_1 \neq 0 \wedge \alpha_2 \neq 0$

$$A^{-1}: \quad \left(\begin{array}{cc|cc} \alpha_1 & 0 & 1 & 0 \\ 0 & \alpha_2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{\alpha_1} & 0 \\ 0 & 1 & 0 & \frac{1}{\alpha_2} \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix}}$$

Př. 6 Určete zda jsou dané vektory LZ/LN?

$\vec{a} = (1; 0; 2; 3)^T \quad \vec{b} = (-2; 3; 4; 1)^T$

$\vec{c} = (0; 0; 0; 3)^T \quad \vec{d} = (2; 1; 5; 0)^T$

Použijte při řešení vlastnosti determinantu matice.

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} + \delta\vec{d} = \vec{o}$$

$$\left(\vec{a} \,|\, \vec{b} \,|\, \vec{c} \,|\, \vec{d}\right) \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \vec{o}$$

$$\boxed{A \cdot \vec{x} = \vec{o}}$$

LN: $\alpha = \beta = \gamma = \delta = 0$
$\Rightarrow$ matice regulární, tj $\det A \neq 0$

LZ: $\exists!$ nenulový potomek
$\Rightarrow$ matice má LZ sloupce
$\Rightarrow \det A = 0$

$$\begin{vmatrix} 1 & -2 & 0 & 2 \\ 0 & 3 & 0 & 1 \\ 2 & 4 & 0 & 5 \\ 3 & 1 & 3 & 0 \end{vmatrix} = (-1)^{4+3} \cdot 3 \cdot \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 1 \\ 2 & 4 & 5 \end{vmatrix} =$$

$$= -3 \cdot \left[15 + 0 + (-4) - 12 - 4 - 0 \right] =$$

$$= \underline{\underline{15}} \implies \text{Vektory } \vec{a}, \vec{b}, \vec{c}, \vec{d} \text{ jsou LN}$$

Př. 7 $A = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}$ Existuje A^{-1}? Pokud ano, určete ji.

$$\det A = \cos^2(x) - (-\sin(x))(\sin(x)) = 1 \implies A \text{ je regulární, tedy } A^{-1} \text{ existuje}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} =$$

$$= \frac{1}{1} \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \overset{!}{=} \begin{pmatrix} \cos(-x) & -\sin(-x) \\ \sin(-x) & \cos(-x) \end{pmatrix}$$

Neboť $\boxed{\begin{aligned} \cos(-x) &= \cos(x) \\ \sin(-x) &= -\sin(x) \end{aligned}}$

Př. 8 Určete zda dané vektory tvoří bázi E_3.

$\vec{u} = (1, 0, 6)$ $\vec{v} = (2, 2, 2)$ $\vec{w} = (-1, -1, 2)$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 6 & 2 & 2 \end{pmatrix} ; \det A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 6 & 2 & 2 \end{vmatrix} = +4 + 0 + (-12) - (-12) - (-2) - 0 = \underline{\underline{8}}$$

$\det A \neq 0 \iff$ matice A je regulární $\iff$ $(h(A) = 3)$

$\iff$ sloupcové (řádkové) vektory matice A jsou LN.

Máme skupinu 3 LN vektorů z E_3 $\Rightarrow$ tvoří bázi E_3.

Cvičení 5.

Soustavy lineárních algebraických rovnic

Soustava rovnic:

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}} \Rightarrow \boxed{A\vec{x} = \vec{b}}$$

$$\underset{m\times n}{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}; \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}; \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right) \rightarrow$$ rozšířená matice soustavy $A\vec{x} = \vec{b}$

Frobeniova věta

1. Soustava lin. algeb. rovnic má řešení $\Leftrightarrow$ $\boxed{h(A) = h(A|\vec{b})}$

2. $h(A) = h(A|\vec{b}) = n$ (počet neznámých) $\Rightarrow$ má soustava 1 řešení

   $h(A) = h(A|\vec{b}) < n \Rightarrow$ má soustava ∞ řešení (závislé na parametrech)

Gaussova eliminační metoda:

Př. 1 Pomocí Frobeniovy věty vyšetřete, kolik má daná soustava řešení a tato řešení najděte.

(Pozn.: „soustava" $\Leftrightarrow$ soustava lineárních algebraických rovnic)

$$
\begin{aligned}
x - 2y + 3z + 9u &= 20 \\
3x + 5y - 3z - 19u &= -31 \\
2x - 3y + 2z + 8u &= 23
\end{aligned}
$$

$$
\left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 20 \\ 3 & 5 & -3 & -19 & -31 \\ 2 & -3 & 2 & 8 & 23 \end{array}\right) \begin{array}{c} (-3) \\ + \\ (-2) \\ + \end{array} \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 20 \\ 0 & 11 & -12 & -46 & -91 \\ 0 & 1 & -4 & -10 & -17 \end{array}\right) \begin{array}{c} - \\ (11) \end{array} \sim
$$

$$
\sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 20 \\ 0 & 11 & -12 & -46 & -91 \\ 0 & 0 & -32 & -64 & -96 \end{array}\right) \left(\cdot \tfrac{1}{-32}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 20 \\ 0 & 11 & -12 & -46 & -91 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)
$$

Diskuze: Počet neznámých $n = 4$ $\left((x, y, z, u)^T = \vec{x} \right)$

$$h(A) = 3 \ ; \ h(A|\vec{b}) = 3$$

$\rightarrow$ Frob. věta: $h(A) = h(A|\vec{b}) = 3 < (4 = n)$

$\Rightarrow \infty$ řešení $\Rightarrow$ závislost na parametrech

Kolik parametrů?

Věta: Množina všech řešení homogenní soustavy je podprostor aritmetického n-rozměrného prostoru $\mathbb{R}^n$, který má dimenzi rovnou: $\boxed{n - h(A)}$

Pozn. Řešení soustavy $A\vec{x} = \vec{b}$ lze napsat jako

$$\boxed{\vec{x} = \underbrace{c_1 \vec{x}_1 + \dots + c_{n-h(A)} \vec{x}_{n-h(A)}}_{\vec{x}_H \text{ (homogenní)}} + \underbrace{\vec{x}_P}_{\vec{x}_P \text{ - partikulární}}}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 20 \\ 0 & 11 & -12 & -46 & -91 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

$$n - h(A) = 4 - 3 = 1 \Rightarrow 1 \text{ parametr}$$

I. („Klasický postup“)

a) zvolme parametr (3. řádek), např. $u = t \in \mathbb{R}$

3ř. $\rightarrow$ $\boxed{z = 3 - 2t}$ (3. řádek)

b) zpětně dosadíme ...

2ř. $\rightarrow$ $11y - 12z - 46u = -91$

$\rightarrow$ $11y = -91 + 12\underbrace{(3-2t)}_{z} + 46\underbrace{t}_{u} = -55 + 22t$

ř. $\} \longrightarrow \quad y = -5 + 2t$

1.ř. $\left\{ \begin{array}{l} 1x - 2y + 3z + 9u = 20 \\ x = 20 + 2(\underbrace{-5+2t}_{y}) - 3\cdot(\underbrace{3-2t}_{z}) - 9\cdot \underset{u}{\underset{\shortparallel}{t}} = 1 + t \end{array} \right.$

Řešení soustavy $A\vec{x} = \vec{b}$ je tedy:
$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 1+t \\ -5+2t \\ 3-2t \\ t \end{pmatrix} \quad t \in \mathbb{R}$$

nebo-li:
$$\boxed{\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 3 \\ 0 \end{pmatrix} + t\cdot \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} \quad t \in \mathbb{R}}$$

?

II. přístup ($\vec{x} = \vec{x}_H + \vec{x}_P$)

$\vec{x}_P$ $\qquad\qquad$ $\vec{x}_H$ $\quad$ ($\boxed{A\vec{x} = \vec{o}}$)

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 20 \\ 0 & 11 & -12 & -46 & -91 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 0 \\ 0 & 11 & -12 & -46 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$2 \; \} -5 \; \} \, 1 \; 1 \qquad\qquad \} -2 \; \} \, 2 \; -1$

$$\vec{x}_P = \begin{pmatrix} 2 \\ -3 \\ 1 \\ 1 \end{pmatrix} \qquad\qquad \vec{x}_H = t\cdot \begin{pmatrix} -1 \\ -2 \\ 2 \\ -1 \end{pmatrix} \quad t \in \mathbb{R}$$

$11y = -91 + 58 = -33$ $\quad x = 20 - 9 - 3 - 6$
$y = -3$ $\qquad\qquad = 2$

$11y = -46 + 24 = -22$
$y = -2$

Tedy řešení soustavy $A\vec{x} = \vec{b}$ je $\vec{x} = \vec{x}_H + \vec{x}_p$

$$\boxed{\vec{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \\ -1 \end{pmatrix} \quad t \in \mathbb{R}}$$

$$\parallel$$

nebo: $$\boxed{\vec{x} = \begin{pmatrix} 2 \\ -3 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ -2 \\ 2 \\ -1 \end{pmatrix} \right\rangle}$$

značení pro lineární obal

Jsou oba přístupy ekvivalentní?

$$\begin{pmatrix} 1 \\ -5 \\ 3 \\ 0 \end{pmatrix} + t_I \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 2 \\ -3 \\ 1 \\ 1 \end{pmatrix} + t_{II} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \\ -1 \end{pmatrix}$$

$t_I, t_{II} \in \mathbb{R}$

$\rightarrow$ ANO. Pro $t_I = 1 + t_{II}$ uvedená rovnost platí.

Pozn.: Řešením soustavy je tedy přímka (1 parametr) v $\mathbb{E}_4$.

Př.: 2 Rozhodněte (na základě Frobeniovy věty) zda mají následující soustavy řešení a případně jaký je jejich počet. (dimenze)

A)
$$\begin{aligned} x - 2y &= -3 \\ 2x - y &= 0 \\ 4x - 5y &= -6 \end{aligned}$$

B)
$$\begin{aligned} x - 2y + 2z &= -9 \\ 3x + 5y + 4z &= 10 \\ 5x + 12y + 6z &= 29 \end{aligned}$$

C)
$$\begin{aligned} 6x - 9y + 7z + 10u &= 3 \\ 2x - 3y - 3z - 4u &= -1 \\ 2x - 3y + 13z + 18u &= 1 \end{aligned}$$

A)
$$\left(\begin{array}{cc|c} 1 & -2 & -3 \\ 2 & -1 & 0 \\ 4 & -5 & -6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 3 & 6 \end{array}\right)$$

$$h(A) = h(A|\vec{b}) = 2 \stackrel{?}{=} n \implies \exists!\ \text{řešení}$$

B)
$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & -9 \\ 3 & 5 & 4 & 10 \\ 5 & 12 & 6 & 29 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 2 & -9 \\ 0 & 11 & -2 & 37 \\ \cancel{0} & \cancel{22} & \cancel{-4} & \cancel{74} \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 2 & -9 \\ 0 & 11 & -2 & 37 \end{array}\right) \qquad h(A) = h(A|\vec{b}) = 2 < (n=3)$$

$$\implies \exists\ \infty\ \text{řešení}$$

c) $\left(\begin{array}{cccc|c} 6 & -9 & 7 & 10 & 3 \\ 2 & -3 & -3 & -4 & -1 \\ 2 & -3 & 13 & 18 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 6 & -9 & 7 & 10 & 3 \\ 0 & 0 & 16 & 22 & 6 \\ 0 & 0 & -32 & -44 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 6 & -9 & 7 & 10 & 3 \\ 0 & 0 & 16 & 22 & 6 \\ 0 & 0 & 0 & 0 & 12 \end{array}\right) \longrightarrow \left.\begin{array}{l} h(A) = 2 \\ h(A|\vec{b}) = 3 \end{array}\right\} \begin{array}{l} h(A) \neq \\ h(A|\vec{b}) \end{array}$

$\Longrightarrow$ řešení neexistuje

Př. 3 Zjistěte počet řešení soustavy v závislosti na $\alpha \in \mathbb{R}$.

$$x - y + z = 1$$
$$x + y + 3z = 1$$
$$(2\alpha-1)x + (\alpha+1)y + z = 1-\alpha$$

$\begin{matrix} -1 \\ \\ \\ \end{matrix}\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 2\alpha-1 & \alpha+1 & 1 & 1-\alpha \end{array}\right)\begin{matrix} -(2\alpha-1) \\ \\ + \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 3\alpha & -2\alpha+2 & -3\alpha+2 \end{array}\right)\begin{matrix} \\ (3\alpha) \\ (-2)\ + \end{matrix} \sim$

$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 10\alpha-4 & 6\alpha-4 \end{array}\right)\begin{matrix} \\ \cdot\left(\frac{1}{2}\right) \\ \cdot\left(\frac{1}{2}\right) \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 5\alpha-2 & 3\alpha-2 \end{array}\right)$

$\exists!$ řešení

$\nearrow$

$\Longrightarrow \alpha = \frac{2}{5}$: $h(A) = 2$; $h(A|\vec{b}) = 3 \longrightarrow$ neex. řešení

$\alpha = \frac{2}{3}$: $h(A) = 3$; $h(A|\vec{b}) = 2 \longrightarrow$ neex. řešení ;

$\left.\begin{array}{l} \alpha \neq \frac{2}{3} \\ \alpha \neq \frac{2}{5} \end{array}\right\} \Rightarrow$ $h(A) = h(A|b) = 3$

Př. 4 Gaussovou el. metodou řešte soustavy :

A)
$$\begin{aligned} x+4y-3z &= 0\\ x-3y-z &= 0\\ 2x+y-4z &= 0 \end{aligned}$$

B)
$$\begin{aligned} x_1+3x_2+2x_3 &= 0\\ 2x_1-x_2+3x_3 &= 0\\ 3x_1-5x_2+4x_3 &= 0\\ x_1+17x_2+4x_3 &= 0 \end{aligned}$$

C)
$$\begin{aligned} x+2y+3z &= 4\\ 2x+y-z &= 3\\ 3x+3y+2z &= 10 \end{aligned}$$

D)
$$\begin{aligned} x-2y+z+v &= 1\\ x-2y+z-v &= -1\\ x-2y+z+5v &= 5 \end{aligned}$$

A)
$$\left(\begin{array}{ccc|c} 1 & 4 & -3 & 0\\ 1 & -3 & -1 & 0\\ 2 & 1 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & -3 & 0\\ 0 & -7 & 2 & 0\\ 0 & -7 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & -3 & 0\\ 0 & -7 & 2 & 0 \end{array}\right)$$

$$\vec{x} = \begin{pmatrix} x\\ y\\ z \end{pmatrix} = t\cdot\begin{pmatrix} 13\\ 2\\ 7 \end{pmatrix} = \left\langle \begin{pmatrix} 13\\ 2\\ 7 \end{pmatrix} \right\rangle$$

$t \in \mathbb{R}$

B)
$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & -5 & 4 & 0 \\ 1 & 17 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & -14 & -2 & 0 \\ 0 & 14 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 7 & 1 & 0 \end{array}\right)$$

$$\vec{x} = \left\langle \begin{pmatrix} 11 \\ 1 \\ -7 \end{pmatrix} \right\rangle$$

C)
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 3 & 3 & 2 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -7 & -5 \\ 0 & -3 & -7 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

$\rightarrow h(A) \neq h(A|\vec{b}) \Rightarrow$ řešení neex.

D)
$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t_1 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad t_1 ; t_2 \in \mathbb{R}$$

$$\left.\begin{array}{l} v = 1 \\ z = t_1 \in \mathbb{R} \\ y = t_2 \in \mathbb{R} \end{array}\right\} \begin{array}{l} x = 1 - 1 - 1 \cdot t_1 + 2 \cdot t_2 \\ x = -t_1 + 2t_2 \end{array}$$

Př 5. Řešte soustavu v závislosti na parametru $a \in \mathbb{R}$

$$
\begin{aligned}
ax + y + z &= 4 \\
x + 2y + z &= 3 \\
x + 4y + z &= 4
\end{aligned}
$$

$$
\left(\begin{array}{ccc|c} a & 1 & 1 & 4 \\ 1 & 2 & 1 & 3 \\ 1 & 4 & 1 & 4 \end{array}\right)
\begin{array}{c} (2.) \to \\ (3.) \to \\ (1.) \to \end{array}
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 4 & 1 & 4 \\ a & 1 & 1 & 4 \end{array}\right)
\sim
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 1-2a & 1-a & 4-3a \end{array}\right)
\begin{array}{c} \sim \\ (1-2a) \\ - \\ 2 \end{array}
$$

$$
\sim
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2-2a & 8-6a-(1-2a) \end{array}\right)
\sim
\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2-2a & 7-4a \end{array}\right)
$$

Pro : $\boxed{a = 1}$: $\left.\begin{array}{l} h(A) = 2 \\ h(A|\vec{b}) = 3 \end{array}\right\}$ řešení neex.

není nutné (oprava)

$\left(\boxed{a = \frac{7}{4}} : h(A) = h(A|\vec{b}) = 3 \Longrightarrow \exists!\ \text{řešení} \right)$

Celkem: Pro $a \neq 1$: $h(A) = h(A|\vec{b}) = 3 \longrightarrow \exists!$ řešení

Tedy pro $a \neq 1$ existuje právě 1 řešení ve tvaru :

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 - 1\frac{7-4a}{2-2a} - 2 \cdot \frac{1}{2} \\ \frac{1}{2} \\ \frac{7-4a}{2-2a} \end{pmatrix} = \begin{pmatrix} 2 - \frac{7-4a}{2-2a} \\ \frac{1}{2} \\ \frac{7-4a}{2-2a} \end{pmatrix} \quad a \in \mathbb{R}
$$

Cvičení 6

Př. 1 Pomocí Frobeniovy věty vyšetřete, kolik řešení mají v závislosti na hodnotách vyskytujících se parametrů dané soustavy:

A)
$$\begin{aligned} 2x - y + z + u &= 1 \\ x + 2y - z + 4u &= 2 \\ x + 7y - 4z + 11u &= \lambda \end{aligned}$$

B)
$$\begin{aligned} ax + y + z &= 1 \\ x + ay + z &= 1 \\ x + y + az &= 1 \end{aligned}$$

A)
$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 7 & -4 & 11 & \lambda \end{array}\right) \begin{array}{l} \\ (2) \\ (2) \end{array} \sim \left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 0 & 5 & -3 & 7 & 3 \\ 0 & 15 & -9 & 21 & 2\lambda-1 \end{array}\right) (3) \sim$$

$$\sim \left(\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 0 & 5 & -3 & 7 & 3 \\ 0 & 0 & 0 & 0 & 2\lambda-10 \end{array}\right) \Rightarrow \begin{cases} \lambda = 5 \Rightarrow \infty \text{ řešení } (2 \text{ parametry}, h(A)=2) \\ \lambda \neq 5 \Rightarrow \left.\begin{array}{l} h(A)=2 \\ h(A|\vec{b})=3 \end{array}\right\} \text{neex.} \end{cases}$$

B)
$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 1 & a & 1 & 1 \\ a & 1 & 1 & 1 \end{array}\right) (a) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 1-a & 1-a^2 & 1-a \end{array}\right) + \sim \left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & 2-a-a^2 & 1-a \end{array}\right) \sim$$

②

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & 2-a-a^2 & 1-a \end{array}\right) \longrightarrow \quad a=1: \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow h(A)=1; \; n-h(A)=3-1=2$

$\Rightarrow \infty$ řešení (2 parametry)

$a \neq 1$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & 1-a^2+1-a & 1-a \end{array}\right) \begin{array}{l} \\ \left(\cdot \frac{1}{1-a}\right) \\ \left(\cdot \frac{1}{1-a}\right) \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & \underbrace{1+a+1}_{2+a} & 1 \end{array}\right)$$

$$a=-2: \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left.\begin{array}{l} h(A)=2 \\ h(A|\vec{b})=3 \end{array}\right\} \text{řešení neex.}$$

Pro $a \neq 1 \wedge a \neq -2$ existuje právě 1 řešení.

$$z = \frac{1}{2+a} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2+a} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad a \in \mathbb{R} \setminus \{1; -2\}$$

$$y = \frac{1}{2+a}$$

$$x = 1 - \frac{a}{2+a} - \frac{1}{2+a} = \frac{2+a-a-1}{2+a} = \frac{1}{2+a}$$

Cramerovo pravidlo

Je-li matice soustavy $A_{n \times n}$ regulární, pak lze řešení zapsat jako: $\boxed{x_i = \dfrac{\Delta_i}{\Delta}}$ $i = 1, \ldots, n$

kde $\Delta = \det(A)$; $\Delta_i = \det\left(\vec{a}_1 | \vec{a}_2 | \ldots | \vec{a}_{i-1} | \vec{b} | \vec{a}_{i+1} | \ldots | \vec{a}_n\right)$

kde $\vec{a}_i$ – i-tý sloupec (sloupcový vektor) matice A

$\vec{b}$ – vektor pravé strany

resp. Δ_i : determinant z matice vzniklé nahrazením i-tého sloupce vektorem pravé strany.

Př. 2 Řešte danou soustavu pomocí Cramerova pravidla. (Ověřte jeho použitelnost!)

$$A = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a^3 + 1 + 1 - a - a - a = a^3 - 3a + 2$$

$$\begin{matrix} a & 1 & 1 \\ 1 & a & 1 \end{matrix}$$

!

$\boxed{\Delta \neq 0}$ $\Longrightarrow a \neq 1$ (zkusíme uhádnout kořen...)

$\Longrightarrow (a^3 - 3a + 2) : (a - 1) = a^2 + a - 2$

$\varnothing \quad a^2 - 3a + 2 \quad -2a + ...$

$$\Delta = (a-1)\cdot(a^2+a-2)$$

$$a \neq 1 \qquad a \neq \frac{-1 \pm \sqrt{\overbrace{1-4\cdot 1\cdot(-2)}^{3}}}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$$\Rightarrow a \notin \{1; -2\} \quad ; \quad \boxed{\Delta = (a-1)(a-1)(a+2)}$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a^2+1+1-a-1-a = a^2-2a+1 = (a-1)^2$$

$$\Delta_2 = \begin{vmatrix} a & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a \end{vmatrix} = a^2+1+1-1-a-a = a^2-2a+1 = (a-1)^2$$

$$\Delta_3 = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{vmatrix} = a^2+1+1-a-a-1 = a^2-2a+1 = (a-1)^2$$

$$\left.\begin{aligned} x &= \frac{\Delta_1}{\Delta} = \frac{(a-1)^2}{(a-1)(a-1)(a+2)} = \frac{1}{a+2} \\ y &= \frac{\Delta_2}{\Delta} = \frac{\Delta_1}{\Delta} = \frac{1}{a+2} \\ z &= \frac{\Delta_3}{\Delta} = \frac{\Delta_1}{\Delta} = \frac{1}{a+2} \end{aligned}\right\} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{a+2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$a \in \mathbb{R} \,|\, \{1; -2\}$$

Př 3 Cramerovo pravidlo (+ ověřit použitelnost)

$$x + 2y + az = 0$$
$$-x + 3y + az = -8 \qquad a \in \mathbb{R}$$
$$3x - y + 2z = 13$$

$$A = \begin{pmatrix} 1 & 2 & a \\ -1 & 3 & a \\ 3 & -1 & 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ -8 \\ 13 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 2 & a \\ -1 & 3 & a \\ 3 & -1 & 2 \end{vmatrix} = 6 + a + 6a - 9a + a + 4 = \underline{10 - a}$$

$$\begin{matrix} 1 & 2 & a \\ -1 & 3 & a \end{matrix}$$

$$\boxed{a \neq 10}$$

$$\Delta_1 = \begin{vmatrix} 0 & 2 & a \\ -8 & 3 & a \\ 13 & -1 & 2 \end{vmatrix} = 0 + 8a + 26a - 39a - 0 + 32 = \underline{32 - 5a}$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & a \\ -1 & -8 & a \\ 3 & 13 & 2 \end{vmatrix} = -16 - 13a + 0 + 24a - 13a - 0 = \underline{-16 - 2a}$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & -8 \\ 3 & -1 & 13 \end{vmatrix} = 3 \cdot 13 + 0 - 48 - 0 - 8 + 2 \cdot 13 = 5 \cdot 13 - 56 =$$
$$= \underline{9}$$

$$\boxed{x = \frac{32 - 5a}{10 - a}} \; ; \; \boxed{y = \frac{-16 - 2a}{10 - a}} \; ; \; \boxed{z = \frac{9}{10 - a}}$$

Př 4 Určete vzájemnou polohu 2 rovin ρ_1 a ρ_2 v $\mathbb{E}_3$:

ρ_1: $x + 2y - z = 10$

ρ_2: $3x - y + 2z = 2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ 3 & -1 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ 0 & -7 & 5 & -28 \end{array}\right)$$

$z = t \; ; \; t \in \mathbb{R}$

$$y = \frac{-28 - 5t}{-7} = \frac{28 + 5t}{7} = 4 + \frac{5}{7}t$$

$$x = 10 + t - 2\frac{28 + 5t}{7} = 2 + \frac{7 - 10}{7}t = 2 - \frac{3}{7}t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3/7 \\ 5/7 \\ 1 \end{pmatrix} \quad t \in \mathbb{R} \qquad \text{resp.} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix}$$

Závěr: Roviny ρ_1 a ρ_2 se protínají na přímce zadané parametricky jako:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \quad t \in \mathbb{R}$$

Cvičení 7.

Př. 1. Řešte soustavy lin. alg. rovnic s parametrem:
Pokud řešení existuje, určete ho.

A)
$$\begin{aligned} -7y - 5z &= -1 \\ (2m-1)x - y &= 1 \\ 4mx - 7y - 5z &= 0 \end{aligned}$$
$m \in \mathbb{R}$

B)
$$\begin{aligned} (\lambda+1)x + y &= \lambda \\ (\lambda-1)y + z &= 1-\lambda \\ (\lambda+1)x + \lambda y + z &= 1 \end{aligned}$$
$\lambda \in \mathbb{R}$

A)

$$(*)\left(\begin{array}{ccc|c} 0 & -7 & -5 & -1 \\ 2m-1 & -1 & 0 & 1 \\ 4m & -7 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4m & -7 & -5 & 0 \\ 0 & -7 & -5 & -1 \\ 2m-1 & -1 & 0 & 1 \end{array}\right)\begin{array}{l} \cdot(1-2m) \\ + \\ \cdot(4m);\ m \neq 0 \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 4m & -7 & -5 & 0 \\ 0 & -7 & -5 & -1 \\ 0 & (10m-7) & (10m-5) & 4m \end{array}\right)\begin{array}{l} \\ \cdot(10m-7) \\ \cdot(7)\ + \end{array} \sim \left(\begin{array}{ccc|c} 4m & -7 & -5 & 0 \\ 0 & -7 & -5 & -1 \\ 0 & 0 & 20m & 18m+7 \end{array}\right)$$

Pro $m \neq 0$ máme $h(A) = h(A|\vec{b}) = 3 \Rightarrow \exists!$ řešení (viz dále)

Pro $m = 0$ máme: $(*)$

$$\left(\begin{array}{ccc|c} 0 & -7 & -5 & -1 \\ -1 & -1 & 0 & 1 \\ 0 & -7 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & -1 & 0 & 1 \\ 0 & -7 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left.\begin{array}{l} h(A) = 2 \\ h(A|\vec{b}) = 3 \end{array}\right\} \text{řešení neex.}$$

$m \neq 0$

$$z = \frac{18m+7}{20m} \; ; \quad y = \frac{-1 + 5\cdot\left(\frac{18m+7}{20m}\right)}{-7} = \frac{-1}{7} \cdot \frac{-20m + 90m + 35}{20m} =$$

$$= \frac{-10m-5}{20m} = -\frac{1}{2} - \frac{1}{4m}$$

$$x = \left[0 + 5\cdot\left(\frac{18m+7}{20m}\right) + 7\cdot\left(-\frac{1}{2} - \frac{1}{4m}\right)\right]\cdot \frac{1}{4m} = \frac{1}{4m}\cdot\left(\frac{18m+7-14m-7}{4m}\right)$$

$$= \frac{1}{4m}\cdot\left(\frac{4m}{4m}\right) = \frac{1}{4m}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{4m} \\ -\frac{1}{2} - \frac{1}{4m} \\ \frac{18m+7}{20m} \end{pmatrix} \quad m \in \mathbb{R}$$

B)
$$\left(\begin{array}{ccc|c} \lambda+1 & 1 & 0 & \lambda \\ 0 & \lambda-1 & 1 & 1-\lambda \\ \lambda+1 & \lambda & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} \lambda+1 & 1 & 0 & \lambda \\ 0 & \lambda-1 & 1 & 1-\lambda \\ 0 & \lambda-1 & 1 & 1-\lambda \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} \lambda+1 & 1 & 0 & \lambda \\ 0 & \lambda-1 & 1 & 1-\lambda \end{array}\right)$$

$x = p \; ; \; p \in \mathbb{R}$

$y = \lambda - (\lambda+1)p$

$z = 1-\lambda - (\lambda-1)\left(\lambda - (\lambda+1)p\right) = 1-\lambda^2 - (1-\lambda^2)p$

Celkem: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ 1-\lambda^2 \end{pmatrix} + p\cdot\begin{pmatrix} 1 \\ -(\lambda+1) \\ -(1-\lambda^2) \end{pmatrix}$

$p \in \mathbb{R}$

Př. 2 Řešte soustavu lin. alg. rovnic s parametrem.
Pokud řešení existuje, určete ho.

$$\lambda x + \lambda y + (\lambda+1)z = \lambda$$
$$\lambda x + \lambda y + (\lambda-1)z = \lambda \qquad \lambda \in \mathbb{R}$$
$$(\lambda+1)x + \lambda y + (2\lambda+3)z = 1$$

$$*)\ \left(\begin{array}{ccc|c} \lambda & \lambda & \lambda+1 & \lambda \\ \lambda & \lambda & \lambda-1 & \lambda \\ \lambda+1 & \lambda & 2\lambda+3 & 1 \end{array}\right) \begin{array}{l} \cdot(\lambda+1) \\ - \\ + \quad \cdot(-\lambda) \end{array} \boxed{\lambda \overset{!}{\neq} 0} \sim \left(\begin{array}{ccc|c} \lambda & \lambda & \lambda+1 & \lambda \\ 0 & 0 & -2 & 0 \\ 0 & -\lambda^2+\lambda^2+\lambda & -2\lambda^2-3\lambda+\lambda^2+2\lambda+1 & -\lambda+\lambda^2+\lambda \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} \lambda & \lambda & \lambda+1 & \lambda \\ 0 & \lambda & -\lambda^2-\lambda+1 & \lambda^2 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

$\lambda \neq 0$: $h(A) = h(A|\vec{b}) = 3$ (existuje 1 řešení – závislé na parametru $\lambda \in \mathbb{R} \setminus \{0\}$)

$$z = 0\ ;\ y = \lambda\ ;\ x = \frac{\lambda - \lambda^2}{\lambda} = 1 - \lambda\ ;\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-\lambda \\ \lambda \\ 0 \end{pmatrix} \quad \lambda \in \mathbb{R} \setminus \{0\}$$

$\lambda = 0$: *)
$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad h(A) = h(A|\vec{b}) = 2$$
(1 parametr. řešení)

$$z = 0\ ;\ x = 1\ ;\ y = p \in \mathbb{R} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + p \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad p \in \mathbb{R}$$

Vlastní čísla a vlastní vektory čtvercových matic

Def.: Komplexní číslo $\lambda \in \mathbb{C}$ nazýváme vlastním číslem čtvercové matice A typu $n \times n$, existuje-li nenulový vektor $\vec{x}$ takový, že: $\boxed{A\vec{x} = \lambda\vec{x}}$

Vektor $\vec{x}$ nazýváme vlastním vektorem odpovídajícím vl. číslu λ.

jednotková matice

Výpočet: $\boxed{A\vec{x} = \lambda\vec{x}, \ \vec{x} \neq \vec{o}} \longrightarrow \boxed{A\vec{x} = \lambda \cdot E\vec{x}} \longrightarrow$

(pozn. $\boxed{E\vec{x} \equiv \vec{x}}$) $\longrightarrow \boxed{(A - \lambda E)\vec{x} = \vec{o}, \quad x \neq \vec{o}} \longrightarrow$

$\rightarrow$ existuje-li $\vec{x} \neq o$ řešení homogenní soustavy $\Rightarrow$ matice soustavy musí být singulární!

