General one-step method

First-order initial value (Cauchy) problem for a system of ODEs:

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)) \quad \text{with an initial condition} \;\; \mathbf{Y}(x_0) = \mathbf{Y}^{(0)}\,, \tag{1}$$

In this text, we assume that the problem (1) is *well-posed*, which means (roughly speaking) that it's solution depends continuously on the given data: that a small perturbation of $\mathbf{F}$ or $\mathbf{Y}^{(0)}$ leads to a small change of the solution.

We consider here *one-step methods* only, which have a form of

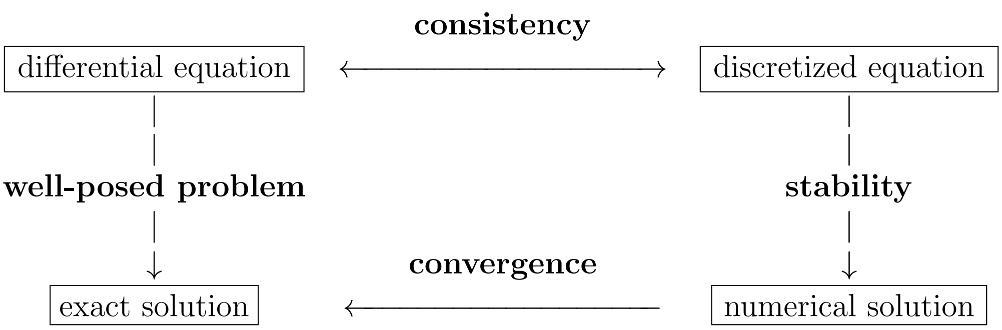
$$\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\;\Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)}, \, x_k\,, h)\;, \tag{2}$$

where h is the step-size, $x_k = x_0 + k\,h$ and $\mathbf{Y}^{(k)}$ is the numerical approximation of the exact solution $\mathbf{Y}(x_k)$ of the problem (1).

Examples of one-step methods:

- Explicit Euler's method: $\Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)}, \, x_k\,, h) = \mathbf{F}(\,x_k\,, \mathbf{Y}^{(k)})$
- Implicit Euler's method: $\Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)}, \, x_k\,, h) = \mathbf{F}(\,x_k + h\,, \mathbf{Y}^{(k+1)})$
- Midpoint method: $\Phi(\,\mathbf{Y}^{(k)}, \mathbf{Y}^{(k+1)}, \, x_k\,, h) = \mathbf{F}(\,x_k + \frac{h}{2}\,, \mathbf{Y}^{(k)} + \frac{h}{2}\,\mathbf{F}(\,x_k\,, \mathbf{y}_k))$

Convergence, consistency, stability



Convergence

Global truncation error (GTE) of the approximate solution $\mathbf{Y}^{(k)}$ at x_k is defined as

$$e_k = \|\,\mathbf{Y}^{(k)} \;-\; \mathbf{Y}(x_k)\|\,. \tag{3}$$

The numerical solution on a given interval $< a, b >$ should approach the exact one (converge to it) as the mesh-size h tends to zero, which means that some norm of the vector $\mathbf{e} = (e_1, \ldots e_n)^T$ of global errors, where $n = |b - a|/h$, should tend to zero:

$$\|\mathbf{e}\| \to 0 \;\; \text{as} \;\; h \to 0\,.$$

Order of a method: A method is of p-th order of accuracy, if maximum norm of a vector $\mathbf{e} = (e_1, \ldots e_n)^T$ of global errors (3) is $\mathcal{O}(h^p)$: $\|\mathbf{e}\| = \max(|e_1|, \ldots |e_n|) = \mathcal{O}(h^p)$.

Local truncation error (LTE) at x_{k+1} is defined as

$$E_{k+1} = \| \, \mathbf{Y}^{(k+1)} \; - \; \widehat{\mathbf{Y}}(x_{k+1}) \| \, , \tag{4}$$

where $\widehat{\mathbf{Y}}(x_{k+1})$ is the value of the exact solution for initial condition $\widehat{\mathbf{Y}}(x_k) = \mathbf{Y}^{(k)}$.

