A system of first-order ODEs, initial value problem

First-order initial value (Cauchy) problem for a system of ODEs:

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)) \quad \text{with an initial condition } \mathbf{Y}(x_0) = \mathbf{Y}^{(0)}, \tag{1}$$

where

$$\mathbf{Y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \ \mathbf{Y}'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}, \ \mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} f_1(x, y_1, y_2, \ldots, y_n) \\ f_2(x, y_1, y_2, \ldots, y_n) \\ \vdots \\ f_n(x, y_1, y_2, \ldots, y_n) \end{bmatrix}$$

The system of (1) is called *linear*, if functions f_i are linear with respect to all variables y_j , i.e., they have the form

$$f_i(x, y_1, \ldots, y_n) = g_{i0}(x) + g_{i1}(x)\, y_1 + g_{i2}(x)\, y_2 + \ldots + g_{in}(x)\, y_n\,, \quad i = 1, \ldots n\,. \tag{2}$$

In matrix form this can be written as

$$\mathbf{Y}' = \mathbf{G}\,\mathbf{Y} + \mathbf{G}_0, \quad \text{where } \mathbf{G} = \{g_{ij}\}_{i,j=1}^{n} \text{ and } \mathbf{G}_0 = (g_{1\,0}, g_{2\,0}, \ldots g_{n0})^T.$$

Existence and uniqueness of the (exact) solution

There are two standard theorems that formulate *sufficient* conditions for existence and uniqueness of the solution of the initial value problem (1):

Theorem 1

Suppose that for $i, j = 1, \ldots n$, functions f_i and their partial derivatives $\frac{\partial f_i}{\partial y_j}$ are continuous in a domain $\Omega \subset R^{n+1}$. Then every point $[x_0, \mathbf{Y}^{(0)}] \in \Omega$ determines a unique maximal solution $\mathbf{Y}(x)$ of (1), $[x, \mathbf{Y}(x)] \subset \Omega$.

Theorem 2

Suppose that for $i, j = 1, \ldots n$, functions f_i are continuous in some region D defined by $a \le x \le b$, $a_j < y_j < b_j$ and that F satisfies *Lipschitz condition* with respect to $\mathbf{Y}$ in D, i. e., there exists a constant L such that

$$\|F(x, \mathbf{Y}) - F(x, \mathbf{Z})\| \le L\,\|\mathbf{Y} - \mathbf{Z}\| \quad \forall\ (x, \mathbf{Y}),\ (x, \mathbf{Z}) \in D\,. \tag{3}$$

Then for any $(x_0, \mathbf{Y}^{(0)}) \in D$ there exists a unique maximal solution $\mathbf{Y}(x) \subset D$ of the problem (1). Moreover, if $D = I \times R^n$, then the unique maximal solution is defined on the whole interval I.

A special case – the linear system

For a linear system (2) the following holds:

Suppose that all functions $g_{ij}(x)$ are continuous on an interval $I = < a, b >$. Then the assumptions of Theorem 2 are satisfied in $D = I \times R^n$ with $L = \max_{x \in I} |g_{ij}(x)|$, so the unique maximal solution is defined on the whole interval I.

Numerical solution

Throughout the text, we assume that the problem (1) satisfies the assumptions either of Theorem 1 or of Theorem 2 in some region D and $(x_0, \mathbf{Y}^{(0)}) \in D$.

It also implicates that the problem is *well-posed*, which means (roughly speaking) that it's solution depends continuously on the given data: that a small perturbation of $\mathbf{F}$ or $\mathbf{Y}^{(0)}$ leads to a small change of the solution.

Notation:
x_k . . . discretization points on x-axis
$h = x_{k+1} - x_k$ . . . the discretization step
$\mathbf{Y}^{(k)}$ . . . approximation of the exact value $\mathbf{Y}(x_k)$ of the solution at x_k

Explicit Euler's method (or Euler's **forward** method)

choose a step size h and for $k = 0, 1, 2, \ldots$

1. compute the derivative $\mathbf{K}$ of the vector function $\mathbf{Y}$ as

   $\mathbf{K} = \mathbf{F}(x_k,\, \mathbf{Y}^{(k)})$

2. set

   $x_{k+1} = x_k + h$

   $\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\,\mathbf{K}$

Implicit Euler's method (or Euler's **backward** method)

choose a step size h and for $k = 0, 1, 2, \ldots$

1. set $x_{k+1} = x_k + h$
2. compute $\mathbf{Y}^{(k+1)}$ from the equation (using FPI or Newton's method)

   $\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\,\mathbf{F}(x_{k+1},\, \mathbf{Y}^{(k+1)})$

   For linear system $\mathbf{Y}' = \mathbf{G}\,\mathbf{Y} + \mathbf{G}_0$, the equation above represents a linear system

   $(\mathbf{I} - h\,\mathbf{G})\,\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\,\mathbf{G}_0$ (both $\mathbf{G}$ and $\mathbf{G}_0$ generally depend on x_{k+1}).

