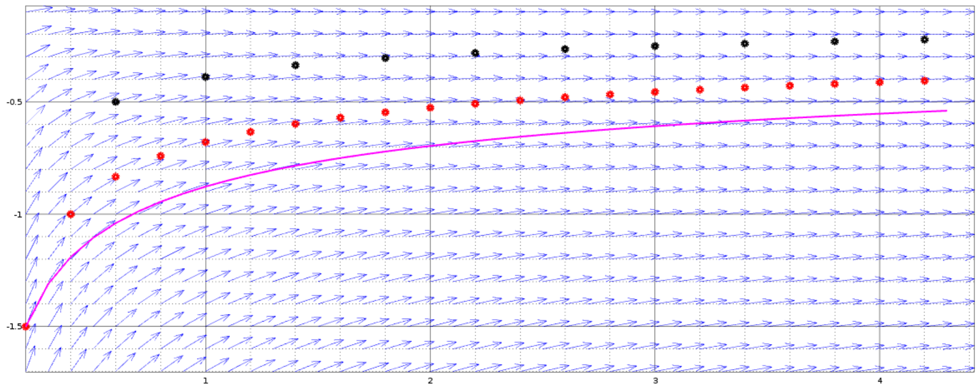
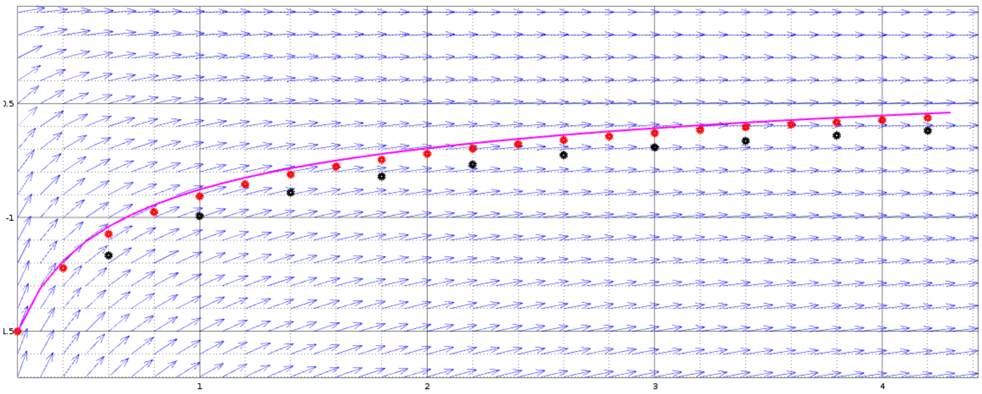
One-step methods errors – graphically

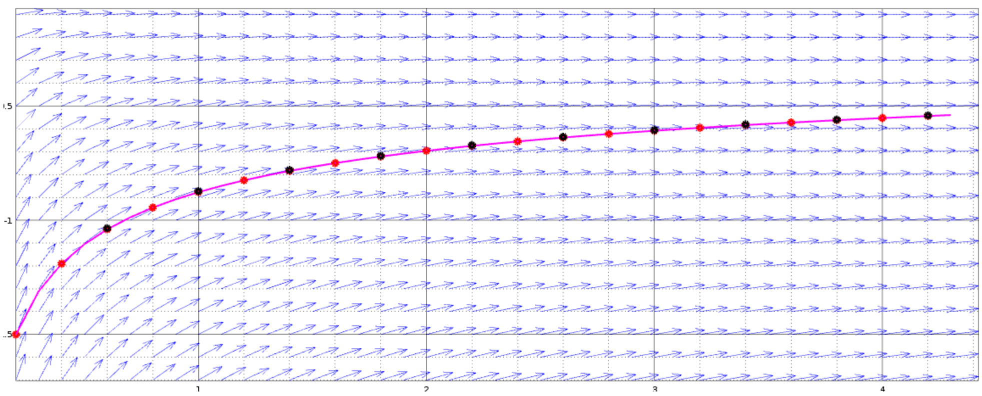
Bellow, three explicit RK methods are illustrated on an example of $y' = -\frac{1}{3}\frac{y}{x}$ with the initial condition $y(0.2) = -1.5$, using step-sizes $h = 0.4$ (black) and $h = 0.2$ (red). Exact solution: $-1.5 \cdot 0.2^{\frac{1}{3}} \cdot x^{-\frac{1}{3}}$ (magenta).



