ODE - second-order boundary value problems

We want to find a solution $y(x)$ of a linear, second-order, self-adjoint boundary value problem with Dirichlet boundary condition on the interval $\langle a, b\rangle$:

$$-(p(x)\,y'(x))' + q(x)\,y(x) = f(x)\;, \quad y(a) = y_0\,,\; y(b) = y_n\,, \tag{1}$$

or its special simple case with $p(x) \equiv 1$:

$$-y''(x) + q(x)\,y(x) = f(x)\;, \quad y(a) = y_0\,,\; y(b) = y_n\,. \tag{2}$$

Existence and uniqueness of the (exact) solution

Sufficient conditions for existence of a unique solution of the problem (1):

- functions $p(x)$, $p'(x)$, $q(x)$, $f(x)$ are continuous on the interval $\langle a, b\rangle$, and
- $p(x) > 0$, $q(x) \geq 0$ on $\langle a, b\rangle$

Numerical solution by the finite-difference method

Choose a suitable step size h, set $n = \frac{b-a}{h}$ and define $n-1$ equidistant nodes inside the interval $\langle a, b\rangle$: $a = x_0 < x_1 < \ldots x_n = b$, $x_{i+1} - x_i = h$ for all $i = 0, 1, 2, \ldots n-1$. Denote by y_i an approximate value of $y(x_i)$. Values y_0 and y_n are given by the boundary conditions, remaining values y_i for $i = 1, 2, \ldots n-1$ can be computed from a system of linear equations

$$-p_{i-\frac{1}{2}}\;y_{i-1} + (p_{i-\frac{1}{2}} + h^2 q_i + p_{i+\frac{1}{2}})\;y_i - p_{i+\frac{1}{2}}\;y_{i+1} = h^2 f_i \tag{3}$$

using the notation $\quad q_i = q(x_i)\;, \quad f_i = f(x_i)\;, \quad p_{i\pm\frac{1}{2}} = p(x_i \pm \frac{h}{2})\;.$

Specially in the case (2), the formula (3) is simplified to

$$-y_{i-1} + (2 + h^2 q_i)\;y_i - y_{i+1} = h^2 f_i\;. \tag{4}$$

If $p(x) > 0$ and $q(x) > 0$ on $\langle a, b\rangle$, then the matrix of the system is strictly diagonal dominant – so it is nonsingular and the system can be solved by Jacobi or Gauss-Seidel iterative method.

Inference of systems (3) or (4) from the equations (1) or (2), respectively

Let us start with the simple case (2). The equation (2) is expressed in every inner node x_i for $i = 1, 2, \ldots n-1$ as

$$-y''(x_i) \; + \; q(x_i)\, y(x_i) \; = \; f(x_i) \; ,$$

then the second derivatives are approximated by the second central differences:

$$-\frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{h^2} + \mathcal{O}(h^2) \; + \; q(x_i)\; y(x_i) = f(x_i) \; .$$

Omitting the consistence error, using the approximate values y_i instead of $y(x_i)$ and notation $q_i = q(x_i)$, $f_i = f(x_i)$ leads to

$$-\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + q_i \; y_i = f_i$$

and after multiplying by h^2 and rearranging, the formula (4) is obtained.

Inference of the numerical formula (3) for the more general equation (1) is less straightforward. Again, the first step is to express the equation in every inner node x_i for $i = 1, 2, \ldots n-1$:

$$-(p(x_i)\, y'(x_i))' \; + \; q(x_i)\, y(x_i) \; = \; f(x_i) \; . \tag{5}$$

Then the first derivatives are approximated by central differences with $\frac{h}{2}$, in two steps:

1. Denote $z(x) = p(x)\, y'(x)$ and use the central difference: $z'(x_i) \approx \frac{z(x_i+h/2) - z(x_i-h/2)}{h}$, then the first term of equation (5) can be approximated as
$$-(p(x_i)\, y'(x_i))' = -z'(x_i) \approx -\tfrac{1}{h}\, [\, p(x_i + \tfrac{h}{2})\, y'(x_i + \tfrac{h}{2}) - p(x_i - \tfrac{h}{2})\, y'(x_i - \tfrac{h}{2})\,] =$$
$$= \tfrac{1}{h}\, [\, p_{i-\frac{1}{2}}\, y'(x_i - \tfrac{h}{2}) - p_{i+\frac{1}{2}}\, y'(x_i + \tfrac{h}{2})\,] \; .$$
2. Use central differences at mid-points:
$y'(x_i + \frac{h}{2}) \approx \frac{y(x_i+h) - y(x_i)}{h}$, $\; y'(x_i - \frac{h}{2}) \approx \frac{y(x_i) - y(x_i-h)}{h}$, then
$$-(p(x_i)\, y'(x_i))' \approx \frac{1}{h^2}\, [-p_{i+\frac{1}{2}}\; y(x_i{+}h) + (p_{i-\frac{1}{2}} {+} p_{i+\frac{1}{2}})\; y(x_i) - p_{i-\frac{1}{2}}\; y(x_i{-}h)\,] \; . \tag{6}$$

System (3) is obtained by substituting (6) into (5), using $y_i \approx y(x_i)$ and rearranging.