Midpoint (Collatz) method

choose a step size h and for $k = 0, 1, 2, \ldots$

1. compute an auxiliary point $[x_p, \mathbf{Y}_p]$ using forward Euler's method with half-step:

   $\mathbf{K}_1 = \mathbf{F}(x_k,\, \mathbf{Y}^{(k)}), \quad x_p = x_k + \frac{1}{2}h, \quad \mathbf{Y}_p = \mathbf{Y}^{(k)} + \frac{1}{2}h\,\mathbf{K}_1$

2. compute the derivative $\mathbf{K}_2$ at the auxiliary point $[x_p, \mathbf{Y}_p]$ as

   $\mathbf{K}_2 = \mathbf{F}(x_p,\, \mathbf{Y}_p)$

3. set

   $x_{k+1} = x_k + h$

   $\mathbf{Y}^{(k+1)} = \mathbf{Y}^{(k)} + h\,\mathbf{K}_2$

Example 1

Consider Cauchy problem

$$\mathbf{Y}' = \begin{bmatrix} y_1 \sin(x) + y_3 \\ y_2 \ln(x+1) - 4 \\ 2y_1 \ - \ \dfrac{y_3}{x-2} \end{bmatrix}, \qquad \mathbf{Y}(1) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

a) Verify that the given problem has unique solution and find the interval I of its maximal solution.

b) Choose the step size $h = 0.1$ and compute an approximate value of $\mathbf{Y}(1.2)$ using Euler's (explicit) method.

c) Choose the step size $h = 0.2$ and compute an approximate value of $\mathbf{Y}(1.2)$ using midpoint method.

Solution

a) The system of equations is linear, so the continuity of its coefficients has to be checked only:

$x+1>0 \ \Rightarrow \ x>-1, \quad x-2 \neq 0 \ \Rightarrow \ x \neq 2 \qquad I_1 = (-1,2), \ I_2 = (2,\infty)$

$x_0 = 1 \in I_1 \Rightarrow$ interval of maximal solution is $(-1,2)$.

b) $x_0 = 1$, $\mathbf{Y}^{(0)} = (-1,1,2)^T$, $h = 0.1$:

$$\mathbf{K} = \mathbf{F}(x_0, \mathbf{Y}^{(0)}) = \begin{bmatrix} -1 \cdot \sin(1) + 2 \\ 1 \cdot \ln(1+1) - 4 \\ 2 \cdot (-1) \ - \ \dfrac{2}{1-2} \end{bmatrix} = \begin{bmatrix} -0.84147 + 2 \\ 0.69315 - 4 \\ -2 + 2 \end{bmatrix} = \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix}$$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.8842 \\ 0.6693 \\ 2 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{F}(x_1, \mathbf{Y}^{(1)}) = \begin{bmatrix} -0.8842 \cdot \sin(1.1) + 2 \\ 0.6693 \cdot \ln(1.1+1) - 4 \\ 2 \cdot (-0.8842) \ - \ \dfrac{2}{1.1-2} \end{bmatrix} = \begin{bmatrix} -0.7880 + 2 \\ 0.6693 \cdot 0.74194 - 4 \\ -1.7684 + 2.2222 \end{bmatrix} = \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.45380 \end{bmatrix}$$

$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$$\mathbf{Y}^{(2)} = \mathbf{Y}^{(1)} + h\,\mathbf{K} = \begin{bmatrix} -0.8842 \\ 0.6693 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.4538 \end{bmatrix} = \begin{bmatrix} -0.7630 \\ 0.3190 \\ 2.0454 \end{bmatrix}$$

The value of $\mathbf{Y}(1.2)$ is approximately $\mathbf{Y}^{(2)} = (-0.7630, 0.3190, 2.0454)^T$.