$\Longleftrightarrow \boxed{\det(A - \lambda E) = 0}$ $\leftarrow$ z této "rovnice" pak vypočteme vl. čísla

Máme-li vl. čísla (ozn. $\lambda_1, \lambda_2, \dots$) pak řešíme:

$(A - \lambda_1 E)\vec{x} = \vec{o}$ $\longrightarrow$ řešením je vl. vektor příslušný λ_1

$(A - \lambda_2 E)\vec{x} = \vec{o}$ $\longrightarrow$ —— // —— λ_2

$\vdots \qquad\qquad \vdots$

Př. 3 Najděte vlastní čísla a odpovídající vlastní vektory zadaných matic: (5)

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(A)

$$|A - \lambda E| = \begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 20 = \lambda^2 - 5\lambda + 6 - 20 =$$

$$= \lambda^2 - 5\lambda - 14 = P(\lambda) \rightarrow \text{charakteristický polynom}$$

Chceme aby $\boxed{P(\lambda) = 0}$

$$\Rightarrow \quad \lambda^2 - 5\lambda - 14 = 0$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-14)}}{2} = \frac{5 \pm 9}{2} = \begin{cases} \lambda_1 = -2 \\ \lambda_2 = 7 \end{cases}$$

$$\lambda_1 : \begin{pmatrix} 3-(-2) & 4 \\ 5 & 2-(-2) \end{pmatrix} \vec{x}_1 = \vec{o}$$

$$\left(\begin{array}{cc|c} 5 & 4 & 0 \\ 5 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & 4 & 0 \end{array}\right) \Rightarrow \vec{x}_1 = c_1 \cdot \begin{pmatrix} 4 \\ -5 \end{pmatrix}; \; c_1 \overset{!}{\in} \mathbb{C} \setminus \{0\}$$

$$\lambda_2 : \begin{pmatrix} 3-7 & 4 \\ 5 & 2-7 \end{pmatrix} \vec{x}_2 = \vec{o}$$

$$\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 5 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -4 & 4 & 0 \end{array}\right) \Rightarrow \vec{x}_2 = c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \; c_2 \overset{!}{\in} \mathbb{C} \setminus \{0\}$$

Celkem: Vlastní čísla matice A jsou $\lambda_1 = -2$ a $\lambda_2 = 7$ s příslušnými vl. vektory $\vec{x}_1 = c_1 \cdot \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ a $\vec{x}_2 = c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
$c_1, c_2 \in \mathbb{C}$.

6

B

$$|B - \lambda E| = \begin{vmatrix} -\lambda & 5 \\ -5 & -\lambda \end{vmatrix} = \lambda^2 + 25 = p(\lambda)$$

$$p(\lambda) = 0 \iff \lambda^2 + 25 = 0$$

$$\lambda_{1,2} = \begin{cases} \boxed{-5i = \lambda_1} \\ \boxed{+5i = \lambda_2} \end{cases}$$

$$\lambda_1: \left(\begin{array}{cc|c} 5i & 5 & 0 \\ -5 & 5i & 0 \end{array}\right) \cdot(-i) \sim \left(\begin{array}{cc|c} 5i & 5 & 0 \\ 5i & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 5i & 5 & 0 \end{array}\right)$$

$$\vec{x}_1 = c_1 \cdot \begin{pmatrix} -5 \\ 5i \end{pmatrix} \quad c_1 \in \mathbb{C} \setminus \{0\}$$

$$\lambda_2: \left(\begin{array}{cc|c} -5i & 5 & 0 \\ -5 & -5i & 0 \end{array}\right) \cdot(i) \sim \left(\begin{array}{cc|c} -5i & 5 & 0 \\ -5i & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -5i & 5 & 0 \end{array}\right)$$

$$\vec{x}_2 = c_2 \cdot \begin{pmatrix} -5 \\ -5i \end{pmatrix} \quad c_2 \in \mathbb{C} \setminus \{0\}$$

Celkem: Vlastní čísla matice B jsou $\lambda_1 = -5i$ a $\lambda_2 = +5i$ s příslušnými vl. vektory $\vec{x}_1 = c_1 \begin{pmatrix} -5 \\ 5i \end{pmatrix}$ a $\vec{x}_2 = c_2 \begin{pmatrix} -5 \\ -5i \end{pmatrix}$; $c_1, c_2 \in \mathbb{C} \setminus \{0\}$

Pozn. ! Pokud jsou vl. čísla komplexní $\Rightarrow$ pak jsou komplexně sdružená!
Totéž platí i pro odpovídající vl. vektory.

c

$$p(\lambda) = \det(C - \lambda E) = |C - \lambda E| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} =$$

$$= \lambda^2 \cdot (1-\lambda) - (1-\lambda)$$ (zde lépe dále neupravovat!)

$$p(\lambda) = 0 \iff \lambda^2 \cdot (1-\lambda) - (1-\lambda) = 0$$

$$(\lambda^2 - 1) \cdot (1-\lambda) = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \\ \lambda_3 &= 1 \end{aligned}} \rightarrow \text{2-násobné vl. č. } \lambda_1 = \lambda_3 = 1$$

$$\lambda_1; \lambda_3: \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} h(A) = h(A|\vec{b}) = 1 \\ n = 3 \end{array}\right\} \text{2 parametry}$$

$$\vec{x}_{1,3} = \begin{pmatrix} p \\ q \\ p \end{pmatrix}; \ p, q \in \mathbb{C}; \ |p| + |q| \neq 0$$ ($\vec{x}_{1,3}$ nesmí být nulový vektor) !

$$\lambda_2: \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \quad h(A) = h(A|\vec{b}) = 2$$

(1 parametr)

$$\vec{x}_2 = \begin{pmatrix} -p \\ 0 \\ p \end{pmatrix}; \ p \in \mathbb{C}; \ |p| \neq 0$$

Cvičení 8.

Př. 1 Najděte vlastní čísla a odpovídající vl. vektory zadaných matic:

$$A = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 0 \\ -13 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$$

Ⓐ

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 2 & 1 \\ -2 & -\lambda & 3 \\ -1 & -3 & -\lambda \end{vmatrix} = -\lambda^3 + 6 - 6 - \lambda - 9\lambda - 4\lambda =$$

$$= -\lambda^3 - 14\lambda = -\lambda \cdot (\lambda^2 + 14) = P(\lambda)$$

$$P(\lambda) = 0 \iff \boxed{\lambda_1 = 0}\ ;\ \boxed{\lambda_2 = +i\sqrt{14}}\ ;\ \boxed{\lambda_3 = -i\sqrt{14}}$$

$$\lambda_1: \left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ -2 & 0 & 3 & 0 \\ -1 & -3 & 0 & 0 \end{array}\right) \overset{(3)}{\underset{-}{}} \sim \left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ -2 & -6 & 0 & 0 \\ \cancel{-1} & \cancel{-3} & \cancel{0} & \cancel{0} \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ -2 & -6 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \longrightarrow \left.\begin{array}{l} z = -2p\ ;\ p \in \mathbb{C} \\ y = p\ ;\ x = -3p \end{array}\right\} \overset{!}{|p| \neq 0}$$

$$\Rightarrow \vec{x}_1 = c_1 \cdot \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \quad c_1 \in \mathbb{C}\ ;\ |c_1| \neq 0$$

nebo: $\vec{x}_1 = \begin{pmatrix} -3p \\ p \\ -2p \end{pmatrix}\ ;\ p \in \mathbb{C}\ ;\ |p| \neq 0$

$$\lambda_2: \quad \left(\begin{array}{ccc|c} -i\sqrt{14} & 2 & 1 & 0 \\ -2 & -i\sqrt{14} & 3 & 0 \\ -1 & -3 & -i\sqrt{14} & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & i\sqrt{14} & 0 \\ 2 & i\sqrt{14} & -3 & 0 \\ -i\sqrt{14} & 2 & 1 & 0 \end{array}\right) \begin{array}{l} \\ (2)\,- \\ (i\sqrt{14})\,+ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & i\sqrt{14} & 0 \\ 0 & i\sqrt{14}-6 & -2i\sqrt{14}-3 & 0 \\ 0 & 3i\sqrt{14}+2 & 14i^2+1 & 0 \end{array}\right) \begin{array}{c} \\ (3i\sqrt{14}+2) \\ \downarrow + \\ (6-i\sqrt{14}) \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & i\sqrt{14} & 0 \\ 0 & i\sqrt{14}-6 & -2i\sqrt{14}-3 & 0 \\ 0 & 0 & \underbrace{(-2i\sqrt{14}-3)\cdot(3i\sqrt{14}+2) - 13\cdot(6-i\sqrt{14})} & 0 \end{array}\right) \sim$$

$$-6i^2\cdot 14 - 4i\sqrt{14} - 9i\sqrt{14} - 6$$

$$-84 + 13i\sqrt{14} = 0i + 0 = 0$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & i\sqrt{14} & 0 \\ 0 & i\sqrt{14}-6 & -2i\sqrt{14}-3 & 0 \end{array}\right)$$

$$\vec{x}_2 = c_2 \cdot \begin{pmatrix} -\left[3(3+2i\sqrt{14}) + i\sqrt{14}(-6+i\sqrt{14})\right]^{*} \\ 3+2i\sqrt{14} \\ -6+i\sqrt{14} \end{pmatrix} = c_2 \cdot \begin{pmatrix} 5 \\ 3+2i\sqrt{14} \\ -6+i\sqrt{14} \end{pmatrix}$$

$$* \quad -\left[3(3+2i\sqrt{14}) + i\sqrt{14}(-6+i\sqrt{14})\right] =$$

$$= -(9 + 6i\sqrt{14} - 6i\sqrt{14} + i^2\cdot 14) = -(9-14) = 5 \quad \left\{ \begin{array}{l} c_2 \in \mathbb{C} \\ |c_2| \neq 0 \end{array} \right.$$

λ_3: Zde využijeme vlastností vlastních čísel a vlastních vektorů:

→ Je-li λ vlastním číslem matice A a $\vec{x}$ příslušným vlastním vektorem, pak $\overline{\lambda}$ je také vl. číslem matice A a $\overline{\vec{x}}$ je jeho příslušným vl. vektorem.

Pozn.: $\overline{\lambda}$ ... komplexně sdružené číslo

$\overline{\vec{x}}$ ... komplexně sdružený vektor

→ $$\lambda_3 = \overline{\lambda_2} = -i\sqrt{14}$$

$$\vec{x}_3 = \overline{\vec{x}_2} = c_3 \cdot \begin{pmatrix} 5 \\ 3-2i\sqrt{14} \\ -6-i\sqrt{14} \end{pmatrix} \quad c_3 \in \mathbb{C}, \ |c_3| \neq 0$$

B

$$\det(B-\lambda E) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ -13 & -1-\lambda & 0 \\ 4 & -8 & -2-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)(-2-\lambda) - (-2-\lambda)\cdot 1\cdot(-13) =$$

$$= (2+\lambda)\left[(3-\lambda)(1+\lambda) - 13\right] = P(\lambda)$$

$$P(\lambda)=0 \Leftrightarrow \begin{cases} \lambda = -2 = \lambda_1 \\ -\lambda^2+2\lambda-10=0 \Leftrightarrow \lambda_{2,3} = \dfrac{-2 \pm \sqrt{2^2-4\cdot(-1)\cdot(-10)}}{-2} \end{cases}$$

$\lambda_1 = -2 \;;\; \lambda_2 = 1+3i \;;\; \lambda_3 = 1-3i$

$$\lambda_1 : \left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ -13 & -1 & 0 & 0 \\ 4 & -8 & 0 & 0 \end{array}\right)^{(8)} \sim \left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ -18 & 0 & 0 & 0 \\ \cancel{44} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \cdot \left(\frac{1}{-18}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \;;\; \vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} \quad p \in \mathbb{C} \;;\; |p| \neq 0$$

resp. $\vec{x}_1 = c_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;;\; c_1 \in \mathbb{C}$, $|c_1| \neq 0$

$$\lambda_2 : \left(\begin{array}{ccc|c} 3-(1+3i) & 1 & 0 & 0 \\ -13 & -1-(1+3i) & 0 & 0 \\ 4 & -8 & -2-(1+3i) & 0 \end{array}\right) \begin{array}{c} (1+(1+3i)) \\ + \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} (2-3i)(2+3i)-13 & 0 & 0 & 0 \\ -13 & -2-3i & 0 & 0 \\ 4 & -8 & -3-3i & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ -13 & -2-3i & 0 & 0 \\ 4 & -8 & -3-3i & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} -13 & -2-3i & 0 & 0 \\ 4 & -8 & -3-3i & 0 \end{array}\right)$$

$$\vec{x}_2 = c_2 \cdot \begin{pmatrix} 2+3i \\ -13 \\ \dfrac{4(2+3i)-8\cdot(-13)}{3+3i} \end{pmatrix} = c_2 \begin{pmatrix} 6+9i+6i+9i^2 \\ -39-39i \\ 8+12i+8\cdot 13 \end{pmatrix} =$$

$$= C_2 \cdot \begin{pmatrix} -3+15i \\ -39-39i \\ 112+12i \end{pmatrix} \qquad C_2 \in \mathbb{C}; |C_2| \neq 0$$

$$\lambda_3: \quad \vec{x}_3 = \overline{\vec{x}_2} = C_3 \cdot \begin{pmatrix} -3-15i \\ -39+39i \\ 112-12i \end{pmatrix} \qquad C_3 \in \mathbb{C}; |C_3| \neq 0$$

Př. 2 Najděte vl. čísla matice A:

$$A = \begin{pmatrix} 2 & 5 & -6 \\ 4 & 6 & -9 \\ 3 & 6 & -8 \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 5 & -6 \\ 4 & 6-\lambda & -9 \\ 3 & 6 & -8-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda)(-8-\lambda) + 4 \cdot 6 \cdot (-6) + 5 \cdot (-9) \cdot 3 - (-6)(6-\lambda) \cdot 3 - (-9) \cdot 6 \cdot (2-\lambda) - (-8-\lambda) \cdot 5 \cdot 4$$

$$= \underbrace{-(\lambda^2 - 8\lambda + 12)(8+\lambda)} \underbrace{- 279 + 108 - 18\lambda + 108 - 54\lambda + 160 + 20\lambda} =$$

$$= \underbrace{-\lambda^3 + 8\lambda^2 - 12\lambda - 8\lambda^2 + 64\lambda - 96} \underbrace{-52\lambda + 97} =$$

$$= -\lambda^3 + 1 = P(\lambda)$$

$$p(\lambda)=0 \iff (1-\lambda^3)=(1-\lambda)(1+\lambda+\lambda^2)=0$$ 6

$$\iff \begin{cases} \lambda=1=\lambda_1 \\ \lambda_{2,3}=\dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2} = \begin{cases} -\frac{1}{2}+i\sqrt{3}=\lambda_2 \\ -\frac{1}{2}-i\sqrt{3}=\lambda_3 \end{cases} \end{cases}$$

Pozn.: Pro λ_1 máme:

$$\left(\begin{array}{ccc|c} 1 & 5 & -6 & 0 \\ 4 & 5 & -9 & 0 \\ 3 & 6 & -9 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 5 & -6 & 0 \\ 0 & -15 & 15 & 0 \\ 0 & -9 & 9 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & -6 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \Rightarrow \vec{x}_1 = c_1\begin{pmatrix}1\\1\\1\end{pmatrix} \quad c_1\in\mathbb{C}, \ |c_1|\neq 0$$

Cvičení 9. (1)

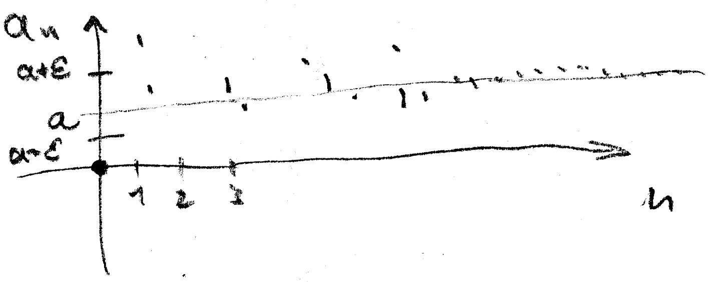
Posloupnosti reálných čísel

- *Limita posloupnosti*:

Číslo $a \in \mathbb{R}^*$ nazýváme limitou posloupnosti $\{a_n\}$ jestliže:

$\forall \varepsilon \in \mathbb{R}; \varepsilon > 0 \ \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N}; n \geq n_0$ platí:

$a_n \in (a-\varepsilon, a+\varepsilon)$



- Posloupnosti:
  - → omezená shora / zdola / omezená
  - → rostoucí / klesající
  - → neklesající / nerostoucí
  - → monotónní
  - → ryze monotónní

→ konvergentní (vlastní limita)

→ divergentní (nevlastní limita)

Věta o limitě součtu, rozdílu, součinu a podílu posloupností:

- Nechť $\lim_{n\to\infty} a_n = a$ a $\lim_{n\to\infty} b_n = b$ kde $\{a_n\}_{n=1}^{\infty}$ a $\{b_n\}_{n=1}^{\infty}$

jsou posloupnosti reálných čísel.

Pak platí: $\lim_{n\to\infty} (a_n * b_n) = a * b$ pokud výraz

$a * b$ má smysl, za $*$ můžeme dosadit $(\cdot), (+), (-), (/)$.

Pozn.: Pro $\frac{a_n}{b_n}$ musí mít výraz smysl $\forall n \in \mathbb{N}$!

Př. 1. O následujících posloupnostech:

a) $\{2+3^n\}$ b) $\{\frac{n}{n+1}\}$ c) $\{\frac{(-1)^n}{n^2+1}\}$ d) $\{\frac{1+(-1)^n}{2}\}$

e) $\{-\frac{n^2}{n+1}\}$ f) $\{\frac{n+5}{n+2}\}$

Rozhodněte zda jsou:

	rostoucí	kles.	nerost.	nekles.	mon.	ryze mon.	omez. zdola	omez. shora	omez.
a)	+	−	−	+	+	+	+	−	−
b)	+	−	−	+	+	+	+	+	+
c)	−	−	−	−	−	−	+	+	+
d)	−	−	−	−	−	−	+	+	+
e)	−	+	+	−	+	+	−	+	−
f)	−	+	+	−	+	+	+	+	+

Př 2. Je dána posloupnost $\{a_n\}$ a kladné číslo ε. Najděte limitu L posloupnosti $\{a_n\}$ a dále přirozené číslo $n_0 \in \mathbb{N}$ s tou vlastností, že $\forall n \in \mathbb{N}: n \geq n_0$ je $a_n \in U_\varepsilon(L)$:

a) $a_n = \frac{1}{n^2}$; $\varepsilon = 0.05$

b) $a_n = 1 + 2^{-n}$; $\varepsilon = 0.1$

c) $a_n = -1 + \frac{\sin(n)}{n}$; $\varepsilon = 0.2$

d) $a_n = \frac{3}{n}$; $\varepsilon = 0.05$

---

a) $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}\right) = 0 = L$; $\quad \frac{1}{n_0^2} \in (L-\varepsilon, L+\varepsilon) = (-0.05; 0.05)$

resp. $|a_{n_0} - L| < \varepsilon \implies \left|\frac{1}{n_0^2}\right| < 0.05$

$$\frac{1}{n_0^2} < \frac{5}{100} = \frac{1}{20} \implies \boxed{n_0 = 5}$$

b) $\lim\limits_{n\to\infty}\left(1 + 2^{-n}\right) = \lim\limits_{n\to\infty}(1) + \lim\limits_{n\to\infty}(2^{-n}) = 1 + 0 = 1$

$$\left|\left(1 + \frac{1}{2^{n_0}}\right) - 1\right| < 0.1 \longrightarrow \frac{1}{2^{n_0}} < 0.1 = \frac{1}{10}$$

$\boxed{n_0 = 4}$ resp. $10 < 2^{n_0} \longrightarrow \log_2(10) < n_0$

$n_0 \in \mathbb{N}$

c)

$$\lim_{n\to\infty}\left(-1+\frac{\sin(n)}{n}\right) = \lim_{n\to\infty}(-1) + \lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right) =$$

$$= \left| \begin{array}{l} \text{*} \\ \underline{\text{Limita sevřené posloupnosti}} \\ \frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \\ \downarrow \qquad\qquad\qquad \downarrow \\ \lim_{n\to\infty}\left(\frac{-1}{n}\right)=0 \qquad \lim_{n\to\infty}\left(\frac{1}{n}\right)=0 \end{array} \right\} \lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)=0 \Bigg| =$$

$$= -1+0 = \underline{-1} \; ; \quad \left|\left(-1+\frac{\sin(n_0)}{n_0}\right)-(-1)\right| < 0.2$$

$$\frac{|\sin(n_0)|}{n_0} < 0.2 = \frac{1}{5} \; ; \quad \frac{|\sin(n_0)|}{n_0} < \frac{1}{5}$$

$\boxed{n_0 = 5}$ $\quad |\sin(n_0)| < 1 \;\; \forall n_0 \in \mathbb{N}$ $\left(k\frac{\pi}{2} \text{ a } k\frac{3}{2}\pi \text{ jsou iracionální čísla} \; ; \; k \in \mathbb{Z} \right)$

d)

$$\lim_{n\to\infty}\left(\frac{3}{n}\right) = 0 \; ; \quad \left|\frac{3}{n_0} - 0\right| < 0.05 = \frac{1}{20}$$

$$\rightarrow \frac{3}{60+1} < \frac{3}{60} = \frac{1}{20} \rightarrow \boxed{n_0 = 61}$$

Př. 3 Vypočtěte následující limity

1) $\lim\limits_{n\to+\infty}\left(1+\frac{3}{n}\right)^n = e^3$ → dle definice Eulerova čísla

2) $\lim\limits_{n\to+\infty}\frac{2n^2-3n+5}{3n^2-2n+1} = \lim\limits_{n\to+\infty}\frac{2-\frac{3}{n}+\frac{5}{n^2}}{3-\frac{2}{n}+\frac{1}{n^2}} = \frac{2}{3}$

$\frac{3}{n}\to 0$, $\frac{5}{n^2}\to 0$, $\frac{2}{n}\to 0$, $\frac{1}{n^2}\to 0$

3) $\lim\limits_{n\to+\infty}\frac{(2n-3)(1-2n)}{5n^2-1} = \lim\limits_{n\to+\infty}\frac{2n-4n^2-3+6n}{5n^2-1} = \lim\limits_{n\to+\infty}\frac{-4+\frac{8}{n}-\frac{3}{n^2}}{5-\frac{1}{n^2}} =$

$= \frac{-4}{5}$

4) $\lim\limits_{n\to+\infty}\frac{n^2-n+3}{n^3+2n+2} = \lim\limits_{n\to+\infty}\frac{\frac{1}{n}-\frac{1}{n^2}+\frac{3}{n^3}}{1+\frac{2}{n^2}+\frac{2}{n^3}} = \frac{0}{1} = 0$

5) $\lim\limits_{n\to+\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+3}} = \lim\limits_{n\to\infty}\frac{\frac{1}{\sqrt{n}}\left(\sqrt{n+1}+\sqrt{n}\right)}{\frac{1}{\sqrt{n}}\left(\sqrt{n+3}\right)} = \lim\limits_{n\to+\infty}\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{3}{n}}} =$

$= \frac{\sqrt{1}+1}{\sqrt{1}} = 2$

6) $\lim\limits_{n\to+\infty}\left(\frac{4}{n}-\frac{3n}{n^2+1}\right) = \lim\limits_{n\to+\infty}\frac{4(n^2+1)-3n^2}{n\cdot(n^2+1)} = \lim\limits_{n\to+\infty}\frac{n^2+1}{n\cdot(n^2+1)}$

$= \lim\limits_{n\to+\infty}\frac{1}{n} = 0$

7) $$\lim_{n\to+\infty}\left(\sqrt{n^2-3n+2}-n\right) = \lim_{n\to+\infty}\frac{(n^2-3n+2)-n^2}{\sqrt{n^2-3n+2}+n} =$$

$$= \lim_{n\to+\infty}\frac{\overbrace{-\frac{3n}{n}}^{\to -3}+\overbrace{\frac{2}{n}}^{\to 0}}{\sqrt{\underbrace{\frac{n^2}{n^2}}_{1}-\underbrace{\frac{3n}{n^2}}_{0}+\underbrace{\frac{2}{n^2}}_{0}}+1} = \frac{-3}{2}$$

8) $$\lim_{n\to+\infty}\left(\sqrt{n+2}-\sqrt{n+5}\right) = \lim_{n\to+\infty}\frac{\left(\sqrt{n+2}+\sqrt{n+5}\right)}{\left(\sqrt{n+2}+\sqrt{n+5}\right)}\cdot\left(\sqrt{n+2}-\sqrt{n+5}\right) =$$

$$= \lim_{n\to+\infty}\frac{(n+2)-(n+5)}{\sqrt{n+2}+\sqrt{n+5}} = \lim_{n\to+\infty}\frac{-3}{\sqrt{n+2}+\sqrt{n+5}} = 0$$

9) $$\lim_{n\to+\infty}\frac{(2n+1)!+(2n+2)!}{(2n+3)!} = \lim_{n\to+\infty}\frac{\cancel{(2n+1)!}\left[1+(2n+2)\right]}{\cancel{(2n+1)!}\left[(2n+3)(2n+2)\right]} =$$

$$= \lim_{n\to+\infty}\frac{2n+3}{(2n+3)(2n+2)} = 0$$

10) $$\lim_{n\to+\infty}\frac{1+2+3+\dots+n}{\sqrt{9n^4+1}} = \lim_{n\to+\infty}\frac{(n+1)\cdot\frac{n}{2}}{\sqrt{9n^4+1}} = \lim_{n\to+\infty}\frac{\frac{n^2}{2}+\frac{n}{2}}{\sqrt{9n^4+1}} =$$

$$= \lim_{n\to+\infty}\frac{\frac{1}{2}+\overbrace{\frac{1}{2n}}^{\to 0}}{\sqrt{9+\underbrace{\frac{1}{n^2}}_{0}}} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

11) $\lim\limits_{n\to+\infty} \dfrac{1+3+5+\dots+(2n-1)}{1+2+3+\dots+n} = \lim\limits_{n\to+\infty} \dfrac{[1+(2n-1)]\cdot\frac{n}{2}}{[1+n]\cdot\frac{n}{2}} =$

$= \lim\limits_{n\to+\infty} \dfrac{2n}{n+1} = \lim\limits_{n\to+\infty} \dfrac{2}{1+\underbrace{\frac{1}{n}}_{\to 0}} = 2$

12) $\lim\limits_{n\to+\infty} \underbrace{\dfrac{1+\sin(n)}{n+1}}_{= b_n} = \left| \begin{array}{l} 0 \leq \dfrac{1+\sin(n)}{n+1} \leq \dfrac{2}{n+1} \\ \forall n \in \mathbb{N} \\ a_n = \{0\};\ c_n = \left\{\dfrac{2}{n+1}\right\} \end{array} \right|$

$\Rightarrow$ Pokud $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \ a_n \leq b_n \leq c_n$

a $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} c_n = L \Rightarrow \lim\limits_{n\to+\infty} b_n = L$

$\lim\limits_{n\to+\infty} 0 = 0$; $\lim\limits_{n\to+\infty} \dfrac{2}{n+1} = 0 \Rightarrow \lim\limits_{n\to+\infty} \dfrac{1+\sin(n)}{n+1} = \underline{\underline{0}}$

13) $\lim\limits_{n\to+\infty} \dfrac{\overbrace{\operatorname{arctg}(n^2)}^{\to\frac{\pi}{2}}}{n+1} = 0$

$|\cos(n!)| \leq 1$

⊕ limita sevřené posl.

14) $\lim\limits_{n\to+\infty} \dfrac{n+\cos(n!)}{2n+1} = \lim\limits_{n\to+\infty} \dfrac{n}{2n+1} + \lim\limits_{n\to+\infty} \dfrac{\cos(n!)}{2n+1} =$

$= \dfrac{1}{2} + 0 = \underline{\underline{\dfrac{1}{2}}}$

Eulerovo číslo:

Chceme aby $\boxed{\frac{d}{dx}(e^x) \overset{*}{=} e^x}$ → využijeme definice derivace funkce jedné proměnné

$$\rightarrow \frac{d}{dx}(e^x) \overset{\text{def.}}{=} \lim_{\varepsilon \to 0} \frac{e^{x+\varepsilon} - e^x}{\varepsilon} = \lim_{\varepsilon \to 0} e^x \cdot \frac{e^{\varepsilon} - 1}{\varepsilon}$$

$$\boxed{= \underbrace{e^x \cdot \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - 1}{\varepsilon}}_{\frac{d}{dx}(e^x)} \overset{*}{=} e^x} \rightarrow$$

$$\rightarrow \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - 1}{\varepsilon} = 1$$

$$\rightarrow \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - 1}{\varepsilon} = \left| \begin{array}{l} \text{sub.} \\ \varepsilon \to \frac{1}{n} \\ \varepsilon \to 0 \\ n \to +\infty \end{array} \right| = \lim_{n \to +\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1$$

→ Pokud zvolíme v limitě: $e = \left(1 + \frac{1}{n}\right)^n$:

$$\lim_{n \to +\infty} \frac{\left[\left(1 + \frac{1}{n}\right)^n\right]^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_{n \to +\infty} 1 = 1$$

Tedy $\boxed{e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n}$

Cvičení 10.