Problem 1

Consider the equation $\quad -(x^2y'(x))' + \dfrac{x^3}{2+x}y(x) = 4 + x$

with boundary conditions $y(-5) = -2,\ y(-3) = 2.$

a) Prove the existence of a unique solution of the given problem.

b) Choose step size of $h = 0.4$ and using the finite difference method put together the system of linear equations for computing an approximate solution at the nodes of the mesh.

Solution

Using the notation of the general problem (1) we have

$p(x) = x^2 \quad , \quad q(x) = \dfrac{x^3}{2+x} \quad , \quad f(x) = 4 + x \quad .$

a) Verify the conditions sufficient for the existence of an unique solution of the problem:

- functions x^2, $2x$, $\dfrac{x^3}{2+x}$, $4 + x$ are continuous in the interval $\langle -5, -3 \rangle$,
- $x^2 > 0, \quad \dfrac{x^3}{2+x} \geq 0$ in $\langle -5, -3 \rangle$,

so there exists a unique solution of the given problem.

b) First of all, let us divide the interval with step size $h = 0.4$ (it has to be chosen so that both endpoints of the interval are nodes of the mesh), compute coefficients needed for assembling the system of equations and store them into Table 1:

$p_{\frac{1}{2}} = p(-4.8) = (-4.8)^2 = 23.04$
$p_{1\frac{1}{2}} = p(-4.4) = (-4.4)^2 = 19.36$
$\ldots$

$h^2q_1 = 0.4^2 \cdot q(-4.6) = 0.16 \cdot \frac{(-4.6)^3}{2-4.6} = 5.9899$
$h^2q_2 = 0.4^2 \cdot q(-4.2) = 0.16 \cdot \frac{(-4.2)^3}{2-4.2} = 5.3882$
$\ldots$

$h^2f_1 = 0.4^2 \cdot f(-4.6) = 0.16 \cdot (4 - 4.6) = -0.096$
$h^2f_2 = 0.4^2 \cdot f(-4.2) = 0.16 \cdot (4 - 4.2) = -0.032$
$\ldots$

i	x_i	$p_{i\pm\frac{1}{2}}$	$h^2 q_i$	$h^2 f_i$
	-4.8	23.04		
1	-4.6		5.9899	-0.096
	-4.4	19.36		
2	-4.2		5.3882	-0.032
	-4.0	16.00		
3	-3.8		4.8775	0.032
	-3.6	12.96		
4	-3.4		4.4919	0.096
	-3.2	10.24		

Table 1: Coefficients needed for assembling the system of equations for Problem 1.

Use the coefficitents for assembling 4 equations for 4 unknowns y_1 až y_4 :

the first equation $(i = 1)$:

$-23.04\ y_0 + (23.04 + 5.9899 + 19.36)\ y_1 - 19.36\ y_2 = -0.096$

substitute the boundary value $y_0 = -2$ and move to the right hand side:

$48.3899\ y_1 - 19.36\ y_2 = -0.096 + 23.04 \cdot (-2) = -46.176$

the second equation $(i = 2)$:

$-19.36\ y_1 + (19.36 + 5.3882 + 16.00)\ y_2 - 16.00\ y_3 = -0.032$

$-19.36\ y_1 + 40.7482\ y_2 - 16.00\ y_3 = -0.032$

the third equation $(i = 3)$:

$-16.00\ y_2 + (16.00 + 4.8775 + 12.96)\ y_3 - 12.96\ y_4 = 0.032$

$-16.00\ y_2 + 33.8375\ y_3 - 12.96\ y_4 = 0.032$

the fourth equation $(i = 4)$:

$-12.96\ y_3 + (12.96 + 4.4919 + 10.24)\ y_4 - 10.24\ y_5 = 0.096$

substitute the boundary value $y_5 = 2$ and move to the right hand side:

$-12.96\ y_3 + 27.6919\ y_4 = 0.096 + 10.24 \cdot 2 = 20.576$

Resulting system of equations in matrix notation:

$$\begin{bmatrix} 48.3899 & -19.36 & 0 & 0 \\ -19.36 & 40.7482 & -16.00 & 0 \\ 0 & -16.00 & 33.8375 & -12.96 \\ 0 & 0 & -12.96 & 27.6919 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -46.176 \\ -0.032 \\ 0.032 \\ 20.576 \end{bmatrix}$$

The solution of the system of equations is the vector

Y = (-1.1719, -0.5440, 0.0345, 0.7592)^T

representing approximate values of the solution at inner nodes of the mesh, it means approximate values of $y(-4.6)$, $y(-4.2)$, $y(-3.8)$ and $y(-3.4)$.

Problem 2 - a harmonic oscilator (damped oscillations)
Consider the equation $y'' + 2y' + y = e^{-t}$ with boundary conditions $y(0) = 2,\ y(2) = 0$. Find numerically an approximate value of the solution $y(0.2)$ at the time $t = 0.2$.