c) $x_0 = 1,\ \mathbf{Y}^{(0)} = (-1, 1, 2)^T,\ h = 0.2$:

$$\mathbf{K}_1 = \mathbf{F}(x_0, \mathbf{Y}^{(0)}) = \begin{bmatrix} -1 \cdot \sin(1) + 2 \\ 1 \cdot \ln(1+1) - 4 \\ 2 \cdot (-1) \ - \ \dfrac{2}{1-2} \end{bmatrix} = \begin{bmatrix} -0.84147 + 2 \\ 0.69315 - 4 \\ -2 + 2 \end{bmatrix} = \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix}$$

$x_p = x_0 + \frac{1}{2}\,h = 1 + 0.1 = 1.1$

$$\mathbf{Y}_p = \mathbf{Y}^{(0)} + \frac{1}{2}\,h\,\mathbf{K}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 1.1585 \\ -3.3068 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.8842 \\ 0.6693 \\ 2 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{F}(x_p, \mathbf{Y}_p) = \begin{bmatrix} -0.8842 \cdot \sin(1.1) + 2 \\ 0.6693 \cdot \ln(1.1+1) - 4 \\ 2 \cdot (-0.8842) \ - \ \dfrac{2}{1.1-2} \end{bmatrix} = \begin{bmatrix} -0.7880 + 2 \\ 0.6693 \cdot 0.74194 - 4 \\ -1.7684 + 2.2222 \end{bmatrix} = \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.45380 \end{bmatrix}$$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0.2 \begin{bmatrix} 1.2120 \\ -3.5034 \\ 0.4538 \end{bmatrix} = \begin{bmatrix} -0.7576 \\ 0.2993 \\ 2.091 \end{bmatrix}$$

The value of $\mathbf{Y}(1.2)$ is approximately $\mathbf{Y}^{(1)} = (-0.7576,\ 0.2993,\ 2.091)^T$.

When using **implicit** Euler's method instead of the explicit one (Euler's or midpoint), then at every iteration, a system of equations has to be solved:

$x_0 = 1,\ \mathbf{Y}^{(0)} = (-1, 1, 2)^T$, choose $h = 0.2$

$x_1 = x_0 + h = 1 + 0.2 = 1.2$

$\mathbf{Y}^{(1)}$ has to be computed from the system of equations $\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{F}(x_1,\ \mathbf{Y}^{(1)})$:

$$\mathbf{Y}^{(1)} \equiv \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y_3^{(1)} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0.2 \begin{bmatrix} y_1^{(1)} \sin(1.2) + y_3^{(1)} \\ y_2^{(1)} \ln(1.2+1) - 4 \\ 2y_1^{(1)} \ - \ \dfrac{y_3^{(1)}}{1.2-2} \end{bmatrix}$$

This system can be solved using FPI for unknown $\mathbf{Y}^{(1)}$.

Moreover, this system is linear (because the given system is linear) and so it can also be solved using Gauss elimination (after reorganizing).

Higher-order initial value problems

Differential equation of the n-th order:

$$y^n(x) = f(x, y, y', y'', \dots y^{n-1}) \quad \text{with initial conditions}$$
$$y(x_0) = y_1^{(0)}, \quad y'(x_0) = y_2^{(0)}, \quad \dots \quad y^{n-1}(x_0) = y_n^{(0)} \tag{4}$$

In order to be able to use methods for first-order problems, we need to represent this diferential equation of n-th order as n first-order diferential equations. Introducing auxiliary variables $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $\dots y_n = y^{n-1}$ into equation (4) leads to a system (1) with

$$\mathbf{F}(x, \mathbf{Y}) = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ f(x, y_1, y_2, \dots, y_n) \end{bmatrix}, \qquad \mathbf{Y}(x_0) = \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ \vdots \\ y_n^{(0)} \end{bmatrix}$$

Example 2 - a harmonic oscilator (damped oscillations)

Consider the equation $y'' + 2y' + y = e^{-t}$ with initial cond. $y(0) = 2,\ y'(0) = -4.$
Find the approximate solution at time $t = 0.2$. Use Euler's method with $h = 0.1$.