Note: Sometimes a method is said to be of p-th order, if LTE is $\mathcal{O}(h^{p+1})$. Both definitions are compatible for a well-posed problem: the order of the global error at the n-th time step can be reasonably guessed to be n-times the order of LTE; for fixed $a = x_0$ and $b = x_n$, n is proportional to $1/h$ and order of GTE at b can be expected to be proportional to $1/h$ times order of LTE, that is $\mathcal{O}(h^p)$.

Consistency of discretized equation with differential equation

The discretization of the differential equation should become exact, as the mesh-size tends to zero. Consistency error (for one-step method) at x_k is

$$\eta_k = \left\| \, \frac{\mathbf{Y}(x_{k+1}) - \mathbf{Y}(x_k)}{h} - \; \Phi(\, \mathbf{Y}(x_k), \mathbf{Y}(x_{k+1}), \, x_k \, , h) \right\| \tag{5}$$

– it is the error in discretized equation, if the exact solution evaluated at mesh-points is substituted into it. It measures the extent to which the true solution satisfies the discrete equation. Consistency errors should vanish, as the mesh-size tends to zero.

Stability

Numerical errors that are generated as a consequence of using discretized equation, should be held under control. There are several different definitions of stability which put this idea into more specific terms. We will not go into further details here.

Lax equivalence theorem

For *linear* well-posed initial value problem and *consistent* finite difference approximation of it, *stability* is necessary and sufficient condition for *convergence*.

Note: for nonlinear problems this equivalence does not hold. However, we probably cannot expect good behaviour of any method which is not convergent for linear problems, so consistency and stability of methods are important even if nonlinear problems are solved, when these two properties cannot guarantee convergency.

Examples

The theory will be illustrated on a scalar equation as the simplest case of the system (1):

$$y'(x) = f(x, y(x)) \; , \qquad y(x_0) = y_0 \; . \tag{6}$$

Then the consistency error (5) is

$$\eta_k = \left| \, \frac{y(x_{k+1}) - y(x_k)}{h} - \; \Phi(\, y(x_k), y(x_{k+1}), \, x_k \, , h) \, \right| \, ,$$

the limit of the first term is $\lim\limits_{h\to 0} \frac{y(x_k+h)-y(x_k)}{h} = y'(x_k) = f(x_k, y(x_k))$, so the consistency error tends to zero if and only if $\lim\limits_{h\to 0} \Phi(\, y(x_k), y(x_{k+1}),\, x_k\,, h) = f(x_k, y(x_k))$ for all x_k.

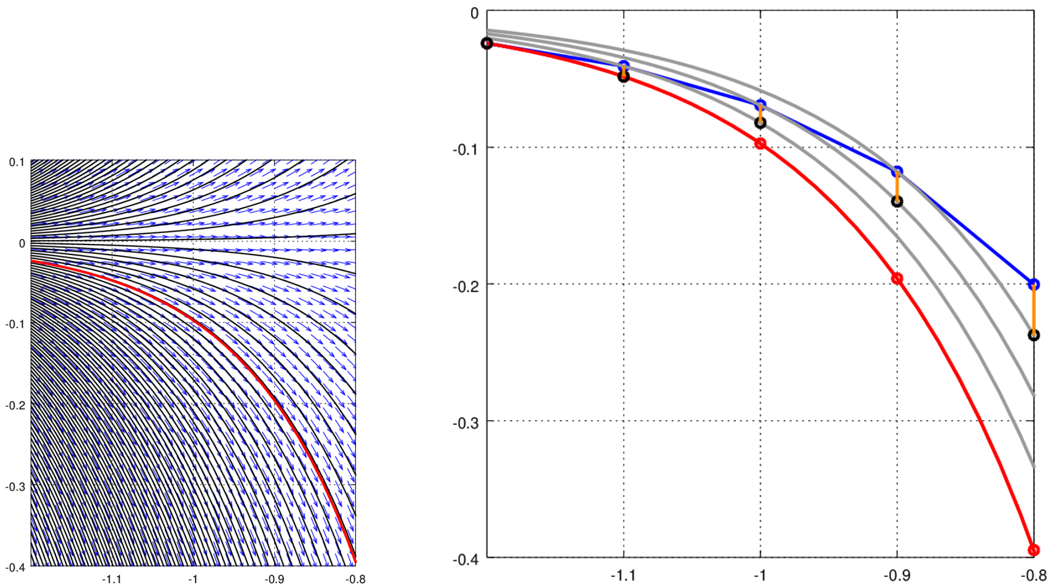
Explicit Euler's method $x_{k+1} = x_k + h, \quad y_{k+1} = y_k + h\, f(x_k, y_k)$