Př. 1 $\lim\limits_{n\to+\infty}\left(\frac{n+3}{n+5}\right)^{n-1}$

$$\lim_{n\to+\infty}\left(\frac{n+3}{n+5}\right)^{n-1} = \lim_{n\to+\infty}\left(\frac{n+3+2-2}{n+5}\right)^{n-1} =$$

$$\lim_{n\to+\infty}\left(1-\frac{2}{n+5}\right)^{n-1} = \left|\begin{array}{c}\text{substituce}\\ n \to m-5\end{array}\right| = \lim_{m\to+\infty}\left(1-\frac{2}{m}\right)^{m-6} =$$

$$= \lim_{m\to+\infty}\underbrace{\left(1-\frac{2}{m}\right)^{m}}_{e^{-2}}\cdot\underbrace{\left(1-\frac{2}{m}\right)^{-6}}_{1^{-6}} = \underline{\underline{e^{-2}}}$$

Př. 2 $\lim\limits_{n\to+\infty}\frac{2^n+(-2)^n}{2\cdot 4^n}$

$$\lim_{n\to+\infty}\frac{2^n+(-2)^n}{2\cdot \underbrace{4^n}_{2^{2n}=(2^n)^2}} = \lim_{n\to+\infty}\frac{1+(-1)^n}{2\cdot 2^n} = \underline{\underline{0}}$$

$0 \leq 1+(-1)^n \leq 2$

Funkce

Př. 3 Stanovte definiční obory následujících funkcí:

a) $y = \arcsin\left(\frac{x}{4}\right)$

b) $y = \sqrt{1-|x|}$

c) $y = \arccos\left(\frac{1-2x}{4}\right)$

d) $y = \sqrt{\frac{x-2}{x+2}} + \sqrt[3]{\frac{x-2}{x+2}}$

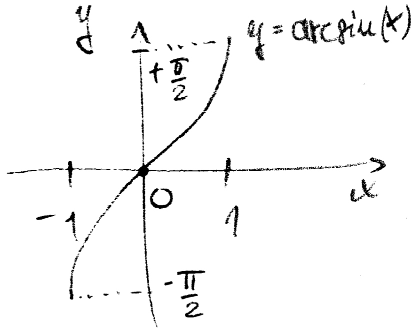
e) $y = \ln(x+3) + \sqrt{5-2x}$

f) $y = \sin(\arcsin(x))$

$y = \arcsin(\sin(x))$

---

a)

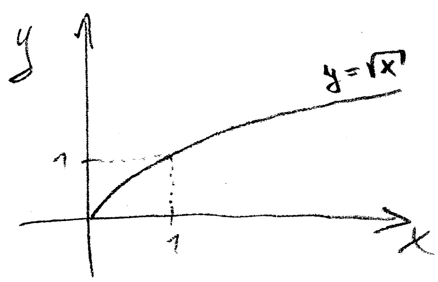


$D(\arcsin) = \langle -1; 1 \rangle$

$\Rightarrow \frac{x}{4} \in \langle -1; 1 \rangle$

$\Rightarrow \boxed{x \in \langle -4; 4 \rangle}$

b) $y = \sqrt{1-|x|}$



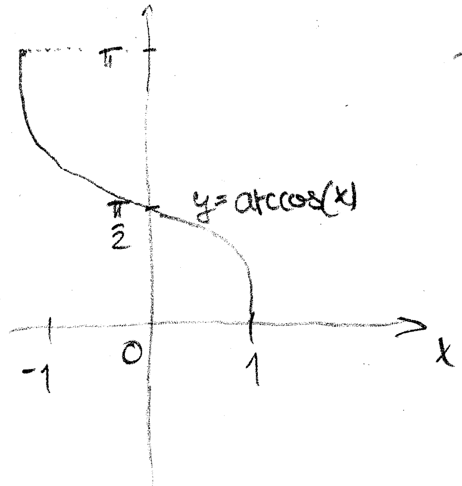
$1 - |x| \geq 0$

$-|x| \geq -1$

$|x| \leq 1$

$\Rightarrow \boxed{x \in \langle -1; 1 \rangle}$

c)



$D(\arccos) = \langle -1; 1 \rangle$

$\Rightarrow \frac{1-2x}{4} \in \langle -1; 1 \rangle$

$\Rightarrow \frac{1-2x}{4} = 1 \longrightarrow x = -\frac{3}{2}$

$\frac{1-2x}{4} = -1 \longrightarrow x = \frac{5}{2}$

Celkem: $x \in \langle -\frac{3}{2}; \frac{5}{2} \rangle$

---

d) $y = \sqrt{\frac{x-2}{x+2}} + \sqrt[3]{\frac{x-2}{x+2}}$

$\longrightarrow$ Pro $f(x) = \sqrt[3]{x}$ máme $D_f = (-\infty; +\infty)$

$\longrightarrow$ Pro $f(x) = \sqrt[2]{x}$ máme $D_f = \langle 0; +\infty)$

$\longrightarrow$ Rozhodující bude tedy výraz: $\sqrt{\frac{x-2}{x+2}}$

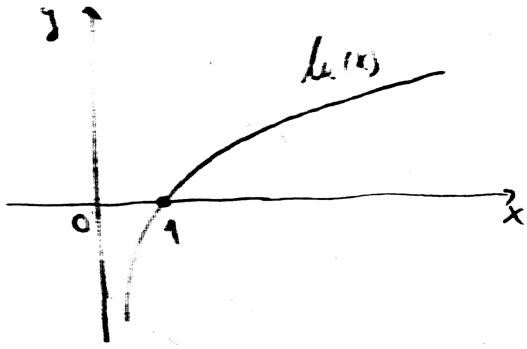
$\longrightarrow \frac{x-2}{x+2} \geq 0$

$x \in$	$(-\infty; -2)$	$(-2; 2\rangle$	$(2; +\infty)$
$\frac{x-2}{x+2}$	> 0	≤ 0 ($= 0$; $x = 2$)	> 0

$\frac{x-2}{x+2} \geq 0 \quad \forall x \in (-\infty; -2) \cup \langle +2; +\infty)$

$D_y = (-\infty; -2) \cup \langle +2; +\infty)$

e) $y = \ln(x+3) + \sqrt[2]{5-2x}$



$D(\ln) = (0; +\infty)$

$$\Rightarrow \begin{cases} 1)\ \ x+3 > 0 \\ 2)\ \ 5-2x \geq 0 \end{cases}$$

$$\left.\begin{array}{l} 1)\ x \in (-3; +\infty) = I_1 \\ 2)\ x \in (-\infty; \frac{5}{2}\rangle = I_2 \end{array}\right\} \Rightarrow I_1 \cap I_2 = (-3; \tfrac{5}{2})$$

$D_y = (-3; \frac{5}{2}\rangle$

f)
$$\left.\begin{array}{l} D(\sin) = \mathbb{R} \\ D(\arcsin) = \langle -1; 1\rangle \\ f_1 = \sin(\arcsin(x)) \\ f_2 = \arcsin(\sin(x)) \end{array}\right\} \begin{array}{l} D_{f_1} = \langle -1; 1\rangle \\ D_{f_2} = \left| H(\sin) = \langle -1; 1\rangle \right| = \mathbb{R} \end{array}$$

Př. 4 Jsou dány fce f_1 a f_2. Sestavte $g = f_1 * f_2$ a $h = f_2 * f_1$.

Pozn.: $\big(g(x) = f_1(f_2(x)) \; ; \; h(x) = f_2(f_1(x)) \big)$

a) $f_1 = x^2 + 5x + 2$

$f_2 = 3x - 1$

b) $f_1 = \ln(x)$

$f_2 = e^{2x}$

c) $f_1 = x^3 + 1$

$f_2 = \cos(x)$

a) $g(x) = f_1(f_2(x)) = (3x-1)^2 + 5(3x-1) + 2 = 9x^2 + 9x - 2$

$h(x) = f_2(f_1(x)) = 3(x^2 + 5x + 2) - 1 = 3x^2 + 15x + 5$

b) $g(x) = \ln(e^{2x}) = 2x \cdot \ln(e) = 2x$

$h(x) = e^{2(\ln(x))} = [e^{\ln(x)}]^2 = x^2$

c) $g(x) = \cos^3(x) + 1$

$h(x) = \cos(x^3 + 1)$

Př 5 Které z následujících funkcí jsou sudé, které liché?

a) $y = \frac{\sin(x)}{x}$ b) $y = x^3 + x \cdot \cos(x)$ c) $y = tg(4x)$

d) $y = x + x^2$ e) $y = \frac{x^2 - 1}{x + x^3}$ f) $y = \sqrt[3]{x}$

Pozn.: S – sudá, l – lichá $\Rightarrow$

$$\boxed{\begin{aligned} S * S &= S \\ S * l &= l \\ l * l &= S \end{aligned}}$$

kde $* = (\cdot\,;\,\div)$

$$\boxed{\begin{aligned} S + S &= S \\ l + l &= l \\ S + l &= \text{ani } S \text{ ani } l \end{aligned}}$$

Př.: x^1 ... lichá, x^2 ... sudá, x^3 ... lichá $\quad \frac{x^3}{x^1} \ldots \quad x^3 \cdot x^2 = \ldots$ apod.

a) sudá b) lichá c) lichá

d) — e) lichá f) lichá

Cvičení 11.

Př. 1 Které z následujících funkcí jsou periodické a jakou mají periodu?

a) $y = \sin(x) + \cos(2x)$ b) $y = \operatorname{arctg}(\operatorname{tg}(x))$

c) $y = \cos^2\left(\frac{x}{2}\right)$ d) $y = \operatorname{sgn}(\sin(x))$

e) $y = |\sin(x-1)|$

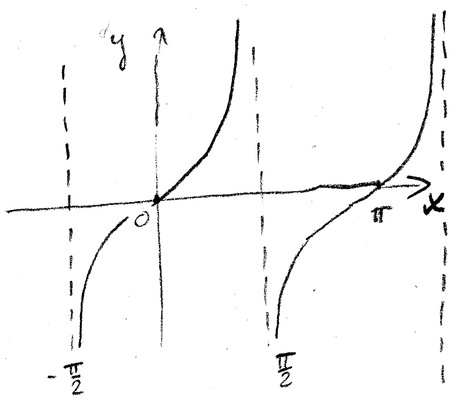
---

a) $\left.\begin{array}{l} \sin(x): \quad T = 2\pi \\ \cos(2x): \quad T = \pi \end{array}\right\}$ $\boxed{T\big(\sin(x)+\cos(2x)\big) = \pi}$

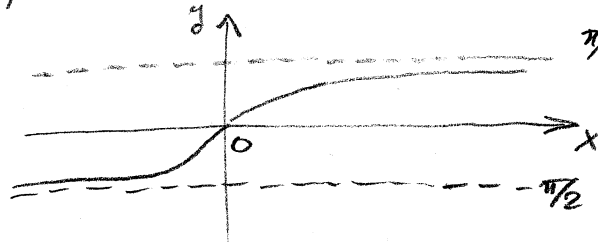
; graf: viz např. www.wolframalpha.com

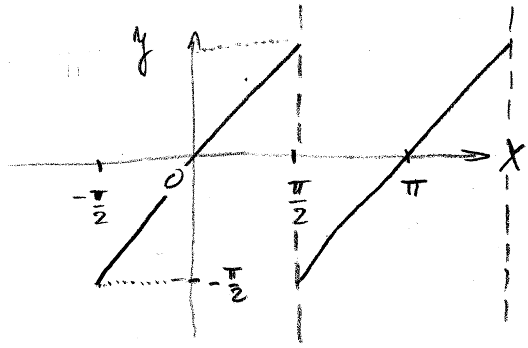
---

b) $\operatorname{tg}(x)$: $T = \pi$



$\operatorname{arctg}(x)$:

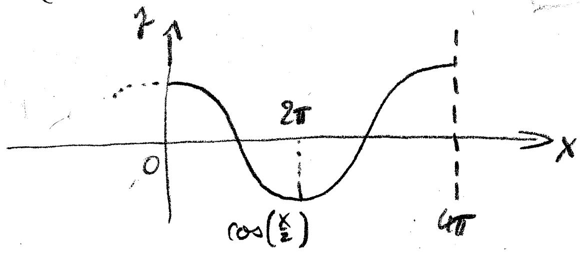




$\operatorname{arctg}(\operatorname{tg}(x))$

$\boxed{T\big(\operatorname{arctg}(\operatorname{tg})\big) = \pi}$

c) $\cos\left(\frac{x}{2}\right)$: $T = 4\pi$



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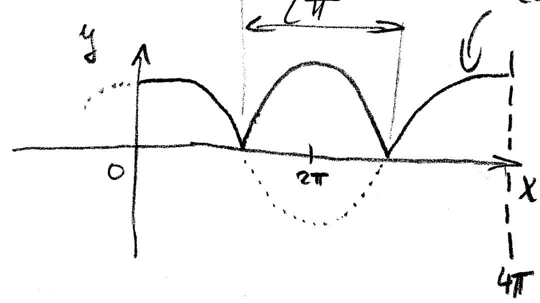
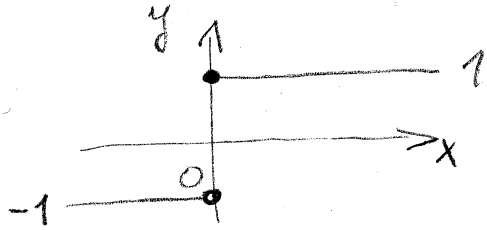


$\Rightarrow \quad T\left(\cos^2\left(\frac{x}{2}\right)\right) = 2\pi$

d) sgn(sin(x))



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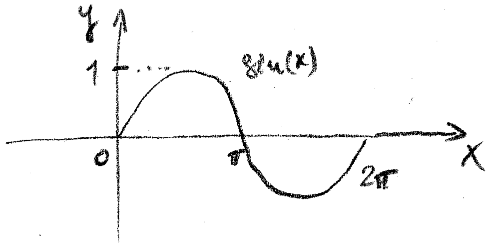
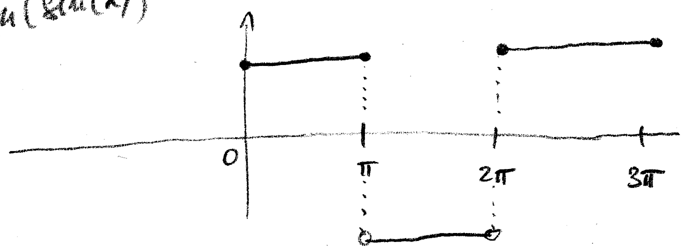


sgn(sin(x))

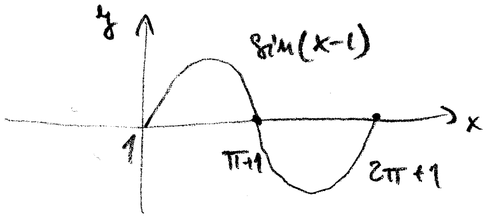


$\Rightarrow \boxed{T(\operatorname{sgn}(\sin(x))) = 2\pi}$

---

e) $y = |\sin(x-1)|$

$\sin(x-1)$:



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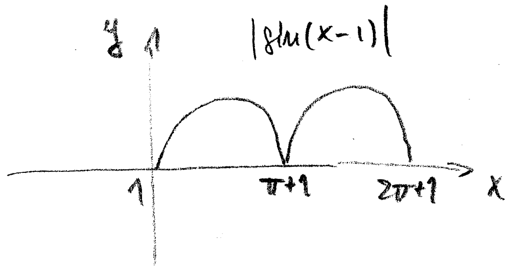


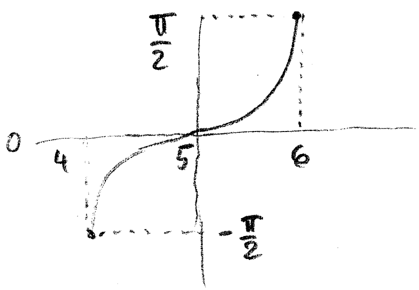
$\Rightarrow \boxed{T(|\sin(x-1)|) = \pi}$

Př2. Zjistěte zda jsou následující funkce omezené (zdola / shora / omezené) a najděte jejich supremum / infimum a pokud existují tak i maximum / minimum.

a) $y = \arcsin(x-5)$ b) $y = 2\operatorname{arctg}(x) + \pi$

c) $y = e^{|x|}$ d) $y = \sqrt[3]{x-1}$

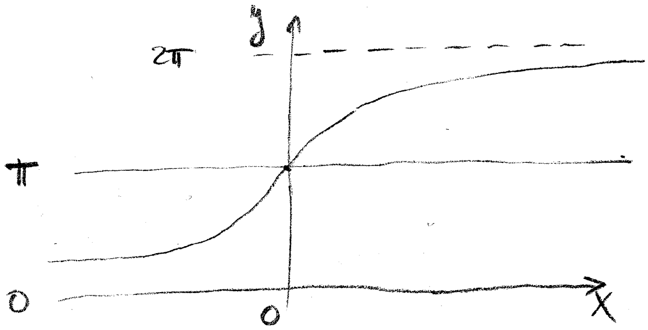
a) $y = \arcsin(x-5)$



sup : $\frac{\pi}{2}$ max : $\frac{\pi}{2}$ omezená (shora i zdola)

inf : $-\frac{\pi}{2}$; min : $-\frac{\pi}{2}$;

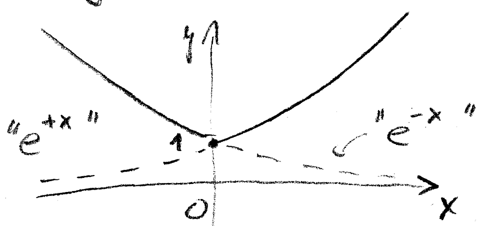
b) $y = 2\operatorname{arctg}(x) + \pi$



sup : 2π max : ne

inf : 0 ; min : 0 ; omezená (shora i zdola)

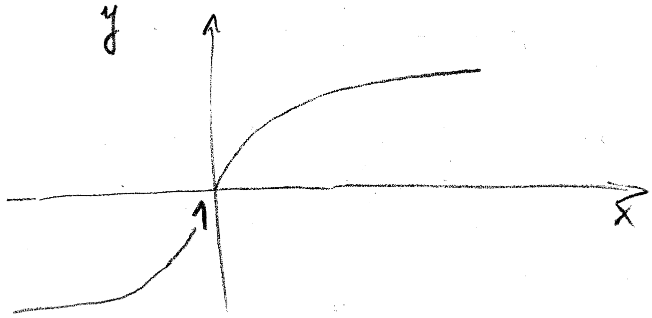
c) $y = e^{|x|}$



sup : $+\infty$ max : ne omezená zdola

inf : 1 ; min : 1 ;

d) $y = \sqrt[3]{x-1}$



sup : $+\infty$ max/min : ne

inf : $-\infty$; neomezená

Př. 3 Určete maximální intervaly, na kterých je daná funkce f ryze monotónní a najděte k ní na těchto intervalech inverzní funkce a stanovte jejich def. obory.

a) $f(x) = 1+\sqrt{3+e^{2x}}$ b) $f(x) = \ln(2-3x)$

c) $f(x) = x^2$

---

a) $I = (-\infty; +\infty)$ (ryze monotónní)

$\overset{-1}{f}: \quad x = 1+\sqrt{3+e^{2y}} \longrightarrow (x-1)^2 - 3 = e^{2y}$ $\boxed{\overset{!}{(x-1) \geq 0}}$

$\longrightarrow 2y = \ln(x^2-2x-2)$

$y = \frac{1}{2}\ln(x^2-2x-2)$

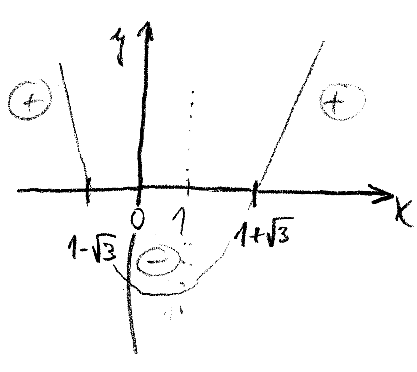
$\overset{-1}{f}(x) = \frac{1}{2}\ln(x^2-2x-2)$

$D(\overset{-1}{f}) = (1+\sqrt{3}; +\infty)$

$D(f) = I$

$x^2-2x-2 > 0$

$x_{1,2} = \frac{2 \pm \sqrt{(-2)^2-4\cdot 1\cdot(-2)}}{2} = \begin{cases} 1+\frac{\sqrt{12}}{2} & ✓ \\ 1-\frac{\sqrt{12}}{2} & ✗ \end{cases}$



b) $f(x) = \ln(2-3x)$ $\quad f^{-1}: \; x = \ln(2-3y) \rightarrow e^x = 2-3y$

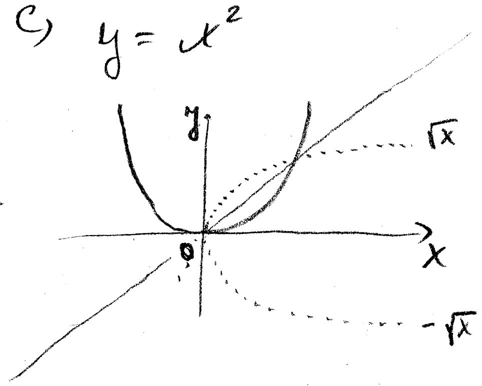
$I = (-\infty; \frac{2}{3}) = D(f)$ $\quad \rightarrow y = \frac{1}{3}(2-e^x)$

$f^{-1}(x) = \frac{1}{3}(2-e^x)$

$D(f^{-1}) = (-\infty; +\infty)$

---

c) $y = x^2$



ryze monotónní: $I_1 = (-\infty; 0\rangle$

$I_2 = \langle 0; +\infty)$

$I_1: \; f^{-1}: \; x = y^2 \rightarrow y = \sqrt{x} \quad (x \geq 0)$

$D(f^{-1}) = \langle 0; +\infty)$

$I_2: \; f^{-1}: \; x = y^2 \rightarrow y = -\sqrt{x} \quad (x \geq 0)$

$D(f^{-1}) = \langle 0; +\infty)$

Cvičení 12.

Limita funkce:

- Předpokládejme, že $x_0 \in \mathbb{R}^*$ a $P(x_0) \subset D(f)$.

Pokud $\boxed{x_n \to x_0 \Rightarrow f(x_n) \to a}$ pro každou posloupnost $\{x_n\}$ v $P(x_0)$

říkáme, že funkce f má v bodě x_0 limitu rovnou a.

$$\left(\lim_{x \to x_0} f(x) = a\right)$$

Př. 1. Je dána funkce f a reálná čísla x_0 a ε. Vypočítejte limitu L funkce f v bodě x_0 a najděte číslo $\delta > 0$: $\forall x \in P_\delta(x_0)$ je $f(x) \in U_\varepsilon(L)$.

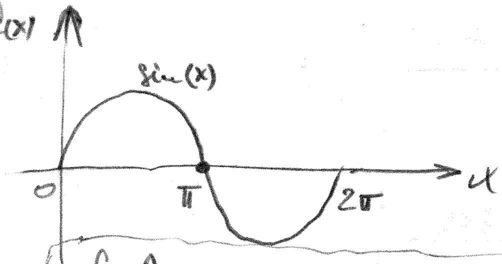
a) $f(x) = \sin(x)$; $x_0 = \pi$; $\varepsilon = 0.01$

b) $f(x) = x^2$; $x_0 = 2$; $\varepsilon = 0.005$

c) $f(x) = tg(x)$; $x_0 = 0$; $\varepsilon = 0.1$

a) $f(x) = \sin(x)$

$$\lim_{x \to \pi} \left(\sin(x)\right) = 0$$

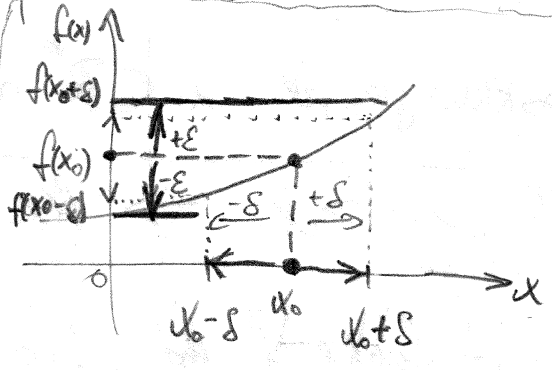


$\varepsilon = 0.01$; $\boxed{\delta > 0}$

Hledáme : $\forall x \in (x_0 - \delta ; x_0 + \delta)$

platí $f(x) \in \left(f(x_0) - \varepsilon ; f(x_0) + \varepsilon\right)$

$\delta = ?$

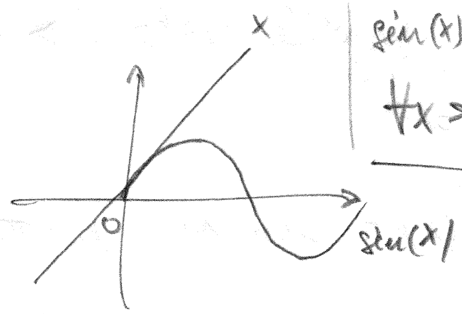


δ : $\left|f(x_0+\delta) - f(x_0)\right| < \varepsilon \quad \wedge \quad \left|f(x_0-\delta) - f(x_0)\right| < \varepsilon$

$\left|\sin(\pi+\delta)\right| < 0.01 \quad \wedge \quad \left|\sin(\pi-\delta)\right| < 0.01$

jelikož $\sin(\pi + x) = -\sin(x)$ máme 1 podmínku :
$\sin(\pi - x) = \sin(x)$

- $\sin(\delta) < 0.01$; $\boxed{\delta > 0}$
- např. : $\delta = 0.01$ $\longrightarrow$



$\sin(x) < x$
$\forall x > 0$

b) $f(x) = x^2$; $\varepsilon = 0.005$; $x_0 = 2$; $\boxed{\delta > 0}$

$$\lim_{x \to 2} x^2 = 4$$

Chceme: $\Big(\left| (x_0+\delta)^2 - x_0^2 \right| < 0.005 \Big) \wedge \Big(\left| (x_0-\delta)^2 - x_0^2 \right| < 0.005 \Big)$

$$\left| 2\delta x_0 + \delta^2 \right| < 0.005 \qquad\qquad \left| -2\delta x_0 + \delta^2 \right| < 0.005$$

$$4\delta + \delta^2 < 0.005 \qquad\qquad \left| -4\delta + \delta^2 \right| < 0.005$$

$\Rightarrow$ Lze řešit buď exaktně (výpočtem) nebo stačí pouze odhad:

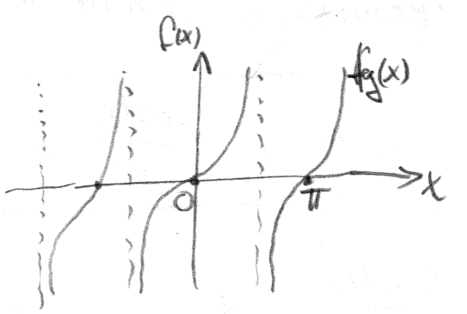
Pokud např.: $\boxed{\delta = 0.001}$ $\Rightarrow 4 \cdot 0.001 + (0.001)^2 < 0.005$

$\Rightarrow \left| -4 \cdot 0.001 + (0.001)^2 \right| < 0.005$

---

c) $f(x) = tg(x)$; $\varepsilon = 0.1$; $x_0 = 0$

$$\lim_{x \to 0} tg(x) = 0 \ ; \ \boxed{\delta > 0}$$



$$\left| tg(x_0 - \delta) - tg(x_0) \right| < \varepsilon \quad \wedge \quad \left| tg(x_0 + \delta) - tg(x_0) \right| < \varepsilon$$

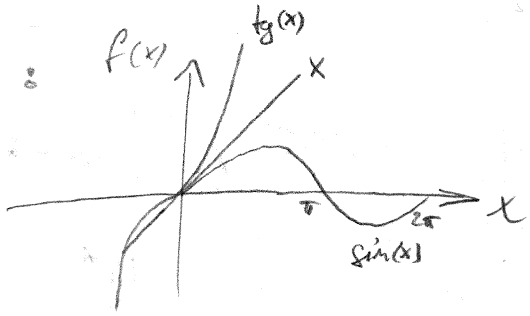
$$\Rightarrow \left| tg(-\delta) \right| < 0.1 \quad \wedge \quad \left| tg(\delta) \right| < 0.1$$

$$\Rightarrow \left. \begin{array}{l} tg(\delta) < 0.1 \\ \delta \in \langle 0 ; \pi/2) \end{array} \right\} \quad \delta < tg^{-1}(0.1) = 0.0996....$$

např.: $\boxed{\delta < 0.05}$

Př. 2 Spočítejte $\lim\limits_{x\to 0} \frac{\sin(x)}{x}$ (bez použití l'Hospitalova pravidla) ④

→ Limita sevřené fce:



$$\overset{(1)}{\sin(x) \leq} x \overset{(2)}{\leq} \mathrm{tg}(x) \quad \forall x \in (0;1)$$

$$\overset{(2)^*}{\Rightarrow} \frac{1}{x} \geq \frac{1}{\mathrm{tg}(x)}$$

$x \in (0;1)$:

$$\frac{\sin(x)}{\mathrm{tg}(x)} \overset{(2)^*}{\leq} \frac{\sin(x)}{x} \overset{(1)}{\leq} \frac{x}{x}$$

$$\boxed{\cos(x) \leq \frac{\sin(x)}{x} \leq 1}$$

$$\Rightarrow \underbrace{\lim_{x\to 0^+} \cos(x)}_{1} \leq \lim_{x\to 0^+} \frac{\sin(x)}{x} \leq \underbrace{\lim_{x\to 0^+} 1}_{1} \Rightarrow \boxed{\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1}$$

⇒ Obdobná geometrická úvaha i pro $x \in (-1;0)$:

$x \in (-1;0)$: $\boxed{\cos(x) \leq \frac{\sin(x)}{x} \leq 1}$

⇒ $\cos(x)$; $\frac{\sin(x)}{x}$; 1 jsou sudé funkce tj.

Pro $x \to -x$ (a tedy $(0;1) \to (-1;0)$) se výše uvedená nerovnost nezmění!

$$\Rightarrow \lim_{x\to 0^-} \frac{\sin(x)}{x} = 1. \quad \text{Celkem: } \boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1}$$

Př. 3 Vypočítejte následující limity:

- $$\lim_{x\to 2} \frac{x^2-4x+1}{2x+1} = \frac{4-8+1}{4+1} = \frac{-3}{5}$$

- $$\lim_{x\to +\infty} \frac{2x+\operatorname{arctg}(x)}{x+1} = \lim_{x\to +\infty}\left(\frac{2x+2}{x+1} - \frac{2}{x+1} + \frac{\operatorname{arctg}(x)}{x+1}\right) =$$

$$= \lim_{x\to +\infty} 2 - \underbrace{\lim_{x\to +\infty}\frac{2}{x+1}}_{0} + \underbrace{\lim_{x\to +\infty}\frac{\operatorname{arctg}(x)}{x+1}}_{0} = 2$$

- $$\lim_{x\to +\infty} \frac{3x-1}{x^2+1} = \lim_{x\to +\infty} \frac{3-\overbrace{\frac{1}{x}}^{\to 0}}{\underbrace{x}_{+\infty}+\underbrace{\frac{1}{x}}_{\to 0}} = 0$$

- $$\lim_{x\to +\infty} x\cdot\left(\sqrt{x^2+1}-x\right)^{\text{"}\infty\cdot 0\text{"}} = \lim \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\cdot\left(\sqrt{x^2+1}-x\right)\cdot x =$$

$$= \lim_{x\to +\infty} \frac{\left[(x^2+1)-x^2\right]\cdot x}{x+\sqrt{x^2+1}} = \lim_{x\to +\infty} \frac{x}{x+\sqrt{x^2+1}} = \lim_{x\to +\infty} \frac{1}{1+\sqrt{1+\frac{1}{x^2}}} =$$

$$= \frac{1}{1+\sqrt{1+0}} = \frac{1}{2}$$

- $$\lim_{x\to 1} \frac{x^2-2x+1}{2x^2-x-1} = „\frac{0}{0}" = \left| \begin{array}{l} x^2-2x+1 = (x-1)(x-1) \\ 2x^2-x-1 = (x-1)(2x+1) \\ \left\{ x_{1,2} = \frac{1\pm\sqrt{(-1)^2-4\cdot 2\cdot(-1)}}{4} = \begin{array}{l} \nearrow 1 \\ \searrow -\frac{1}{2} \end{array} \right\} \end{array} \right| =$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}\underbrace{(x-1)}_{\to 0}}{\cancel{(x-1)}\underbrace{(2x+1)}_{\to 3}} = \frac{0}{3} = \underline{\underline{0}}$$

- $$\lim_{x\to 0} \frac{x^2\cdot\cos(x)}{\cos(x)-1} \overset{„\frac{0}{0}"}{=} \lim_{x\to 0} \frac{\cos(x)+1}{\cos(x)+1}\cdot\frac{x^2\cdot\cos(x)}{\cos(x)-1} =$$

$$= \lim_{x\to 0} \frac{x^2\cdot(\cos^2(x)+\cos(x))}{\cos^2(x)-1} = \lim_{x\to 0} \frac{-x^2}{\sin^2(x)} \cdot \underbrace{\lim_{x\to 0}\left(\cos^2(x)+\cos(x)\right)}_{1+1}$$

$$= \left| \lim_{x\to 0} \frac{\sin(x)}{x} = 1 \right| = \underline{\underline{-2}}$$

- $$\lim_{x\to 0} \operatorname{arctg}\left(\frac{1}{x^2}\right) = \left| \begin{array}{l} \lim\limits_{x\to 0^+} \frac{1}{x^2} = \lim\limits_{x\to 0^-} \frac{1}{x^2} = +\infty \\ + \text{věta o limitě složené fce} \end{array} \right| =$$

$$= \lim_{y\to +\infty} \operatorname{arctg}(y) = \underline{\underline{\frac{\pi}{2}}}$$

Př. 4 Vypočtěte následující jednostranné limity:

- $\lim\limits_{x \to 1^+} \dfrac{x+2}{x-1} = \left| \dfrac{3^+}{0^+} \right| = +\infty$

- $\lim\limits_{x \to 0^+} x \cdot \ln(x) = \left| \begin{matrix} "0 \cdot (-\infty)" \\ \to \text{l'Hospital} \end{matrix} \right| = \lim\limits_{x \to 0^+} \dfrac{\ln(x)}{\frac{1}{x}} \overset{l'H}{=} \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} =$

$= \lim\limits_{x \to 0^+} (-x) = 0$

- $\lim\limits_{x \to 0^+} \dfrac{5+x}{x \cdot (x-1)} = \left| \dfrac{5^+}{0^+ \cdot (-1^+)} \sim -\dfrac{5^+}{0^+} \right| = -\infty$

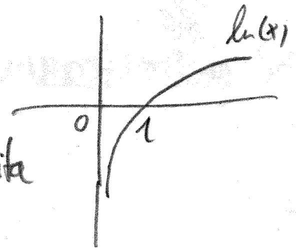
např.: $-\dfrac{5}{0{,}000009} \ldots$

Př. 5 Zdůvodněte proč dané limity neexistují:

a) $\lim\limits_{x \to 0} \ln(x)$ b) $\lim\limits_{x \to 0} \dfrac{1}{x}$ c) $\lim\limits_{x \to \frac{\pi}{2}} \operatorname{tg}(x)$

d) $\lim\limits_{x \to +\infty} \sin(x)$ e) $\lim\limits_{x \to 2} \cos^{-1}(x)$ f) $\lim\limits_{x \to 2} \dfrac{x+1}{x-2}$

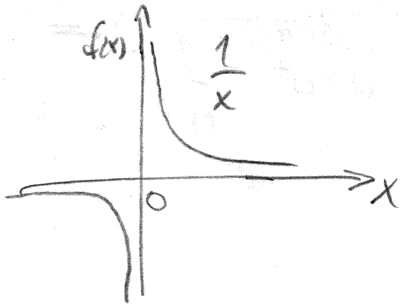
a) $D(\ln(x)) = (0; +\infty)$!



