Matrix properties

Vector norms:

A *vector norm* on R^n is a real valued function $x \to \|x\|$ such that $\forall\, x,\, y \in R^n$, $\forall\, \alpha \in R$:

1. $\|x\| \geq 0,\ \|x\| = 0 \ \Leftrightarrow\ x = 0$
2. $\|\alpha x\| = |\alpha|\, \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

column norm: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

Euclidean norm: $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$

row norm: $\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$

Matrix norms:

A *p-norm* $\|A\|_p$ of a matrix A (we will use $p = 1,\ 2$ or ∞) is a norm induced by the vector norm $\|x\|_p$ as a maximal value of $\|Ax\|_p$ on the unit sphere:

$$\|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p \equiv \max_{\|x\|_p \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Frobenius norm $\|A\|_F$ is a square root of the sum of all diagonal elements of $A^{\mathrm{T}}A$:

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

Matrix norm $\|\,.\,\|_a$ on $R^{n,m}$ is said to be *consistent* with a vector norm $\|.\|_b$ on R^n, R^m, if $\|Ax\|_b \leq \|A\|_a \|x\|_b$ for all $A \in R^{n,m}$, $x \in R^m$.

Definitions for square matrices:

Spectral radius $\rho(A)$ of a matrix A is the maximum modulus of eigenvalues:

$$\rho(A) = \max_{i=1,\ldots n} |\lambda_i|, \quad \text{where } \lambda_i \text{ are eigenvalues of } A$$

A is *symmetric*, if $A = A^{\mathrm{T}}$.

A is *positive definite*, if $x^{\mathrm{T}}Ax > 0\ \forall x \neq 0$.

A is *diagonally dominant* (abbreviation *d.d.*), if in every row the absolute value (or modulus) of the diagonal element is greater or equal than the sum of absolute values of all other elements in that row (or if this holds for A^{T}).

A is *strictly diagonally dominant* (abbreviation *s.d.d.*), if all these inequalities are strict.

Trace $tr(A)$ of a matrix A is the sum of all its diagonal elements.

Condition number $\kappa(A)$ of A (relative to norm $\|\,.\,\|$) is $\kappa(A) = \|A\| \cdot \|A^{-1}\|$.

Properties of matrix norms:

1. $\|Ax\|_p \leq \|A\|_p \|x\|_p \;\; \forall x$ – induced p-norms are consistent
2. $\|A\|_1 = \max(\sum_i |a_{i1}|, \sum_i |a_{i2}|, \cdots \sum_i |a_{in}|)$ – *column norm*
3. $\|A\|_\infty = \max(\sum_j |a_{1j}|, \sum_j |a_{2j}|, \cdots \sum_j |a_{nj}|)$ – *row norm*
4. $\|A\|_2 = \sqrt{\rho(A^T A)}$ – *spectral norm*
5. $\|A\|_2 \leq \|A\|_F$ – Frobenius norm is consistent with Euclidean norm on R^n

Properties of square matrices:

1. $\rho(A) \leq \|A\|$ for *any* norm of A .
2. If A is symmetric, then it has *real* eigenvalues and $\|A\|_2 = \rho(A)$.
3. If A is symmetric, then it is positive definite $\iff$ all its leading principal minors are positive. (Sylvester's criterion)
4. If A is symmetric, then it is positive definite $\iff$ all its eigenvalues are (real) positive numbers.
5. If A is symmetric and s.d.d. and $a_{ii} > 0 \; \forall i$, then A is positive definite.
6. General convergence result: A^k converges to zero $\iff$ $\rho(A) < 1$.

   Corollary: If $\|A\| < 1$ for some matrix norm, then A^k converges to zero.

Examples

Matrix and vector norms

Problem 1

A matrix A and a vector y are given as

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -4 \\ -3 & -2 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Compute all theirs norms (a vector is in fact a matrix of type $n \times 1$).
For the given matrix and vector, confirm the following inequality for consistent norms:

$$\|Ay\| \leq \|A\| \cdot \|y\| \tag{1}$$

Solution

$Ay = (4, 14, -2)^T$

1-norm, or *column norm* (the maximum of the column sums):
$\|A\|_1 = \max(\ |2| + |3| + |-3|,\ 0 + |-4| + |-2|) = \max(\ 8,\ 6) = 8$
$\|y\|_1 = |2| + |-2| = 4$
$\|Ay\|_1 = |4| + |14| + |-2| = 20$
$20 \leq 8 \cdot 4 = 32$

