A first-order ODE, initial value problem

A first-order ordinary differential equation, initial value problem:

$$y' = f(x, y) \ , \quad \text{with the initial condition } \ y(x_0) = y_0 \, . \tag{1}$$

The equation (1) is called *linear*, if the function f is linear with respect to the variable y, i.e. it has a form $f(x, y) = g_0(x) + g_1(x)\, y$.

Existence and uniqueness of the (exact) solution

Before we try to solve the problem numerically, we should verify that the problem has a unique solution. This topic is covered here by a short recapitulation from a course on ordinary differential equations.

There are two standard theorems that formulate *sufficient* conditions for existence and uniqueness of the solution of the initial value problem (1):

Theorem 1
Suppose that the function $f(x, y)$ and its partial derivative $\frac{\partial f}{\partial y}$ are continuous in a domain $\Omega \subset R^2$. Then every point $[x_0, y_0] \in \Omega$ determines a unique maximal solution $y(x)$ of (1), $[x, y(x)] \subset \Omega$.

Theorem 2
Let the function $f(x, y)$ be continuous on a rectangle $D =< a, b > \times < c, d > \subset R^2$ and let it be *Lipschitz* continuous with respect to y in D, i.e. there exists a constant L such that for all $x \in< a, b >$ and $y_1, y_2 \in< c, d >$:

$$|f(x, y_1) - f(x, y_2)| \le L\, |y_1 - y_2| \, . \tag{2}$$

Then every point $[x_0, y_0] \in D$ determines a unique maximal solution $y(x)$ of (1), $[x, y(x)] \subset D$. Moreover, if $D =< a, b > \times R$, then the unique maximal solution is defined on the whole interval $< a, b >$.

The Lipschitz condition (2) can be interpreted as requiring a little more than continuity but a little less than differentiability. (Inequality (2) with $L = \max\limits_{x \in <a,b>} |\frac{\partial f}{\partial y}(x)|$ follows from differentiability, using the mean value theorem).

A special case – the linear equation

For a linear equation $y' = g_0(x) + g_1(x)\, y$, the following holds:
Suppose that the functions $g_0(x)$ and $g_1(x)$ are continuous on an interval $< a, b >$. Then every point $[x_0, y_0] \in< a, b > \times R$ determines a unique maximal solution $y(x)$ of (1), defined on the whole interval $< a, b >$.

Proof: Follows from Theorem 2 with $L = \max\limits_{x \in <a,b>} |g_1(x)|$.

Numerical solution

Throughout this text we assume that the assumptions either of Theorem 1 or of Theorem 2 hold.

Notation:
x_k ... discretization points on x-axis
$h = x_{k+1} - x_k$ ... the discretization step
y_k ... approximation of the exact value $y(x_k)$ of the solution at x_k

Explicit Euler's method (or Euler's **forward** method)

choose a step size h and for $k = 0, 1, 2, \dots$ compute

$$x_{k+1} = x_k + h$$
$$y_{k+1} = y_k + h\, f(x_k, y_k)$$

Where this formula comes from - one step of Euler's forward method:

- express the equation at x_k:
  $$y'(x_k) = f(x_k, y(x_k))$$
- substitute the **forward** difference instead of the derivative:
  $$\frac{y(x_{k+1}) - y(x_k)}{h} + \mathcal{O}(h) = f(x_k, y(x_k))$$
- multiply by h and rearange the equation:
  $$y(x_{k+1}) = y(x_k) + h\, f(x_k, y(x_k)) + \mathcal{O}(h^2)$$
- omit the error at the step – switch to the approximate value of $y(x_{k+1})$:
  $$y_{k+1} = y(x_k) + h\, f(x_k, y(x_k))$$

Implicit Euler method (or Euler **backward** method)

choose a step size h and for $k = 0, 1, 2, \dots$ compute

$$x_{k+1} = x_k + h$$
$$y_{k+1} = y_k + h\, f(x_{k+1}, y_{k+1}) \quad \text{– an implicit equation for } y_{k+1}$$

Inference:

- express the equation at x_{k+1}:
  $$y'(x_{k+1}) = f(x_{k+1}, y(x_{k+1}))$$
- substitute the **backward** difference instead of the derivative:
  $$\frac{y(x_{k+1}) - y(x_k)}{h} + \mathcal{O}(h) = f(x_{k+1}, y(x_{k+1}))$$
- multiply by h and rearange the equation:
  $$y(x_{k+1}) = y(x_k) + h\, f(x_{k+1}, y(x_{k+1})) + \mathcal{O}(h^2)$$
- omit the error at the step – switch to the approximate value of $y(x_{k+1})$:
  $$y_{k+1} = y(x_k) + h\, f(x_{k+1}, y_{k+1})$$

Example 1

Consider Cauchy problem $y' = \dfrac{y}{x^2}\,, \quad y(1) = 2\ .$

1) Find the domain of the existence and uniqueness of the solution of the problem.