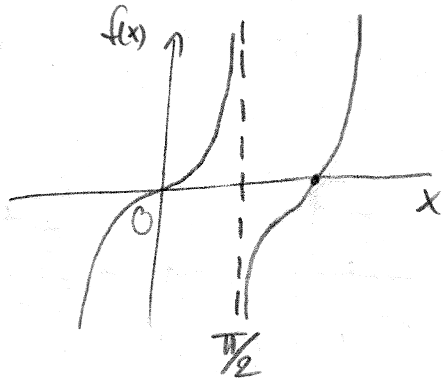
→ existuje pouze 1-stranná limita

$$\lim_{x \to 0^+} \ln(x) = -\infty$$

b) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$; $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ → jednostranné limity se nerovnají



c) $\lim_{x \to \frac{\pi}{2}+} tg(x) = -\infty$



→ jednostranné limity jsou různé

$$\lim_{x \to \frac{\pi}{2}^-} tg(x) = +\infty$$

d) $\lim_{x \to +\infty} (\sin(x)) = \Big|$ zvolíme-li: $\left.\begin{array}{l} x_n = \frac{\pi}{2} + \pi n \\ n \in N \\ n \to \infty \end{array}\right\} \Rightarrow x_n \to +\infty$ ale $\lim_{x_n \to +\infty} \sin(x_n) = \text{neex.} \Big|$

$= \text{neex.}$

(limita funkce musí platit $\forall \{x_n\}$!

Cvičení 13

Derivace: Mějme funkci f definovanou v jistém okolí bodu x_0.
Je-li
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
vlastní, pak ji

nazýváme derivací funkce f v bodě x_0 a označujeme ji $f'(x_0)$ nebo $\frac{d}{dx}f(x_0)$

Př 1. Výpočet derivací dle definice:

- $\frac{d}{dx}(x^m)$; $m\in\mathbb{N}$ $\to$ $\lim\limits_{h\to 0}\frac{(x+h)^m - x^m}{h} =$

$$= \lim_{h\to 0}\frac{\left[\binom{m}{0}x^m + \binom{m}{1}x^{m-1}\cdot h + \binom{m}{2}x^{m-2}h^2 + \dots + \binom{m}{m}x^0 h^m\right] - x^m}{h} =$$

$$= \lim_{h\to 0}\left[m\cdot x^{m-1} + \underbrace{\binom{m}{2}x^{m-2}\cdot h}_{\to 0} + \underbrace{\dots}_{\to 0} + \underbrace{\binom{m}{m}x^0\cdot h^{m-1}}_{\to 0}\right] =$$

$$= \boxed{m\cdot x^{m-1}}$$

Pozn.: Lze odvodit i $\frac{d}{dx}(x^\alpha) = \alpha\cdot x^{\alpha-1}$; $\forall \alpha\in\mathbb{R}$; $\alpha\neq 0$

(2)

$$\frac{d}{dx}\left(\sin(x)\right) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \Big| \text{goniometrické vztahy } \sin(a+b) \Big|$$

$$= \lim_{h\to 0} \frac{\left[\sin(x)\cdot\cos(h) + \cos(x)\cdot\sin(h)\right] - \sin(x)}{h} =$$

$$= \cos(x)\cdot \underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{\to 1 \text{ (víme z minula...)}} + \sin(x)\cdot \underbrace{\overset{(*)}{\lim_{h\to 0}}\frac{\cos(h)-1}{h}}_{\to 0} = \boxed{\cos(x)}$$

(*) $$\lim_{h\to 0}\frac{\cos(h)-1}{h} = \lim_{h\to 0}\frac{\cos^2(h)-1}{(\cos(h)+1)h} = \lim_{h\to 0}\Big[\underbrace{-\frac{\sin(h)}{h}}_{\to -1}\cdot\underbrace{\frac{\sin(h)}{\cos(h)+1}}_{\to \frac{0}{2}}\Big] =$$

$$= -1\cdot\frac{0}{2} = \underline{\underline{0}}$$

$$\frac{d}{dx}\left(\cos(x)\right) = \dots \text{ obdobně } = \boxed{-\sin(x)}$$ (Zkuste si jako cvičení)

Př. 2 Popište chování funkce f v okolí bodu x_0 (rostoucí / klesající), jak rychle.

a) $f(x) = 5x^2 + 7x - 2$; $x_0 = 1$

b) $f(x) = \sqrt{2x^2 - x + 5}$; $x_0 = 1$

c) $f(x) = \dfrac{x+1}{x-1}$; $x_0 = 2$

d) $f(x) = e^{3x}$; $x_0 = 1$

e) $f(x) = \ln\left(\dfrac{x + \sqrt{x}}{\sin(x)}\right)$; $x_0 = \dfrac{\pi}{2}$

f) $f(x) = 3^x + 5x$; $x_0 = 1$

g) $f(x) = \operatorname{arctg}(x^2)$; $x_0 = 5$

h) $f(x) = 2x - 3\sqrt[3]{x^2}$; $x_0 = 1$

i) $f(x) = e^{\left[\frac{\sin(x)}{x}\right]}$ $x_0 = \pi$

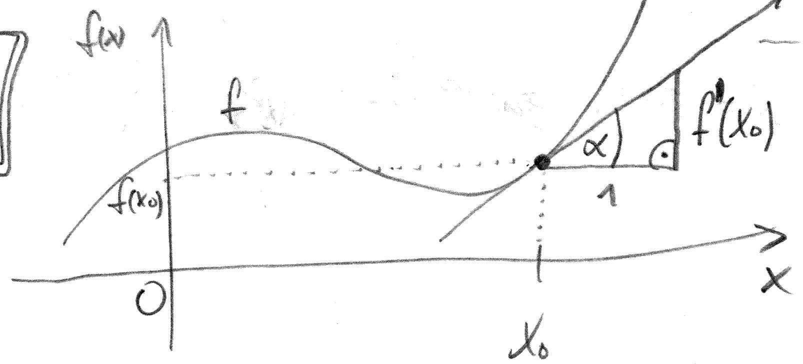
Pozn.:

$(f \cdot g)' = f' \cdot g + f \cdot g'$	$[f(g)]' = f'(g) \cdot g'$
$\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$	$(f \pm g)' = f' \pm g'$

Geom. interpretace:

tečna : $y = f'(x_0) \cdot x + b$

$\boxed{\operatorname{tg} \alpha = f'(x_0)}$



$\Rightarrow f'(x_0)$ odpovídá směrnici přímky a pro $\boxed{y = ax + b}$

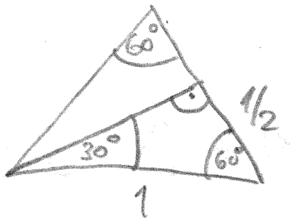
④

a) $f'(x) = 10x+7$; $f'(1) = 17 \rightarrow \alpha = \mathrm{arctg}(17) \doteq 86^\circ$

b) $f'(x) = \boxed{\frac{1}{2}\left(2x^2 - x + 5\right)^{-1/2} \cdot (4x-1)}$; $f'(1) = \frac{1}{2} \cdot \frac{1}{\sqrt{6}} \cdot 3 =$

$= \sqrt{\frac{9}{24}} = \sqrt{\frac{3}{8}} \doteq 0.61$; $\alpha \doteq \mathrm{arctg}(0.61) \doteq 31^\circ$

Pozn.:



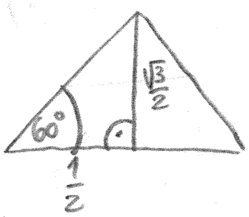
$\mathrm{tg}\, 30^\circ = \frac{1}{\sqrt{3}} \doteq 0.58$

$\mathrm{arctg}(0.58) = 30^\circ$

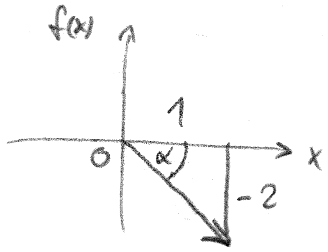
c) $f'(x) = \left[\frac{x+1}{x-1}\right]' = \frac{(x+1)' \cdot (x-1) - (x+1) \cdot (x-1)'}{(x-1)^2} = \frac{x-1-(x+1)}{(x-1)^2} =$

$= \boxed{\frac{-2}{(x-1)^2}}$; $f'(2) = -2$; $\alpha = \mathrm{arctg}(-2) \doteq -63^\circ$

Pozn.:



$\rightarrow \mathrm{tg}(60^\circ) = \sqrt{3} \doteq 1.73$



d) $f(x) = e^{3x}$; $f'(x) = e^{3x} \cdot 3 = \boxed{3e^{3x}}$;

$f'(1) = 3e^3$; $\alpha = \mathrm{arctg}(3e^3) \doteq \ldots 89^\circ$

e) $f(x) = \ln\left(\frac{x+\sqrt{x}}{\sin(x)}\right)$

$$f'(x) = \frac{1}{\left(\frac{x+\sqrt{x}}{\sin(x)}\right)} \cdot \frac{(x+\sqrt{x})' \cdot \sin(x) - (x+\sqrt{x}) \cdot \sin'(x)}{\sin^2(x)} =$$

$$= \frac{1}{\left(\frac{x+\sqrt{x}}{\sin(x)}\right)} \cdot \frac{\left(1+\frac{1}{2\sqrt{x}}\right) \cdot \sin(x) - (x+\sqrt{x}) \cdot \cos(x)}{\sin^2(x)} \overset{\text{algeb. úpravy}}{=}$$

$$= \boxed{\frac{1+\frac{1}{2\sqrt{x}}}{x+\sqrt{x}} - \operatorname{cotg}(x)} \; ; \quad f'\left(\frac{\pi}{2}\right) = \frac{1+\frac{1}{2\sqrt{\frac{\pi}{2}}}}{\frac{\pi}{2}+\sqrt{\frac{\pi}{2}}} \doteq$$

$$\doteq 0.5 \; ; \quad \alpha \doteq \operatorname{arctg}(0.5) = 30^\circ$$

---

f) $f(x) = 3^x + 5x$

$$f'(x) = \left[e^{\ln(3^x)} + 5x\right]' = \left[e^{x \cdot \ln(3)} + 5x\right]' =$$

$$= e^{x \cdot \ln(3)} \cdot \ln(3) + 5 = \boxed{\ln(3) \cdot 3^x + 5}$$

$$f'(1) = 3 \cdot \ln(3) + 5 = \ln(9) + 5 \doteq 7.2$$

$$\alpha = \operatorname{arctg}(7.2) \doteq 82^\circ$$

g) $f(x) = \mathrm{arctg}(x^2)$

Pozn.: Jak nalézt derivaci inverzní funkce?

1) $y = \mathrm{tg}(x)$; $\frac{dy}{dx} = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\sin'(x)\cdot\cos(x) - \sin(x)\cdot\cos'(x)}{\cos^2(x)} =$

2) $x = \mathrm{arctg}(y)$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

goniom. vzorce

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \rightarrow \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{1}{\cos^2(x)}} = \cos^2(x) =$$

$$= \frac{1}{1+\mathrm{tg}^2(x)} = \left[x = \mathrm{arctg}(y)\right] = \frac{1}{1 + \underbrace{\mathrm{tg}(\mathrm{arctg}(y))}_{y} \cdot \underbrace{\mathrm{tg}(\mathrm{arctg}(y))}_{y}} =$$

$$= \boxed{\frac{1}{1+y^2}} = f'(y) \quad \text{pro} \quad f(y) = \mathrm{arctg}(y)$$

$$\Rightarrow \boxed{\mathrm{arctg}'(x) = \frac{1}{1+x^2}}$$

$$f'(x) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}; \quad f'(5) = \frac{10}{1+625} \doteq 0.016$$

$$\alpha = \mathrm{arctg}(0.016) \doteq 1^\circ$$

h) $f(x) = 2x - 3\sqrt[3]{x^2}$

$$f'(x) = 2 - 3\left(x^{\frac{2}{3}}\right)' = 2 - 3 \cdot \frac{2}{3} x^{\frac{2}{3}-1} =$$

$$= 2 - 2x^{-1/3} = \boxed{2 - \frac{2}{\sqrt[3]{x}}}$$

$$f'(1) = 2 - \frac{2}{1} = 0 \; ; \quad \alpha = \operatorname{arctg}(0) = 0$$

i) $f(x) = e^{\left[\frac{\sin(x)}{x}\right]}$; $f'(x) = e^{\frac{\sin(x)}{x}} \cdot \left(\frac{\sin(x)}{x}\right)' =$

$$= e^{\frac{\sin(x)}{x}} \cdot \frac{\sin'(x) \cdot x - \sin(x) \cdot x'}{x^2} = \boxed{e^{\frac{\sin(x)}{x}} \cdot \frac{\cos(x) \cdot x - \sin(x)}{x^2}}$$

$$f'(\pi) = e^{\frac{0}{\pi}} \cdot \frac{-1 \cdot \pi - 0}{\pi^2} =$$

$$= \frac{-1}{\pi} \; ; \quad \alpha = \operatorname{arctg}\left(\frac{-1}{\pi}\right) \doteq -17{,}6^\circ$$

Cvičení 14

Limity – l'Hospitalovo pravidlo:

Př. 1 Vypočítejte následující limity. Použijte l'Hospitalovo pravidlo.

Pozn.: l'Hospitalovo pravidlo:

Předpokládejme, že $c \in \mathbb{R}^*$ a že limity $\lim_{x\to c} f(x)$ a $\lim_{x\to c} g(x)$ jsou buď obě nulové, nebo obě nekonečné. Pak platí:

$$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}}$$

a) $\lim_{x\to 0} \dfrac{\operatorname{arctg}(x)}{x}$

b) $\lim_{x\to 0} \dfrac{\operatorname{tg}(5x)}{3x}$

c) $\lim_{x\to 0} \dfrac{\arcsin(3x)}{\sqrt{2+x} - \sqrt{2}}$

d) $\lim_{x\to 0} \dfrac{e^x - 1}{x}$

e) $\lim_{x\to +\infty} \dfrac{e^x + e^{-x}}{x \cdot \ln(x)}$

f) $\lim_{x\to 4} \dfrac{2^x - 16}{\sin(\pi x)}$

a) $$\lim_{x\to 0} \frac{\operatorname{arctg}(x)}{x} \overset{[l'H;\ \frac{0}{0}]}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = \underline{\underline{1}}$$

2

b) $\lim\limits_{x\to 0} \dfrac{tg(5x)}{3x} \overset{[l'H; \frac{0}{0}]}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{[\cos(5x)]^2}\cdot 5}{3} = \dfrac{\frac{1}{1^2}\cdot 5}{3} = \underline{\underline{\dfrac{5}{3}}}$

---

c) $\lim\limits_{x\to 0} \dfrac{\arcsin(3x)}{\sqrt{2+x}-\sqrt{2}} = \left[l'H; \text{"}\frac{0}{0}\text{"}\right] = \lim\limits_{x\to 0} \dfrac{\frac{1}{\sqrt{1-(3x)^2}}\cdot 3}{\frac{1}{2}\cdot\frac{1}{\sqrt{2+x}}\cdot 1} = \underline{\underline{6\cdot\sqrt{2}}}$

---

d) $\lim\limits_{x\to 0} \dfrac{e^x-1}{x} = \left[l'H; \text{"}\frac{0}{0}\text{"}\right] = \lim\limits_{x\to 0} \dfrac{e^x}{1} = \underline{\underline{1}}$

---

e) $\lim\limits_{x\to +\infty} \dfrac{e^x+e^{-x}}{x\cdot \ln(x)} = \left[l'H; \frac{+\infty}{+\infty}\right] = \lim\limits_{x\to +\infty} \dfrac{e^x-e^{-x}}{\ln(x)+x\cdot\frac{1}{x}} = \left[l'H; \frac{+\infty}{+\infty}\right] =$

$= \lim\limits_{x\to +\infty} \dfrac{e^x+e^{-x}}{\frac{1}{x}} = \lim\limits_{x\to +\infty} x\cdot(e^x+e^{-x}) = \underline{\underline{+\infty}}$

---

f) $\lim\limits_{x\to 4} \dfrac{2^x-16}{\sin(\pi x)} = \left[l'H; \text{"}\frac{0}{0}\text{"}\right] = \lim\limits_{x\to 4} \dfrac{\ln(2)\cdot 2^x}{\cos(\pi x)\cdot\pi} = \underline{\underline{\dfrac{16\cdot\ln(2)}{\pi}}}$

<u>Př. 2</u> Vypočítejte limitu: $\lim\limits_{x\to \pi/4} [tg(x)]^{tg(2x)}$

$\lim\limits_{x\to \pi/4} [tg(x)]^{tg(2x)} = [\text{"}1^\infty\text{"}] \overset{\text{"trik"}}{=} \lim\limits_{x\to \pi/4} e^{\ln\left(tg(x)^{tg(2x)}\right)} = e^{\lim\limits_{x\to \pi/4}\left[tg(2x)\cdot\ln(tg(x))\right]}$

$\lim\limits_{x\to \pi/4} tg(2x)\cdot\ln(tg(x)) = \left[\text{"}\infty\cdot 0\text{"} \sim \text{"}\frac{\infty}{\infty}\text{"}\right] = \lim\limits_{x\to \pi/4} \dfrac{\ln(tg(x))}{\frac{1}{tg(2x)}} = \left[l'H; \text{"}\frac{\infty}{\infty}\text{"}\right]$

$=$

$$\lim_{x\to\pi/4} \frac{\ln(\operatorname{tg}(x))}{\frac{1}{\operatorname{tg}(2x)}} \overset{l'H}{=} \lim_{x\to\pi/4} \frac{\frac{1}{\operatorname{tg}(x)} \cdot \frac{1}{\cos^2(x)}}{\frac{-\sin(2x)\cdot 2\cdot \sin(2x) - \cos(2x)\cdot\cos(2x)\cdot 2}{\sin^2(2x)}} =$$

$$\frac{\cos(2x)}{\sin(2x)}$$

$$= \lim_{x\to\pi/4} \frac{\frac{1}{\sin(x)\cdot\cos(x)}}{\frac{-2\cdot\left[\sin^2(2x)+\cos^2(2x)\right]}{\left[2\cdot\sin(x)\cdot\cos(x)\right]^2}} = \lim_{x\to\pi/4} \frac{\frac{4\cdot\sin^2(x)\cdot\cos^2(x)}{\sin(x)\cdot\cos(x)}}{-2} =$$

$$= \lim_{x\to\pi/4} -\left(2\sin(x)\cdot\cos(x)\right) = \lim_{x\to\pi/4} -\sin(2x) = -1$$

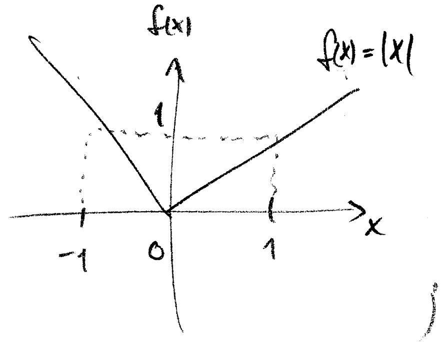
$$\Rightarrow e^{\lim\limits_{x\to\pi/4}\left[\operatorname{tg}(2x)\cdot\ln(\operatorname{tg}(x))\right]} = \boxed{e^{-1} = \lim_{x\to\pi/4}\left[\operatorname{tg}(x)\right]^{\operatorname{tg}(2x)}}$$

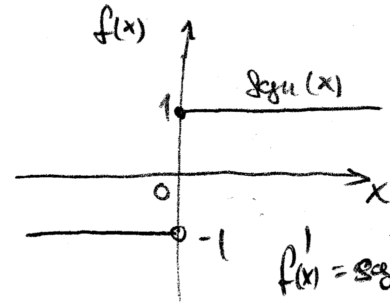
Př. 3 Najděte derivaci dané funkce f a nakreslete grafy funkcí f i f'.

a) $f(x) = |x|$ b) $f(x) = \ln|x|$

a) $f(x) = |x|$

$f'(x) = \mathrm{sgn}(x);\ x \neq 0$





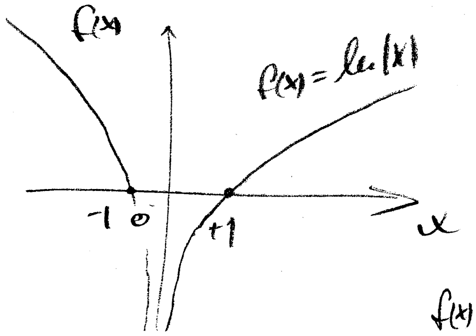
$f'(x) = \mathrm{sgn}(x)$

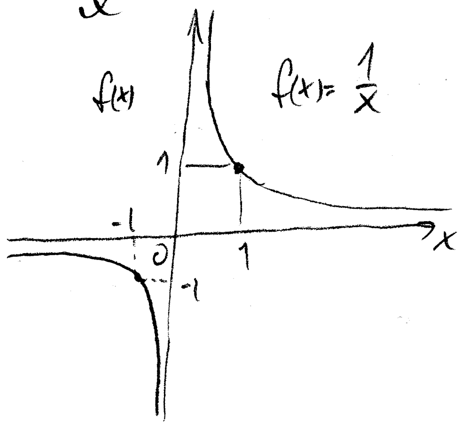
$x \neq 0$

b) $f(x) = \ln|x|$

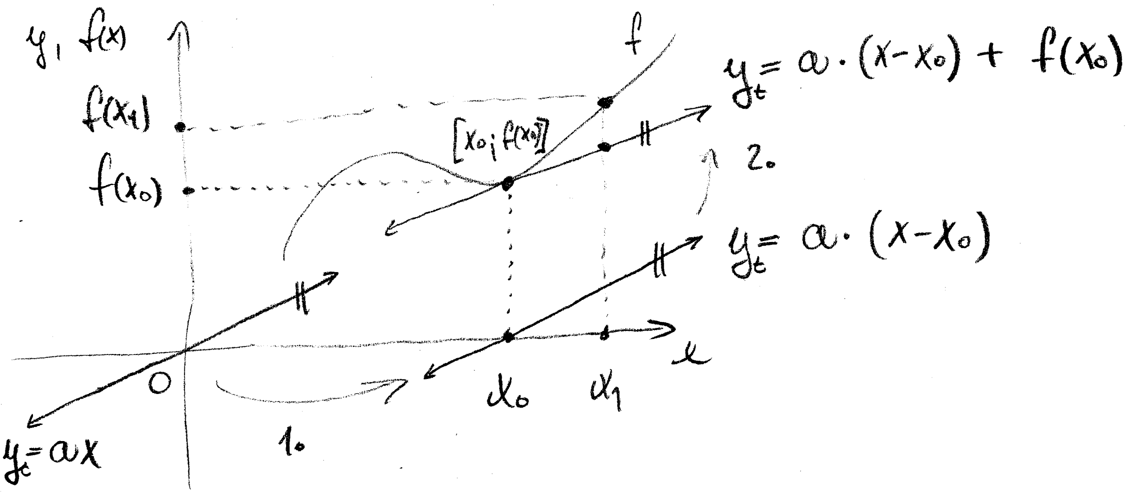
$$f'(x) = \frac{1}{|x|} \cdot \mathrm{sgn}(x) = \frac{1}{x}$$

$x \neq 0$





<u>Tečna a normála ke grafu funkce f</u> :



$a = f'(x_0)$

Celkem máme pro rovnici tečny ke grafu funkce f v bodě $[x_0; f(x_0)]$:

$$\boxed{y_t = f'(x_0) \cdot (x - x_0) + f(x_0)}$$

Rovnice normály: Je-li směrový vektor tečny: $\vec{t} = (1; f'(x_0))$

pak směrový vektor normály bude: $\vec{n} = \left(1; \frac{-1}{f'(x_0)}\right)$

$$\Rightarrow \quad y_n = \frac{-1}{f'(x_0)} \cdot (x - x_0) + f(x_0)$$

Př. 4 Napište rovnici tečny a normály ke grafu funkce f v bodě $[x_0; f(x_0)]$. Rovnici tečny použijte pro přibližný výpočet hodnoty f v bodě x_1.

a) $f(x) = \frac{1}{3}x^3$; $x_0 = -1$; $x_1 = -2/3$

b) $f(x) = \frac{e^x}{x+1}$; $x_0 = 0$; $x_1 = 0.2$

c) $f(x) = \sqrt{2x+3} - x$; $x_0 = 3$; $x_1 = 3.2$

---

a) $f'(x) = x^2$; $f'(x_0) = 1$; $\frac{-1}{f'(x_0)} = \frac{-1}{1}$

$$y_t = 1 \cdot (x - (-1)) + \frac{1}{3}(-1)^3 = x + \frac{2}{3}$$

$$y_n = \frac{-1}{1} \cdot (x - (-1)) + \frac{1}{3}(-1)^3 = -x - \frac{2}{3}$$

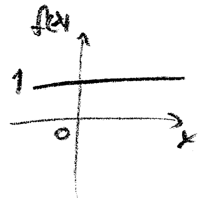
$$f(x_1) \simeq y_t(x_1) = \frac{-2}{3} + \frac{2}{3} = \underline{\underline{0}}$$

b) $f(x) = \frac{e^x}{x+1}$; $f'(x) = \frac{e^x \cdot (x+1) - e^x \cdot 1}{(x+1)^2} = \frac{x \cdot e^x}{(x+1)^2}$

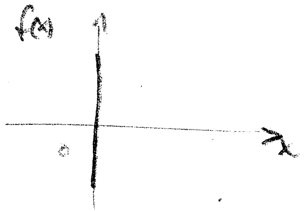
$f'(x_0) = 0$; $\frac{-1}{f'(x_0)} \to \pm\infty$ $(x_0 = 0)$

$f(x_1) \cong y_t(x_1) = \underline{\underline{1}}$

$y_t = 0 \cdot (x - 0) + 1 = 1$



$y_n: \ x = 0$



---

c) $f(x) = \sqrt{2x+3} - x$; $f'(x) = \frac{1}{2\sqrt{2x+3}} \cdot 2 - 1$

$f'(x_0) = f'(3) = \frac{1}{\sqrt{9}} - 1 = -\frac{2}{3}$; $\frac{-1}{f'(x_0)} = \frac{3}{2}$

$y_t = -\frac{2}{3}(x-3) + \left(\sqrt{2 \cdot 3 + 3} - 3\right) = -\frac{2}{3}x + 2$

$y_n = \frac{3}{2}(x-3) + \left(\sqrt{2 \cdot 3 + 3} - 3\right) = \frac{3}{2}x - \frac{9}{2}$

$f(x_1) \cong y_t(x_1) = y_t(3.2) = -\frac{2}{3} \cdot 3.2 + 2 = \underline{\underline{-0.1\overline{3}}}$

Cvičení 15.

Př. 1 Napište rovnice tečen k hyperbole $xy = 4$ v bodech $[x_1, y_1]$ a $[x_2, y_2]$ pro $x_1 = 1$ a $x_2 = 4$.

Vypočítejte velikost úhlu, sevřeného mezi oběma tečnami.

- $[x_1, y_1] = [1; 4]$
  $[x_2, y_2] = [4; 1]$; $\quad y = \frac{4}{x}$; $x \in \mathbb{R} \setminus \{0\}$

- $y'(x) = \frac{-4}{x^2}$; $x \in \mathbb{R} \setminus \{0\}$

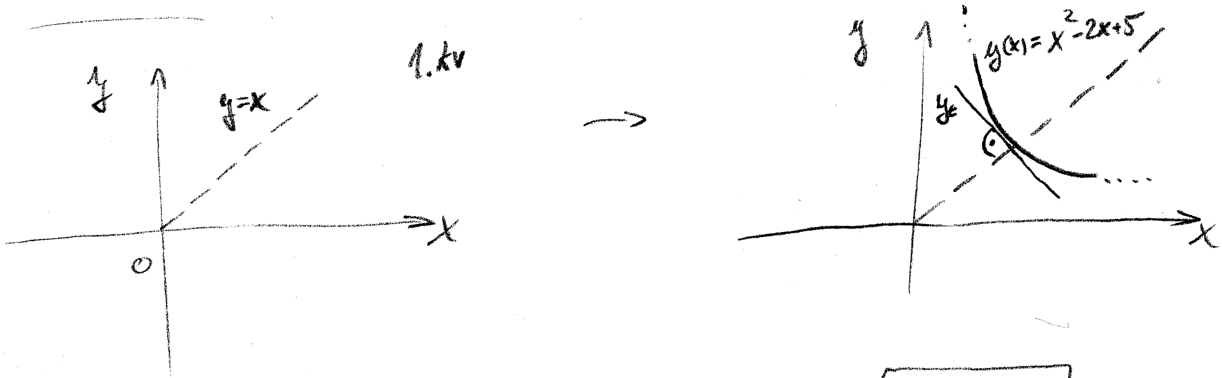
- $$\left.\begin{aligned} y'(x_1) &= -4 \\ y'(x_2) &= \frac{-1}{4} \end{aligned}\right\} \quad \begin{aligned} y_t|_{x_1} &= y'(x_1) \cdot (x - x_1) + y(x_1) = -4(x-1) + 4 \\ y_t|_{x_2} &= y'(x_2) \cdot (x - x_2) + y(x_2) = \frac{-1}{4}(x-4) + 1 \end{aligned}$$

- Velikost úhlu mezi $y_t|_{x_1}$ a $y_t|_{x_2}$ je dána polohou jejich směrových vektorů:

$$\left.\begin{aligned} \vec{t}_1 &= (1; y'(x_1)) = (1; -4) \\ \vec{t}_2 &= (1; y'(x_2)) = \left(1; -\frac{1}{4}\right) \end{aligned}\right\} \quad \cos\alpha = \frac{\vec{t}_1 \cdot \vec{t}_2}{\|\vec{t}_1\| \cdot \|\vec{t}_2\|} = \frac{(1; -4) \cdot \left(1; -\frac{1}{4}\right)}{\sqrt{1^2 + (-4)^2} \cdot \sqrt{1^2 + \left(-\frac{1}{4}\right)^2}} =$$

$$= \frac{2}{\sqrt{17} \cdot \sqrt{\frac{17}{16}}} = \frac{2 \cdot 4}{17} = \frac{8}{17} \implies \alpha = \arccos\left(\frac{8}{17}\right) \doteq 62^\circ$$

Př. 2 Ve kterém bodě paraboly $y(x) = x^2 - 2x + 5$ je její tečna kolmá k ose 1. kvadrantu?



