Fixed point iterations for real functions

Motivation example: the logistic map – a very simple model of evolution of a population

$x_{n+1} = r\, x_n\, (1 - x_n)$, where x_n is a number in $< 0, 1 >$

– x_n represents the ratio of existing population to the maximum possible one, at n-th year

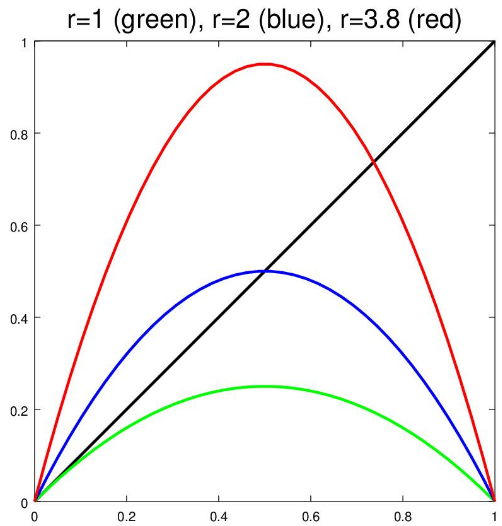
– r is a parameter of the model ($0 < r < 4$, so that if $x_n \in< 0, 1 >$, then $x_{n+1} \in< 0, 1 >$)

Question 1: Does there exist a *fixed point* x^* that solves the *logistic equation*

$x^* = r\, x^*\, (1 - x^*)$?

– x^* represents some steady state, or equilibrium

The graphical solution:



Question 2: Does the logistic map (the "evolution") converge to a fixed point?

This is a special example of a more general formulation that will be discussed in this text.

Fixed point iterations

The goal: find the solution of a fixed point equation

$x = g(x), \quad$ where $x \in R,\ g : R \to R.$

Fixed point iterations (fpi): choose x_0, then for $k = 0, 1, 2, \ldots$ compute

$x_{k+1} = g(x_k)$

Example 1: $\quad g(x) = 0.2\,x^2 + 1, \quad$ use $x_0 = 0.2$ and precision up to 4 decimal places

$x_1 = g(x_0) = 0.2\,{x_0}^2 + 1 = 0.2 \cdot 0.2^2 + 1 = 0.0080 + 1 = 1.0080$

$x_2 = g(x_1) = 0.2\,{x_1}^2 + 1 = 0.2 \cdot 1.0080^2 + 1 = 1.2032$

$x_3 = g(x_2) = 0.2\,{x_2}^2 + 1 = 0.2 \cdot 1.2032^2 + 1 = 1.2895$

$x_4 = 0.2\,{x_3}^2 + 1 = 0.2 \cdot 1.2895^2 + 1 = 1.3326$

$\ldots$

$x_{13} = 0.2\,{x_{12}}^2 + 1 = 1.3818$

$x_{14} = 0.2\,{x_{13}}^2 + 1 = 1.3819$

$x_{15} = 0.2\,{x_{14}}^2 + 1 = \underbrace{0.2 \cdot 1.3819^2 + 1}_{g(1.3819)} = 1.3819$

$g(1.3819) = 1.3819 \quad \Rightarrow \quad x = g(x)$ holds with accuracy up to 4 decimal places

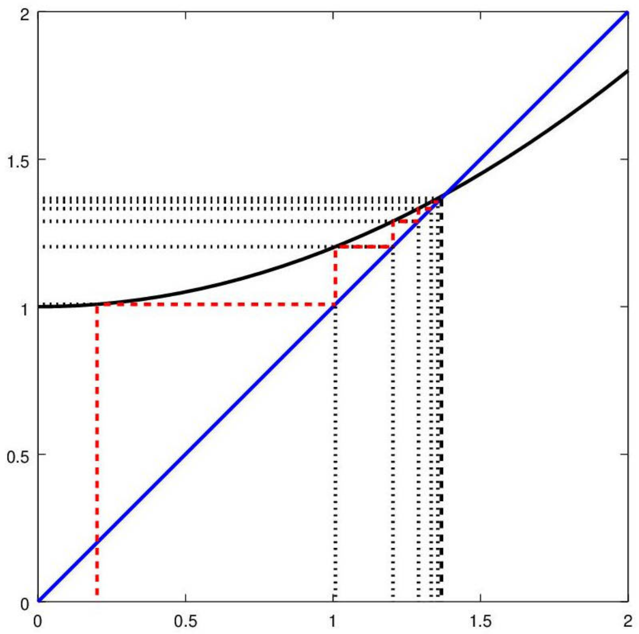


Figure 1: Graphical illustration of iterations $x_{k+1} = g(x_k)$: black: $y = g(x) = 0.2\,x^2 + 1$, blue: $y = x$.