Convergence

Example: Consider Cauchy problem $y' = 7\,y$, $y(-1.2) = -0.024$. On the figure bellow there is the exact solution (red), several other solutions for different initial conditions and the approximate solution obtained by forward Euler's method with step-length $h = 0.1$ (blue circles connected by a broken line). Exact solution of the problem $y' = a\,y$, $y(x_0) = y_0$ is $y = c \cdot \mathrm{e}^{ax}$, where $c = y_0\, \mathrm{e}^{-ax_0}$.

Global error: At the given x_k, the distance between the approximation point (blue circle) and the exact solution of the problem (red circle on the red curve).

Local error: At the given x_k, the orange segment between the blue and the black circles, i.e. the (vertical) distance betwen the approximation point and the trajectory of the solution which passes through the previous approximation point.



Left: Direction field of the equation with different trajectories. **Right:** Local and global errors.

k	x_k	$y(x_k)$	y_k	$e_k = \lvert y_k - y(x_k)\rvert$	E_k
0	-1.2	-0.0240	-0.0240	0.0000	-
1	-1.1	-0.0483	-0.0408	0.0075	0.0075
2	-1.0	-0.0973	-0.0694	0.0280	0.0128
3	-0.9	-0.1960	-0.1179	0.0781	0.0218
4	-0.8	-0.3947	-0.2005	0.1942	0.0370

Note: $e_4 = 0.1942 \neq \sum\limits_{k=1}^{4} E_k = 0.0791$ … GTE is **not equal** to sum of LTE's

How to compute the order of LTE and GTE: The technique is allways the same – use expansion in Taylor series.

LTE (4) at x_{k+1} is $E_{k+1} = |\, y_{k+1} - \; y(x_{k+1})|$, where $y(x_{k+1})$ is the value of the exact solution $y' = f(x,y)$ for initial condition $y(x_k) = y_k$.

Euler's method: $y_{k+1} = y_k + \; h \; f(x_k, y_k)$

Taylor expansion of y at $y(x_k)$: $y(x_{k+1}) \equiv y(x_k + h) = y(x_k) + h\, y'(x_k) + \mathcal{O}(h^2)$.

Using $y(x_k) = y_k$ and $y'(x_k) = f(x_k, y(x_k)) = f(x_k, y_k)$: $y(x_{k+1}) = y_k + h\, f(x_k, y_k) + \mathcal{O}(h^2)$,

inserting this to the expression for E_{k+1} leads to

$E_{k+1} = |\, y_k + \; h \; f(x_k, y_k) \; - (\; y_k + h\, f(x_k, y_k) + \mathcal{O}(h^2)\,)| = \mathcal{O}(h^2)$.

GTE (3) at x_{k+1} is $e_{k+1} = |\, y_{k+1} \; - \; y(x_{k+1})|$.

