# Iterative methods for linear systems

## The theory (short excerpts from lectures)

### Fixed-point iteration (abbreviation *fpi*)

The system $x = Ux + v$ is solved by the following algorithm:
1. choose $x^{(0)}$,
2. for $k = 0, 1, 2, \dots$ compute $x^{(k+1)} = Ux^{(k)} + v$, until $||x^{(k+1)} - x^{(k)}|| < \epsilon$ .

Convergence conditions:
$\rho(U) < 1 \quad \Leftrightarrow \quad$ *fpi* converges
$||U|| < 1 \quad \Rightarrow \quad$ *fpi* converges

### Solving the system Ax = b

The main idea: the system $Ax = b$ is transformed to a system $x = Ux + v$, which is then solved by *fpi*. The splitting $A = L + D + U$ is used, where $L$ is the lower triangular part, $D$ is the main diagonal and $U$ is the upper triangular part.

### Jacobi method (abbreviation $J$)

The system $Ax = b$ is expressed as $x = U_J\, x + v_J$, where
$U_J = -D^{-1}(L + U)$ and $v_J = D^{-1}b$.
This system is then solved by *fpi*.

Convergence conditions:
$\rho(U_J) < 1 \quad \Leftrightarrow \quad J$ converges
$A$ is s.d.d. $\quad \Rightarrow \quad J$ converges

### Gauss-Seidel method (abbreviation $GS$)

The system $Ax = b$ is expressed as $x = U_G\, x + v_G$, where
$U_G = -(L + D)^{-1}U$ and $v_G = (L + D)^{-1}b$.
This system is then solved by *fpi*.

Convergence conditions:
$\rho(U_G) < 1 \quad \Leftrightarrow \quad GS$ converges
$A$ is s.d.d. $\quad \Rightarrow \quad GS$ converges
$A$ is symmetric and positive definite $\quad \Rightarrow \quad GS$ converges

## Fixed-point iteration

**Problem 1**

Suppose the system $x = Ux + v$ is given as

$$U = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

a) Choose $x^{(0)} = (\,0,\,0)^T$ and compute the first three iterations by *fpi*.
b) Prove that *fpi* converges for the given system.

**The solution**

a)

$$x^{(1)} = Ux^{(0)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$x^{(2)} = Ux^{(1)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5/2 \end{bmatrix}$$

$$x^{(3)} = Ux^{(2)} + v = \begin{bmatrix} 1/2 & 1 \\ -5/4 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ -5/2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b) First of all check the sufficient conditions, as they are easier to compute:
$||U||_1 = \max(\ |1/2| + |-5/4|,\ |\ 1\ | + |-3/2|) = \max(\ 7/4,\ 10/4) = 10/4 > 1$
$||U||_\infty = \max(\ |1/2| + |\ 1\ |,\ |-5/4| + |-3/2|) = \max(\ 3/2,\ 11/4) = 11/4 > 1$
$||U||_F = \sqrt{(1/2)^2 + 1^2 + (-5/4)^2 + (-3/2)^2} = \sqrt{81/16} = 9/4 > 1$
All norms are greater or equal than 1 and so we cannot conclude anything. Now we have to check the necessary and sufficient condition, it means to compute $\rho(U)$:
$\det(U - \lambda I) = (1/2 - \lambda)(-3/2 - \lambda) + 5/4 = \lambda^2 + \lambda + 1/2 = 0$
$\Rightarrow \quad \lambda_{1,2} = -1/2 \pm 1/2i$
$||\lambda_{1,2}|| = \sqrt{(1/2)^2 + (\pm 1/2)^2} = \sqrt{1/2} = \sqrt{2}/2$
$\rho(U) = \max(||\lambda_1||, ||\lambda_2||) = \sqrt{2}/2 < 1 \quad \Rightarrow \quad$ *fpi* converges.

## Jacobi method

**Problem 2**

Consider following matrices:

$$A = \begin{bmatrix} -5 & -1 & 0 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

a) Consider the system $Ax = b$ with $b = (2, 1, -1)^T$.
Choose $x^{(0)} = (\,0,\,0,\,0)^T$ and compute the first two iterations by $J$ method.
b) Prove that $J$ converges for the system $Ax = b$.
c) Is there any easy-to-check condition to find out whether $J$ converges for matrices $B$ and $C$?

**The solution**

a) The iterations of $J$ method are easier to compute by elements than in the vector form. The algorithm:

1. Write the system as equations:

$$\begin{aligned} -5x_1 - x_2 \quad &= 2 \\ 3x_1 + 3x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

2. Express the diagonal unknown from every equation:

$$\begin{aligned} x_1 &= -(2 + x_2)/5 \\ x_2 &= (1 - 3x_1 - x_3)/3 \\ x_3 &= (-1 - x_1 + x_2)/2 \end{aligned} \tag{1}$$

3. For $k = 0, 1, 2, \ldots$ compute $x^{(k+1)}$ by substituting the elements $x^{(k)}$ from the previous iteration to the right hand side of (1):

$$\begin{aligned} x_1^{(k+1)} &= -(2 + x_2^{(k)})/5 \\ x_2^{(k+1)} &= (1 - 3x_1^{(k)} - x_3^{(k)})/3 \\ x_3^{(k+1)} &= (-1 - x_1^{(k)} + x_2^{(k)})/2 \end{aligned}$$

For $k = 0$ we have

$$\begin{aligned} x_1^{(1)} &= -(2 + x_2^{(0)})/5 = -(2 + 0)/5 = -2/5 \\ x_2^{(1)} &= (1 - 3x_1^{(0)} - x_3^{(0)})/3 = (1 - 0 - 0)/3 = 1/3 \\ x_3^{(1)} &= (-1 - x_1^{(0)} + x_2^{(0)})/2 = (-1 - 0 + 0)/2 = -1/2 \end{aligned}$$