Graphical solution – the algorithm:

- choose x_0 and set $\mathrm{P}_0 = [x_0,\ x_0]$
- for $k = 0, 1, 2, \ldots$, repeat
  1. move vertically to the curve $y = g(x)$: set $\mathrm{Q}_k = [x_k,\ g(x_k)]$ and $x_{k+1} = g(x_k)$
  2. move horizontally to the line $y = x$: set $\mathrm{P}_{k+1} = [x_{k+1},\ x_{k+1}]$

  until $|x_{k+1} - x_k| <$ some given precision

The resulting broken line $\mathrm{P}_0, \mathrm{Q}_0, \mathrm{P}_1, \mathrm{Q}_1, \ldots \mathrm{P}_k, \mathrm{Q}_k \ldots$ (red, dashed) converges to the graphical solution.

Question: under which assumptions fpi converge (i.e. the loop in the algorithm above is finite)?

Example 2: $g(x) = 0.2\,x^2 + 1$ as in ex. 1; use $x_0 = 3.4$ (below left) and then $x_0 = 3.8$ (below right)

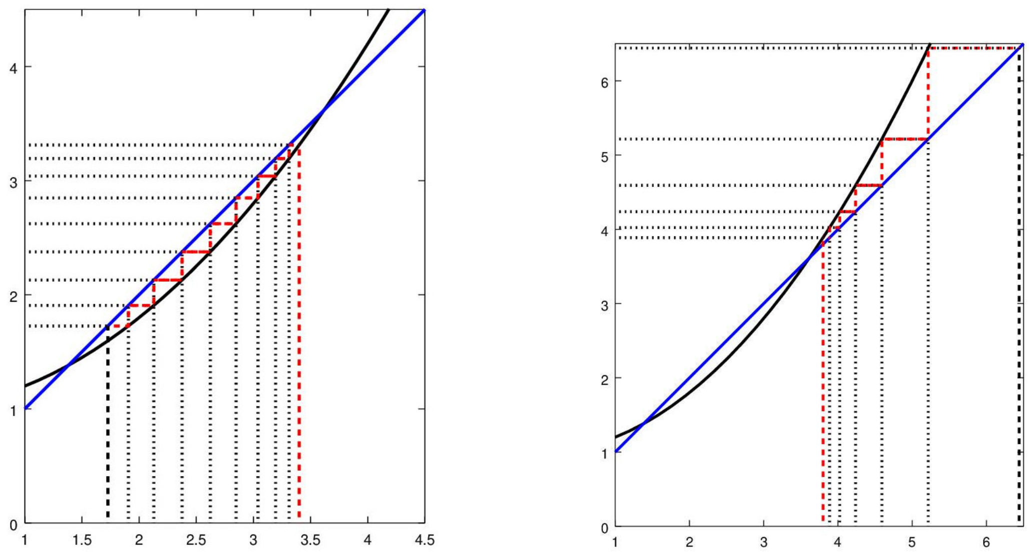


Figure 2: Ex. 2 – convergent staircase, to the left solution (left), divergent staircase (right).

Example 3: $g(x) = \cos x$, use $x_0 = 0.4$ (below left)

Example 4: $g(x) = -1.4\,(x-2) + 0.8\,\cos x + 20$, use $x_0 = 8.9$ (below right)

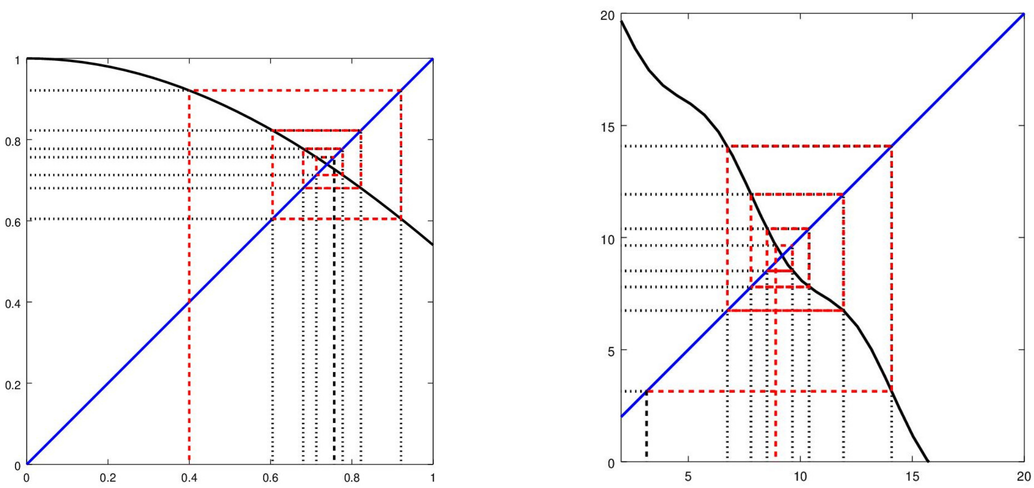


Figure 3: Left: ex. 3 – spiral inwards (convergent). Right: ex. 4 – spiral outwards (divergent).