(a) $y(x_{k+1}) = y(x_k) + h\, y'(x_k) + \mathcal{O}(h^2)$ ... Taylor expansion of exact solution

(b) $y_{k+1} = y_k + \; h \; f(x_k, y_k)$ ... Euler's method

(a) - (b):

$$
\begin{aligned}
y(x_{k+1}) - y_{k+1} \; &= \; y(x_k) - y_k \; + h\,(f(x_k, y(x_k)) - f(x_k, y_k)) + \mathcal{O}(h^2) \\
\underbrace{|y(x_{k+1}) - y_{k+1}|}_{e_{k+1}} &\le \underbrace{|y(x_k) - y_k|}_{e_k} \; + h\,|f(x_k, y(x_k)) - f(x_k, y_k)| + \mathcal{O}(h^2) \\
e_{k+1} &\le e_k + h\,|f(x_k, y(x_k)) - f(x_k, y_k)| + c\,h^2 \text{ for some } c \in R.
\end{aligned}
$$

The second term at the right hand side can be bounded using Lipschitz condition as $|f(x_k, y(x_k)) - f(x_k, y_k)| \le L\,|y(x_k) - y_k| = L\,e_k$, and so $e_{k+1} \le e_k\,(1 + h\,L) + c\,h^2$.

By recursion and using notation $a = (1 + h\,L)$ it follows

$e_1 \le c\,h^2$ (the local error at the first step)

$e_2 \le e_1\,a + c\,h^2 \le a\,c\,h^2 + c\,h^2 = (a+1)\,c\,h^2$

$e_3 \le e_2\,a + c\,h^2 \le (a+1)\,c\,h^2\,a + c\,h^2 = (a^2 + a + 1)\,c\,h^2$

...

$e_n \le (a^{n-1} + \ldots \; a^2 + a + 1)\,c\,h^2 = \frac{a^n - 1}{a-1}\,c\,h^2 = \frac{(1+h\,L)^n - 1}{L}\,c\,h \le (\mathrm{e}^{Lhn} - 1)\,\frac{1}{L}\,c\,h$

– the last inequality follows from the fact $x + 1 \le \mathrm{e}^x$. So if $hn = |x_0 - x_n|$ is constant, i. e., if we consider some bounded interval only, then the global error is $\mathcal{O}(h)$.

Conclusion: **Explicit Euler's method is of the first order**.

Consistency

Consistency error (5) is

$$
\eta_k = \left| \frac{y(x_{k+1}) - y(x_k)}{h} - \; f(x_k, y(x_k)) \right|
$$

Taylor expansion gives

$y(x_{k+1}) \equiv y(x_k + h) = y(x_k) + h\, y'(x_k) + \mathcal{O}(h^2) = y(x_k) + h\, f(x_k, y(x_k)) + \mathcal{O}(h^2)$

so $y(x_{k+1}) - y(x_k) = h\, f(x_k, y(x_k)) + \mathcal{O}(h^2)$ and after substitution we have

$$\eta_k = \left| \frac{h\, f(x_k, y(x_k)) + \mathcal{O}(h^2)}{h} - f(x_k, y(x_k)) \right| = |\, f(x_k, y(x_k)) + \mathcal{O}(h) - f(x_k, y(x_k))\,| = \mathcal{O}(h)\,,$$

which converges to zero as $h \to 0$.

Stability

Stability will be studied only on a standard model equation

$$y'(x) = -a\, y(x)\,, \quad y(x_0) = y_0\,, \quad a > 0 \tag{7}$$

However, if a method is not stable on this simple linear equation, we probably cannot expect its good behaviour on other equations, too.

The exact solution of equation (7) is $y_0\, \mathrm{e}^{-a(x-x_0)}$, which tends to zero as $x \to \infty$.

Euler's method:

$y_{k+1} = y_k + h(-a\, y_k) = (1 - h\, a)\, y_k = (1 - h\, a)^2\, y_{k-1} = \cdots = (1 - h\, a)^{k+1}\, y_0$

y_{k+1} converges to zero if and only if the modulus of the *growth factor* $(1 - h\, a)$ is less than 1, which means $|ha| < 2$ (for $1 < |ha| < 2$ it alternates sign, however it is stable).