∞-*norm*, or *row norm* (the maximum of the row sums):
$\|A\|_\infty = \max(\ |2| + 0,\ |3| + |-4|,\ |-3| + |-2|) = \max(\ 2,\ 7,\ 5) = 7$
$\|y\|_\infty = \max(\ |2|,\ |-2|) = 2$
$\|Ay\|_\infty = \max(\ |4|,\ |14|,\ |-2|) = 14$
$14 \leq 7 \cdot 2 = 14$

Frobenius norm:
$\|A\|_F = \sqrt{2^2 + 0^2 + 3^2 + (-4)^2 + (-3)^2 + (-2)^2} = \sqrt{42} = 6.4807$
($\|A\|_2 \leq \|A\|_F$ – Frobenius norm is the upper limit, easier to compute)
$\|y\|_2 = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2.8284$
$\|Ay\|_2 = \sqrt{4^2 + 14^2 + (-2)^2} = 14.6969$
$14.6969 \leq 6.4807 \cdot 2.8284 = 18.3300$

Caution: the relationship (1) does not hold if different norms are mixed together!
For instance, compare $\|Ay\|_1$ and $\|A\|_F$, $\|y\|_2$.

Spectral radius

Problem 2
Compute spectral radius $\rho(A)$ of the matrix

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$$

and confirm that $\rho(A) \leq \|A\|$ holds for column, row and Frobenius norm of A.

Solution

$\det(A - \lambda I)\ =\ (-2 - \lambda)^2 + 1 = \lambda^2 + 4\lambda + 5 = 0 \Leftrightarrow \lambda_{1,2} = -2 \pm i$
$|\lambda_{1,2}| = \sqrt{(-2)^2 + (\pm 1)^2} = \sqrt{5}$
$\rho(A) = \max(\,|\lambda_1|, |\lambda_2|) = \max(\ \sqrt{5},\ \sqrt{5}) = \sqrt{5} = 2.2361$
$\|A\|_F = \sqrt{(-2)^2 + (-1)^2 + 1^2 + (-2)^2} = \sqrt{10} = 3.1623 \quad \geq \rho(A)$
$\|A\|_1 = \|A\|_\infty = \max(\ |-2| + |-1|,\ |1| + |-2|) = \max(\ 3,\ 3) = 3 \quad \geq \rho(A)$

Symmetry, positive definiteness, diagonal dominance

Problem 3
Consider the following matrices:

$$A = \begin{bmatrix} -5 & -1 & 0 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

a) Which ones of them are symmetric?
b) Which ones of them are strictly diagonally dominant?
c) Which ones of them are symmetric positive definite?

Solution

a) The matrices B and C are symmetric, the matrix A is not.

b) Matrix A:

$$\text{rows:} \quad \begin{array}{ccccc} |-5| & > & |-1| & + & |0| \\ |3| & < & |3| & + & |1| \\ |2| & \geq & |1| & + & |-1| \end{array} \qquad \text{columns:} \quad \begin{array}{ccccc} |-5| & > & |3| & + & |1| \\ |3| & > & |-1| & + & |-1| \\ |2| & > & |1| & + & |0| \end{array}$$

The condition is violated at the second row $\Rightarrow$ A is not d.d. by rows. Let us try the columns: the condition holds for all three columns, moreover, the inequalities are strict $\Rightarrow$ A is s.d.d. by columns. It follows that A is strictly diagonally dominant.
The matrix B is symmetric - we can check the rows only (the columns would give the same result):

$$\begin{array}{ccccc} |1| & \geq & |-1| & + & |0| \\ |2| & \geq & |-1| & + & |-1| \\ |2| & > & |0| & + & |-1| \end{array}$$

The condition holds for all rows, however the inequality is not always strict $\Rightarrow$ B is diagonally dominant, although not strictly.
The matrix C is symmetric - we can check the rows only:

$$\begin{array}{ccccc} |1| & \geq & |-1| & + & |0| \\ |1| & < & |-1| & + & |1| \\ |2| & > & |0| & + & |1| \end{array}$$

The matrix C is not d.d. by rows (neither by columns). It follows that the matrix C is not diagonally dominant.

c) A symmetric matrix is *positive definite*, if and only if all its leading principal minors are positive.
The matrix A is not symmetric.
The matrix B:

$$\det(1) = 1 > 0 \ , \quad \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 1 > 0 \ , \quad \det \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 1 > 0$$

All leading principal minors are positive and so the matrix B is positive definite.

The matrix C:
$\det(1) = 1 > 0$, $\det\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0$
The second leading principal minor is not positive, so we can stop investigating. The matrix C is not positive definite.

Conclusions: The matrix A is strictly diagonally dominant, it is not symmetric, so there is no need to check its positive definiteness. The matrix B is diagonally dominant (not strictly), it is symmetric and positive definite. The matrix C is symmetric, it is not diagonally dominant, nor positive definite.