Finite differences for second order linear PDE in 2 variables

The mathematical nature of solutions of a given partial differential equation depends on its type. Numerical method for solving the equation should be chosen accordingly.

PDE are classified into three types:

- **elliptic** (example: Poisson equation)
- **parabolic** (example: heat equation)
- **hyperbolic** (example: wave equation)

Discretization of PDE (inside the given domain) consists of the three following steps:

1. Choosing the step-size in both directions and constructing the grid.
2. Expressing the equation at every grid node (inside the domain).
3. Substitution of derivatives with the finite differences.

Caution: All terms of the equation have to be expressed or approximated at the same grid node.

Poisson equation

Dirichlet problem for Poisson equation

We are seeking a function $u \equiv u(x, y)$ which satisfies

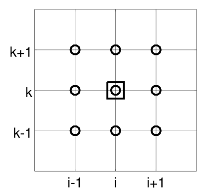
$$-\Delta u = f(x, y), \qquad \text{where} \quad \Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

in the domain Ω and has prescribed values $u(x, y) = \phi(x, y)$ on its boundary Γ.

Discretization:

1. Both x and y in the equation are treated the same way (they usually represent spatial directions). So it is natural to choose the same step-size h in both directions and construct a rectangular grid of nodes over Ω with equal mesh spacing h in both directions.

Scheme of the grid around a grid node P_i^k:



Notation:

$P_i^k \equiv [x_i, y_k]$ ... grid nodes, where

x_i ... x-coordinates of the nodes: $h = x_{i+1} - x_i$

y_k ... y-coordinates of the nodes: $h = y_{k+1} - y_k$

$u(x, y)$ ... function of two variables defined in Ω, $u(P_i^k) \equiv u(x_i, y_k)$

$U_i^k \approx u(P_i^k)$ ... approximate value of $u(x, y)$ at a grid node P_i^k

2. Express the equation (1) at every interior node $P_i^k = [x_i, y_k]$:

$$-\frac{\partial^2 u}{\partial x^2}(P_i^k) - \frac{\partial^2 u}{\partial y^2}(P_i^k) \;=\; f(P_i^k) \tag{2}$$

3. Partial derivatives at the node P_i^k then can be approximated by the second central differences with respect to x and y, respectively (see Figure 1):

$$\frac{\partial^2 u}{\partial x^2}(P_i^k) \;=\; \frac{u(P_{i-1}^k) - 2u(P_i^k) + u(P_{i+1}^k)}{h^2} + \mathcal{O}(h^2)$$

$$\frac{\partial^2 u}{\partial y^2}(P_i^k) \;=\; \frac{u(P_i^{k-1}) - 2u(P_i^k) + u(P_i^{k+1})}{h^2} + \mathcal{O}(h^2)$$

Substitution of these differences into (2)

$$-\frac{u(P_{i-1}^k) - 2u(P_i^k) + u(P_{i+1}^k)}{h^2} - \frac{u(P_i^{k-1}) - 2u(P_i^k) + u(P_i^{k+1})}{h^2} + \mathcal{O}(h^2) \;=\; f(P_i^k)$$

and omitting the term $\mathcal{O}(h^2)$, so exact values $u(P_i^k)$ have to be substituted with approximate ones U_i^k:

$$-\frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} - \frac{U_i^{k-1} - 2U_i^k + U_i^{k+1}}{h^2} \;=\; f(P_i^k)\;.$$

After rearrangig, this leads to equation for 5 unknowns:

$$4U_i^k - U_{i-1}^k - U_{i+1}^k - U_i^{k-1} - U_i^{k+1} \;=\; h^2\, f(P_i^k)\;. \tag{3}$$

This discretization scheme is called *five-point stencil*. The discretization is performed at every inner node, which leads to a system of linear equations for unknowns U_i^k.