Observations:

- $g(x)$ is increasing $\Rightarrow$ staircase (ex. 1, 2, Figures 1, 2),
  $g(x)$ is decreasing $\Rightarrow$ spiral (ex. 3, 4, Figure 3).
- the slope of $g(x)$ is small $\Rightarrow$ fpi converge (ex. 1, 2, 3, left figures),
  the slope of $g(x)$ is large, steep $\Rightarrow$ fpi diverge (ex. 2, 4, right figures)
  the treshold $\approx 45^o$

Theorem 1 – contraction mapping theorem in R:

Let $D \subset R,\ D$ be closed, let $g : D \to R$ be a real function. Assume

1. if $x \in D$, then $g(x) \in D$
2. the function g is a *contraction* on D: there exists a number $q < 1$ such that

$$|g(x) - g(y)| \le q \cdot |x - y| \quad \forall\ x, y \in D \tag{1}$$

Then

- there exists unique $x^* \in D$ such that $x^* = g(x^*)$ and the fixed point iterations converge to x^* for any choice of $x_0 \in D$,
- x_k satisfies the a-priori error estimate $|x_k - x^*| \le \frac{q^k}{1-q}\,|x_1 - x_0|$ and the a-posteriori error estimate $|x_k - x^*| \le \frac{q}{1-q}\,|x_k - x_{k-1}|$.

Proof: see [1]

Theorem 2 – the sufficient condition for the contraction property in R:

Let $g(x)$ be a differentiable function on interval $I \subset R$, such that $|g'(x)| \le q < 1\ \forall x \in I$.
Then g is a contraction in I and satisfies (1), i.e. $|g(x) - g(y)| \le q \cdot |x - y| \quad \forall\ x, y \in I$.

Proof: from Taylor's expansion for function $g(x)$, we have $\forall\ x, y \in I$:

$$\begin{aligned} g(y) &= g(x) + g'(\xi)\cdot(x-y)\ , \quad \text{where } \xi \in (x,y) \\ g(y) - g(x) &= g'(\xi)\cdot(x-y) \\ |g(x) - g(y)| &= |g'(\xi)|\cdot|x-y| \le q\cdot|x-y| \end{aligned}$$

Fixed point iterations for general equation $f(x) = 0$

Transform the equation to the previous form:

$$\begin{aligned} 0 &= f(x) \quad / \cdot \alpha \\ 0 &= \alpha\, f(x) \\ x &= \underbrace{x + \alpha\, f(x)}_{g(x)} \end{aligned}$$

The coefficient α has to be chosen so that $g(x) = x + \alpha\, f(x)$ is a contraction.

Example 5: solve numerically $2\,\mathrm{arctg}\,x + \log x = 0$.

The function $f(x) = 2\,\mathrm{arctg}\,x + \log x$ is defined for $x \in (0, \infty)$, values of $f(x)$ are growing from $-\infty$ to ∞, so the equation has unique solution x^*. In this case, the solution can be easily located to, say, $x^* \in (0.3, 1)$, as $f(0.3) < 0$ and $f(1) > 0$.

The function $f(x)$ is differentiable – we will ty to find α so that the assumptions of Theorem 2 hold:
$|g'(x)| = |1 + \alpha\, f'(x)| \le q < 1$ for $x \in I = (0.3, 1)$ and $f(I) \subset I$.
Let us use some guesswork: $f'(x) > 0$, so α has to be some negative number, small enough so that $1 + \alpha\, f'(x) > -1$. Choose $x_0 = 1$ and examine some values of α. We can check that values $\alpha = -0.1$, or $\alpha = -0.2$, or $\alpha = -0.3$ work perfectly ($x^* \approx 0.438$ after few iterations), value $\alpha = -0.4$ is not so good and for $\alpha < -0.5$ the fpi method does not converge.

References

[1] Tobias von Petersdorff: Contraction mapping theorem
`http://terpconnect.umd.edu/~petersd/666/fixedpoint.pdf`