Examples of theoretical problems

Give reasons for all your answers.

1. Consider a linear system $X = UX + V$, where

$$U = \begin{bmatrix} 0.5 & -0.6 \\ -0.5 & 0.2 \end{bmatrix}, \qquad V = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Will Fixed Point iterations converge for the given system?
- Give some sufficient conditions for convergence of Fixed Point iterations.
- What are necessary and sufficient conditions for convergence of Fixed Point iterations?
- Can any of the eigenvalues of the given U be equal to 1.5? (Give some reason for your answer other than computation of the eigenvalues).
- Give some upper limit on eigenvalues of U.

2. Consider a linear system $AX = B$, where

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

- Is the matrix A diagonally dominant (or strictly d.d., or symmetric positive definite)?
- Derive Jacobi iterative method (in the matrix form $X^{i+1} = UX^i + V$) for the system $AX = B$.
- Derive Gauss-Seidel iterative method (in the matrix form $X^{i+1} = UX^i + V$) for the system $AX = B$.
- Will Gauss-Seidel iterative method converge for the given system?
- Will Jacobi iterative method converge for the given system?
- What are sufficient conditions (on matrix A) for convergence of Jacobi (or Gauss-Seidel) method?
- What are necessary and sufficient conditions for convergence of Jacobi (or Gauss-Seidel) method?

3. Consider the nonlinear system

$$\begin{aligned} x^2 + y^2 &= 4 \\ yx &= 1 \end{aligned}$$

- Derive the linear system which is to be solved in every step of Newton method for solution of a nonlinear system $F(x, y) = 0$, where $F = (f_1(x, y), f_2(x, y))^{\mathrm{T}}$.

4. Consider the following table of x_i and y_i coordinates of 5 points:

x_i	-1	-1	0	1	2
y_i	2.9	2.9	2.1	4	3.6

- Derive the system of equations for computing coefficients of the first degree polynomial $p_1(x)$, which approximates N given points $[x_i, y_i]$, $i = 1, \dots N$, using the least squares method.
- Write down the matrix form of the linear system of normal equations for computing the coefficients of the polynomial of the first (or second) degree. List some properties of the system matrix.

5. Consider Cauchy problem
$$y'' = x\,y' - \sqrt{y}\ , \qquad y(0) = 4, \quad y'(0) = 1\ .$$

- Find a domain G where the conditions of existence and uniqueness of the solution are satisfied.
- What are sufficient conditions for existence and uniqueness of solution for this problem?
- Show that substitution of the first forward (or backward) difference instead of the first derivative leads to $\mathcal{O}(h)$ error.
- Show that the consistency error of explicit Euler method for the problem $y' = f(x, y),\ y(x_0) = y_0$ is $\mathcal{O}(h)$.
- What is the order of accuracy of explicit Euler method for the problem $y' = f(x, y),\ y(x_0) = y_0$?
- What is the order of accuracy of implicit Euler method for the problem $y' = f(x, y),\ y(x_0) = y_0$?
- Show that substitution of the first central difference instead of the first derivative leads to $\mathcal{O}(h^2)$ error.
- What is the order of accuracy of the midpoint method for the problem $y' = f(x, y),\ y(x_0) = y_0$?
- Under which assumptions substitution of the first central difference instead of the first derivative leads to $\mathcal{O}(h^2)$ error?
- What is the order of error when using substitution of the first central difference instead of the first derivative, if the function has only second continuous derivative?
- What is the order of error when using substitution of the first central difference instead of the first derivative, if the function has six continuous derivatives?

6. Consider Cauchy problem
$$y'' = \frac{2}{x-7} + x\,y\ , \qquad y(0) = 0, \quad y'(0) = 2\ .$$

- Find an interval I of the maximal solution.
- What are sufficient conditions for existence and uniqueness of solution for this problem?

7. Consider a boundary value problem

$$y'' = \frac{2}{x-7} + x\,y\ , \qquad y(0) = 0, \quad y'(2) = 2\ .$$

- Prove that this problem has an unique solution on the given interval.
- Derive the finite difference scheme for this equation
(hint: use the central difference for approximation of y'').
- Derive the finite difference scheme for a boundary value problem

  $-y''(x) + q(x)\,y(x) = f(x),\ y(a) = \alpha,\ y(b) = \beta$
- Show that substitution of the second central difference instead of the second derivative leads to $\mathcal{O}(h^2)$ error.
- Under which assumptions substitution of the second central difference instead of the second derivative leads to $\mathcal{O}(h^2)$ error?
- What are the sufficient conditions for existence and uniqueness of solution of
a boundary value problem $-(p(x)y'(x))' + q(x)y(x) = f(x),\ y(a) = \alpha,\ y(b) = \beta$?

8. Consider Dirichlet problem for Poisson equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x - y$$

in the domain given by its vertices $A = [0,\ 0]$, $B = [1.5,\ 0]$, $C = [1,\ 1.5]$, $D = [0,\ 1.5]$
with prescribed value $u(x, y) = 2y$ on its boundary.

- Under which assumptions on the given function, substitution of the second central difference instead of the second derivative leads to $\mathcal{O}(h^2)$ error? Justify your answer.
- Derive the finite difference scheme for the equation $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y)$.
What is the consistency error of this scheme, if only regular nodes are present?
- What is the type of this equation (elliptic, parabolic or hyperbolic)? Justify your answer.

9. Consider mixed problem for heat equation

$$\frac{\partial u}{\partial t} = 0.3\,\frac{\partial^2 u}{\partial x^2} + x + t^2 \quad \text{in} \quad \Omega = \{[x,t] : x \in (0,1),\ t \in (0,T)\}\ ,$$

$u(x,0) = x^2, \quad u(0,t) = \sin(t), \quad u(1,t) = \frac{1}{2t+1}$.

- Derive the explicit scheme for the equation $\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + f(x,t)$.
- Will the explicit method be stable for a choice of time step $\tau = 0.01$ and space step $h = 0.1$?
- What is the consistency error of the explicit scheme?
- Derive the implicit scheme for the equation $\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + f(x,t)$.
- Will the implicit method be stable for a choice of time step $\tau = 0.01$ and space step $h = 0.1$?
- Consider a given spatial step $h = 0.1$. From what interval can the time step τ be selected, for the explicit method to be stable?
- Consider a given time step $\tau = 0.01$. From what interval the spatial step h can be selected, for the explicit method to be stable?
- What is the type of this equation (elliptic, parabolic or hyperbolic)? Justify your answer.

10. Consider mixed problem for wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2} + 2x - t \quad \text{in} \quad \Omega = \{[x,t] : x \in (0,1),\ t \in (0,T)\}\ ,$$

$u(x,0) = x^2, \quad u(0,t) = \sin(t), \quad u(1,t) = \frac{1}{2t+1}, \quad \frac{\partial u}{\partial t}u(x,0) = 1 - 2x$.

- Derive the explicit scheme for the equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + f(x,t)$.
- What is the consistency error of the explicit scheme?
- Will the explicit method be stable for a choice of time step $\tau = 0.01$ and space step $h = 0.1$?
- Consider a given spatial step $h = 0.1$. From what interval can the time step τ be selected, for the explicit method to be stable?
- Consider a given time step $\tau = 0.01$. From what interval the spatial step h can be selected, for the explicit method to be stable?
- What is the type of this equation (elliptic, parabolic or hyperbolic)? Justify your answer.