2) Compute an approximate value of $y(1.4)$ using:
   a) Explicit Euler method with step size $h = 0.2$,
   b) Implicit Euler method with step size $h = 0.2$,
   c) Explicit and Implicit Euler method with step size $h = 0.1$.

Solution

1) This is linear differential equation with coefficients $g_0(x) = 0$ and $g_1(x) = \frac{1}{x^2}$, continuous in the intervals $I_1 = (-\infty, 0)$ and $I_2 = (0, \infty)$. As $x_0 = 1$ lies in I_2, the interval of maximal solution of the given problem is I_2.

2) The results are summarized in Table 1 and for explicit Euler method also in Figure 1.

   Computation:

   a) $h = 0.2, \quad x_0 = 1, \quad y_0 = 2$

   $k \equiv f(x_0, y_0) = \dfrac{y_0}{(x_0)^2} = \dfrac{2}{1^2} = 2$

   $x_1 = x_0 + h = 1 + 0.2 = 1.2, \quad y_1 = y_0 + h\,k = 2 + 0.2 \cdot 2 = 2.4$

   $k \equiv f(x_1, y_1) = \dfrac{y_1}{(x_1)^2} = \dfrac{2.4}{(1.2)^2} = 1.6667$

   $x_2 = x_1 + h = 1.2 + 0.2 = 1.4, \quad y_2 = y_1 + h\,k = 2.4 + 0.2 \cdot 1.6667 = 2.7333$

   $y(1.4)$ is approximately equal to $y_2 = 2.7333$.

   b) There is no general explicit formula: in every iteration, we have to solve the equation $y_{k+1} = y_k + h\,f(x_{k+1}, y_{k+1})$; for this problem it is

$$y_{k+1} = y_k + h\,\frac{y_{k+1}}{(x_{k+1})^2}\ .$$

   In the case of *linear* differential equation like this, however, we can express y_{k+1} from the equation above explicitly:

$$y_{k+1} = \frac{(x_{k+1})^2}{(x_{k+1})^2 - h}\,y_k\ .$$

   $h = 0.2, \quad x_0 = 1, \quad y_0 = 2$

   $x_1 = x_0 + h = 1 + 0.2 = 1.2\ , \quad y_1 = \frac{(x_1)^2}{(x_1)^2 - h}\,y_0 = \frac{1.2^2}{1.2^2 - 0.2} \cdot 2 = 2.3226$

   $x_2 = x_1 + h = 1.2 + 0.2 = 1.4\ , \quad y_2 = \frac{(x_2)^2}{(x_2)^2 - h}\,y_1 = \frac{1.4^2}{1.4^2 - 0.2} \cdot 2.3226 = 2.5865$

   $y(1.4)$ is approximately equal to $y_2 = 2.5865$.

c) Using similar process as in a), we obtain values presented at the second column of Table 1.

   $y(1.4)$ is approximately equal to $y_4 = 2.6979$ for explicit Euler method and to $y_4 = 2.6241$ for the implicit one.

Observations and questions

A) The errors of both explicit and implicit Euler methods are very similar. From the last row in the Table 1, the errors of explicit and implicit Euler methods can be computed: they are 0.0756 and 0.0753 for step $h = 0.1$ and 0.1506 and 0.1497 for step $h = 0.2$.

Why should then we bother using the implicit method, which generally requires to solve an implicit equation in every step? Because of *stability* (will be covered in next lectures).

B) Using half step size, the errors are reduced approximately twice. Does this hold for any numerical method of solving ODE? Could it be improved? See the midpoint method bellow. The speed of *convergence* will be discussed in next lectures.