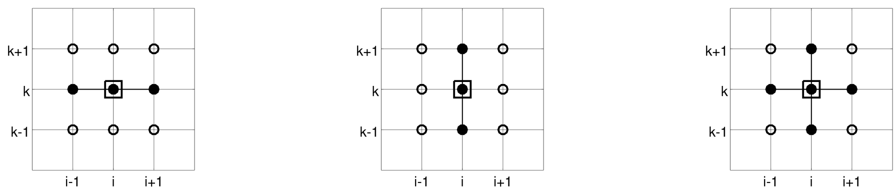


Figure 1: Grid nodes used for finite differences centered at the node P_i^k. Left: 2-nd central difference with respect to x. Center: 2-nd central difference with respect to y. Right: Five-point stencil for Δu.

The equation (3) above can be used for **regular nodes** only: a node is called regular, if all its 4 neighbors in the five-point stencil lie inside Ω or on its boundary Γ (if some of its neighbors lie on Γ, use corresponding prescribed boundary value and substitute it to the equation).

A **non-regular** node P is treated by linear interpolation, see Fig. 2: Here, the node R is one of the neigbors of P and the point Q represents an intersection of the grid-line PR and the boundary Γ. This 2D graph is a section of the 3D graph of the approximate solution, by the plane P, Q, $u(Q)$. The distance between P and Q is expressed as a multiple of the step-size h, i.e. dist$(P,Q)= d\cdot h$, where $d \in (0,1)$.

The equation corresponding to the node P is obtained from the assumption that the three points $[Q, u(Q)]$, $[P, U_P]$ and $[R, U_R]$ are colinear. Then the similarity of triangles leads to

$$\frac{U_P - U_R}{h} = \frac{u(Q) - U_R}{h + d\cdot h}\;, \qquad \text{or} \qquad (1+d)\cdot U_P - d\cdot U_R = u(Q)\,. \tag{4}$$

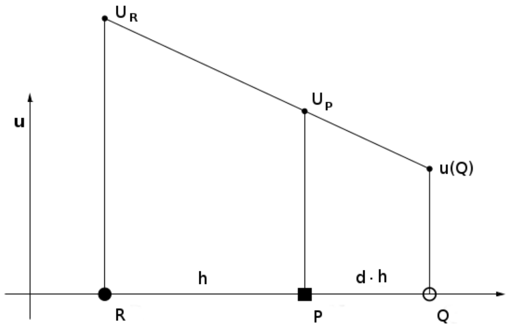


Figure 2: Treatment of the non-regular node P.

Problem 1

Consider Dirichlet problem for Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y - x \quad \text{in } \Omega\,, \quad u(x,y) = y \;\text{ on } \Gamma,$$

where Ω is a pentagon with vertices [- 1, 0], [- 1, 1.5], [0, 1.5], [0.5, 1] and [- 0.5, 0].

Choose the step-size $h = 0.5$ and compute approximate solution at the point [- 0.5, 1] using finite difference method.

Solution

In order to compute approximate value at the given point, we have to design a mesh over the domain, so that the point is one of the nodes of the mesh, and compute values at all nodes of the mesh. First of all, let us sketch the picture of the domain and the mesh and denote the nodes which we will use:

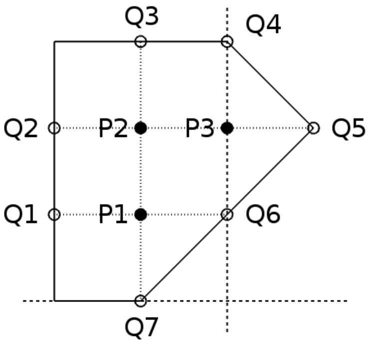


Figure 3: The given domain with interior nodes denoted as P_i and boundary nodes denoted as Q_j .