Euler's method – the 1-st order; max GTE is 0.54 and 0.21, respectively



Midpoint method – the 2-nd order; max GTE is 0.13 and 0.03, respectively



RK4 method – the 4-th order; max GTE is 0.005 and 0.0006, respectively

TEs for methods of different order and for different steps

Expected behavior of GTEs after halving the step (as h approaches zero):

Euler method – the 1-st order method: norm of the global error $\|e(h)\| = \mathcal{O}(h)$
$\|e(h)\| < M\,|h|$, $\|e(\frac{h}{2})\| < M\,|\frac{h}{2}| = \frac{M}{2}\,|h|$ ... the error can be expected 2-times less

Midpoint (Collatz) method – the 2-nd order method: $\|e(h)\| = \mathcal{O}(h^2)$
$\|e(h)\| < M\,|h|^2$, $\|e(\frac{h}{2})\| < M\,|\frac{h}{2}|^2 = \frac{M}{4}\,|h|^2$ ... 4-times less

RK4 method – the 4-st order method: $\|e(h)\| = \mathcal{O}(h^4)$
$\|e(h)\| < M\,|h|^4$, $\|e(\frac{h}{2})\| < M\,|\frac{h}{2}|^4 = \frac{M}{16}\,|h|^4$ ... 16-times less

Illustration on the previous example:

	Euler		midpoint		RK4	
h	max GTE	ratio	max GTE	ratio	max GTE	ratio
0.8	1.38		0.40		0.03	
0.4	0.54	2.5-2.8	0.13	3.4-3.5	5e-3	$\approx$ 6.8
0.2	0.21	2.3-2.6	0.03	3.8-3.9	6e-4	$\approx$ 9.7
0.1	0.09	2.2-2.4	0.008	4.12-4.15	4e-5	$\approx$ 12.8
0.05	0.04	2.1-2.2	0.002	4.16-4.18	3e-6	$\approx$ 14.9
0.025	0.02	2.0-2.1	0.0005	4.10-4.13	2e-7	$\approx$ 15.9

In this table, for all the methods: In the left column there is the row norm of vector of GTEs in interval $(0.2, 4.2)$, in the right column there is the extent of values of the vector of ratios of GTE errors computed for the value of h on the previous line (twice as big) to the current one. As h approaches to zero, this ratio converges to the prediction based on the order of the method.

The dependence of errors on both step-length and order of the method can be used for estimation of errors and that can be further used for optimization of step-length (as usual, we assume well-posed problems: then LTE is $\mathcal{O}(h^{p+1})$ for p-th order method).

Estimation of LTE based on halving the step:
Let the last approximation $[x_k, y_k]$ be the starting point, let y_{k+1} be the next approximation at x_{k+1} with step h and let $\widehat{y}_{k+1}$ be the approximation at x_{k+1} with step $\frac{h}{2}$ (i.e. two steps are needed from x_k to x_{k+1}). Then for a method of p-th order we can roughly estimate LTE for the half-step as $E_{k+1} \approx \dfrac{|y_{k+1} - \widehat{y}_{k+1}|}{2^p - 1}$.

Estim. of LTE based on 2 methods of different order: (as in Matlab ODE45)
Let the last approximation $[x_k, y_k]$ be the starting point, let y_{k+1} be the next approximation at x_{k+1} using a method of p-th order and let $\widehat{y}_{k+1}$ be the approximation at x_{k+1} using a method of $(p+1)$-th order (both with the same step h). Then LTEs are $E_{k+1} = |y_{k+1} - y(x_{k+1})| = \mathcal{O}(h^{p+1})$, $\widehat{E}_{k+1} = |\widehat{y}_{k+1} - y(x_{k+1})| = \mathcal{O}(h^{p+2})$
and $E_{k+1} = |y_{k+1} - \widehat{y}_{k+1} + \widehat{y}_{k+1} - y(x_{k+1})| = |y_{k+1} - \widehat{y}_{k+1}| + \mathcal{O}(h^{p+2})$, so
$E_{k+1} \approx |y_{k+1} - \widehat{y}_{k+1}|$.