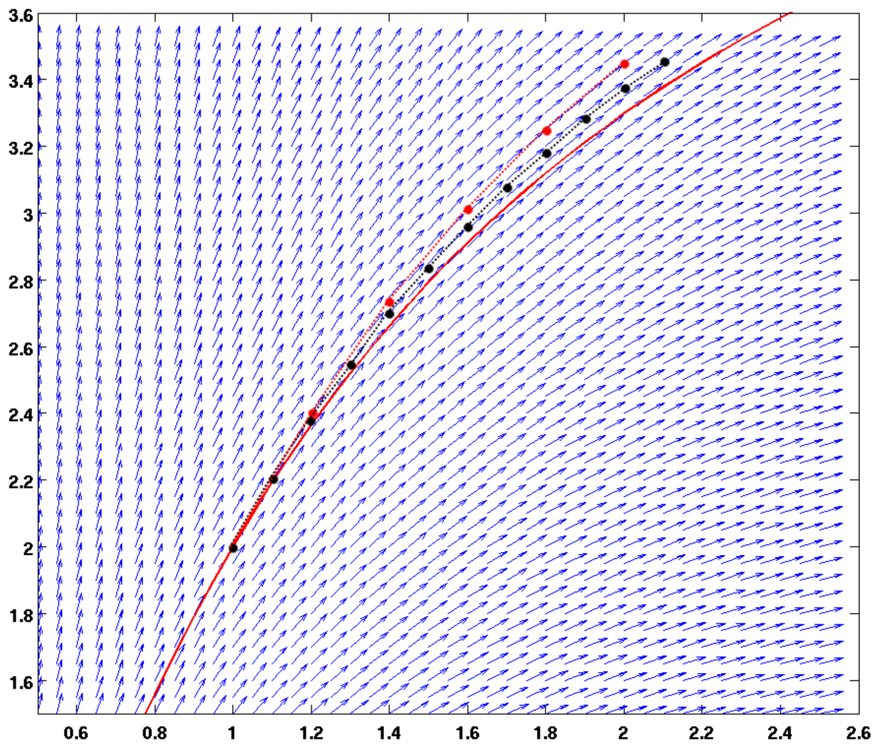

Figure 1: **Example 1**. Horizontal axis is x, vertical axis is y. Every blue arrow represent tangent vector to the integral curve passing through the matching point. The full red line represents the exact solution of the problem with a given initial condition $y(x) = 2\,e^{1-1/x}$ (it can be computed using a separation of variables). Red and black points represent approximation of the solution computed by Euler method with step size 0.2 and 0.1, respectively (see also Table 1).

Midpoint (or Collatz) method

Motivation: Euler (explicit) method is deduced by substitution of the first forward difference instead of the derivative on the left hand side of the differential equation.

What if we used the first central difference with $\mathcal{O}(h^2)$ error instead? As we do not want to involve values at other nodes than at x_k and x_{k+1}, we need to use half-step for the central difference. The inference:

- express the equation at $x_k + \frac{h}{2}$:

  $y'\left(x_k + \frac{h}{2}\right) = f\left(x_k + \frac{h}{2},\ y\left(x_k + \frac{h}{2}\right)\right)$

- substitute the **central** difference (at $x_k + \frac{h}{2}$) instead of the derivative:

  $$\frac{y(x_{k+1}) - y(x_k)}{h} + \mathcal{O}(h^2) = \ f\left(x_k + \tfrac{h}{2},\ y\left(x_k + \tfrac{h}{2}\right)\right)$$

- multiply by h and rearange the equation:

  $y(x_{k+1}) = y(x_k) + h\ f\left(x_k + \frac{h}{2},\ y\left(x_k + \frac{h}{2}\right)\right) + \mathcal{O}(h^3)$

  On the right hand side, there is an unknown value of $y\left(x_k + \frac{h}{2}\right)$ which we want to approximate by values of x_k and $y(x_k)$ in order to obtain an explicit method. Using forward Euler method with half the step we have

  $y\left(x_k + \frac{h}{2}\right) = y(x_k) + \frac{h}{2}\, f(x_k, y(x_k)) + \mathcal{O}(h^2)$

  and so

  $f\left(x_k + \frac{h}{2},\ y\left(x_k + \frac{h}{2}\right)\right) = f\left(x_k + \frac{h}{2},\ y(x_k) + \frac{h}{2}\, f(x_k, y(x_k)) + \mathcal{O}(h^2)\right)$

  from assumptions of Theorem 1 or Theorem 2 it follows

  $f\left(x_k + \frac{h}{2},\ y\left(x_k + \frac{h}{2}\right)\right) = f\left(x_k + \frac{h}{2},\ y(x_k) + \frac{h}{2}\, f(x_k, y(x_k))\right) + \mathcal{O}(h^2)$

  because

  $|f\left(x_k + \frac{h}{2},\ y(x_k) + \frac{h}{2}\, f(x_k, y(x_k)) + \mathcal{O}(h^2)\right) - f\left(x_k + \frac{h}{2},\ y(x_k) + \frac{h}{2}\, f(x_k, y(x_k))\right)| \leq$
  $\leq L\, |\mathcal{O}(h^2)| = \mathcal{O}(h^2)$

  which leads to

  $y(x_{k+1}) = y(x_k) + h\ f\left(x_k + \frac{h}{2},\ y(x_k) + \frac{h}{2}\, f(x_k, y(x_k))\right) + \mathcal{O}(h^3)$