$\Rightarrow$ tečna kolmá k ose 1. kvadrantu má směrnici $\boxed{a = -1}$

$\Rightarrow$ Směrnice tečny ke grafu fce y je : $y'(x)$

$\Rightarrow$ $\boxed{y'(x) = -1}$ $\Rightarrow$ $y'(x) = 2x - 2 = -1$ $\Rightarrow$ $x = \frac{1}{2}$

$$y = \left(\frac{1}{2}\right)^2 - 2\cdot\left(\frac{1}{2}\right) + 5 = \frac{17}{4}$$

• V bodě $\left[\frac{1}{2} ; \frac{17}{4}\right]$ je tečna fce y kolmá k ose 1. kvadrantu.

Př. 3 Najděte číslo α tak, aby přímka $y_1 = 2x$ byla tečnou grafu funkce $y_2 = e^x + \alpha$. Určete bod dotyku.

• Požadujeme opět stejné směrnice tečen k funkcím y_1 a y_2 :

(x-ovou souřadnici bodu dotyku označme x_0)

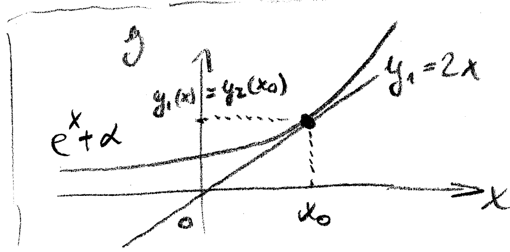
$\Rightarrow$ $y_1'(x) = y_2'(x)$ $\rightarrow$ $2 = e^x$

$x = \boxed{\ln(2) = x_0}$

$\Rightarrow$ Funkční hodnoty $y_1(x_0)$ a $y_2(x_0)$ se musí rovnat (bod dotyku) :

$$y_1(x_0) = 2\ln(2) = e^{\ln(2)} + \alpha$$

$$\boxed{\alpha = 2(\ln(2) - 1)}$$



Př. 4. Pro jaká $x \in D(f)$ existuje tečna ke grafu funkce

$$f(x) = \ln\left[\frac{1+\sqrt{x^2-1}}{x}\right] \text{ v bodě } [x, f(x)] \text{ ?}$$

Existuje tečna rovnoběžná s osou x?

Nejdéle rovnici tečny v bodě $[x_0, f(x_0)]$ pro $x_0 = \sqrt{5}$.

- Tečna existuje na množině $\boxed{I = D(f') \cap D(f)}$
  (tam, kde je definovaná 1. derivace, tj. směrnice tečny ...)

$$\Longrightarrow f'(x) = \frac{1}{\frac{1+\sqrt{x^2-1}}{x}} \cdot \frac{\frac{1}{2\sqrt{x^2-1}} \cdot 2x \cdot x - \left(1+\sqrt{x^2-1}\right) \cdot 1}{x^2} =$$

$$= \frac{x}{1+\sqrt{x^2-1}} \cdot \frac{1}{\sqrt{x^2-1}} - \frac{x}{1+\sqrt{x^2-1}} \cdot \frac{1+\sqrt{x^2-1}}{x^2} =$$

$$= \frac{x}{x^2-1+\sqrt{x^2-1}} - \frac{1}{x}$$

- $$D(f) = \left[\begin{array}{l} \frac{1+\sqrt{x^2-1}}{x} > 0 \\ x^2-1 \geq 0 \end{array}\right\} x \geq 1 \Bigg] = \langle 1; +\infty)$$

- $$D(f') = \left[\begin{array}{l} x^2-1+\sqrt{x^2-1} \neq 0 \\ x^2-1 \geq 0 \\ x \neq 0 \end{array}\right\} x^2 > 1 \Bigg] = (-\infty; -1) \cup (1; +\infty)$$

Tečna je tedy definovaná pro $x > 1$.

- Existuje tečna rovnoběžná s osou x?

$\rightarrow \boxed{f'(x) = 0} \rightarrow \dfrac{x}{x^2-1+\sqrt{x^2-1}} - \dfrac{1}{x} = 0 \qquad x \in (1; +\infty)$

$\rightarrow x^2 = x^2-1+\sqrt{x^2-1}$

$1 = \sqrt{x^2-1}$

$x^2 = 2 \longrightarrow x = \sqrt{2}$

$(x > 1)$

$\Longrightarrow$ Tečna rovnoběžná s osou x existuje v bodě $[\sqrt{2}; f(\sqrt{2})] =$

$= [\sqrt{2}; \ln(\sqrt{2})]$

- Rovnice tečny v bodě $[x_0; f(x_0)]$; $x_0 = \sqrt{5}$:

$$y_t = f'(\sqrt{5})\cdot(x-\sqrt{5}) + f(\sqrt{5}) = \left[\frac{\sqrt{5}}{(\sqrt{5})^2-1+\sqrt{(\sqrt{5})^2-1}} - \frac{1}{\sqrt{5}}\right](x-\sqrt{5})$$

$$+ \ln\left(\frac{1+\sqrt{(\sqrt{5})^2-1}}{\sqrt{5}}\right) = \underline{\left(\frac{\sqrt{5}}{6} - \frac{1}{\sqrt{5}}\right)(x-\sqrt{5}) + \ln\left(\frac{3}{\sqrt{5}}\right)}$$

Př. 5 Je dána funkce f a bod x_0. Vypočítejte a navzájem porovnejte přesnou hodnotu přírůstku $f(x_0+h)-f(x_0)$ a jeho přibližnou hodnotu pomocí diferenciálu.

$$\boxed{df(x_0)(h) \equiv f'(x_0)\cdot h}$$

a) $f(x) = \frac{1}{2}x^2 - 2x + 2$

$x_0 = 0$

$h_1 = 0.5$

$h_2 = 0.1$

$h_3 = 0.01$

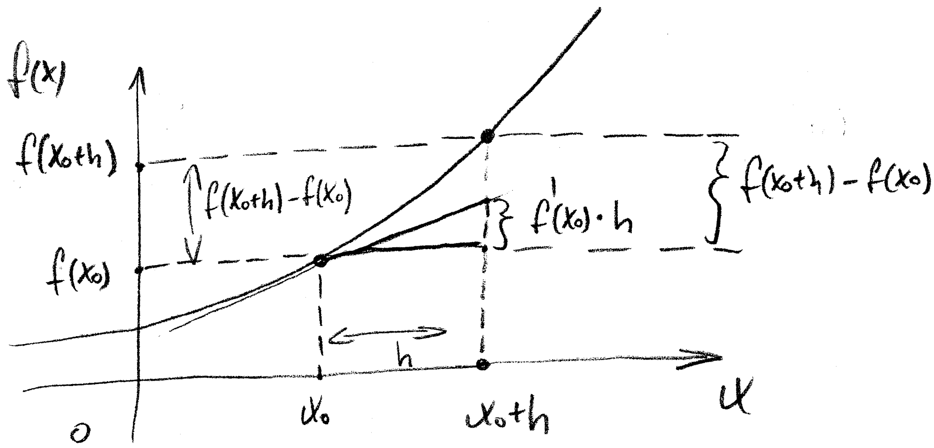
b) $f(x) = \ln(x)$

$x_0 = 1$

$h_1 = 1$

$h_2 = 0.1$

Pozn.:



a) $df(x_0)(h) = (x-2)\Big|_{x=x_0=0} \cdot h = -2\cdot h$

$h_1 = 0.5$: $f(x_0+0.5) - f(x_0) = \left(\frac{1}{2}(0.5)^2 - 2\cdot 0.5 + 2\right) - (2) = -0.875$

$df(x_0)(0.5) = -2\cdot 0.5 = -1$

$h_2 = 0.1$: $f(x_0+0.1) - f(x_0) = \left(\frac{1}{2}(0.1)^2 - 2\cdot 0.1 + 2\right) - (2) = -0.195$

$df(x_0)(0.1) = -2\cdot 0.1 = -0.2$

$h_3 = 0.01: \quad f(x_0 + 0.01) - f(x_0) = \left(\frac{1}{2}(0.01)^2 - 2\cdot 0.01 + 2\right) - (2) = -0.01995$

$df(x_0)(0.01) = -2\cdot 0.01 = -0.02$

---

b) $df(x_0)(h) = \left(\frac{1}{x}\right)\Big|_{x=x_0=1} \cdot h = h$

$h_1 = 1: \quad f(x_0+1) - f(x_0) = \ln(2) - \ln(1) = \ln(2) \doteq 0.6931$

$df(x_0)(1) = 1$

$h = 0.1: \quad f(x_0+0.1) - f(x_0) = \ln(1.1) - \ln(1) = \ln(1.1) \doteq 0.0953$

$df(x_0)(0.1) = 0.1$

Př. 6. Užitím diferenciálu vypočítejte přibližně následující hodnoty:

a) $e^{0.05}$ b) $\sqrt{101}$

---

a) $f(x) = e^x; \; x_0 = 0$

$f(x_0 + 0.05) - f(x_0) \cong df(x_0)(0.05)$; $df(x_0)(h) = (e^x)'\Big|_{x=x_0=0} \cdot h = [e^0]\cdot h = h$

$e^{0.05} \cong e^0 + 0.05 = \underline{1.05}$

---

b) $f(x) = \sqrt{x}; \; x_0 = 100; \quad df(x_0) = (\sqrt{x})'\Big|_{x=x_0=100} \cdot h = \frac{1}{2\sqrt{100}} \cdot h = \frac{1}{20}\cdot h$

$f(x_0+1) - f(x_0) \cong df(x_0)(1)$

$\sqrt{101} \cong \sqrt{100} + \frac{1}{20}\cdot 1 = \underline{10.05}$

Funkce rostoucí a klesající, lokální extrémy, funkce konvexní a konkávní, inflexní body

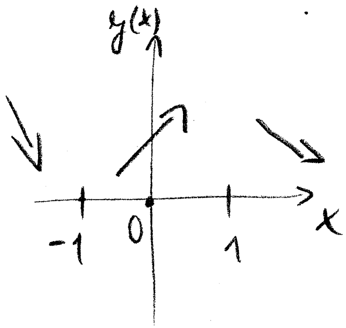
Př. 7 Najděte oblasti monotonie funkce $y(x) = \dfrac{2x}{1+x^2}$.

- $D(y) = \mathbb{R}$
- $$y'(x) = \frac{(2x)' \cdot (1+x^2) - 2x \cdot (1+x^2)'}{(1+x^2)^2} = \frac{2(1+x^2) - 2x \cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$
- Body podezřelé z lok. extrémů: → zde nejsou ...

→ $y'(x) = 0$ (+ krajní body intervalu $D(f)$)

→ $$\frac{2(1-x^2)}{(1+x^2)^2} = 0 \iff x = \pm 1$$

$x \in$	$(-\infty; -1)$	$(-1; 1)$	$(1; +\infty)$
$y'(x)$	$-$	$+$	$-$



$\Rightarrow$ $y(x)$ je klesající pro $x \in (-\infty; -1)$

rostoucí pro $x \in (-1; 1)$

klesající pro $x \in (1; +\infty)$

$\Rightarrow$ V $x = -1$ nabývá $y(x)$ lokálního minima

$x = 1$ nabývá $y(x)$ lokálního maxima

Př. 8 Nalezněte lokální extrémy funkce $\boxed{y = e^x \cdot \sin(x)}$

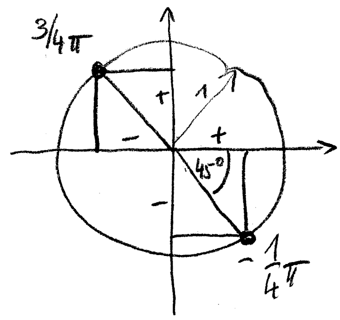
- $D(y) = \mathbb{R}$

- $y'(x) = \left(e^x \cdot \sin(x)\right)' = e^x \cdot \sin(x) + e^x \cdot \cos(x) = e^x \cdot \left(\sin(x) + \cos(x)\right)$

- $\boxed{y'(x) = 0} \iff e^x \cdot \left(\sin(x) + \cos(x)\right) = 0$

$\iff \sin(x) = -\cos(x)$

$\iff \boxed{x = \frac{3}{4}\pi + k\pi;\ k \in \mathbb{Z}}$



- $y''(x) = \left[e^x \cdot \left(\sin(x) + \cos(x)\right)\right]' = e^x \cdot \left(\sin(x) + \cos(x)\right) + e^x \cdot \left(\cos(x) - \sin(x)\right) =$

$= 2e^x \cdot \cos(x)$

$y''(x)\Big|_{x = \frac{3}{4}\pi + k\pi} = 2e^{\frac{3}{4}\pi + k\pi} \cdot \cos\left(\frac{3}{4}\pi + k\pi\right)$

$\nearrow\ y''(x) < 0 : x = \frac{3}{4}\pi + 2k\pi$

$\searrow\ y''(x) > 0 : x = -\frac{1}{4}\pi + 2k\pi$

$\Rightarrow$ Funkce $y(x)$ má v bodech $x = \frac{3}{4}\pi + k\pi,\ k \in \mathbb{Z}$ nulovou 1. derivaci.

Druhá derivace je v těchto bodech:

- $y''\left(\frac{3}{4}\pi + 2k\pi\right) < 0 \Rightarrow y(x)$ zde nabývá lok. maxima:

$y\left(\frac{3}{4}\pi + 2k\pi\right) = \frac{e^{\left(\frac{3}{4}\pi + 2k\pi\right)}}{\sqrt{2}}$

$k \in \mathbb{Z}$

- $y''\left(-\frac{1}{4}\pi + 2k\pi\right) > 0 \Rightarrow y(x)$ zde nabývá lok. minima:

$y\left(-\frac{1}{4}\pi + 2k\pi\right) = -\frac{e^{\left(-\frac{1}{4}\pi + 2k\pi\right)}}{\sqrt{2}}$

Cvičení 16. ①

Vyšetření průběhu funkce:

1. definiční obor ; obor hodnot
2. spojitost ; charakteristika bodů nespojitosti ; krajní body $D(f)$
3. sudost, lichost, periodičnost
4. průsečíky s osami x, y ; $f(x)>0$ / $f(x)<0$
5. $f'(x)=0$; $D(f')$ $\Rightarrow$ monotonie ; extrémy
6. $f''(x)=0$; $D(f'')$ $\Rightarrow$ konvexnost ; konkávnost ; inflexní body
7. asymptoty (bez směrnice / se směrnicí)
8. graf funkce

Př. 1 Vyšetřete průběh funkce $f(x) = \dfrac{x^3}{x^2-1}$

1) $D(f)$: $x^2-1 \neq 0 \Rightarrow D(f) = (-\infty; -1) \cup (-1; 1) \cup (1; +\infty)$

$x \neq \pm 1$

$H(f)$: viz později

2) f je spojitá v $D(f)$. Krajní body $D(f)$:

$$\lim_{x\to-\infty} \frac{x^3}{x^2-1} = -\infty \; ; \; \lim_{x\to+\infty} \frac{x^3}{x^2-1} = +\infty \; ; \; \lim_{x\to1^-} \frac{x^3}{x^2-1} = -\infty$$

$$\lim_{x\to1^+} \frac{x^3}{x^2-1} = +\infty \; ; \; \lim_{x\to-1^-} \frac{x^3}{x^2-1} = -\infty \; ; \; \lim_{x\to-1^+} \frac{x^3}{x^2-1} = +\infty$$

• $H(f) = \mathbb{R}$ $\left(\text{např. } f \text{ je spojitá v } x \in (-1;1) \text{ a } \lim_{x\to-1^+} f(x) = -\infty \text{ a } \lim_{x\to1^-} f(x) = +\infty \right)$

3) sudost / lichost / periodičnost :

- $f(-x) = \dfrac{-x^3}{x^2-1} = -f(x) \Rightarrow$ lichá funkce

- f je zřejmě neperiodická

4) průsečíky s osami x, y ; $f(x) > 0$ / $f(x) < 0$:

x: $f(x) = 0 \Rightarrow \dfrac{x^3}{x^2-1} = 0 \Rightarrow x = 0 \Rightarrow [0;0]$

y: $x = 0 \Rightarrow y = 0 \Rightarrow [0;0]$

→ Průsečík s osou x i y nastává v jediném bodě : $[0;0]$

→ Nyní získáme intervaly, kde je $f(x) > 0$ a $f(x) < 0$:

x	$(-\infty;-1)$	$(-1;0)$	$(0;1)$	$(1;+\infty)$
$f(x)$	$-$	$+$	$-$	$+$

5) monotonie ; extrémy

$$f'(x) = \left(\frac{x^3}{x^2-1}\right)' = \frac{3x^2(x^2-1) - x^3 \cdot 2x}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2}$$

$D(f') = (-\infty;-1) \cup (-1;1) \cup (1;+\infty)$

Stacionární body : $\boxed{f'(x) = 0} \Leftrightarrow x^2 \cdot (x^2-3) = 0$

$\Leftrightarrow x_1 = 0$; $x_2 = -\sqrt{3}$; $x_3 = +\sqrt{3}$

x	$(-\infty; -\sqrt{3})$	$(-\sqrt{3}; -1)$	$(-1; 0)$	$(0; 1)$	$(1; \sqrt{3})$	$(\sqrt{3}; +\infty)$
$f'(x)$	+	−	−	−	−	+
$f(x)$	↗	↘	↘	↘	↘	↗

Z tabulky vidíme, že f má lokální maximum pro $x = -\sqrt{3}$ a lok. minimum pro $x = +\sqrt{3}$.

$$f(-\sqrt{3}) = \frac{-(\sqrt{3})^3}{2} = -\frac{3}{2}\sqrt{3} \quad ; \quad f(+\sqrt{3}) = \frac{(\sqrt{3})^3}{2} = \frac{3}{2}\sqrt{3}$$

• globální extrémy: neex. (viz limity v krajních bodech)

6) konvexnost / konkávnost:

$$\boxed{f''(x) = 0} \longrightarrow f''(x) = \left(\frac{x^4 - 3x^2}{(x^2-1)^2}\right)' = \frac{(4x^3 - 6x)(x^2-1)^2 - (x^4 - 3x^2)\cdot 2(x^2-1)(2x)}{(x^2-1)^4} =$$

$$= \frac{(4x^5 - 6x^3 - 4x^3 + 6x) - (4x^5 - 12x^3)}{(x^2-1)^3} = \frac{2x^3 + 6x}{(x^2-1)^3}$$

$$D(f'') = \mathbb{R} \setminus \{+1; -1\}$$

kritické body: $2x \cdot (x^2+3) = 0 \implies x_1 = 0$

x	$(-\infty; -1)$	$(-1; 0)$	$(0; 1)$	$(1; +\infty)$
$f''(x)$	−	+	−	+
$f(x)$	∩	∪	∩	∪

∩ … konkávní

∪ … konvexní

→ Bod $[0; 0]$ je inflexním bodem. Směrnice tečny v tomto bodě činí:

$f'(x)\big|_{x=0} = 0 \implies$ tečna je rovnoběžná s osou x.

7, Asymptoty

bez směrnice: $\lim\limits_{x\to x_0^\pm} f(x) = \pm\infty$; $x_0 \neq \pm\infty$ $\to$ viz limity v krajních bodech $D(f)$

$\Longrightarrow$ svislé asymptoty v bodech: $\boxed{\begin{matrix} x=-1 \\ x=+1 \end{matrix}}$

se směrnicí: Přímka $y = kx+q$ je tzv. šikmou asymptotou grafu funkce f pro $x\to\pm\infty$, jestliže: $\lim\limits_{x\to\pm\infty}\left(f(x)-kx-q\right)=0$

$$\boxed{\lim_{x\to\pm\infty}\frac{f(x)}{x}=k} \qquad \boxed{\lim_{x\to\pm\infty}\left(f(x)-kx\right)=q}$$

Tedy: $\lim\limits_{x\to-\infty}\frac{f(x)}{x} = \lim\limits_{x\to-\infty}\frac{x^2}{x^2-1} = 1$; $\lim\limits_{x\to-\infty}\left(f(x)-1\cdot x\right) = \lim\limits_{x\to-\infty}\left(\frac{x^3}{x^2-1}-x\right) =$

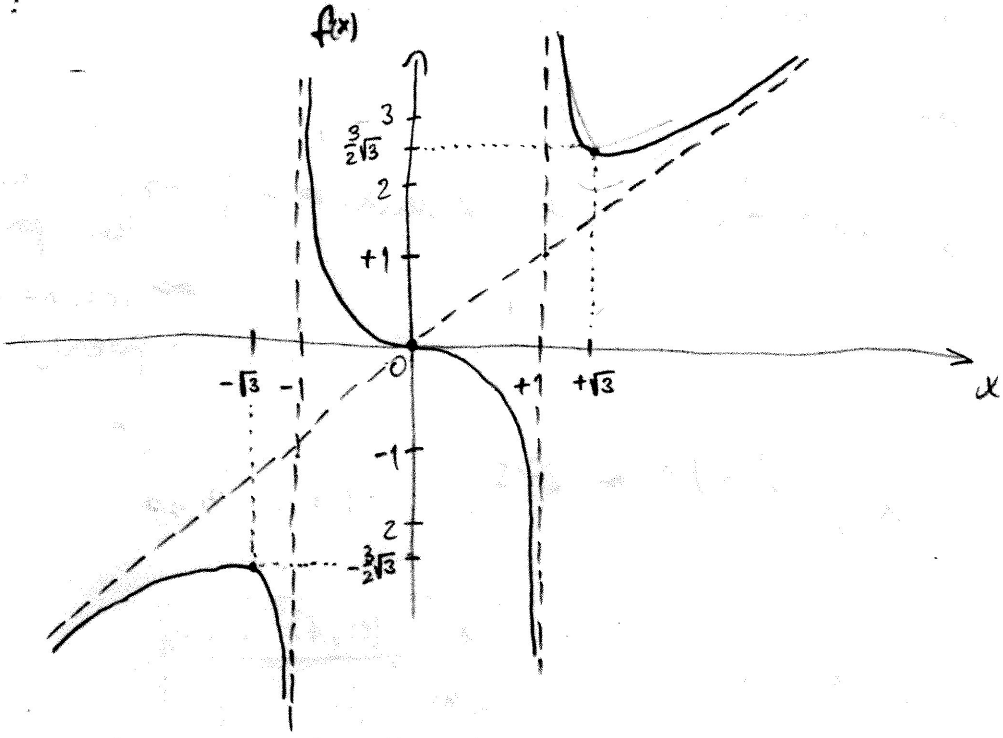
$= \lim\limits_{x\to-\infty}\frac{x}{x^2-1} = 0$ $\Rightarrow$ Pro $x\to-\infty$ máme asymptotu $\boxed{y=x}$

Obdobně pro $x\to+\infty$

$\lim\limits_{x\to+\infty}\frac{f(x)}{x}=1$; $\lim\limits_{x\to+\infty}\left(f(x)-kx\right)=0$ $\Rightarrow$ Opět máme pro $x\to+\infty$ asymptotu $\boxed{y=x}$

Toto souvisí i s tím, že f je lichá.
(Stejná asymptota pro $x\to-\infty$ i $x\to+\infty$)

8) Graf funkce:



Př. 2 Vyšetřete průběh funkce $f(x) = \frac{1}{x} + \ln(x)$

1) $D(f) = (0; +\infty)$

2) Funkce f je zřejmě spojitá v $D(f)$. Krajní body $D(f)$:

- $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{1}{x} + \ln(x) = \lim\limits_{x \to 0^+} \frac{1 + x \cdot \ln(x)}{x} = \left[\lim\limits_{x \to 0^+} x \cdot \ln(x) = \lim\limits_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{l'H}{=} \lim\limits_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0 \right] =$

$$= \frac{1 + 0^-}{0^+} = +\infty$$

- $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \frac{1}{x} + \ln(x) = 0 + \infty = +\infty$

3) Vzhledem ke tvaru $D(f)$ funkce f není ani sudá, ani lichá, ani periodická.

4, Průsečíky s x, y. $f(x) > 0$; $f(x) < 0$

$x = 0$: nedef.

$f(x) = 0$: $\frac{1}{x} + \ln(x) = 0 \longrightarrow x \cdot \ln(x) = -1$ → neumíme řešit přímo

⇒ vrátíme se k tomu později *

5) $f'(x) = \frac{-1}{x^2} + \frac{1}{x}$; $D(f') = \mathbb{R} \setminus \{0\}$

$f'(x) = 0 \iff \frac{1}{x} - \frac{1}{x^2} = 0$

$x - 1 = 0$

$\boxed{x_1 = 1}$

x	$(0;1)$	$(1;+\infty)$
$f'(x)$	$-$	$+$
$f(x)$	↘	↗

→ V bodě $x=1$ má f lokální minimum: $f(1) = 1 + \ln(1) = 1$

* Víme, že $\lim\limits_{x \to 0^+} f(x) = +\infty$ a $\lim\limits_{x \to +\infty} f(x) = +\infty$

navíc pro $x=1$ je $f(1)=1$ lokálním minimem ⇒ $f(x) > 0$ na celém $D(f)$.

- Bod $x=1$ je zároveň globálním minimem. Glob. maxima neexistují.

- $H(f) = \langle 1; +\infty)$

6) Konvexnost / konkávnost :

$\boxed{f''(x)=0}$ $\longrightarrow$ $f''(x) = \left(\frac{1}{x}-\frac{1}{x^2}\right)' = -\frac{1}{x^2}+\frac{2}{x^3} = \frac{2-x}{x^3}$

$D(f'') = \mathbb{R}\setminus\{0\}$; $f''(x)=0 \Leftrightarrow \boxed{x_1=2}$

x	$(0;2)$	$(2;+\infty)$
$f''(x)$	$+$	$-$
$f(x)$	$\cup$	$\cap$

$\rightarrow$ Funkce f má v $x_1=2$ inflexní bod.

$f(2) = \frac{1}{2}+\ln(2) \doteq 1{,}19$

7) Asymptoty :

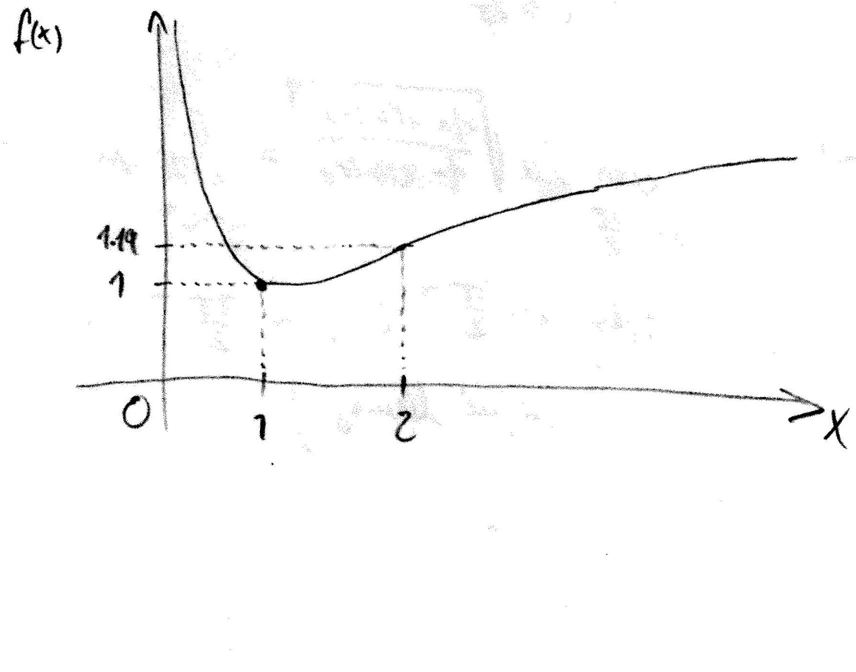
- $\lim\limits_{x\to 0^+} f(x) = +\infty$ $\rightarrow$ svislá asymptota v $x=0$.

$$k = \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\left(\frac{1}{x^2}+\frac{\ln(x)}{x}\right) = 0 + \lim_{x\to+\infty}\frac{\ln(x)}{x} \overset{l'H;\ \frac{\infty}{\infty}}{=} \lim_{x\to+\infty}\frac{\frac{1}{x}}{1} = 0$$

$$q = \lim_{x\to+\infty}\left(f(x)-0\cdot x\right) = +\infty$$

$\Rightarrow$ f nemá šikmou asymptotu pro $x\to+\infty$.

7) Graf funkce f :



Př. 3 Vyšetřete průběh funkce $f(x) = \ln\left(\sqrt{\frac{1+\sin(x)}{1-\sin(x)}}\right)$ v intervalu $x \in \langle 0; 2\pi \rangle$.

1) $D(f)$: Pro $x \in \langle 0; 2\pi \rangle$ musí dále platit:

α) $\frac{1+\sin(x)}{1-\sin(x)} > 0$ a β) $\sin(x) \neq 1 \Rightarrow x \neq \frac{\pi}{2}$

A) $\left.\begin{array}{l} 1+\sin(x) > 0 \\ \wedge \; 1-\sin(x) > 0 \end{array}\right\} x \neq \frac{\pi}{2};\; x \neq \frac{3}{2}\pi$

B) $1+\sin(x) < 0$
$1-\sin(x) < 0$
nemá řešení

$\Rightarrow D(f) = \langle 0; \frac{\pi}{2}) \cup (\frac{\pi}{2}; \frac{3\pi}{2}) \cup (\frac{3\pi}{2}; 2\pi \rangle$

$H(f)$ ... viz později *

2) f je zřejmě spojitá v $D(f)$. Limity v krajních bodech $D(f)$:

$$\lim_{x\to 0^+} \ln\sqrt{\frac{1+\sin(x)}{1-\sin(x)}} = \ln\sqrt{1} = 0 \; ; \; \lim_{x\to 2\pi^-} \ln\sqrt{\frac{1+\sin(x)}{1-\sin(x)}} = \ln\sqrt{1} = 0$$

$$\lim_{x\to \frac{\pi}{2}} \ln\sqrt{\frac{1+\sin(x)}{1-\sin(x)}} = +\infty \qquad \lim_{x\to \frac{3\pi}{2}} \ln\sqrt{\frac{1+\sin(x)}{1-\sin(x)}} = \lim_{y\to 0^+} \ln(y) = -\infty$$

zde limita zprava i zleva je totožná. ; (opět $x \to \frac{3\pi}{2}^+$ i $x \to \frac{3\pi}{2}^-$ vedou na stejné limity)

*

$\Rightarrow H(f) = \mathbb{R}$

3) Vzhledem k $D(f) = \langle 0; 2\pi \rangle / \{\frac{\pi}{2}; \frac{3\pi}{2}\}$ není f sudá, lichá ani periodická

4) Průsečíky s x, y. $f(x) > 0$; $f(x) < 0$:

$x = 0 \longrightarrow f(x \to 0^+) = 0$ viz obrázek

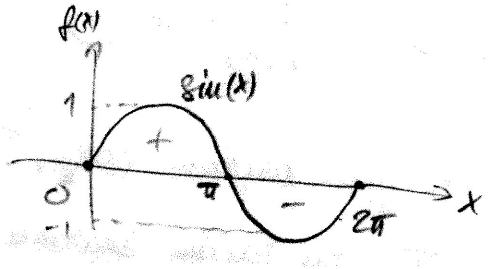
$$f(x) = 0 \longrightarrow 0 = \ln\sqrt{\frac{1+\sin(x)}{1-\sin(x)}} \iff 1 = \frac{1+\sin(x)}{1-\sin(x)}$$

$$2\sin(x) = 0$$

Graf funkce f protíná osu x v bodech: $[0;0]$ a $[\pi;0]$

$$x_1 = 0 \; ; \; x_2 = \pi$$

x	$(0; \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3}{2}\pi)$	$(\frac{3\pi}{2}; 2\pi)$
$f(x)$	+	+	−	−



5) monotonie/extrémy:

$$f'(x) = \frac{d}{dx}\left(\ln\sqrt{\frac{1+\sin(x)}{1-\sin(x)}}\right) = \frac{1}{\sqrt{\frac{1+\sin(x)}{1-\sin(x)}}} \cdot \frac{1}{2\sqrt{\frac{1+\sin(x)}{1-\sin(x)}}} \cdot$$

$$\cdot \frac{\cos(x)\cdot(1-\sin(x)) - (1+\sin(x))(-\cos(x))}{(1-\sin(x))^2} = \frac{1}{2}\,\frac{1-\sin(x)}{1+\sin(x)} \cdot \frac{2\cos(x)}{(1-\sin(x))^2} =$$

$$= \frac{\cos(x)}{(1+\sin(x))(1-\sin(x))} = \frac{1}{\cos(x)} \; ; \quad D(f') = \langle 0; 2\pi \rangle / \{\frac{\pi}{2}; \frac{3\pi}{2}\}$$

stacionární body:

x	$(0; \frac{\pi}{2})$	$(\frac{\pi}{2}; \frac{3}{2}\pi)$	$(\frac{3}{2}\pi; 2\pi)$
$f'(x)$	+	−	+
$f(x)$	↗	↘	↗

→ Jelikož body $x = \frac{\pi}{2}$ a $x = \frac{3}{2}\pi$ nepatří do definičního oboru $D(f)$, tak funkce f nemá žádné lokální extrémy ve vnitřních bodech $D(f)$.