The result of the first iteration is $x^{(1)} = (-2/5,\, 1/3, -1/2)^T$. The second iteration:

$$\begin{aligned} x_1^{(2)} &= -(2 + x_2^{(1)})/5 = -(2 + 1/3)/5 = -7/15 \\ x_2^{(2)} &= (1 - 3x_1^{(1)} - x_3^{(1)})/3 = (1 - 3(-2/5) - (-1/2))/3 = 9/10 \\ x_3^{(2)} &= (-1 - x_1^{(1)} + x_2^{(1)})/2 = (-1 - (-2/5) + 1/3)/2 = -2/15 \end{aligned}$$

The result of the second iteration is $x^{(2)} = (-7/15,\, 9/10, -2/15)^T$.

b) $A$ is strictly diagonally dominant, which is a sufficient condition for convergence of $J$ method.

c) Matrices $B$ and $C$ are not s.d.d., so the sufficient condition for convergence of $J$ method is not fulfilled and so we do not know anything about the convergence. If we want to find out whether $J$ method converges, we need to compute spectral radius of the matrix $U_J$. Generally, that is not easy for $3 \times 3$ matrices.

**Problem 3**

Consider any system $Ax = b$ with the following matrix and decide, whether $J$ method converges:

$$A = \begin{bmatrix} 6 & 11 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

**The solution**

The convergence of the $J$ method does not depend on the right hand side $b$.

The matrix $A$ is not s.d.d., the sufficient convergence condition for $J$ method does not hold and we have to compute spectral radius of the matrix $U_J$:

$$U_J = -D^{-1}(L+U) = -\begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \left( \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 11 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & -\frac{11}{6} & \frac{1}{6} \\ -\frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$det(U_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{11}{6} & \frac{1}{6} \\ -\frac{1}{3} & -\lambda & 0 \\ \frac{1}{2} & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{1}{12}\lambda + \frac{11}{18}\lambda = \lambda(-\lambda^2 + \frac{25}{36}) = 0$$

$$\Rightarrow \quad \lambda_1 = 0,\ \lambda_{2,3} = \pm 5/6 \quad \Rightarrow \quad \rho(U_J) = 5/6 < 1.$$

Spectral radius of the matrix $U_J$ is less than 1, so $J$ method converges.

## Gauss-Seidel method

**Problem 4**

Consider the system $Ax = b$ given in Problem 2.
a) Choose $x^{(0)} = (\,0,\,0,\,0)^T$ and compute the first two iterations by $GS$ method.
b) Prove that $GS$ converges for the given system.
c) Is there any easy-to-check condition to find out whether $GS$ converges for matrices $B$ and $C$ given in Problem 2?

**The solution**

a) The iterations of $GS$ method are easier to compute by elements than in the vector form. The algorithm:

1. Rewrite the system as equations - as in $J$ method, see Problem 2.

2. From every equation, express the diagonal unknown - as in $J$ method, see Problem 2.

3. For $k = 0, 1, 2, \ldots$ compute $x^{(k+1)}$ one by one, starting with the first equation. The first element $x_1^{(k+1)}$ is computed as in $J$ method, however, as soon as you compute a new unknown, substitute it immediatelly to the right hand sides of

all remaining equations of (1):

$$\begin{aligned}
x_1^{(k+1)} &= -(2 + x_2^{(k)})/5 \\
x_2^{(k+1)} &= (1 - 3x_1^{(k+1)} - x_3^{(k)})/3 \\
x_3^{(k+1)} &= (-1 - x_1^{(k+1)} + x_2^{(k+1)})/2
\end{aligned}$$

For $k = 0$ we have

$$\begin{aligned}
x_1^{(1)} &= -(2 + x_2^{(0)})/5 = -(2+0)/5 = -2/5 \\
x_2^{(1)} &= (1 - 3x_1^{(1)} - x_3^{(0)})/3 = (1 - 3(-2/5) - 0)/3 = 11/15 \\
x_3^{(1)} &= (-1 - x_1^{(1)} + x_2^{(1)})/2 = (-1 - (-2/5) + 11/15)/2 = 1/15
\end{aligned}$$

The result of the first iteration is $x^{(1)} = (-2/5,\ 11/15,\ 1/15)^T$. The second iteration:

$$\begin{aligned}
x_1^{(2)} &= -(2 + x_2^{(1)})/5 = -(2 + 11/15)/5 = -41/75 \\
x_2^{(2)} &= (1 - 3x_1^{(2)} - x_3^{(1)})/3 = (1 - 3(-41/75) - 1/15)/3 = 193/225 \\
x_3^{(2)} &= (-1 - x_1^{(2)} + x_2^{(2)})/2 = (-1 - (-41/75) + 193/225)/2 = 91/450
\end{aligned}$$

The result of the second iteration is $x^{(2)} = (-41/75,\ 193/225, 91/450)^T$.

b) $A$ is strictly diagonally dominant, which is sufficient condition for convergence of *GS* mehod. *GS* method converges.

c) Check that $B$ is symmetric and positive definite, which is sufficient condition for convergence of *GS* mehod.
The matrix $C$ is not s.d.d. nor symmetric, and so neither condition sufficient for convergence of *GS* method is satisfied for the matrix $C$.
Conclusion: *GS* method converges for the matrix $B$, nothing is known about convergence of *GS* method for matrix $C$. If we want to find out whether *GS* method converges for the matrix $C$, we need to compute spectral radius of the matrix $U_G$.