Solution

The second-order problem has to be formulated as two first-order equations: set $y_1 = y$ and $y_2 = y'$ (i.e. use 2 scalar functions: y_1 represents the amplitude and y_2 the velocity). We have $y_1' = y_2$ and $y_2' = e^t - 2y_2 - y_1$:

$$\mathbf{Y}' = \begin{bmatrix} y_2 \\ e^{-t} - 2y_2 - y_1 \end{bmatrix}, \qquad \mathbf{Y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$h = 0.1,\ \ t_0 = 0,\ \ \mathbf{Y}^{(0)} = (2, -4)^T$

$$\mathbf{K} = \mathbf{F}(t_0, \mathbf{Y}^{(0)}) = \begin{bmatrix} -4 \\ e^0 - 2\cdot(-4) - 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$$

$t_1 = t_0 + h = 0.1$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + 0.1 \begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.6 \\ -3.3 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{F}(t_1, \mathbf{Y}^{(1)}) = \begin{bmatrix} -3.3 \\ e^{-0.1} - 2\cdot(-3.3) - 1.6 \end{bmatrix} = \begin{bmatrix} -3.3000 \\ 5.9048 \end{bmatrix}$$

$t_2 = t_1 + h = 0.2$

$$\mathbf{Y}^{(2)} = \mathbf{Y}^{(1)} + h\,\mathbf{K} = \begin{bmatrix} 1.6 \\ -3.3 \end{bmatrix} + 0.1 \begin{bmatrix} -3.3000 \\ 5.9048 \end{bmatrix} = \begin{bmatrix} 1.2700 \\ -2.7095 \end{bmatrix}$$

At time $t = 0.2$, the amplitude $y(0.2)$ is approximately 1.2700 and the velocity $y'(0.2)$ is approximately -2.7095. (The exact solution: $y(t) = (2-2t+0.5t^2)\,e^{-t}$, $y(0.2) = 1.3263$.)

Example 3

Consider Cauchy problem $(x-1)y''' + 2xy'' + 5 \;=\; 2x^2y'' + (x-1)\sqrt{(y')^2 - 2}$ with initial conditions $y(0) = 0,\; y'(0) = 2,\; y''(0) = -1$.

a) Find a domain where existence of a unique solution of the problem is guaranteed.

b) Compute an approximate value of $y'(0.1)$ using Euler's method.

Solution

a) First of all, express the equation in normal (canonical) form:

$$y''' \;=\; \sqrt{(y')^2 - 2} \;+\; 2x\,y'' \;-\; \frac{5}{x-1}$$

Now set $y_1 = y,\; y_2 = y',\; y_3 = y''$ and transform it to the first-order system:

$$\mathbf{Y}' = \begin{bmatrix} y_2 \\ y_3 \\ \sqrt{(y_2)^2 - 2} \;+\; 2x\,y_3 \;-\; \dfrac{5}{x-1} \end{bmatrix}, \qquad \mathbf{Y}(0) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

Functions $y_2,\; y_3$ and $\sqrt{(y_2)^2 - 1} + 2x\,y_3 - \frac{5}{x-1}$ and their derivatives with respect to y_i ($\frac{\partial f_3}{\partial y_2} = \frac{y_2}{\sqrt{(y_2)^2-2}}$) are continuous for $x \neq 1$ a $y_2 \notin \langle -\sqrt{2}, \sqrt{2} \rangle$, i.e on the domains

$$\Omega_1 = (-\infty, 1) \times R \times (-\infty, -\sqrt{2}) \times R \quad , \quad \Omega_2 = (-\infty, 1) \times R \times (\sqrt{2}, \infty) \times R$$

$$\Omega_3 = (1, \infty) \times R \times (-\infty, -\sqrt{2}) \times R \quad , \quad \Omega_4 = (1, \infty) \times R \times (\sqrt{2}, \infty) \times R$$

The initial condition $[0, 0, 2, -1]$ is situated in the domain Ω_2, and so the domain, where existence of a unique solution is guaranteed, is Ω_2 .

b) We have $x_0 = 0$, $\mathbf{Y}^{(0)} = (0, 2, -1)^T$ and we choose $h = 0.1$:

$$\mathbf{K} = \mathbf{F}(x_0, \mathbf{Y}^{(0)}) = \begin{bmatrix} 2 \\ -1 \\ \sqrt{2^2 - 2} \;+\; 2 \cdot 0 \cdot (-1) \;-\; \dfrac{5}{0-1} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ \sqrt{2} \;+\; 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6.4142 \end{bmatrix}$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$\mathbf{Y}^{(1)} = \mathbf{Y}^{(0)} + h\,\mathbf{K} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 0.1 \begin{bmatrix} 2 \\ -1 \\ 6.4142 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.9 \\ -0.3586 \end{bmatrix}$$

The value of $y'(0.1)$ is approximately $y_2^{(1)} = 1.9$.