6) konvexnost / konkávnost :

$$f''(x) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \frac{-1\cdot(-\sin(x))}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)}$$

$$D(f'') = D(f') = \langle 0; 2\pi \rangle / \left\{\frac{\pi}{2}; \frac{3}{2}\pi\right\}$$

$$f''(x) = 0 \iff \frac{\sin(x)}{\cos^2(x)} = 0 \longrightarrow x_1 = 0,\quad x_2 = \pi,\quad x_3 = 2\pi$$

Jelikož x_1 a x_3 jsou krajními body, tak zbývá $x_2 = \pi$. Zde určíme směrnici tečny a funkční hodnotu:

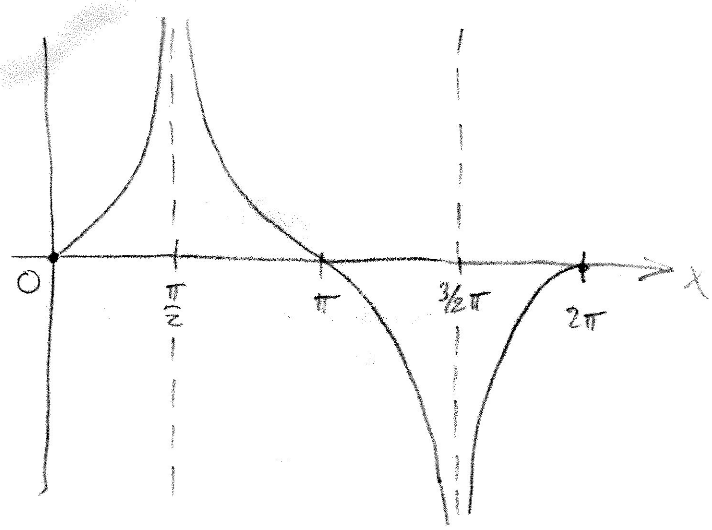
$$f'(\pi) = -1 \; ; \; f(\pi) = 0$$

x	$(0; \frac{\pi}{2})$	$(\frac{\pi}{2}; \pi)$	$(\pi; \frac{3}{2}\pi)$	$(\frac{3}{2}\pi; 2\pi)$
$f''(x)$	+	+	−	−
$f(x)$	$\cup$	$\cup$	$\cap$	$\cap$

7) Asymptoty: Jelikož $\lim\limits_{x \to \pi/2} f(x) = +\infty$ a $\lim\limits_{x \to \frac{3\pi}{2}} f(x) = -\infty$ má zde f svislé asymptoty.

Jelikož jsme na omezeném intervalu $x \in \langle 0; 2\pi \rangle$, nemá smysl určovat šikmé asymptoty.

8) Graf funkce



Cvičení 17.

Taylorovy polynomy:

- Nechť funkce f má derivace až do řádu $(n+1)$ včetně v intervalu $(a;b)$ a nechť $x_0 \in (a;b)$. Pak ke každému $x \in (a;b)$ existuje bod ξ ležící mezi x a x_0 tak, že:

$$\boxed{f(x) = T(f, n, x_0)(x) + R(f, n+1, x_0)(x)}$$

kde

$$\boxed{T(f, n, x_0) \overset{\text{ozn.}}{:=} T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}$$

$$\boxed{R(f, n+1, x_0) \overset{\text{ozn.}}{:=} R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}$$

(Lagrangeův tvar zbytku)

Pozn.: $$\boxed{T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}$$

Př1. Sestavte Taylorův polynom $T_n(x)$ dané funkce f v bodě x_0. Napište, jak lze vyjádřit zbytek po n-tém členu, tj. $R_{n+1}(x)$. Užijte $T_n(x)$ pro výpočet přibližné hodnoty $f(x_1)$ v bodě x_1. Pomocí zbytku $R_{n+1}(x_1)$ odhadněte chybu, které se touto aproximací dopustíte.

a) $f(x) = -2x^3 + 5x^2 + 3x + 1$; $n = 3$; $x_0 = -1$; $x_1 = 4$

$$f(-1) = 5$$

$$f'(-1) = -6x^2 + 10x + 3\Big|_{x=-1} = -13$$

$$f''(-1) = -12x + 10\Big|_{x=-1} = 22$$

$$f'''(-1) = -12\Big|_{x=-1} = -12$$

$$f^{IV}(-1) = 0$$

$$f(x) = 5 + \frac{-13}{1!}(x-(-1)) + \frac{22}{2!}(x-(-1))^2 + \frac{-12}{3!}(x-(-1))^3 + R_4(x) =$$

$$= 5 - 13(x+1) + 11(x+1)^2 - 2(x+1)^3 + R_4(x)$$

kde $R_4(x) = \dfrac{0}{4!}(x-(-1))^4 = 0$

Tedy $f(x_1) = f(4) = 5 - 13(4+1) + 11(4+1)^2 - 2(4+1)^3 =$

$= -35$

(Kontrola: $f(4) = -2\cdot 4^3 + 5\cdot 4^2 + 3\cdot 4 + 1 = -35$)

$\Rightarrow$ Zde, jelikož f je polynom 3. stupně, tak nahrazením (aproximací) Taylorovým polynomem 3. stupně se nedopustíme žádné chyby

$(R_4(x) = 0)$!

b) $f(x) = e^x$; $x_0 = 0$; $n = 5$; $x_1 = 1$

$$\left.\begin{aligned} f'(0) &= e^x\big|_{x=0} = 1 \\ f''(0) &= e^x\big|_{x=0} = 1 \\ &\vdots \\ f^{(n)}(0) &= 1 \quad \forall n \in \mathbb{N} \end{aligned}\right\}$$

$$\left(f(0) = 1 \right)$$

$$f(x) = 1 + \frac{1}{1!}(x-0)^1 + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4 + \frac{1}{5!}(x-0)^5 + R_6(x) =$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + R_6(x)$$

kde $R_6(x) = \dfrac{e^\xi}{6!} x^6$; $\xi \in (0; x)$

$x_1 = 1$: $$f(1) = \underbrace{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}}_{T_5(1)} + \frac{e^\xi}{6!} \qquad \xi \in (0;1)$$

nepřesnost : $R_6(1)$

(nevíme ale kde přesně leží ξ) !

Odhad nepřesnosti: $$|R_6(1)| = \left| \frac{e^\xi}{6!} \right|_{\xi \in (0;1)} \underset{(2<e<3)}{<} \frac{1}{1 \cdot 2 \cdot 4 \cdot 5 \cdot 6} = \frac{1}{240}$$

dále víme, že $\dfrac{1}{6!} < |R_6(1)|$ a navíc zde $R_6(1) > 0$. $\left(\dfrac{e^\xi}{6!} > 0 \right)$

Tedy můžeme např. psát, že:

chyba aproximace

$$f(1) = e^1 = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \delta \ ; \quad \delta \in \left(\frac{1}{6!} ; \frac{3}{6!} \right)$$

$$\delta \in \left(\frac{1}{720} ; \frac{1}{240} \right)$$

c) $f(x) = \sqrt{6-3x}$; $n=3$; $x_0=-1$; $x_1=-2$

$$f(-1) = 3$$

$$f'(-1) = \frac{1}{2\sqrt{6-3x}}(-3)\Big|_{x=-1} = -\frac{1}{2}$$

$$f''(-1) = \frac{-3}{2}\cdot\left(-\frac{1}{2}\right)\cdot(6-3x)^{-3/2}\cdot(-3)\Big|_{x=-1} = \frac{3}{4}\cdot\frac{-3}{3^3} = \frac{-1}{12}$$

$$f'''(-1) = -\frac{9}{4}\cdot\left(-\frac{3}{2}\right)\cdot(6-3x)^{-5/2}\cdot(-3)\Big|_{x=-1} = \frac{-3^4}{8}\cdot\frac{1}{3^5} = -\frac{1}{24}$$

$$f^{(4)}(\xi) = \frac{-3^4}{8}\cdot\left(\frac{-5}{2}\right)\cdot(6-3\xi)^{-7/2}\cdot(-3) = \frac{-5\cdot 3^5}{16}(6-3\xi)^{-7/2}$$

$$T_3(x) = 3 + \frac{-\frac{1}{2}}{1!}(x+1)^1 + \frac{-\frac{1}{12}}{2!}(x+1)^2 + \frac{-\frac{1}{24}}{3!}(x+1)^3$$

$$R_4(x) = \frac{\frac{-5\cdot 3^5}{16}(6-3\xi)^{-7/2}}{4!}(x+1)^4$$; $\xi \in (x;-1)$ $\forall x \leq -1$ nebo $\xi \in (-1;x)$ $\forall x \geq -1$

$$f(x) \cong T_3(x) = 3 - \frac{(x+1)}{2} - \frac{(x+1)^2}{24} - \frac{(x+1)^3}{144}$$

$$f(-2) = \sqrt{12} \overset{T_3(-2)}{\cong} 3 - \frac{(-2+1)}{2} - \frac{(-2+1)^2}{24} - \frac{(-2+1)^3}{144} \cong 3.4653$$

(Přesná hodnota (resp. přesnější): $\sqrt{12} \cong 3.46410161 5$)

Tj. "přesná" hodnota chyby: $\left|\sqrt{12} - T_3(-2)\right| \doteq 0.00118$

Odhad chyby $R_4(-2)$: (5)

$$|R_4(-2)| = \left| \frac{\frac{-5\cdot 3^5}{16}\cdot(6-3\xi)^{-7/2}}{4!}(-2+1)^4 \right| = \frac{3.125}{\left(\sqrt{6-3\xi}\right)^7} < \frac{3.125}{\left(\sqrt{6-3(-1)}\right)^7} =$$

$\xi \in (-2;-1)$

$$= \frac{3.125}{3^7} < \frac{3}{3^7} = \frac{1}{3^6} \doteq \underline{\underline{0.0013}}$$

Př2. Vypočítejte $\sqrt[4]{83}$ s přesností na 4 desetinná místa pomocí Taylorova polynomu.

$$f(x) = x^{1/4} \; ; \; x_0 = 81 \quad \left(\sqrt[4]{81} = 3\right) ; \; x_1 = 83$$

$$f(x_0) = 3$$

$$f'(x_0) = \frac{1}{4}\cdot x^{-3/4}\Big|_{x=81} = \frac{1}{4}\cdot\frac{1}{3^3}$$

$$f''(x_0) = -\frac{3}{4^2}\cdot x^{-7/4}\Big|_{x=81} = \frac{-3}{4^2}\cdot\frac{1}{3^7}$$

$$f'''(x_0) = \frac{21}{4^3}\cdot x^{-11/4}\Big|_{x=81} = \frac{21}{4^3}\cdot\frac{1}{3^{11}} \; ;$$

Zkusme nyní napsat tvar Lagrangeova tvaru zbytku $R_4(x_1)$: (v abs. hodnotě)

$$|R_4(x_1)| = \left| \frac{f^{(4)}(\xi)}{4!}(x_1-x_0)^4 \right| = \left| \frac{-\frac{11\cdot 21}{4^4}\cdot \xi^{-15/4}}{4!}\cdot 2^4 \right|_{\xi\in(81;83)} =$$

$$= \frac{231}{384}\cdot\frac{1}{\sqrt[4]{\xi^{15}}}\Bigg|_{\xi\in(81;83)} < \frac{231}{384}\cdot\frac{1}{\left(\sqrt[4]{81}\right)^{15}} = \underline{\underline{4.2\cdot 10^{-8}}}$$

Tedy v tomto případě je absolutní velikost výpočetní nepřesnosti menší než $4.2\cdot 10^{-8}$. Zkusme nyní vyjádřit $T_2(x_1)$ a $T_3(x_1)$:

$$T_2(x_1) = f(x_0) + \frac{f'(x_0)}{1!}(x_1-x_0)^1 + \frac{f''(x_0)}{2!}(x_1-x_0)^2 =$$

$$= 3 + \frac{\frac{1}{4}\cdot\frac{1}{3^3}}{1!}\cdot 2 + \frac{\frac{-3}{4^2}\cdot\frac{1}{3^7}}{2!}\,2^2 \doteq 3.018\underset{\sim}{34}7\ldots$$

$$T_3(x_1) = T_2(x_1) + \frac{\frac{21}{4^3}\cdot\frac{1}{3^{11}}}{3!}\cdot 2^3 \doteq 3.018\underset{\sim}{34}8\ldots.$$

Tedy je vidět, že $T_3(x_1) \simeq \sqrt[4]{83}$ na dokonce minimálně 5 desetinných míst! Na 4 desetinná místa by zřejmě stačila aproximace polynomem $T_2(x)$.

Cvičení 18.

Neurčité integrály:

Mají-li funkce f a g neurčité integrály v intervalu I a $\alpha \in \mathbb{R}$, pak:

- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \ ; \ x \in I$
- $\int \alpha f(x)\,dx = \alpha \int f(x)\,dx \ ; \ x \in I$

Tabulkové integrály:

$\int 0\,dx = C$	$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C \ (\alpha \neq -1)$
$\int \cos(x)\,dx = \sin(x) + C$	$\int \sin(x)\,dx = -\cos(x) + C$
$\int \frac{1}{\cos^2(x)}\,dx = \operatorname{tg}(x) + C$	$\int \frac{1}{\sin^2(x)}\,dx = -\operatorname{cotg}(x) + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$	$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg}(x) + C$
$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \ (a>0 ; a \neq 1)$	$\int e^x\,dx = e^x + C$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Př. 1 Za pomoci tabulkových integrálů vypočítejte následující integrály:

(2)

$$\int (1-2u)\,du = \int 1\,du - 2\int u\,du = u - 2\frac{u^2}{2} + C = \underline{\underline{u - u^2 + C}}$$

$$\int r^2(1-r^2)\,dr = \int r^2\,dr - \int r^4\,dr = \underline{\underline{\frac{r^3}{3} - \frac{r^5}{5} + C}}$$

$$\int \left(\frac{1}{2x^2} - \frac{1}{2\sqrt{x}}\right) dx = \frac{1}{2}\int x^{-2}\,dx - \frac{1}{2}\int x^{-1/2}\,dx = \frac{1}{2}\frac{x^{-1}}{-1} - \frac{1}{2}\frac{x^{1/2}}{1/2} + C =$$

$$= \underline{\underline{\frac{-1}{2x} - \sqrt{x} + C}}$$

$$\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3}\,dx = \int x^{-5/2}\,dx - \int e^x\,dx + \int \frac{1}{x}\,dx =$$

$$= \underline{\underline{\frac{x^{-3/2}}{-\frac{3}{2}} - e^x + \ln|x| + C}} = \dots$$

$$\int \left(\frac{1-x}{x}\right)^2 dx = \int \frac{1-2x+x^2}{x^2}\,dx = \int x^{-2}\,dx - 2\int \frac{1}{x}\,dx + \int dx =$$

$$= \underline{\underline{\frac{-1}{x} - 2\ln|x| + x + C}}$$

$$\int \frac{1+2x^2}{x^2(1+x^2)}\,dx = \int \frac{1+x^2}{x^2\cdot(1+x^2)}\,dx + \int \frac{x^2}{x^2\cdot(1+x^2)}\,dx = \int \frac{1}{x^2}\,dx + \int \frac{1}{1+x^2} =$$

$$= \underline{\underline{-\frac{1}{x} + \operatorname{arctg}(x) + C}}$$

$$\int 13^x e^x\,dx = \int (13e)^x\,dx = \underline{\underline{\frac{(13e)^x}{\ln(13e)} + C}} \qquad \left(\begin{array}{l}\ln(13e) = \ln(13) + \ln(e) \\ = \ln(13) + 1\end{array}\right)$$

$$\int \frac{\cos(2x)}{\cos^2(x)\cdot\sin^2(x)}\,dx = \int \frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)\cdot\sin^2(x)}\,dx = \int \frac{1}{\sin^2(x)}\,dx - \int \frac{1}{\cos^2(x)}\,dx =$$

$$= \underline{\underline{-\operatorname{cotg}(x) + \operatorname{tg}(x) + C}}$$

Metoda per-partes:

Vzorec: $\int f'(x)\,dx = f(x) + C \rightarrow \int [f(x)\cdot g(x)]'\,dx = f(x)\cdot g(x) + C$

$$\rightarrow \int [f'(x)\,g(x) + f(x)\cdot g'(x)]\,dx = f(x)\cdot g(x) + C$$

$$\rightarrow \boxed{\int f'\cdot g\,dx = f\cdot g - \int f\cdot g'\,dx}$$

konstanta $C \in \mathbb{R}$ "vtažena" do tohoto integrálu ...

Pozn.: Pro ozn.: $f' := f \quad g' := g$
(substituce) $f := F, \; g := G$
máme $\boxed{\int f\cdot G\,dx = F\cdot G - \int F\cdot g\,dx}$

Př2 Metodou per-partes vypočítejte následující integrály:

$$\int \underset{(u\cdot v')}{x e^x}\,dx \overset{P.P.}{=} \begin{bmatrix} u = x & v' = e^x \\ u' = 1 & v = e^x \end{bmatrix} = \underset{(u\cdot v)}{x\cdot e^x} - \int \underset{(u'\cdot v)}{1\cdot e^x}\,dx =$$

$$= \underline{\underline{e^x\cdot(x-1) + C}}$$

$$\int x\ln(x)\,dx \overset{P.P.}{=} \begin{bmatrix} u = \ln(x) & v' = x \\ u' = \frac{1}{x} & v = \frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx =$$

$$= \underline{\underline{\frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C}}$$

$$\int \underset{(F\cdot g)}{(1-3x)\sin(x)}\,dx \overset{P.P.}{=} \underset{F\cdot G}{(1-3x)(-\cos(x))} - \int \underset{G}{-\cos(x)}\cdot\underset{f}{(-3)}\,dx =$$

$$= \underline{\underline{(3x-1)\cos(x) - 3\sin(x) + C}}$$

$$\int \underset{(f\cdot G)}{x\cdot\mathrm{arctg}(x)}\,dx \overset{P.P.}{=} \underset{(F\cdot G)}{\frac{x^2}{2}\cdot\mathrm{arctg}(x)} - \int \underset{F\cdot g}{\frac{x^2}{2}\cdot\frac{1}{1+x^2}}\,dx =$$

trik!

$$= \frac{x^2}{2}\mathrm{arctg}(x) - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\,dx =$$

$$= \frac{x^2}{2}\mathrm{arctg}(x) - \frac{1}{2}\left[\int 1\,dx - \int \frac{1}{1+x^2}\,dx\right] =$$

$$= \underline{\underline{\frac{x^2}{2}\mathrm{arctg}(x) - \frac{1}{2}\left(x - \mathrm{arctg}(x)\right) + C}} = \ldots$$

$$\int \ln(x^2+1)\,dx = \int \underset{(f\ \cdot\ G)}{1\cdot \ln(x^2+1)}\,dx \overset{P.P.}{=} \underset{(F\ \cdot\ G)}{x\cdot \ln(x^2+1)} - \int \underbrace{x}_{F}\cdot \underbrace{\frac{1}{x^2+1}\cdot 2x}_{g}\,dx =$$

$$= x\cdot \ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\,dx =$$

$$= \underline{\underline{x\cdot \ln(x^2+1) - 2\left(x - \operatorname{arctg}(x)\right) + C}}$$

$$\int [\ln(x)]^2\,dx = \int \underset{(f\ \cdot\ G)}{1\cdot \ln^2(x)}\,dx \overset{P.P.}{=} \underset{(F\ \cdot\ G)}{x\cdot \ln^2(x)} - \int \underbrace{x}_{F}\cdot \underbrace{2\ln(x)\cdot\frac{1}{x}}_{g}\,dx =$$

$$= x\cdot \ln^2(x) - 2\int \underset{(f\ \cdot\ G)}{1\cdot \ln(x)}\,dx \overset{P.P.}{=} x\cdot \ln^2(x) - 2\left[\underset{(F\ \cdot\ G)}{x\cdot \ln(x)} - \int \underset{(F\ \cdot\ g)}{x\cdot \frac{1}{x}}\,dx\right]$$

$$= \underline{\underline{x\cdot \ln^2(x) - 2x\cdot \ln(x) - 2x + C}}$$

$$\int \underset{(f_1\ \cdot\ G_1)}{e^x\cdot \sin(x)}\,dx \overset{P.P.}{=} \underset{(F_1\ \cdot\ G_1)}{e^x\cdot \sin(x)} - \int \underset{\substack{(F_1\ \cdot\ g_1)\\ (f_2\ \cdot\ G_2)}}{e^x\cdot \cos(x)}\,dx \overset{P.P.}{=} e^x\cdot \sin -$$

$$-\left[\underset{(F_2\ \cdot\ G_2)}{e^x\cdot \cos(x)} - \int \underset{(F_2\ \cdot\ g_2)}{e^x\cdot(-\sin(x))}\,dx\right] =$$

$$= e^x\cdot\left(\sin(x) - \cos(x)\right) - \int e^x\cdot \sin(x)\,dx$$

Označíme-li : $\int e^x \cdot \sin(x)\,dx = I$, potom máme rovnici :

$$I = e^x \cdot (\sin(x) - \cos(x)) - I$$

$$2I = e^x \cdot (\sin(x) - \cos(x)) \implies I = \int e^x \cdot \sin(x)\,dx = \frac{1}{2} e^x (\sin(x) + \cos(x)) + C$$

Substituční metoda výpočtu neurčitých integrálů :

Vzorec : $[f(g(x))]' = f'(g(x)) \cdot g'(x) \quad / \int$

$$\int [f(g(x))]'\,dx = \boxed{\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C}$$

Resp., ozn. : $\left.\begin{array}{l} f' \to f \\ f \to F \\ g \to \varphi \end{array}\right\}$ $\boxed{\int f(\varphi(x))\,\varphi'(x)\,dx = F(\varphi(x)) + C}$

$$\frac{d\varphi}{dx} = \varphi' \to \boxed{d\varphi = \varphi'(x)\,dx}$$

Př. 3. Užitím substituční metody určete následující integrály :

$$\int \sin(2\varphi - 3)\,d\varphi = \left| \begin{array}{l} \text{sub.} \\ 2\varphi - 3 = x \\ 2d\varphi = dx \\ d\varphi = \frac{dx}{2} \end{array} \right| = \int \sin(x)\,\frac{dx}{2} = -\frac{1}{2}\cos(x) + C =$$

$$= -\frac{1}{2}\cos(2\varphi - 3) + C$$

$$\int \frac{x^2}{x^3+1}\,dx = \left| \begin{array}{l} \text{sub.} \\ x^3+1 = y \\ 3x^2\,dx = dy \end{array} \right| = \frac{1}{3}\int \frac{dy}{y} = \frac{1}{3}\ln|y| + C =$$

$$= \underline{\underline{\frac{1}{3}\ln|1+x^3| + C}}$$

$$\int \operatorname{cotg}(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx = \left| \begin{array}{l} \text{sub.} \\ \sin(x) = y \\ \cos(x)\,dx = dy \end{array} \right| = \int \frac{1}{y}\,dy =$$

$$= \ln|y| + C = \underline{\underline{\ln|\sin(x)| + C}}$$

$$\int \sqrt[5]{(-3x+8)^6}\,dx = \left| \begin{array}{l} \text{sub.} \\ (-3x+8) = y \\ -3dx = dy \\ dx = -\frac{1}{3}\,dy \end{array} \right| = \int y^{6/5}\left(-\frac{1}{3}\right)dy =$$

$$= -\frac{1}{3}\,\frac{y^{11/5}}{\frac{11}{5}} + C = \underline{\underline{-\frac{5}{33}(-3x+8)^{11/5} + C}}$$

$$\int r\sqrt{1-r^2}\,dr \overset{*}{=} \left| \begin{array}{l} \text{sub.} \\ r = \sin\varphi \\ dr = \cos\varphi\,d\varphi \end{array} \right| = \int \sin\varphi \sqrt{1-\sin^2\varphi}\,\cos\varphi\,d\varphi =$$

$$= \int \sin\varphi \cdot \cos^2\varphi\,d\varphi = \dots$$

$*$ nebo $\left| \begin{array}{l} \text{sub.} \\ 1-r^2 = y \\ -2r\,dr = dy \end{array} \right| = \frac{1}{-2}\int \sqrt{y}\,dy = \frac{1}{-2}\cdot\frac{y^{3/2}}{\frac{3}{2}} + C = \underline{\underline{\frac{-1}{3}(1-r^2)^{3/2} + C}}$

Cvičení 19.

Př. 1 Užitím substituční metody určete následující integrály:

$$\int \frac{x^4}{\sqrt{x^5+4}}\,dx = \left| \begin{array}{l} \text{sub.} \\ x^5+4 = y \\ 4x^4\,dx = dy \end{array} \right| = \frac{1}{4}\int \frac{1}{\sqrt{y}}\,dy = \frac{1}{4}\,\frac{y^{1/2}}{1/2} + C =$$

$$= \underline{\underline{\frac{1}{2}\sqrt{x^5+4} + C}}$$

$$\int \frac{\cos(x)}{\sqrt[3]{\sin^2(x)}}\,dx = \left| \begin{array}{l} \text{sub.} \\ \sin(x) = y \\ \cos(x)\,dx = dy \end{array} \right| = \int \frac{dy}{\sqrt[3]{y^2}} = \int y^{-2/3}\,dy =$$

$$= 3\,y^{1/3} + C = \underline{\underline{3\sqrt[3]{\sin(x)} + C}}$$

$$\int \frac{\sqrt{\ln(x)}}{x}\,dx = \left| \begin{array}{l} \text{sub.}\ \ln(x) = y \\ \frac{1}{x}\,dx = dy \end{array} \right| = \int \sqrt{y}\,dy = \frac{2}{3}\,y^{3/2} + C =$$

$$= \underline{\underline{\frac{2}{3}\left(\ln(x)\right)^{3/2} + C}}$$

$$\int \frac{1}{1+9x^2}\,dx = \left| \begin{array}{l} \text{sub.} \\ 3x = y \\ 3dx = dy \end{array} \right| = \int \frac{1}{1+y^2}\,\frac{dy}{3} =$$

$$= \frac{1}{3}\operatorname{arctg}(y) + C = \underline{\underline{\frac{1}{3}\operatorname{arctg}(3x) + C}}$$

$$\int \frac{1}{\sqrt{1-x^2}\,\arcsin^3(x)}\,dx = \left| \begin{array}{l} \text{sub.} \\ \arcsin(x) = y \\ \frac{1}{\sqrt{1-x^2}}\,dx = dy \end{array} \right| = \int \frac{1}{y^3}\,dy =$$

$$= \frac{1}{-2y^2} + C = \underline{\underline{\frac{1}{-2\cdot\arcsin^2(x)} + C}}$$

$$\int \left(\frac{1}{\sqrt{\ln(x)}} + \frac{1}{1+\ln^2(x)} \right) \frac{1}{x}\,dx = \left| \begin{array}{l} \text{sub.} \\ \ln(x) = y \\ \frac{1}{x}\,dx = dy \end{array} \right| = \int \frac{1}{\sqrt{y}}\,dy + \int \frac{1}{1+y^2}\,dy =$$

$$= \frac{y^{1/2}}{\frac{1}{2}} + \operatorname{arctg}(y) + C =$$

$$= \underline{\underline{2\sqrt{\ln(x)} + \operatorname{arctg}(\ln(x)) + C}}$$

$$\int \frac{x}{x^4+1}\,dx = \left|\begin{matrix}\text{sub.}\\ x^2 = y \\ 2x\,dx = dy\end{matrix}\right| = \frac{1}{2}\int \frac{dy}{1+y^2} = \frac{1}{2}\operatorname{arctg}(y) + C$$

$$= \underline{\underline{\frac{1}{2}\operatorname{arctg}(x^2) + C}}$$

$$\int \frac{2x-1}{x-2}\,dx = \int \frac{x-2+x+1}{x-2}\,dx = \int\left(1 + \frac{x+1}{x-2}\right)dx =$$

$$= \left|\begin{matrix}\text{sub.}\\ x-2 = y \\ dx = dy\end{matrix}\right| = \int\left(1 + \frac{y+3}{y}\right)dy =$$

$$= \int\left(2 + \frac{3}{y}\right)dy = 2y + 3\ln|y| + C = \underline{\underline{2(x-2) + 3\ln|x-2| + C}}$$

$$\int \arcsin(x)\,dx \overset{\text{P.P.}}{=} \left|\begin{matrix} f = 1\ ;\ G = \arcsin(x) \\ F = x\ ;\ g = \dfrac{1}{\sqrt{1-x^2}}\end{matrix}\right| = x\cdot\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx =$$

$$= \left|\begin{matrix}\text{sub.}\\ 1-x^2 = y \\ -2x\,dx = dy\end{matrix}\right| = x\cdot\arcsin(x) + \frac{1}{2}\int \frac{1}{\sqrt{y}}\,dy =$$

$$= x\cdot\arcsin(x) + \frac{1}{2}\,\frac{y^{1/2}}{\frac{1}{2}} + C =$$

$$= \underline{\underline{x\cdot\arcsin(x) + \sqrt{1-x^2} + C}}$$

$$\int (1+e^{3x})^2 e^{3x}\,dx = \left| \begin{matrix} \text{sub.} \\ e^{3x} = y \\ 3e^{3x}dx = dy \end{matrix} \right| = \frac{1}{3}\int (1+y)^2\,dy =$$

$$= \frac{1}{3}\left(y + \frac{2y^2}{2} + \frac{y^3}{3} \right) + C =$$

$$= \underline{\underline{\frac{1}{3}\left(e^{3x} + e^{6x} + \frac{1}{3}e^{9x} \right) + C}}$$

?

nebo: $\left| \begin{matrix} \text{sub.} \\ 1+e^{3x} = y \\ 3e^{3x}dx = dy \end{matrix} \right| = \frac{1}{3}\int y^2\,dy = \frac{1}{9}y^3 + C = \underline{\underline{\frac{1}{9}(1+e^{3x})^3 + C}}$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \left| \begin{matrix} \text{sub.} \\ \sqrt{x} = y \\ \frac{1}{2\sqrt{x}}dx = dy \end{matrix} \right| = 2\int \cos(y)\,dy = 2\sin(y) + C =$$

$$= \underline{\underline{2\sin(\sqrt{x}) + C}}$$

$$\int \frac{\operatorname{cotg}(x)}{\ln(\sin(x))}\,dx = \left| \begin{matrix} \text{sub.} \quad \sin(x) = y \\ \cos(x)dx = dy \end{matrix} \right| = \int \frac{\frac{1}{y}}{\ln(y)}\,dy =$$

$$= \left| \begin{matrix} \text{sub.} \quad \ln(y) = z \\ \frac{1}{y}dy = dz \end{matrix} \right| = \int \frac{1}{z}\,dz = \ln|z| + C$$

$$= \underline{\underline{\ln|\ln(\sin(x))| + C}}$$

(5)

$$\int \frac{\sin^2(x)\cos(x)}{1+\sin^2(x)}\,dx = \left| \begin{array}{l} \text{sub.} \\ \sin(x) = y \\ \cos(x)\,dx = dy \end{array} \right| = \int \frac{y^2 \; (\pm 1)}{1+y^2}\,dy =$$

$$= \int \left(1 - \frac{1}{1+y^2}\right) dy = y + \operatorname{arctg}(y) + C =$$

$$= \underline{\underline{\sin(x) + \operatorname{arctg}(\sin(x)) + C}}$$

$$\int \frac{\arcsin(x)}{\left(\sqrt{1-x^2}\right)^3}\,dx = \int \frac{\arcsin(x)}{\sqrt{1-x^2}} \cdot \frac{1}{1-x^2}\,dx =$$

$$= \left| \begin{array}{l} \text{sub.} \\ \arcsin(x) = y \;;\; x = \sin(y) \\ \frac{1}{\sqrt{1-x^2}}\,dx = dy \;;\; \frac{1}{1-x^2} = \frac{1}{1-\sin^2(y)} \end{array} \right| =$$

$$= \int \frac{y}{1-\sin^2(y)}\,dy = \underbrace{\int \frac{y}{\cos^2(y)}\,dy}_{(f \cdot G)} \overset{\text{P.P}}{=} \left| \begin{array}{l} f = \frac{1}{\cos^2(y)} \;;\; G = y \\ F = \operatorname{tg}(y) \;;\; g = 1 \end{array} \right| =$$

$$= \underbrace{y \cdot \operatorname{tg}(y)}_{(F \cdot G)} - \underbrace{\int \operatorname{tg}(y)\,dy}_{F \cdot g} = \left| \begin{array}{l} \cos(y) = u \\ -\sin(y)\,dy = du \end{array} \right| =$$

$$= y \cdot \operatorname{tg}(y) + \int \frac{1}{u}\,du = y \cdot \operatorname{tg}(y) + \ln|u| + C =$$

$$= \underline{\underline{\arcsin(x) \cdot \operatorname{tg}(\arcsin(x)) + \ln\left|\cos(\arcsin(x))\right| + C}}$$

Pozn.: $tg(x) = \frac{\sin(x)}{\cos(x)} = \frac{\sin(x)}{\sqrt{1-\sin^2(x)}}$

$$tg(\arcsin(x)) = \frac{\sin(\arcsin(x))}{\sqrt{1-\sin^2(\arcsin(x))}} = \frac{x}{\sqrt{1-x^2}}$$

$$\cos(x) = \sqrt{1-\sin^2(x)}$$

$$\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))} = \sqrt{1-x^2}$$

Teď můžeme i upravit na tvar:

$$\int \frac{\arcsin(x)}{(\sqrt{1-x^2})^3}\,dx = \boxed{\frac{x\cdot\arcsin(x)}{\sqrt{1-x^2}} + \ln\left|\sqrt{1-x^2}\right| + C}$$

Cvičení 20.