There are 3 interior nodes P_1 = [- 0.5, 0.5], P_2 = [- 0.5, 1] and P_3 = [0, 1], all of them are regular, and 9 boundary nodes from which we will use just 7 denoted as Q_j. We want to compute approximate values U_1, U_2 and U_3 of the solution at nodes P_i. Values at the nodes Q_j can be computed in advance from the boundary conditions:

$u(Q_1) = u(\ -1\ ,\ 0.5) = 0.5$
$u(Q_2) = u(\ -1\ ,\ 1) = 1$
$u(Q_3) = u(\ -0.5\ ,\ 1.5) = 1.5$
$u(Q_4) = u(\ 0\ ,\ 1.5) = 1.5$
$u(Q_5) = u(\ 0.5\ ,\ 1) = 1$
$u(Q_6) = u(\ 0\ ,\ 0.5) = 0.5$
$u(Q_7) = u(\ -0.5\ ,\ 0) = 0$

Let us prepare also the values of $f(P_i)$:

$f(P_1) = y_1 - x_1 = 0.5 - (-0.5) = 1$
$f(P_2) = y_2 - x_2 = 1 - (-0.5) = 1.5$
$f(P_3) = y_3 - x_3 = 1 - 0 = 1$

Now we can assemble (and rearrange) one equation at every node P_i:

P_1:

$4U_1 - U_2 - u(Q_1) - u(Q_6) - u(Q_7) = -h^2\, f(P_1)$
$4U_1 - U_2 = -h^2\, f(P_1) + u(Q_1) + u(Q_6) + u(Q_7)$
$4U_1 - U_2 = -0.25 \cdot 1 + 0.5 + 0.5 + 0 = 0.75$
$4U_1 - U_2 = 0.75$

P_2:

$4U_2 - U_1 - U_3 - u(Q_2) - u(Q_3) = -h^2\, f(P_2)$
$4U_2 - U_1 - U_3 = -h^2\, f(P_2) + u(Q_2) + u(Q_3)$
$4U_2 - U_1 - U_3 = -0.25 \cdot 1.5 + 1 + 1.5 = 2.125$
$4U_2 - U_1 - U_3 = 2.125$

P_3:

$4U_3 - U_2 - u(Q_4) - u(Q_5) - u(Q_6) = -h^2\, f(P_3)$
$4U_3 - U_2 = -h^2\, f(P_3) + u(Q_4) + u(Q_5) + u(Q_6)$
$4U_3 - U_2 = -0.25 \cdot 1 + 1.5 + 1 + 0.5 = 2.75$
$4U_3 - U_2 = 2.75$

The resulting system of linear equations is

$$\begin{aligned} 4U_1 - U_2 \quad\quad &= 0.75 \\ 4U_2 - U_1 - U_3 &= 2.125 \\ 4U_3 - U_2 \quad\quad &= 2.75 \end{aligned}$$

In matrix form

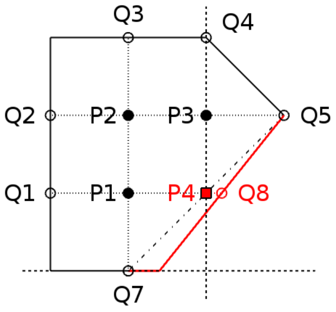
$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 2.125 \\ 2.75 \end{bmatrix}$$

The solution: $U_1 = 0.4018,\ U_2 = 0.8571,\ U_3 = 0.9018$.

The approximate solution at the point $P_2 =$ [- 0.5, 1] is $U_2 = 0.8571$.

Problem 2

Consider the same problem as in Problem 1 with a small change of the domain Ω: move its last vertex a little to the right, so that the domain is now given by the vertices $[-1,\ 0\ ]$, $[-1,\ 1.5\ ]$, [0, 1.5], [0.5, 1] and **[− 0.3, 0]**.



Solution

There are four interior nodes now: the *regular* nodes P_1, P_2, P_3 (regular node has no neighbor outside $\bar{\Omega}$) and the new node $P_4 = [\ 0,\ 0.5\ ]$, which was originally the boundary node Q_6 . The node P_4 is *non-regular* in the sense that its right mesh neighbor does not lie in the domain nor at the boundary (see Fig , where the new part of the boundary and the new nodes P_4 and Q_8 are colored by the red color).