- omit the error at the step – switch to the approximate value of $y(x_{k+1})$:

  $y_{k+1} = y(x_k) + h\ f\left(x_k + \frac{h}{2},\ y(x_k) + \frac{h}{2}\, f(x_k, y(x_k))\right)$.

Algorithm of the midpoint method

choose a step size h and for $k = 0, 1, 2, \ldots$ compute

$x_p = x_k + \frac{h}{2}$
$y_p = y_k + \frac{h}{2}\, f(x_k, y_k)$

$x_{k+1} = x_k + h$
$y_{k+1} = y_k + h\, f(x_p,\ y_p)$

Example 1 – continued

Consider Cauchy problem $\quad y' = \dfrac{y}{x^2}\,, \quad y(1) = 2\ .$

Compute an approximate value of $y(1.4)$ using midpoint method with step size $h = 0.2$ and compare its performance with previous results of Euler methods.

Solution

The results are summarized in Table 1. These results show that midpoint method gives more precise solution than Euler method (both explicit and implicit), even in the case when for midpoint method, step size twice as long as for Euler was used (which represents comparable work).

Computation:

$h = 0.2, \quad x_0 = 1, \quad y_0 = 2$

$k_1 \equiv f(x_0, y_0) = \dfrac{y_0}{x_0^2} = \dfrac{2}{1^2} = 2,$

$x_p = x_0 + \frac{1}{2}h = 1 + 0.1 = 1.1$

$y_p = y_0 + \frac{1}{2}h\,k_1 = 2 + 0.1 \cdot 2 = 2.2$

$k_2 \equiv f(x_p, y_p) = \dfrac{y_p}{x_p^2} = \dfrac{2.2}{1.1^2} = 1.8182$

$x_1 = x_0 + h = 1 + 0.2 = 1.2$

$y_1 = y_0 + h\,k_2 = 2 + 0.2 \cdot 1.8182 = 2.3636$

$k_1 \equiv f(x_1, y_1) = \dfrac{y_1}{x_1^2} = \dfrac{2.3636}{1.2^2} = 1.6414$

$x_p = x_1 + \frac{1}{2}h = 1.2 + 0.1 = 1.3$

$y_p = y_1 + \frac{1}{2}h\,k_1 = 2.3636 + 0.1 \cdot 1.6414 = 2.5278$

$k_2 \equiv f(x_p, y_p) = \dfrac{y_p}{x_p^2} = \dfrac{2.5278}{1.3^2} = 1.4957$

$x_2 = x_1 + h = 1.2 + 0.2 = 1.4$

$y_2 = y_1 + h\,k_2 = 2.3636 + 0.2 \cdot 1.4957 = 2.6628$

$y(1.4)$ is approximately equal to $y_2 = 2.6628$.

x_k	exact $y(x_k)$	Euler Explic.	$h = 0.1$ Implic.	Euler Explic.	$h = 0.2$ Implic.	midpoint $h = 0.2$
1	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
1.1	2.1903	2.2000	2.1802			(2.2000)
1.2	2.3627	2.3818	2.3429	2.4000	2.3226	2.3636
1.3	2.5191	2.5472	2.4902			(2.5278)
1.4	2.6614	2.6979	2.6241	2.7333	2.5865	2.6628
1.5	2.7912	2.8356	2.7462			(2.7986)
1.6	2.9100	2.9616	2.8578	3.0122	2.8057	2.9115
1.7	3.0190	3.0773	2.9602			(3.0253)
1.8	3.1192	3.1838	3.0545	3.2476	2.9903	3.1209
1.9	3.2118	3.2821	3.1415			(3.2172)
2.0	3.2974	3.3730	3.2221	3.4480	3.1477	3.2992

Table 1: **Example 1**. The first column represents values of x, where the approximate solution is computed. At the second column there is exact solution, the third and the fourth columns present solution obtained by Euler method with step size $h = 0.1$ and $h = 0.2$, respectively, and the last column presents approximate solution obtained by midpoint method with step size $h = 0.2$.