Integrace racionálních funkcí: $P(x)/Q(x)$

$\boxed{st(P) \leq 1\ ;\ st(Q) = 2}$

a) $$\frac{P(x)}{Q(x)} = \frac{P(x)}{q_2 x^2 + q_1 x + q_0}$$ (nelze již dále rozložit

b) $$\frac{P(x)}{Q(x)} = \frac{P(x)}{q_0 (x-\alpha)^2} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2}$$

c) $$\frac{P(x)}{Q(x)} = \frac{P(x)}{q_0 (x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$$

$\boxed{st(P) \leq 2\ ;\ st(Q) = 3}$

a) $$\frac{P(x)}{q_0 (x-\alpha)(x^2+rx+s)} = \frac{A}{x-\alpha} + \frac{Bx+C}{x^2+rx+s}$$

b) $$\frac{P(x)}{q_0 (x-\alpha)^3} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \frac{A_3}{(x-\alpha)^3}$$

c) $$\frac{P(x)}{q_0 (x-\alpha)(x-\beta)^2} = \frac{A}{x-\alpha} + \frac{B_1}{x-\beta} + \frac{B_2}{(x-\beta)^2}$$

d) $$\frac{P(x)}{q_0 (x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$$

Př. 1. Vypočítejte neurčité integrály:

$$\int \frac{1}{(x+6)^5}\,dx = \left| \begin{array}{ll} \text{Sub.} & x+6 = y \\ & dx = dy \end{array} \right| = \int y^{-5}\,dy = \frac{y^{-4}}{-4} + C =$$

$$= \underline{\underline{\frac{-1}{4(x+6)^4} + C}}$$

$$\int \frac{2x-1}{(x-1)(x-2)}\,dx = \left| \begin{array}{l} \dfrac{2x-1}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} = \dfrac{Ax-2A+Bx-B}{(x-1)(x-2)} \\ \text{Koeficienty u } x^n\text{:} \\ \left.\begin{array}{ll} x^0\text{:} & -1 = -2A-B \\ x^1\text{:} & 2 = A+B \end{array}\right\} \quad -1 = -2A + A - 2 \\ \boxed{A=-1;\ B=3} \end{array} \right| =$$

$$= \int \frac{-1}{x-1}\,dx + \int \frac{3}{x-2}\,dx = -\ln|x-1| + 3\ln|x-2| + C =$$

$$= \underline{\underline{\ln\left|\frac{(x-2)^3}{x-1}\right| + C}}$$

$$\int \frac{x+1}{x^2+6x+9}\,dx =$$

$x^2+6x+9 = (x+3)^2$ ⟵ odhad, nebo řešíme $x^2+6x+9=0$ a najdeme nulové body

$$\frac{x+1}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

$$\frac{x+1}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)(x+3)}$$

Koeficienty u x^n:

$$\left.\begin{array}{l} x^0: \ 1 = 3A + B \\ x^1: \ 1 = A \end{array}\right\} \quad \boxed{\begin{array}{l} A = 1 \\ B = -2 \end{array}}$$

$$= \int \frac{1}{x+3}\,dx + \int \frac{-2}{(x+3)^2}\,dx = \ln|x+3| + \int \frac{-2}{(x+3)^2}\,dx =$$

$$= \left| \begin{array}{l} \text{sub.} \\ x+3 = y \\ dx = dy \end{array} \right| = \ln|x+3| - 2\int y^{-2}\,dy =$$

$$= \ln|x+3| - 2\,\frac{y^{-1}}{-1} + c = \underline{\underline{\ln|x+3| + \frac{2}{x+3} + c}}$$

$$(x \in \mathbb{R} \setminus \{-3\})$$

4

$$\int \frac{u-1}{u^2+u}\,du = \left| \begin{array}{l} \frac{u-1}{u\cdot(u+1)} = \frac{A}{u} + \frac{B}{u+1} = \frac{A(u+1)+B\cdot u}{u\cdot(u+1)} \\ \text{Koeficienty u } u^n: \\ \left.\begin{array}{l} u^0:\ -1 = A \\ u^1:\ \ 1 = A+B \end{array}\right\} \boxed{\begin{array}{l} A=-1 \\ B=2 \end{array}} \end{array} \right| =$$

$$= \int \frac{-1}{u}\,du + \int \frac{2}{u+1}\,du \overset{*}{=} -\ln|u| + 2\ln|u+1| + C =$$

$$= \underline{\underline{\ln\left|\frac{(u+1)^2}{u}\right| + C}}$$

---

*Pozn.:

$$\int \frac{f'(x)}{f(x)}\,dx = \left| \begin{array}{l} \text{sub. } f(x) = y \\ f'(x)\,dx = dy \end{array} \right| = \int \frac{1}{y}\,dy = \ln|y| + C =$$

$$= \underline{\underline{\ln|f(x)| + C}}$$

$$\int \frac{1+x^2}{1-x^2}\,dx = -\int \frac{-1-x^2}{1-x^2}\,dx = -\int \frac{1-x^2-2}{1-x^2}\,dx =$$

$$= \int \frac{2}{1-x^2} - 1\,dx = -x + \int \frac{2}{1-x^2}\,dx =$$

$$= \left| \frac{2}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x} = \frac{Ax+A-Bx+B}{(1-x)(1+x)} \right| =$$

Koeffizienten x^n:

$$\left.\begin{aligned} x^0 &: \; 2 = A+B \\ x^1 &: \; 0 = A-B \end{aligned}\right\} \boxed{\begin{aligned} A&=1 \\ B&=1 \end{aligned}}$$

$$= -x + \int \frac{1}{1-x}\,dx + \int \frac{1}{1+x}\,dx =$$

$$= -x - \int \frac{-1}{1-x}\,dx + \int \frac{1}{1+x}\,dx = \underline{\underline{-x + \ln\left|\frac{1+x}{1-x}\right| + c}}$$

$$\int \frac{x^3}{x^2+3x+2}\,dx = \int \frac{x\cdot(x^2+3x+2 \quad -3x-2)}{x^2+3x+2}\,dx =$$

$$= \int x - \frac{3x^2+2x}{x^2+3x+2}\,dx = \frac{x^2}{2} - 3\int \frac{x^2+\frac{2}{3}x}{x^2+3x+2}\,dx =$$

$$= \frac{x^2}{2} - 3\int \frac{(x^2+3x+2)+\left(-\frac{7}{3}x-2\right)}{x^2+3x+2}\,dx =$$

$$= \frac{x^2}{2} - 3\int 1\,dx + \int \frac{7x+6}{x^2-3x+2}\,dx =$$

$$x^2+3x+2 = (x+1)(x+2)$$

$$\frac{7x+6}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} =$$

$$= \frac{Ax+2A+Bx+B}{(x+1)(x+2)}$$

Koeficienty u x^n:

x^0: $6 = 2A+B$

x^1: $7 = A+B$

$7 = A+(6-2A)$

$$\boxed{A=-1}$$
$$\boxed{B=8}$$

$$= \frac{x^2}{2} - 3x + \int \frac{-1}{x+1}\,dx + \int \frac{8}{x+2}\,dx =$$

$$= \frac{x^2}{2} - 3x + \ln\left|\frac{(x+2)^8}{x+1}\right| + C$$

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$$\int \frac{x}{(x+1)(x+3)(x+5)}\,dx =$$

$$(+0\cdot x^0 + 0\cdot x^2)$$

$$\frac{1\cdot x}{(x+1)(x+3)(x+5)} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{x+5} = \frac{A(x+3)(x+5) + B(x+1)(x+5) + C(x+1)(x+3)}{(x+1)(x+3)(x+5)}$$

$=$

Koeficienty u x^n:

x^0: $0 = 15A + 5B + 3C$

x^1: $1 = 8A + 6B + 4C$

x^2: $0 = A + B + C$

$$\left(\begin{array}{ccc|c} 15 & 5 & 3 & 0 \\ 8 & 6 & 4 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 1 \\ 0 & -10 & -12 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 1 \\ 0 & 0 & 8 & -5 \end{array}\right) \longrightarrow \quad C = -\frac{5}{8}$$

$$B = \frac{1 + 4\left(-\frac{5}{8}\right)}{-2} = \frac{12}{16} = \frac{6}{8}$$

$$A = -\frac{1}{8}$$

$=$

$$= -\frac{1}{8}\int \frac{1}{x+1}\,dx + \frac{6}{8}\int \frac{1}{x+3}\,dx - \frac{5}{8}\int \frac{1}{x+5}\,dx =$$

$$= \frac{1}{8}\ln\left|\frac{(x+3)^6}{(x+5)^5(x+1)}\right| + C$$

Cvičení 21.

Integrace goniometrických funkcí a jejich mocnin:

$$\boxed{\int \sin^n(x) \cdot \cos^m(x)\, dx \quad ; \quad m, n \in \mathbb{Z}}$$

---

m - liché:

$$\left.\begin{matrix} m = 2k+1 \\ k \in \mathbb{Z} \end{matrix}\right\} \quad \sin^n(x) \cdot \cos^{2k+1}(x)\, dx = \sin^n(x) \cdot \underbrace{\left(1 - \sin^2(x)\right)^k \cdot \cos(x)}_{\cos^{2k+1}(x)}\, dx =$$

$$\rightarrow \left\{\begin{matrix} \sin(x) = t \\ \cos(x)dx = dt \end{matrix}\right\} = t^n \cdot \left(1 - t^2\right)^k dt$$

---

n - liché:

$$\left.\begin{matrix} n = 2k+1 \\ k \in \mathbb{Z} \end{matrix}\right\} \quad \sin^{2k+1}(x) \cdot \cos^m(x)\, dx = \left(1 - \cos^2(x)\right)^k \cdot \cos^m(x) \cdot \sin(x)\, dx$$

$$\rightarrow \left\{\begin{matrix} \cos(x) = t \\ -\sin(x)\,dx = dt \end{matrix}\right\} = -\left(1 - t^2\right)^k \cdot t^m\, dt$$

---

m, n - sudé:

$$\left.\begin{matrix} m = 2k \\ n = 2p \\ k, p \in \mathbb{Z} \end{matrix}\right\}$$

1. upravíme: $\sin^{2k}(x) \cdot \cos^{2p}(x)\, dx = \left[\dfrac{1 - \cos(2x)}{2}\right]^k \cdot \left[\dfrac{1 + \cos(2x)}{2}\right]^p dx$
2. použijeme substituci: $2x = t$
   a upravujeme dále podle 1. dokud není k nebo p liché.
3. Potom postupujeme podle návodu viz výše.

Př. 1.

②

$$\int \cos^3(x)\,dx \overset{\text{Sub.}}{=} \left[\begin{matrix} t = \sin(x) \\ dt = \cos(x)\,dx \end{matrix}\right] = \int \underbrace{(1-t^2)}_{\cos^2(x)}\,dt = t - \frac{t^3}{3} + C =$$

$$= \sin(x) - \frac{\sin^3(x)}{3} + C$$

$$\int \sin^5(x)\,dx = \left[\begin{matrix} t = \cos(x) \\ dt = -\sin(x)\,dx \end{matrix}\right] = -\int \overbrace{(1-t^2)^2}^{\sin^4(x)}\,dt =$$

$$= -\left(t - \frac{2t^3}{3} + \frac{t^5}{5}\right) + C = -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C$$

$$\int \cos^4(x)\sin^3(x)\,dx = \int \cos^4(x)\cdot\left(1-\cos^2(x)\right)\sin(x)\,dx \overset{\text{Sub.}}{=} \left[\begin{matrix} t = \cos(x) \\ dt = -\sin(x)\,dx \end{matrix}\right] =$$

$$= \int t^4\cdot(1-t^2)(-1)\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C =$$

$$= \frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5} + C$$

3

$$\int \cos^3(x)\sin(x)\,dx = \overset{\text{Sub.}}{\begin{bmatrix} t = \cos(x) \\ dt = -\sin(x)\,dx \end{bmatrix}} = \int -t^3\,dt = -\frac{t^4}{4} + C =$$

$$= -\frac{\cos^4(x)}{4} + C$$

$$\int \cos^3(4x)\sin^9(4x)\,dx = \overset{\text{Sub.}}{\begin{bmatrix} \text{Zde můžeme zvolit buď: } t = \sin(4x) \\ \text{nebo} \quad : t = \cos(4x) \\ \Rightarrow \text{Nechť } \boxed{t = \sin(4x)};\ dt = \cos(4x)\cdot 4\,dx \end{bmatrix}} =$$

$$= \int (1-t^2)\cdot t^9 \cdot \frac{1}{4}\,dt = \frac{1}{4}\left(\frac{t^{10}}{10} - \frac{t^{12}}{12}\right) + C =$$

$$= \frac{\sin^{10}(4x)}{40} - \frac{\sin^{12}(4x)}{48} + C$$

$$\int \sin^2(x)\,dx = \left[\sin^2(x) = \frac{1-\cos(2x)}{2}\right] = \int \left(\frac{1}{2} - \frac{\cos(2x)}{2}\right) dx =$$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos(2x)\,dx = \overset{\text{Sub.}}{\begin{bmatrix} 2x = t \\ 2dx = dt \end{bmatrix}} = \frac{1}{2}x - \frac{1}{4}\int \cos(t)\,dt =$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(t) + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int \cos^2(2x)\,dx = \int \frac{1+\cos(4x)}{2}\,dx = \begin{bmatrix} \text{Sub.} \\ 4x = t \\ dx = \frac{1}{4}dt \end{bmatrix} =$$

$$= \frac{1}{4}\int \frac{1+\cos(t)}{2}\,dt = \frac{1}{4}\left(\frac{t+\sin(t)}{2}\right) + C =$$

$$= \frac{1}{4}\left(\frac{4x+\sin(4x)}{2}\right) + C = \underline{\underline{\frac{x}{2} + \frac{\sin(4x)}{8} + C}}$$

$$\int \sin^2(x)\cos^2(x)\,dx = \begin{bmatrix} 2\sin(x)\cos(x) = \sin(2x) \\ \sin^2(x)\cos^2(x) = \frac{\sin^2(2x)}{4} \\ \sin^2(2x) = \frac{1-\cos(4x)}{2} \end{bmatrix} = \frac{1}{4}\int \frac{1-\cos(4x)}{2}\,dx =$$

$$= \begin{bmatrix} \text{Sub.} \\ 4x = t \\ dx = \frac{1}{4}dt \end{bmatrix} = \frac{1}{4}\int \frac{1-\cos(t)}{2}\cdot\frac{1}{4}\,dt =$$

$$= \frac{1}{32}\left(t - \sin(t)\right) + C = \underline{\underline{\frac{1}{32}\left(4x - \sin(4x)\right) + C}}$$

Integrace iracionálních funkcí typu $R\left(x, \sqrt[n]{(ax+b)/(cx+d)}\right)$

→ $R(u,v)$ – racionální funkce 2 proměnných u a v.
Podíl dvou polynomů : $R(u,v) = \dfrac{P(u,v)}{Q(u,v)}$.

→ $\displaystyle\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$. Pro $ad-bc \neq 0$ můžeme použít

substituci : $t = \sqrt{\dfrac{ax+b}{cx+d}}$; $dt = \dfrac{1}{2\sqrt{\dfrac{ax+b}{cx+d}}} \cdot \dfrac{a\cdot(cx+d) - c\,(ax+b)}{(cx+d)^2}\,dx$

Př. 2.

$$\int \frac{1}{1+\sqrt{x+1}}\,dx = \left[\begin{array}{l} \text{Sub} \\ \sqrt{x+1} = t \\ \text{buď} \quad \dfrac{1}{2\sqrt{x+1}}\,dx = dt \\ \text{nebo} \quad x+1 = t^2 \; ; \; x \geq -1 \\ \qquad dx = 2t\,dt \end{array}\right] =$$

$$= \int \frac{2t \overset{(\pm 2)}{}}{1+t}\,dt = \int \left(2 - \frac{2}{1+t}\right) dt = 2t - 2\ln|1+t| + C =$$

$$= \underline{\underline{2\sqrt{x+1} - 2\ln\left(1+\sqrt{1+x}\right) + C}}$$

$$\int \frac{3x}{\sqrt{2x+1}}\,dx = \overset{\text{Sub.}}{\left[\begin{array}{l} 2x+1 = t^2 ;\ t = \sqrt{2x+1} \\ 2dx = 2t\,dt \\ dx = t\,dt \end{array}\right]} = \int \frac{3\cdot\frac{t^2-1}{2}}{t}\,t\,dt =$$

$$= 3\int \frac{t^2-1}{2}\,dt = \cancel{6}\,\frac{3}{2}\left(\frac{t^3}{3} - t\right) + C =$$

$$= 2\left(\sqrt{2x+1}\right)^3 - 6\sqrt{2x+1} + C$$

$$\int \frac{\sqrt{x-1}}{x+2}\,dx = \left[\begin{array}{l} x-1 = t^2 ;\ \sqrt{x-1} = t \\ dx = 2t\,dt \end{array}\right] = \int \frac{t}{t^2+3}\,2t\,dt =$$

$$= 2\int \frac{(t^2+3)-3}{t^2+3}\,dt = 2\int\left(1 - \frac{3}{t^2+3}\right)dt =$$

$$= 2t - 3\int \frac{1}{t^2+3}\,dt = 2t - 3\int \frac{\frac{1}{3}}{\left(\frac{t}{\sqrt{3}}\right)^2+1}\,dt =$$

$$= \overset{\text{Sub.}}{\left[\begin{array}{l} \frac{t}{\sqrt{3}} = u \\ dt = \sqrt{3}\,du \end{array}\right]} = 2t - \sqrt{3}\int \frac{1}{1+u^2}\,du =$$

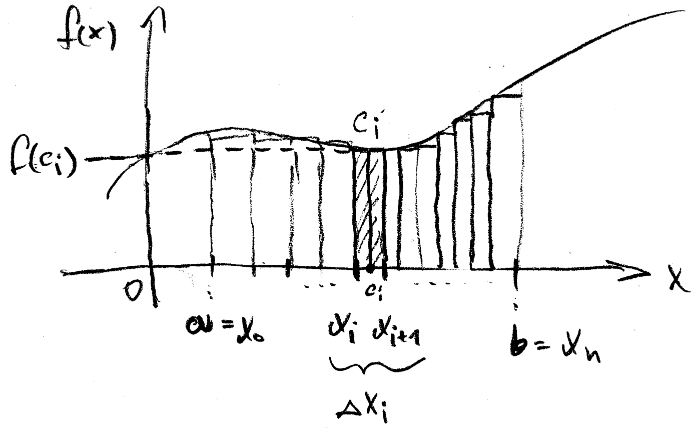
$$= 2t - \sqrt{3}\,\mathrm{arctg}(u) + C = 2t - \sqrt{3}\,\mathrm{arctg}\left(\frac{t}{\sqrt{3}}\right) + C =$$

$$\boxed{= 2\sqrt{x-1} - \sqrt{3}\,\mathrm{arctg}\left(\frac{\sqrt{x-1}}{\sqrt{3}}\right) + C}$$

Cvičení 22.

Riemanův integrál. Newtonova-Leibnitzova formule.

◦ Riemanův integrál

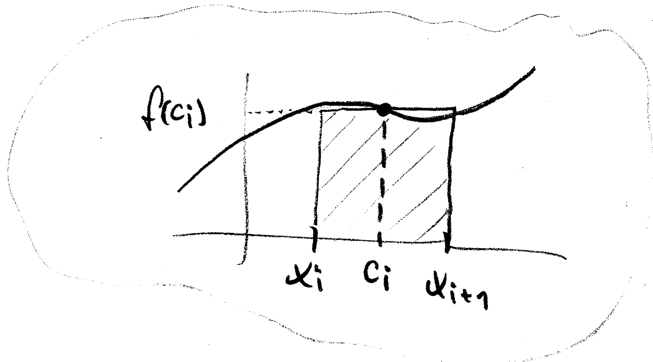


Riemanova suma: $x \in \langle a, b \rangle$

$$\boxed{R(f,D,C) := \sum_{i=0}^{n-1} f(c_i)\,\Delta x_i}$$

$$\boxed{\begin{array}{l} D = (x_0, \dots, x_n) \\ C = (c_0, \dots, c_{n-1}) \\ x_i \leq c_i \leq x_{i+1} \end{array}} \; ; \; \boxed{\Delta x_i = x_{i+1} - x_i}$$

$$\boxed{I = \int_a^b f(x)\,dx = \lim_{\substack{\max(\Delta x_i) \to 0 \\ 0 \leq i \leq n-1}} R(f,D,C)}$$



◦ Newtonova-Leibnitzova formule

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

(Je-li f spojitá funkce v intervalu $\langle a, b \rangle$ a F je primitivní funkce k f v $\langle a, b \rangle$)

„Schéma odvození"

- $$F'(x_i) = f(x_i) \overset{\text{def.}}{=} \lim_{|x_{i+1} - x_i| \to 0} \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i}$$

  Pozn.: $\left(|x_{i+1} - x_i| > 0 ; \; (x_{i+1} - x_i) = \Delta x_i \right)$

- $$\lim_{\Delta x_i \to 0} f(x_i)\,\Delta x_i = \lim_{\Delta x_i \to 0} \left(F(x_{i+1}) - F(x_i) \right)$$

$$\bullet \quad \underbrace{\sum_{\forall i} \lim_{\Delta x_i \to 0} f(x_i)\,\Delta x_i}_{\int_a^b f(x)\,dx} = \sum_{\forall i} \lim_{\Delta x_i \to 0} \left(F(x_{i+1}) - F(x_i) \right)$$

$$\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \Big[\big(F(x_n) - F(x_{n-1})\big) + \big(F(x_{n-1}) - F(x_{n-2})\big) + \dots \big(F(x_2) - F(x_1)\big) + \big(F(x_1) - F(x_0)\big) \Big]$$

$\varnothing$ $\varnothing$

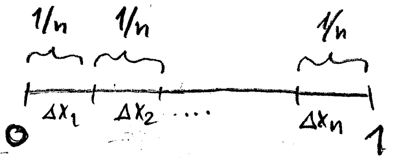
$$\boxed{\int_a^b f(x)\,dx} = \lim_{\Delta x_i \to 0} \big(F(x_n = b) - F(x_0 = a) \big) = \boxed{F(b) - F(a)}$$

Př.:

Ukázka výpočtu určitého integrálu Riemannovým přístupem:

$$\int_0^1 x^2\,dx = \lim_{\Delta x_i \to 0} \sum_{\forall i} f(c_i)\,\Delta x_i \overset{*}{=} \lim_{n \to +\infty} \sum_{i=0}^{n} \frac{i^2}{n^2} \cdot \frac{1}{n} =$$

*



$$c_i = \Delta x_i \cdot i = \frac{i}{n}$$

$$f(c_i) = (\Delta x_i \cdot i)^2 = \frac{i^2}{n^2}$$

$$= \lim_{n \to +\infty} \left[\frac{1}{n^3} \left(\sum_{i=0}^{n} i^2 \right) \right] = \lim_{n \to +\infty} \frac{1}{n^3} \left(\frac{2n^3 + 3n^2 + n}{6} \right)$$

$$= \underline{\underline{\frac{1}{3}}}$$

Newton-Leibnitzova formule

$$\longrightarrow \int_0^1 x^2\,dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \underline{\underline{\frac{1}{3}}}$$

Př. 1

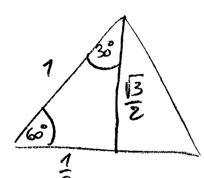
$$\int_1^2 (x^3+3x^2-5)\,dx = \left[\frac{x^4}{4} + x^3 - 5x\right]_1^2 = \left(\frac{16}{4} + 8 - 10\right) - \left(\frac{1}{4} + 1 - 5\right) = 2 - \left(-\frac{15}{4}\right) = \underline{\underline{\frac{23}{4}}}$$

$$\int_1^2 \left(x^2 + \frac{1}{x^4}\right) dx = \left[\frac{x^3}{3} - \frac{1}{3x^3}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{24}\right) - \left(\frac{1}{3} - \frac{1}{3}\right) = \frac{63}{24} = \underline{\underline{\frac{21}{8}}}$$

$$\int_0^{\pi/4} \frac{1}{\cos^2(x)}\,dx = \Big[\operatorname{tg}(x)\Big]_0^{\pi/4} = 1 - 0 = \underline{\underline{1}}$$

$$\int_0^{\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi} = -\Big[(-1) - 1\Big] = \underline{\underline{2}}$$

$$\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \Big[\operatorname{arctg}(x)\Big]_{\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}}^{\sqrt{3}} = \Big[\quad\Big] =$$



$$= \frac{\pi}{3} - \frac{\pi}{6} = \underline{\underline{\frac{\pi}{6}}}$$

$$\int_0^8 \left(\sqrt{2x} + \sqrt[3]{x}\right) dx = \left[\sqrt{2} \cdot \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{4/3}}{\frac{4}{3}}\right]_0^8 =$$

$$= \left(\sqrt{2} \cdot \frac{2}{3} \cdot \left(\sqrt{8}\right)^3 + \frac{3}{4}\left(\sqrt[3]{8}\right)^4\right) - 0 = \frac{64}{3} + 12 = \underline{\underline{\frac{100}{3}}}$$

Př. 2

$$\int_2^3 \frac{x}{x^2+1} dx = \left[\begin{array}{ll} \text{Sub.} & \\ x^2+1 = y & f(x) = \frac{x}{x^2+1} \text{ je spojitá na } x \in \mathbb{R} \\ 2x\,dx = dy & y(x) = x^2+1 \text{ zobrazuje interval } (-\infty; +\infty) \\ x=2 \rightarrow y=5 & \text{do intervalu } (1; +\infty) \\ x=3 \rightarrow y=10; & \end{array}\right] =$$

$$= \int_5^{10} \frac{1}{2}\frac{1}{y} dy = \left[\frac{\ln y}{2}\right]_5^{10} = \frac{1}{2}\left(\ln(10) - \ln(5)\right) = \underline{\underline{\frac{1}{2}\ln(2)}}$$

$$\int_0^\pi \underset{F}{x} \cdot \underset{g}{\sin(x)}\, dx \overset{\text{P.P.}}{=} \underset{F \cdot G}{\left[-x \cdot \cos(x)\right]_0^\pi} - \int_0^\pi \underset{f \cdot G}{-\cos(x)\, dx} = -\pi \cdot (-1) + \left[\sin(x)\right]_0^\pi =$$

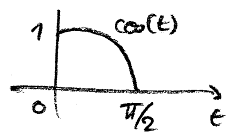
$$= \pi + (0 - 0) = \pi$$

⑤

$$\int_1^{e^3} \frac{1}{x\cdot\sqrt{1+\ln(x)}}\,dx = \left[\begin{array}{ll} \text{sub.} & \\ 1+\ln(x) = y & f(x) = \frac{1}{x\cdot\sqrt{1+\ln(x)}} \text{ je} \\ \frac{1}{x}dx = dy & \text{spojitá (minimálně) na } \langle 1; e^3\rangle \\ x=1 \to y=1 & \\ x=e^3 \to y=4\ ; & [1+\ln(x)] \text{ zobrazuje interval} \\ & \langle 1; e^3\rangle \text{ do intervalu } \langle 1;4\rangle \end{array}\right] =$$

$$= \int_1^4 \frac{1}{\sqrt{y}}\,dy = \left[2\sqrt{y}\right]_1^4 = 4-2 = \underline{\underline{2}}$$

$$\int_0^{\pi/2} \sin^3(t)\sqrt{\cos(t)}\,dt = \left[\begin{array}{ll} \text{sub.} & \\ \cos(t) = y & \cos(t) \text{ zobrazí } \langle 0; \frac{\pi}{2}\rangle \\ -\sin(t)dt = dy & \text{do } \langle 0;1\rangle \\ t=0 \to y=1 & \\ t=\frac{\pi}{2} \to y=0\ ; & \end{array}\right] =$$



$$= \int_1^0 -(1-y^2)\sqrt{y}\,dy = \int_0^1 (1-y^2)\,y^{1/2}\,dy = \left[\frac{2}{3}y^{3/2} - \frac{2}{7}y^{7/2}\right]_0^1 =$$

$$= \left(\frac{2}{3}-\frac{2}{7}\right) - (0-0) = \underline{\underline{\frac{8}{21}}}$$

$$\int_0^1 \frac{\operatorname{arctg}(z)}{1+z^2}\,dz = \left| \int \frac{\operatorname{arctg}(z)}{1+z^2}\,dz \overset{\text{Sub.}}{=} \left[\begin{array}{l} \operatorname{arctg}(z) = y \\ \frac{1}{1+z^2}\,dz = dy \end{array} \right] = \int y\,dy = \right.$$ (6)

$$D\left(\frac{\operatorname{arctg}(z)}{1+z^2}\right) = \mathbb{R} \qquad = \frac{y^2}{2} + C = \left(\frac{\operatorname{arctg}(z)}{2}\right)^2 + C \Bigg|$$

$$= \left[\frac{\operatorname{arctg}^2(z)}{2}\right]_0^1 = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - 0 = \underline{\underline{\frac{\pi^2}{32}}}$$

$$\int_1^2 \frac{1}{x^2+x}\,dx \overset{\text{P.Z.}}{=} \left[\begin{array}{l} \frac{1}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x\cdot(x+1)} \\ x^0:\ 1 = A \\ x^1:\ 0 = A+B \end{array} \left.\begin{array}{l} \\ A=1 \\ B=-1 \end{array}\right. \right] =$$

$$= \int_1^2 \left[\frac{1}{x} - \frac{1}{x+1}\right] dx = \Big[\ln(x)\Big]_1^2 - \Big[\ln(x+1)\Big]_1^2 = \Big(\ln(2) - \ln(1)\Big) -$$

$$- \Big(\ln(3) - \ln(2)\Big) =$$

$$= 2\ln(2) - \ln(3) = \underline{\underline{\ln\left(\frac{4}{3}\right)}}$$

$$\int_1^e \frac{\ln^2(x)}{x}\,dx = \left[\begin{array}{ll} \text{Sub.} & \\ \ln(x) = y & \ln(x): [1;e] \to [0;1] \\ \frac{1}{x}dx = dy & \\ x=1 \to y=0 & \\ x=e \to y=1 & \end{array}\right] =$$

$$= \int_0^1 y^2\,dy = \left[\frac{y^3}{3}\right]_0^1 = \underline{\underline{\frac{1}{3}}}$$

