Partial differential equations – classification

Second order linear PDE in 2 variables

We want to find a function $u \equiv u(x, y)$, which satisfies the equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + hu = f \qquad (1)$$

in a given domain Ω, where $a \equiv a(x, y), b \equiv b(x, y), \ldots h \equiv h(x, y)$ and $f \equiv f(x, y)$ are functions continuous on Ω. Moreover, we demand that $u(x, y)$ fulfills some given conditions on the boundary Γ of the domain Ω.

Classification of the 2-nd order linear PDE in 2 variables

These equations are classified in a similar manner as the conic sections given by the algebraic equation $ax^2+bxy+cy^2+dx+ey+h = 0$. There are three types of equations distingushed by the sign of the discriminant $r(x, y) = (b(x, y))^2 - 4\,a(x, y)\,c(x, y)$:

- **elliptic** $\ldots r(x, y) < 0$ (example: Poisson equation)
- **parabolic** $\ldots r(x, y) = 0$ (example: heat equation)
- **hyperbolic** $\ldots r(x, y) > 0$ (example: wave equation)

The mathematical nature of solutions of the equation (1) depends on its type and numerical method for solving the equation should be chosen accordingly.

Example 1

Determine the type of the equation

$$x^2y^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x \partial y} + \frac{1}{4}\,\frac{\partial^2 u}{\partial y^2} = x + 2y$$

Solution

We are interested in the sign of the discriminant $r(x, y)$:
$r(x, y) = (b(x, y))^2 - 4\,a(x, y)\,c(x, y) = (-xy)^2 - 4\,x^2y^2 \cdot \frac{1}{4} = 0$.
Function $r(x, y)$ is equal to zero for all x, y, so the given equation is classified as parabolic (in any domain).

Example 2

Determine the type of the equation

$$x\,\frac{\partial^2 u}{\partial x^2} - \frac{1}{y}\,\frac{\partial^2 u}{\partial y^2} = x\,y$$

Solution

$r(x, y) = (b(x, y))^2 - 4\,a(x, y)\,c(x, y) = 0 + 4\,\frac{x}{y}$,

$r(x, y) > 0$ in $\Omega_1 = (0, \infty) \times (0, \infty)$ and $\Omega_2 = (-\infty, 0) \times (-\infty, 0)$,

$r(x, y) < 0$ in $\Omega_3 = (-\infty, 0) \times (0, \infty)$ and $\Omega_4 = (0, \infty) \times (-\infty, 0)$.

The given equation is hyperbolic in Ω_1 or Ω_2 and elliptic in Ω_3 or Ω_4.