Method of (Steepest) Gradient Descent

Theorem: Suppose A is a symmetric and positive definite (*spd*) matrix,
b is a vector and $J(x)$ is the quadratic functional $J(x) = \frac{1}{2}x^T A\, x - x^T b\,.$
Then $\quad A\bar{x} = b \iff J(\bar{x}) < J(x) \quad \forall\, x \neq \bar{x}.$

This Theorem says that the solution of a linear system $Ax = b$ with *spd* matrix can be found by minimizing the quadratic functional $J(x)$. To achieve this, *gradient methods* can be used. The most illustrative method of this class is the *Method of Gradient Descent*, sometimes also called *Method of Steepest Descent*.

Method of Steepest Descent
with exact line search for a quadratic function of multiple variables

The main idea: Start at some point x_0, find the direction of the steepest descent of the value of $J(x)$ and move in that direction as long as the value of $J(x)$ descends. At this point, find the new direction of the steepest descent and repeat the whole process.

Note: a direction of the steepest descent of a function at a given point is the direction opposite to its gradient at that point. The gradient is perpendicular to a contour line passing through the given point. See illustration in Figure 1.

Gradient of a quadratic function of multiple variables

$$J(x) = \frac{1}{2}x^T A\, x - x^T b = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\, x_i\, x_j - \sum_{i=1}^{n} b_i\, x_i$$

$$\frac{\partial J(x)}{\partial x_k} = \sum_{i=1}^{n} a_{ki}\, x_i - b_k\, x_k \quad \Rightarrow \quad \mathrm{grad}(J) = A\, x - b.$$

The direction opposite to the gradient of $J(x)$ is equal to the residual
$r = b - A\, x$ of the system $Ax = b$.

Exact line search for a quadratic function

Assume a point $x_0 \in R^n$ and a vector $v \in R^n$ are given. Then the equation
$$x = x_0 + \alpha\, v, \quad \alpha \in R,$$
represents a line going through the point x_0 in the direction of v.

The problem: find the minimum of the functional $J(x)$ on that line, that is
find the minimum of the function $f(\alpha) \equiv J(x_0 + \alpha\, v)$ of one real variable α:

$$\begin{aligned} f(\alpha) = J(x_0 + \alpha\, v) &= \tfrac{1}{2}(x_0 + \alpha\, v)^T A\,(x_0 + \alpha\, v) - (x_0 + \alpha\, v)^T b = \\ &= \tfrac{1}{2}[x_0^T A\, x_0 + \alpha\, x_0^T A\, v + \alpha\, v^T A\, x_0 + \alpha^2\, v^T A\, v] - x_0^T b - \alpha\, v^T b = \\ &= \tfrac{1}{2}[x_0^T A\, x_0 + 2\alpha\, v^T A\, x_0 + \alpha^2\, v^T A\, v] - x_0^T b - \alpha\, v^T b \end{aligned}$$

- the last equality holds due to symmetry of A.

$$f'(\alpha) = v^T A\, x_0 + \alpha\, v^T A\, v - v^T b = 0 \iff \alpha = \frac{v^T (b - A\, x_0)}{v^T A\, v}$$

The algorithm of Method of Steepest Descent:

Choose $x^{(0)}$. For $k = 0, 1, 2, \dots$ compute

1. $r^{(k)} = b - A\,x^{(k)}$
2. $\alpha_k = (r^{(k)})^T\, r^{(k)} / (r^{(k)})^T A\; r^{(k)}$
3. $x^{(k+1)} = x^{(k)} + \alpha_k\, r^{(k)}$

until $||r^{(k)}|| < \varepsilon$ for some small ε of your choice.

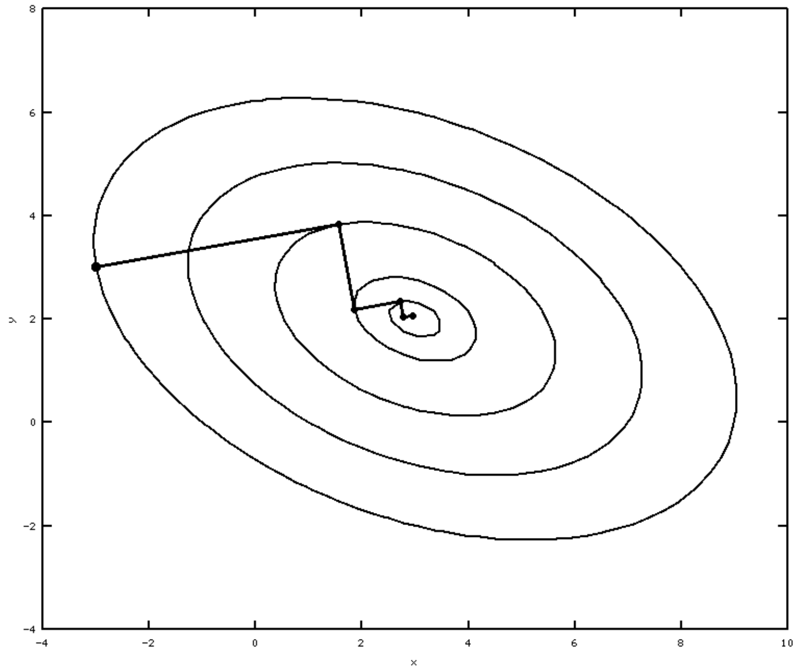

Figure 1: The ellipses represent contour lines of a quadratic functional. The polygonal line starting at the big bullet (on the outermost ellipse) is a path to the lower values of the functional computed by method of steepest descent.