Solution

First of all, the given equation need to be transformed to the self-adjoint form (1). This can be done by the following four steps:

- perform the differentiation of the first term of (1):

$$-p(x)\,y''(x) \;-\; p'(x)\,y'(x) \;+\; q(x)\,y(x) \;=\; f(x) \tag{7}$$

- multiply the given equation by $-p(x)$ (variable t is now renamed to x):

$$-p(x)\,y''(x) \;-\; 2\,p(x)\,y'(x) \;-\; p(x)\,y(x) \;=\; -p(x)\,e^{-x} \tag{8}$$

- compare the coefficients of (7) and (8):

  $p'(x) = 2\,p(x),\ \ q(x) = -p(x),\ \ f(x) = -p(x)\,e^{-x}$

- solve for p, q and f (hint: suppose $p(x) = e^{cx}$, then $p'(x) = c\,e^{cx} = c\,p(x)$):

  $p(x) = e^{2x},\ \ q(x) = -e^{2x},\ \ f(x) = -e^{2x}\,e^{-x} = -e^{x}$

The self-adjoint form is $-(e^{2t}y'(t))' - e^{2t}y(t) = -e^{t}$.

This problem does not comply with the conditions *sufficient* (not *necessary*) for the existence of unique solution listed above, so it is not guaranteed that we obtain a meaningful result when solving such equation numerically. Nevertheless, we try our hand at the numerical solution of this illustrative problem.

Note: For this specific problem, however, we know there exists unique solution, as we are able to solve it analytically – general solution of the equation is $y = (c_1 + c_2\,t + 0.5\,t^2)\,e^{-t}$ and using the boundary conditions we have unique solution $c_1 = 2,\ c_2 = -2$.

In order to find approximate solution at $t = 0.2$, we have to compute approximate solution on the whole interval. First divide the interval with the step size $h = 0.2$ (it has to be chosen so that both endpoints of the interval are nodes of the mesh) and prepare coefficients needed for assembling the system of equations into Table 2:

We have $p(t) = e^{2t}$, $q(t) = -e^{2t}$ a $f(t) = -e^{t}$,

$p_{\frac{1}{2}} = p(0.1) = e^{0.2} = 1.2214$
$p_{1\frac{1}{2}} = p(0.3) = e^{0.6} = 1.8221$
...

$h^2 q_1 = 0.2^2 \cdot q(0.2) = 0.04 \cdot (-e^{0.4}) = -0.0597$
$h^2 q_2 = 0.2^2 \cdot q(0.4) = 0.04 \cdot (-e^{0.8}) = -0.0890$
...

$h^2 f_1 = 0.2^2 \cdot f(0.2) = 0.04 \cdot (-e^{0.2}) = -0.0489$
$h^2 f_2 = 0.2^2 \cdot f(0.4) = 0.04 \cdot (-e^{0.4}) = -0.0597$
...

i	t_i	$p_{i\pm\frac{1}{2}}$	$h^2 q_i$	$h^2 f_i$
	0.1	1.2214		
1	0.2		-0.0597	-0.0489
	0.3	1.8221		
2	0.4		-0.0890	-0.0597
	0.5	2.7183		
3	0.6		-0.1328	-0.0729
	0.7	4.0552		
4	0.8		-0.1981	-0.0890
	0.9	6.0496		
5	1.0		-0.2956	-0.1087
	1.1	9.0250		
6	1.2		-0.4409	-0.1328
	1.3	13.4637		
7	1.4		-0.6578	-0.1622
	1.5	20.0855		
8	1.6		-0.9813	-0.1981
	1.7	29.9641		
9	1.8		-1.4639	-0.2420
	1.9	44.7012		

Table 2: Coefficients needed for assembling the system of equations for Problem 2.

Now we use the prepared coefficients for assembling 9 equations for unknowns y_1 to y_9:

the first equation (for $i = 1$):

$-1.2214\ y_0 + (1.2214 - 0.0597 + 1.8221)\ y_1 - 1.8221\ y_2 = -0.0489$

substitute the boundary value $y_0 = 2$ and move the corresponding term to the rhs:

$2.9838\ y_1 - 1.8221\ y_2 = -0.0489 + 2 \cdot 1.2214 = 2.3939$

the second equation (for $i = 2$):

$-1.8221\ y_1 + (1.8221 - 0.0890 + 2.7183)\ y_2 - 2.7183\ y_3 = -0.0597$

$-1.8221\ y_1 + 4.4514\ y_2 - 2.7183\ y_3 = -0.0597$

. . . etc.

the last equation (for $i = 9$):

$-29.9641\ y_8 + (29.9641 - 1.4639 + 44.7012)\ y_9 - 44.7012\ y_{10} = -0.2420$

substitute the boundary value $y_{10} = 0$:

$-29.9641\ y_8 + 73.2014\ y_9 = -0.2420$

The solution of the system of equations is a vector
$Y = (1.3259,\ 0.8574,\ 0.5373,\ 0.3230,\ 0.1836,\ 0.0961,\ 0.0442,\ 0.0161,\ 0.0033)^T$.

An approximate value of $y(0.2)$ is $y_1 = 1.3259$
(for comparison: exact value of $y(0.2)$ is 1.3263).