At regular nodes, the equations remain the same (with the only exception: there is a new unknown U_4 instead of the given boundary value $u(Q_6)$):

P_1:

$$4U_1 - U_2 - U_4 - u(Q_1) - u(Q_7) = -h^2\, f(P_1)$$
$$4U_1 - U_2 - U_4 = -h^2\, f(P_1) + u(Q_1) + u(Q_7)$$
$$4U_1 - U_2 - U_4 = -0.25 \cdot 1 + 0.5 + 0 = 0.25$$
$$4U_1 - U_2 - U_4 = 0.25$$

P_2:

$$4U_2 - U_1 - U_3 - u(Q_2) - u(Q_3) = -h^2\, f(P_2)$$
$$4U_2 - U_1 - U_3 = -h^2\, f(P_2) + u(Q_2) + u(Q_3)$$
$$4U_2 - U_1 - U_3 = -0.25 \cdot 1.5 + 1 + 1.5 = 2.125$$
$$4U_2 - U_1 - U_3 = 2.125$$

P_3:

$$4U_3 - U_2 - U_4 - u(Q_4) - u(Q_5) = -h^2\, f(P_3)$$
$$4U_3 - U_2 - U_4 = -h^2\, f(P_3) + u(Q_4) + u(Q_5)$$
$$4U_3 - U_2 - U_4 = -0.25 \cdot 1 + 1.5 + 1 = 2.25$$
$$4U_3 - U_2 - U_4 = 2.25$$

The fourth equation corresponding to non-regular node P_4 is given by linear interpolation (4) of the value at P_4 from the values at P_1 and at the auxiliary point Q_8 – the intersection of the line P_1P_4 and the boundary. From similarity of triangles we obtain $Q_8 = [\ 0.1,\ 0.5\ ]$ with the prescribed value of $u(Q_8) = u(\ 0.1\ ,\ 0.5) = 0.5$ and $dist(P_4, Q_8) = 0.1$. Then from linear interpolation we have

$$\frac{U_4 - U_1}{h} = \frac{u(Q_8) - U_1}{h + dist(P_4, Q_8)}\ , \quad \text{after substitution} \quad \frac{U_4 - U_1}{0.5} = \frac{0.5 - U_1}{0.5 + 0.1} \ \Rightarrow\ 0.6\, U_4 - 0.1\, U_1 = 0.25\,.$$

The matrix form of the equations is

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -0.1 & 0 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 2.125 \\ 2.25 \\ 0.25 \end{bmatrix}$$

and the solution: $U_1 = 0.3969,\ U_2 = 0.8547,\ U_3 = 0.8969,\ U_4 = 0.4828$.

Problem 3

Consider Dirichlet problem for Poisson equation

$$-\Delta u \; = 4x - y \quad \text{in } \Omega\,, \quad u(x,y) = 3 - x \;\; \text{on } \Gamma,$$

where $\Omega = \{[x,y] \in R^2 : x > 0, x < 1.5, y < 0, 3y - x + 6 > 0\}$.

a) Sketch the domain Ω and a mesh with step-size $h = 0.5$ with [0 ; 0] being one of the nodes of the mesh. Mark regular and non-regular nodes of the mesh.

b) Use finite differences and assemble the system of discretized equations (use linear interpolation at non-regular nodes.).

Solution

a) See Figure 4: regular nodes are A, B, C, D (denoted by black circles), non-regular E, F (denoted by black squares). Node [0, 0] lies on the boundary (upper left corner).

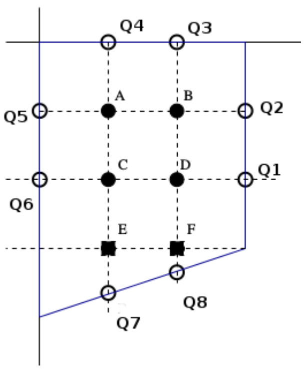


Figure 4: The given domain with interior nodes denoted as $A, \ldots F$ and boundary nodes denoted as Q_j .

b) Let $U_A, U_B, U_C, U_D, U_E, U_F$ denote the approximate values of the solution u at nodes A, B, C, D, E, F, respectively. We need 6 equations for these 6 unknowns.