Example

Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

a) Can the method of steepest descent be used for solving this system?
b) If yes, compute first three iterations by this method, starting from $x^{(0)} = (0, 0, 0)^T$.

Solution:

a) Let us verify the sufficient condition for using the method, i.e. that matrix A is *spd*.

A is symmetric, so we have to check positive definitness:

A is strictly diagonally dominant with positive values on the main diagonal $\Rightarrow$ A is *spd*.

Another (more laborious) way how to prove positive definitness:

$\det(3) = 3 > 0$, $\det \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = 8 > 0$, $\det \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = 20 > 0$

All leading principal minors are positive $\Rightarrow$ A is positive definite.

Conclusion: the method of steepest descent can be used to solve this system.

b) $\mathbf{k = 0}$:

1.
$$r^{(0)} = b - A\,x^{(0)} = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

2.
$$(r^{(0)})^T\, r^{(0)} = \begin{bmatrix} -1 & 7 & -7 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} = 1 + 49 + 49 = 99$$

$$(r^{(0)})^T A\, r^{(0)} = \begin{bmatrix} -1 & 7 & -7 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} = \begin{bmatrix} -1 & 7 & -7 \end{bmatrix} \begin{bmatrix} -17 \\ 29 \\ -29 \end{bmatrix} = 423$$

$\alpha_0 = (r^{(0)})^T\, r^{(0)} / (r^{(0)})^T A\ r^{(0)} = 99/423 = 0.2340$

3.
$$\mathbf{x^{(1)}} = x^{(0)} + \alpha_0\, r^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0.2340 \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} = \begin{bmatrix} -0.2340 \\ 1.6383 \\ -1.6383 \end{bmatrix}$$

$\mathbf{k = 1}$:

1.
$$r^{(1)} = b - A\,x^{(1)} = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -0.2340 \\ 1.6383 \\ -1.6383 \end{bmatrix} = \begin{bmatrix} 2.9787 \\ 0.2128 \\ -0.2128 \end{bmatrix}$$

2.
$$(r^{(1)})^T\, r^{(1)} = \begin{bmatrix} 2.9787 & 0.2128 & -0.2128 \end{bmatrix} \begin{bmatrix} 2.9787 \\ 0.2128 \\ -0.2128 \end{bmatrix} = 8.9633$$

$$(r^{(1)})^T A\ r^{(1)} = \begin{bmatrix} 2.9787 & 0.2128 & -0.2128 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2.9787 \\ 0.2128 \\ -0.2128 \end{bmatrix} =$$

$$= \begin{bmatrix} 2.9787 & 0.2128 & -0.2128 \end{bmatrix} \begin{bmatrix} 8.5106 \\ -2.1277 \\ 2.1277 \end{bmatrix} = 24.4455$$

$\alpha_1 = (r^{(1)})^T\, r^{(1)} / (r^{(1)})^T A\ r^{(1)} = 8.9633/24.4455 = 0.3667$

3.

$$\mathbf{x^{(2)}} = x^{(1)} + \alpha_1\, r^{(1)} = \begin{bmatrix} -0.2340 \\ 1.6383 \\ -1.6383 \end{bmatrix} + 0.3667 \begin{bmatrix} 2.9787 \\ 0.2128 \\ -0.2128 \end{bmatrix} = \begin{bmatrix} 0.8582 \\ 1.7163 \\ -1.7163 \end{bmatrix}$$

$\mathbf{k = 2}$:

1.

$$r^{(2)} = b - A\,x^{(2)} = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0.8582 \\ 1.7163 \\ -1.7163 \end{bmatrix} = \begin{bmatrix} -0.1418 \\ 0.9929 \\ -0.9929 \end{bmatrix}$$

2.

$$(r^{(2)})^T\, r^{(2)} = \begin{bmatrix} -0.1418 & 0.9929 & -0.9929 \end{bmatrix} \begin{bmatrix} -0.1418 \\ 0.9929 \\ -0.9929 \end{bmatrix} = 1.9919$$

$$(r^{(2)})^T A\, r^{(2)} = \begin{bmatrix} -0.1418 & 0.9929 & -0.9929 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -0.1418 \\ 0.9929 \\ -0.9929 \end{bmatrix} =$$

$$= \begin{bmatrix} -0.1418 & 0.9929 & -0.9929 \end{bmatrix} \begin{bmatrix} -2.4113 \\ 4.1135 \\ -4.1135 \end{bmatrix} = 8.5106$$

$\alpha_2 = (r^{(2)})^T\, r^{(2)} / (r^{(2)})^T A\; r^{(2)} = 1.9919/8.5106 = 0.2340$

3.

$$\mathbf{x^{(3)}} = x^{(2)} + \alpha_2\, r^{(2)} = \begin{bmatrix} 0.8582 \\ 1.7163 \\ -1.7163 \end{bmatrix} + 0.2340 \begin{bmatrix} -0.1418 \\ 0.9929 \\ -0.9929 \end{bmatrix} = \begin{bmatrix} 0.8250 \\ 1.9487 \\ -1.9487 \end{bmatrix}$$

$$r^{(3)} = b - A\,x^{(3)} = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0.8250 \\ 1.9487 \\ -1.9487 \end{bmatrix} = \begin{bmatrix} 0.4225 \\ 0.0302 \\ -0.0302 \end{bmatrix}$$

The convergence is quite slow - the exact solution is $\bar{x} = (1,\ 2,\ -2)^T$.