So explicit Euler's method is only *conditionally stable*, see Figure 1.

conditional stability:

- existence of a *critical time step* beyond which numerical instabilities occur,
- is typical for explicit methods

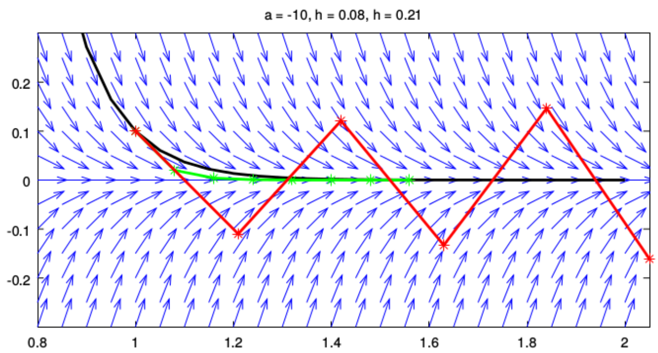


Figure 1: Problem $y' = -10\, y\,,\ y(1) = 0.1$ solved by explicit Euler's method. Black line: exact solution, green line: numerical solution with $h = 0.08$, red line: numerical solution with $h = 0.21$. Blue arrows at background represent directional field for the given equation.

Implicit Euler's method $x_{k+1} = x_k + h, \quad y_{k+1} = y_k + h\ f(x_{k+1}, y_{k+1})$

Convergence and consistency

Using similar techniques as for explicit Euler's method it can be proved that consistency error is $\mathcal{O}(h)$, LTE is $\mathcal{O}(h^2)$ and GTE is $\mathcal{O}(h)$. So the implicit Euler's method is consistent and it is of the first order – the same result as for explicit Euler's method.

Stability is studied on a standard model equation (7) only.

Implicit Euler's method: $y_{k+1} = y_k + h(-a\, y_{k+1})$,
for y_{k+1} the explicit formula can be obtained as

$$y_{k+1} = \frac{1}{1+ha}\, y_k = \left(\frac{1}{1+ha}\right)^2 y_{k-1} = \cdots = \left(\frac{1}{1+ha}\right)^{k+1} y_0$$

y_{k+1} converges to zero for any choice of h, because the growth factor is allways less than 1, so implicit Euler's method is *unconditionally stable.* This is typical behaviour of other implicit methods, too. Compare Figures 1 and 2 – the same problem is solved by explicit (Fig. 1) or implicit (Fig. 2) method.

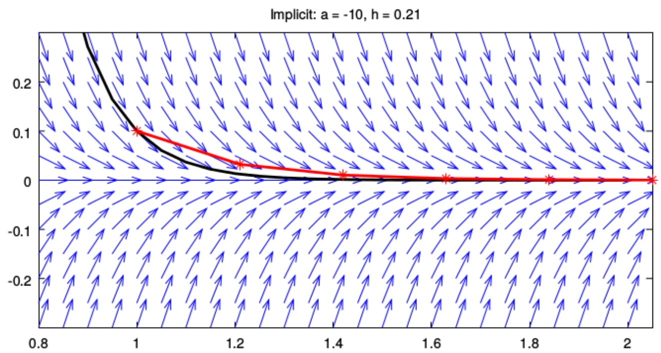


Figure 2: Problem $y' = -10\, y$, $y(1) = 0.1$ solved by implicit Euler's method. Black line: exact solution, red line: numerical solution with $h = 0.21$. Blue arrows at background represent directional field for the given equation.

References

- J. D. Lambert: Numerical Methods for Ordinary Differential Systems, Willey & Sons, 1993
- Gilbert Strang: Computational Science and Engineering, online MIT courses
- D. N. Arnold: Stability, consistency, and convergence of numerical discretizations