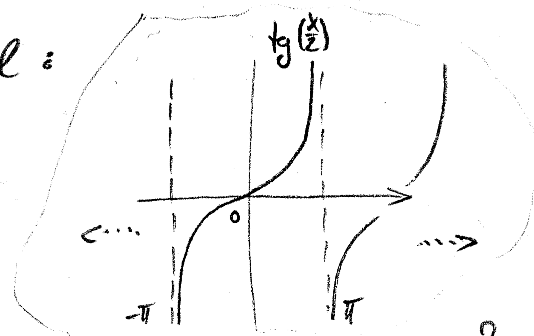
Pozn. (o integraci substitucí)

- Předpokládejme, že funkce $f(x)$ je spojitá v intervalu I.
  Předpokládejme, že funkce $x = g(t)$ má spojitou derivaci v intervalu J a zobrazuje J na I. Pak platí:

$$\int f(x)\,dx = \int f(g(t)) \cdot g'(t)\,dt \qquad \forall x \in I, \forall t \in J, \ x = g(t)$$

Př3. Spočítejte následující určitý integrál:

$$\int_0^{2\pi} \frac{1}{2+\sin(x)-\cos(x)}\,dx$$



subst.

$$t = tg\left(\frac{x}{2}\right) \;;\quad x = 2\,arctg(t) \;;\quad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{1}{2+\sin(x)-\cos(x)}\,dx =$$

Goniom. vzorce:

$$1+tg^2(x) = \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\Longrightarrow \cos^2(x) = \frac{1}{1+tg^2(x)} \;;\quad \sin^2(x) = 1-\cos^2(x) = \frac{tg^2(x)}{1+tg^2(x)}$$

$$\Longrightarrow \cos(x) = \cos^2\left(\tfrac{x}{2}\right) - \sin^2\left(\tfrac{x}{2}\right) = \frac{1-t^2}{1+t^2}$$

$$\sin(x) = 2\sin\left(\tfrac{x}{2}\right)\cdot\cos\left(\tfrac{x}{2}\right) = \frac{2t}{1+t^2}$$

* $\boxed{t = tg\left(\frac{x}{2}\right)}$ zobrazí každý z intervalů $(-\pi+2k\pi\,;\,\pi+2k\pi) = I_k$, $k\in\mathbb{Z}$ na interval $t\in(-\infty;\infty)$. Na těchto intervalech I_k jsou splněny podmínky věty o substituci.

$$= \int \frac{1}{2+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int\frac{2\,dt}{3t^2+2t+1} =$$

$$= \int\frac{2\,dt}{3\left(t+\frac{1}{3}\right)^2+\frac{2}{3}} = \int\frac{3\,dt}{1+\left[\frac{3}{\sqrt{2}}\left(t+\frac{1}{3}\right)\right]^2} = \left[\begin{array}{l}\text{subst.}\\ s = \frac{3}{\sqrt{2}}\left(t+\frac{1}{3}\right)\\ ds = \frac{3}{\sqrt{2}}\,dt\end{array}\right] =$$

$$\sqrt{2}\,arctg(s)+C = \boxed{\sqrt{2}\,arctg\left(\frac{3\,tg\left(\frac{x}{2}\right)+1}{\sqrt{2}}\right)+C}$$

Víme, že primitivní fce $F(x) = \int \frac{1}{2+\sin(x)-\cos(x)}\,dx =$

$= \sqrt{2}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\left(\frac{x}{2}\right)+1}{\sqrt{2}}\right) + C$ existuje na každém z intervalů

$I_k = (-\pi + 2k\pi;\ \pi + 2k\pi)$. Zkusme tedy dodefinovat ("nalepit") v bodech $\{\pi + 2k\pi \mid k \in \mathbb{Z}\}$. Zde, za předpokladu spojitosti $F(x)$ musí platit:

$$\boxed{\lim_{x\to(\pi+2k\pi)^-} F(x) + C_k = \lim_{x\to(\pi+2k\pi)^+} F(x) + C_{k+1}}$$

$$\lim_{x\to(\pi+2k\pi)^-} \sqrt{2}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\left(\frac{x}{2}\right)+1}{\sqrt{2}}\right) = \sqrt{2}\lim_{y\to+\infty}\mathrm{arctg}(y) = \sqrt{2}\,\frac{\pi}{2}$$

$$\lim_{x\to(\pi+2k\pi)^+} \sqrt{2}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\left(\frac{x}{2}\right)+1}{\sqrt{2}}\right) = \sqrt{2}\lim_{y\to-\infty}\mathrm{arctg}(y) = -\sqrt{2}\,\frac{\pi}{2}$$

$$\Rightarrow \boxed{C_{k+1} - C_k = \sqrt{2}\cdot\pi}$$

$$\Rightarrow F(x) = \begin{cases} \sqrt{2}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\left(\frac{x}{2}\right)+1}{\sqrt{2}}\right) + \sqrt{2}\pi\cdot k & \forall x \in (-\pi+2k\pi;\ \pi+2k\pi) \\ \sqrt{2}\pi\cdot k & \forall x \in \{\pi+2k\pi\} \end{cases} \quad k\in\mathbb{Z}$$

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$$\int_0^{2\pi} \frac{1}{2+\sin(x)-\cos(x)}\,dx = F(2\pi) - F(0) =$$

$$= \left(\sqrt{2}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\left(\frac{2\pi}{2}\right)+1}{\sqrt{2}}\right) + \sqrt{2}\pi \right)_{k=1} - \left(\sqrt{2}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\left(\frac{0}{2}\right)+1}{\sqrt{2}}\right) + 0 \right)$$

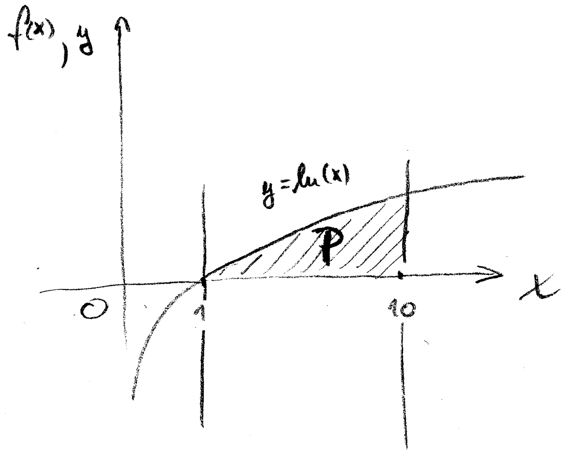
$$= \sqrt{2}\,\mathrm{arctg}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{2}\pi - \sqrt{2}\,\mathrm{arctg}\left(\frac{1}{\sqrt{2}}\right) = \underline{\underline{\sqrt{2}\,\pi}}$$

Cvičení 23.

Geometrické aplikace určitého integrálu:

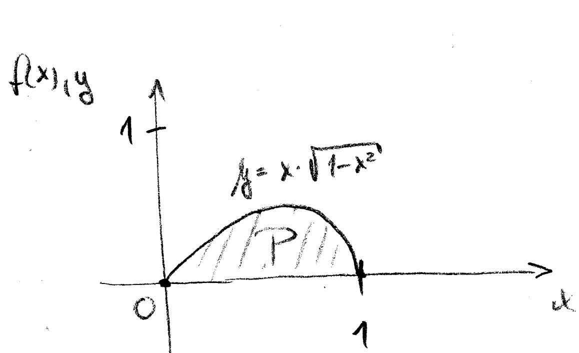
Př. 1 Určete obsahy P křivočarých lichoběžníků ohraničených osou x a křivkami o rovnicích:

a) $y = \ln(x)$; $x = 1$; $x = 10$



$$\Rightarrow P = \int_1^{10} \ln(x)\,dx \overset{P.P.}{=} \Big[x \cdot \ln(x)\Big]_1^{10} - \int_1^{10} x \cdot \frac{1}{x}\,dx = 10 \cdot \ln(10) - 1 \cdot \ln(1) - \Big[x\Big]_1^{10} = \underline{\underline{10 \cdot \ln(10) - 9}}$$

b) $y = x\sqrt{1-x^2}$; $x = 0$; $x = 1$



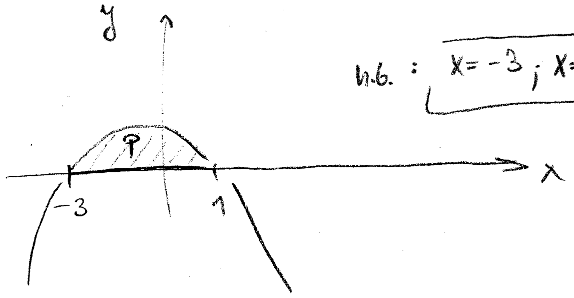
$$P = \int_0^1 x\sqrt{1-x^2}\,dx = \left[\begin{array}{ll} \text{Subst} & \\ 1 - x^2 = y & x = 0 \Rightarrow y = 1 \\ -2x\,dx = dy \,; & x = 1 \Rightarrow y = 0 \end{array}\right]$$

($x = \sqrt{1-y}$ je spojitá a hladká fce na $y \in (0;1)$ a zobrazí tento interval do intervalu $x \in (0;1)$)

$$= \frac{1}{-2}\int_1^0 \sqrt{y}\,dy = \frac{1}{2}\left[\frac{y^{3/2}}{3/2}\right]_0^1 = \underline{\frac{1}{3}}$$

Př 2 Určete obsahy rovinných obrazců ohraničených křivkami o rovnicích:

a) $y = 3 - 2x - x^2$; $y = 0$



n.b.: $\boxed{x = -3 ;\ x = 1}$

$$P = \int_{-3}^{1} (3 - 2x - x^2)\,dx =$$

$$= \left[3x - x^2 - \frac{x^3}{3} \right]_{-3}^{1} =$$

$$= \left(3 - 1 - \frac{1}{3}\right) - \left(-9 - 9 - \left(\frac{-27}{3}\right)\right) =$$

$$= \frac{32}{3}$$

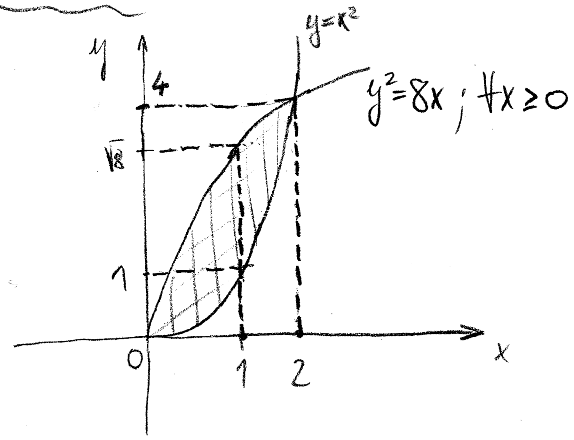
Př 3 Určete objem tělesa vzniklého rotací obrazce ohraničeného křivkami o rovnicích:

a) kolem osy x

b) kolem osy y

$y^2 = 8x$; $y = x^2$

Obr.:



Meze integrace:

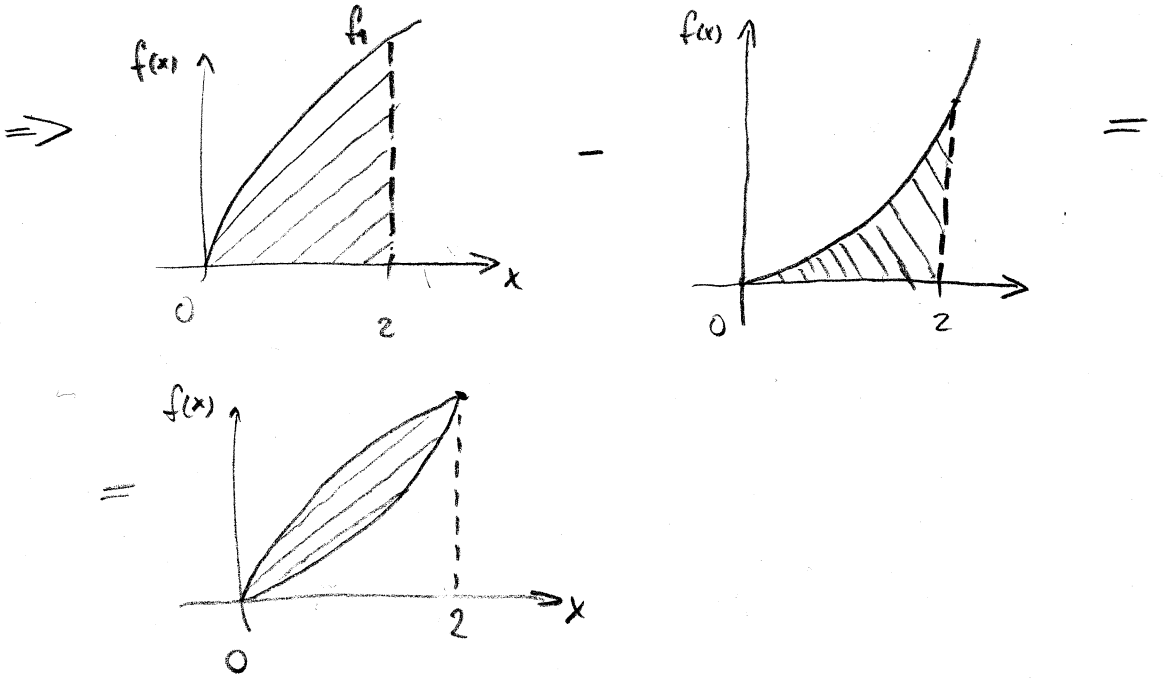
$$(x^2)^2 = 8x \rightarrow x \cdot (8 - x^3) = 0$$

$$\boxed{\begin{aligned} x_1 &= 0 \Rightarrow y_1 = 0 \\ x_2 &= 2 \Rightarrow y_2 = 4 \end{aligned}}$$

Pozn.: Objem rotačního tělesa vzniklého rotací obrazce ohraničeného shora grafem funkce f, zdola osou x a ze stran přímkami $x = a$, $x = b$ okolo osy x je dán vztahem:

$$\boxed{V = \pi \int_a^b f^2(x)\,dx}$$

a) Zde můžeme objem rotačního tělesa vyjádřit jako rozdíl objemů V_1 a V_2 vzniklých rotacemi obrazců ohraničených funkcemi $f_1(x) = \sqrt{8x}$ a $f_2(x) = x^2$ okolo osy x.



Tedy: $$V = V_1 - V_2 = \pi\int_0^2 f_1^2(x)\,dx - \pi\int_0^2 f_2^2(x)\,dx =$$

$$= \pi\int_0^2 (\sqrt{8x})^2\,dx - \pi\int_0^2 (x^2)^2\,dx = \pi\cdot\left[4x^2 - \frac{x^5}{5}\right]_0^2 =$$

$$= \pi\left(16 - \frac{32}{5}\right) = \underline{\underline{\frac{48\pi}{5}}} \quad (= 9{,}6\pi)$$

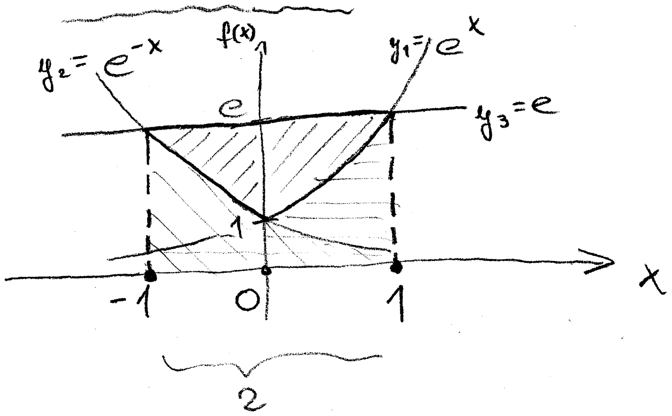
b) Postupujeme obdobně jako v případě a). Nyní však uvažujeme $x_1 = f_1^{-1}(y)$ a $x_2 = f_2^{-1}(y)$. Tedy $x_1 = \frac{1}{8}y^2$ a $x_2 = \sqrt{y}$ $\forall y \in (0;4)$. Objem vzniklý rotací okolo osy y je pak určen rozdílem

$$V = \pi\int_0^4 (\sqrt{y})^2\,dy - \pi\int_0^4 \left(\frac{1}{8}y^2\right)^2 dy = \pi\left[\frac{y^2}{2} - \frac{y^5}{5\cdot 64}\right]_0^4 =$$

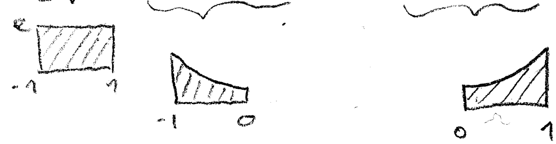
$$= \pi\cdot\left(8 - \frac{16\cdot 64}{5\cdot 64}\right) = \pi\cdot\frac{40-16}{5} = \underline{\underline{4{,}8\pi}}$$

Př. 4 Načrtněte obrazec, který je ohraničen danými křivkami a vypočítejte jeho obsah:

a) $y_1 = e^{x}$; $y_2 = e^{-x}$; $y_3 = e$



$$S = 2e - \int_{-1}^{0} e^{-x}\,dx - \int_{0}^{1} e^{x}\,dx =$$

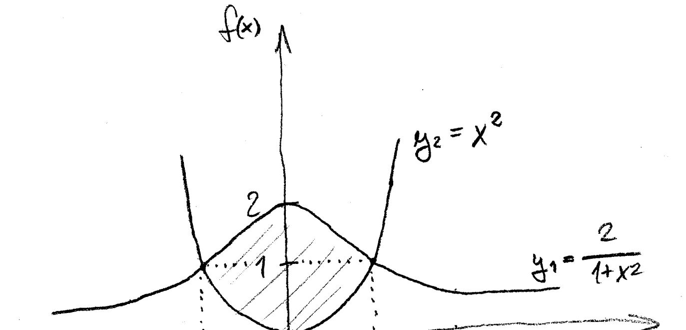


$$= 2e - \left[-e^{-x}\right]_{-1}^{0} - \left[e^{x}\right]_{0}^{1} =$$

$$= 2e - (-1+e) - (e-1) =$$

$$= \underline{\underline{2}}$$

b) $y_1 = \dfrac{2}{1+x^2}$; $y_2 = x^2$



sym.

$$S = 2\int_{0}^{1} \left(\frac{2}{1+x^2} - x^2\right) dx =$$

$$= 2\cdot\left[2\operatorname{arctg}(x) - \frac{x^3}{3}\right]_{0}^{1} =$$

$$= 2\left(2\cdot\frac{\pi}{4} - \frac{1}{3}\right) = \underline{\underline{\pi - \frac{2}{3}}}$$

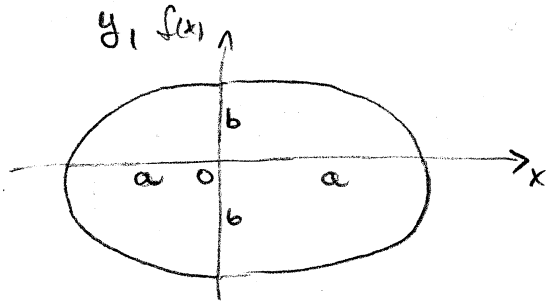
Průsečíky grafů funkcí $y_1(x)$ a $y_2(x)$:

$$x^2 = \frac{2}{1+x^2} \quad \rightarrow \quad x^2 = u \geq 0$$

$$u\cdot(1+u) = 2 \quad \Rightarrow \quad u^2 + u - 2 = 0$$

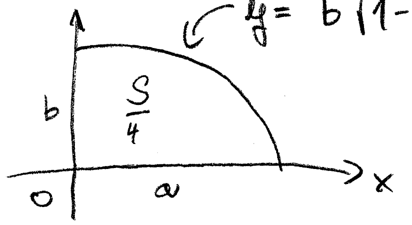
$$u_{1,2} = \frac{-1 \pm \sqrt{1 - 4\cdot 1\cdot(-2)}}{2} = \frac{-1 \pm 3}{2} \quad \begin{matrix} -2 < 0 \\ 1 > 0 \end{matrix} \Bigg\} \; x^2 = 1 \Rightarrow \boxed{x = \pm 1}$$

Př. 5 Odvoďte vzorec pro obsah elipsy určené hlavní poloosou $\underline{a}$ a vedlejší poloosou $\underline{b}$.



$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$ rovnice elipsy

$\rightarrow$



$y = b\sqrt{1-\frac{x^2}{a^2}}$; $x \geq 0$; $y \geq 0$

$$\rightarrow \quad \frac{S}{4} = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \left[\begin{array}{ll} \text{sub.} & \\ x = a\cdot\sin\varphi & x=0 \rightarrow \varphi = 0 \\ dx = a\cdot\cos\varphi\,d\varphi; & x=a \rightarrow \varphi=\frac{\pi}{2} \end{array}\right] =$$

$\left(+ \text{fce } x = a\cdot\sin\varphi \text{ zobrazí } (0;\frac{\pi}{2}) \rightarrow (0;a) \text{ a je na } (0;\frac{\pi}{2}) \text{ hladká}\right)$

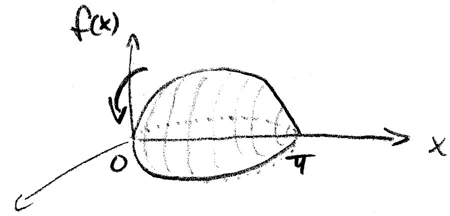
$$= \int_0^{\pi/2} a\cdot b\cdot\cos^2\varphi\,d\varphi =$$

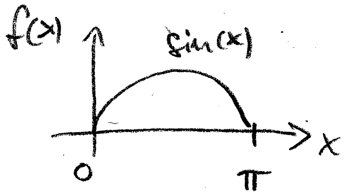
$$= a\cdot b\int_0^{\pi/2}\frac{1+\cos 2\varphi}{2}\,d\varphi = a\cdot b\cdot\left[\frac{1}{2}\varphi + \frac{\sin 2\varphi}{4}\right]_0^{\pi/2} = \frac{\pi}{4}\cdot a\cdot b$$

$$\Rightarrow \boxed{S = \pi\cdot a\cdot b}$$

Př. 6 Určete obsah S rotační plochy vzniklé rotací oblouku křivky dané grafem funkce f kolem osy x pro $x \in \langle a, b \rangle$.

$f(x) = \sin(x) \qquad x \in \langle 0 ; \pi \rangle$





$$\boxed{S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx} = 2\pi \int_0^{\pi} \sin(x) \sqrt{1 + \cos^2(x)}\, dx =$$

subst.

$$= \left[\begin{array}{ll} \cos(x) = u & x = 0 \to u = 1 \\ -\sin(x)\,dx = du \,; & x = \pi \to u = -1 \end{array} \right] = -2\pi \int_1^{-1} \sqrt{1+u^2}\, du = 2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du =$$

+ předpoklady věty o subst. splněny

subst.

$$= \left[\begin{array}{ll} u = tg(\varphi) & u = -1 \to \varphi = -\frac{\pi}{4} \\ du = \frac{1}{\cos^2\varphi} d\varphi \,; & u = +1 \to \varphi = +\frac{\pi}{4} \\ 1 + u^2 = \frac{1}{\cos^2\varphi} & \end{array} \right] = 2\pi \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \frac{1}{\cos\varphi} \cdot \frac{1}{\cos^2\varphi}\, d\varphi \overset{*}{=}$$

$$* \quad \int \frac{1}{\cos^3\varphi}\, d\varphi \overset{P.P.}{=} \left[\begin{array}{ll} \frac{1}{\cos\varphi} = F \,; & \frac{\sin\varphi}{\cos^2\varphi} = f \\ \frac{1}{\cos^2\varphi} = g \,; & tg\,\varphi = G \end{array} \right] = \frac{tg\,\varphi}{\cos\varphi} - \int tg\,\varphi \cdot \frac{\sin\varphi}{\cos^2\varphi}\, d\varphi$$

$$= \frac{tg\,\varphi}{\cos\varphi} - \int \frac{1 - \cos^2\varphi}{\cos^3\varphi}\, d\varphi \implies$$

$$\int \frac{1}{\cos^3\varphi}\,d\varphi = \frac{tg\varphi}{\cos\varphi} + \int \frac{1}{\cos\varphi}\,d\varphi - \int \frac{1}{\cos^3\varphi}\,d\varphi$$

$$\Rightarrow \int \frac{1}{\cos^3\varphi}\,d\varphi = \frac{1}{2}\,\frac{\sin\varphi}{\cos^2\varphi} + \frac{1}{2}\int \frac{1}{\cos\varphi}\,d\varphi \quad **$$

$$** \int \frac{1}{\cos\varphi}\,d\varphi = \left[\begin{array}{ll} \text{Sub.} & \\ tg\left(\frac{\varphi}{2}\right) = t & \cos\varphi = \frac{1-t^2}{1+t^2} \\ d\varphi = \frac{2}{1+t^2}\,dt & \end{array}\right] = \int \frac{1+t^2}{1-t^2}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1-t^2}\,dt = \int \frac{1}{1+t}\,dt + \int \frac{1}{1-t}\,dt = \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln\left|\frac{1+tg\left(\frac{\varphi}{2}\right)}{1-tg\left(\frac{\varphi}{2}\right)}\right| + C$$

$$* \quad 2\pi\int_{-\pi/4}^{+\pi/4} \frac{1}{\cos^3\varphi}\,d\varphi = 2\pi\left[\frac{1}{2}\,\frac{\sin\varphi}{\cos^2\varphi}\right]_{-\pi/4}^{+\pi/4} + 2\pi\cdot\frac{1}{2}\left[\ln\left|\frac{1+tg\left(\frac{\varphi}{2}\right)}{1-tg\left(\frac{\varphi}{2}\right)}\right|\right]_{-\pi/4}^{+\pi/4} =$$

$$= 2\pi\,\frac{1}{2}\left(\sqrt{2}-(-\sqrt{2})\right) + 2\pi\,\frac{1}{2}\left(\ln(1+\sqrt{2}) - \ln(\sqrt{2}-1)\right) =$$

$$2\pi\cdot\left[\sqrt{2} + \frac{1}{2}\ln\left(\frac{(1+\sqrt{2})^2}{2-1}\right)\right] = \underline{\underline{\left[\sqrt{2} + \ln(1+\sqrt{2})\right]\cdot 2\pi}}$$

Př. 7 Určete střední hodnotu funkce na daném intervalu, tj. hodnotu $\boxed{\mu(f)_{\langle a,b\rangle} = \frac{1}{b-a}\int_a^b f(x)\,dx}$

a) $f(x) = \sin^2(x)$; $x \in \langle 0;\pi\rangle$

b) $x\cdot\sin(x)$; $x \in \langle 0;\pi\rangle$

a) $$\mu_{\langle 0;\pi\rangle}(\sin^2(x)) = \frac{1}{\pi}\int_0^\pi \sin^2(x)\,dx = \frac{1}{\pi}\int_0^\pi \frac{1-\cos(2x)}{2}\,dx =$$

$$= \frac{1}{\pi}\left(\frac{1}{2}x - \frac{\sin(2x)}{4}\right)_0^\pi = \underline{\underline{\frac{1}{2}}}$$

b) $$\mu_{\langle 0;\pi\rangle}(x\cdot\sin(x)) = \frac{1}{\pi}\int_0^\pi x\cdot\sin(x)\,dx \overset{\text{P.P.}}{=} \frac{1}{\pi}\Big[-x\cdot\cos(x)\Big]_0^\pi - \frac{1}{\pi}\int_0^\pi -\cos(x)\cdot 1\,dx =$$

$$= \frac{1}{\pi}\cdot\big(-\pi\cdot(-1) - 0\big) + \frac{1}{\pi}\Big[\sin(x)\Big]_0^\pi = \underline{\underline{1}}$$

Př. 8 Ověřte, zda následující nevlastní integrály konvergují.

Pokud ano, určete jejich hodnoty.

$$\int_1^e \frac{1}{x\cdot \ln(x)} = \left| \int \frac{1}{x\cdot \ln(x)}\,dx = \begin{bmatrix} y = \ln(x) \\ dy = \frac{1}{x}\,dx \end{bmatrix} = \int \frac{1}{y}\,dy = -\frac{1}{y^2} + C = \frac{-1}{\ln^2(x)} + C \quad \forall x \in \begin{cases} I_1 = (0;1) \\ I_2 = (1;+\infty) \end{cases} \right| =$$

$$= \lim_{A\to 1^+} \left[-\frac{1}{\ln^2(x)} \right]_A^e = -1 - (-\infty) = \underline{+\infty}$$

→ Zadaný nevlastní integrál diverguje

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{A\to 0^+} \int_A^4 \frac{1}{\sqrt{x}}\,dx = \lim_{A\to 0^+} \left[2\sqrt{x} \right]_A^4 = 4 - 0 = \underline{4}$$

$$\int_1^{+\infty} \frac{1}{x(x+1)^2}\,dx = \lim_{B\to+\infty} \int_1^B \frac{1}{x(x+1)^2}\,dx \overset{\text{P.Z.}}{=} \left| \frac{1}{x\cdot(x+1)^2} = \frac{A}{x} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} \qquad 1 = A(x+1)^2 + B_1\cdot x(x+1) + B_2 x \right.$$

$$\left. \begin{aligned} 1 &= A \\ 0 &= 2A + B_1 + B_2 \\ 0 &= A + B_1 \end{aligned} \right\} \begin{aligned} A &= 1 \\ B_1 &= -1 \\ B_2 &= -1 \end{aligned} \quad \Bigg| = \lim_{B\to+\infty} \int_1^B \left[\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right] dx =$$

$$\lim_{B\to+\infty}\left[\ln\left(\frac{x}{x+1}\right)+\frac{1}{x+1}\right]_1^B = \left(\left(0+0\right)-\left(\ln\left(\tfrac{1}{2}\right)+\tfrac{1}{2}\right)\right) =$$

$$= \underline{\underline{\ln(2)-\frac{1}{2}}}$$

$$\int_0^{\pi/2} tg(x)\,dx = \lim_{B\to\pi/2^-}\int_0^B tg(x)\,dx = \lim_{B\to\pi/2^-}\Big[\ln|\cos(x)|\Big]_0^B =$$

$$= |-\infty|-|0| = \underline{\underline{+\infty}} \quad \longrightarrow \text{divergencija}$$