For **regular nodes** A, B, C, D, the 5-point scheme (3) is used:

$$\begin{aligned}
4U_A - U_B - U_C - u(Q_4) - u(Q_5) &= h^2 \cdot f(A) \\
4U_B - U_A - U_D - u(Q_2) - u(Q_3) &= h^2 \cdot f(B) \\
4U_C - U_A - U_D - U_E - u(Q_6) &= h^2 \cdot f(C) \\
4U_D - U_B - U_C - U_F - u(Q_1) &= h^2 \cdot f(D)
\end{aligned}$$

So we need right hand side values $f(x,y) = 4x - y$ at these nodes

$$\begin{aligned}
f(A) &= f(0.5, -0.5) = 4 \cdot 0.5 + 0.5 = 2.5 \\
f(B) &= f(1, -0.5) = 4 \cdot 1 + 0.5 = 4.5 \\
f(C) &= f(0.5, -1) = 4 \cdot 0.5 + 1 = 3 \\
f(D) &= f(1, -1) = 4 \cdot 1 + 1 = 5
\end{aligned}$$

and prescribed boundary values $u(x,y) = 3 - x$ at nodes $Q_1, \dots Q_6$:

$$\begin{aligned} u(Q_1) = u(Q_2) &= 3 - 1.5 = 1.5 \\ u(Q_3) &= 3 - 1.0 = 2 \\ u(Q_4) &= 3 - 0.5 = 2.5 \\ u(Q_5) = u(Q_6) &= 3 - 0 \ = 3. \end{aligned}$$

Then the following 4 equations are obtained:

$$\begin{aligned} 4U_A - U_B - U_C &= 0.5^2 \cdot 2.5 + 2.5 + 3 = 6.125 \\ 4U_B - U_A - U_D &= 0.5^2 \cdot 4.5 + 1.5 + 2 = 4.625 \\ 4U_C - U_A - U_D - U_E &= 0.5^2 \cdot 3 + 3 = 3.75 \\ 4U_D - U_B - U_C - U_F &= 0.5^2 \cdot 5 + 1.5 = 2.75 \end{aligned}$$

For **non-regular nodes** E, F, the linear interpolation (4) is used:

Node E: The closest intersection of the grid with the boundary Γ is the point Q_7, so the corresponding equation is obtained from the assumption that the points $[Q_7, u(Q_7)]$, $[E, U_E]$ and $[C, U_C]$ are colinear (in R^3). All three points lie in the plane $x = 0.5$, therefore this problem can be restricted to this plane. Then the similarity of triangles and using $u(Q_7) = 3 - 0.5 = 2.5$ and $dist(E, Q_7) = \frac{2}{3}\,h$ lead to the equation $(1 + \frac{2}{3})\, U_E - \frac{2}{3} U_C = 2.5$.

Similarly for node F: points $[Q_8, u(Q_8)]$, $[F, U_F]$ and $[D, U_D]$ should be colinear, $u(Q_8) = 3 - 1 = 2$, $dist(D, Q_8) = \frac{1}{3}\,h$, which leads to the equation $(1 + \frac{1}{3})\, U_F - \frac{1}{3} U_D = 2$.

Representation of these 6 equations in the matrix form (the last two eq. were multiplied by 3):

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & -2 & 0 & 5 & 0 \\ 0 & 0 & 0 & -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} U_A \\ U_B \\ U_C \\ U_D \\ U_E \\ U_F \end{bmatrix} = \begin{bmatrix} 6.125 \\ 4.625 \\ 3.75 \\ 2.75 \\ 7.5 \\ 6 \end{bmatrix}$$