Variant 1

1. Consider a linear system $x = Ux + v$, where

$$U = \begin{bmatrix} 0.8 & -3 & 0.2 \\ 0 & -0.6 & 7 \\ 0 & 0 & 0.5 \end{bmatrix}, \qquad v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

a) Will Fixed point iterations (i.e. simple iteration method) converge for the given system? Justify your answer.

b) Choose $x^{(0)} = (5, 5, -2)^T$ and compute $x^{(1)}$ using Fixed point iterations.

c) Compute column norm $\|x^{(1)} - x^{(0)}\|_1$.

2. Consider Cauchy problem

$$y''' + 2y\,y' - (y')^3 - 2 = 0\ , \qquad y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$$

a) Transform the problem to a system of three equations of the first order.

b) Choose the step size $h = 0.4$ and compute approximate value of $y(0.4)$ and $y'(0.4)$ using the midpoint method.

c) Which difference formula approximates a second derivative with precision $\mathcal{O}(h^2)$? Write down the formula and prove the statement.

3. Consider Cauchy problem

$$-y' + 2y = 5x\ , \qquad y(0) = 1\ .$$

a) Derive the implicit Euler method for an equation $y' = f(x, y)$.

b) Choose step $h = 0.2$ and find an approximate solution of $y(0.2)$ using the implicit Euler method.

4. Consider mixed problem for heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + 3\,t + x \quad \text{in} \quad \Omega = \{[x, t] : x \in (0, 2),\ t \in (0, T)\}\ ,$$

$$u(x, 0) = 2x, \quad u(0, t) = 0, \quad u(2, t) = 4 - 3\,t\ \ .$$

a) Derive the implicit scheme for an equation $\dfrac{\partial u}{\partial t} = p\,\dfrac{\partial^2 u}{\partial x^2} + f(x, t)$.

b) Choose time step $\tau = 0.3$ and space step $h = 0.5$ and assemble equations for numerical solution on the first level using the implicit method. Will the implicit method be stable for this choice of discretization steps?

c) Write equations from b) in a matrix form. Will Jacobi iteration method converge if used for this linear system? Give reasons to your answer.

Variant 2

1. Consider the linear system $AX = B$ with a real parameter p, where

$$A = \begin{bmatrix} -2 & 1 & p \\ 1 & 3 & 1 \\ -1 & p & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}.$$

a) Find all values of the parameter p such that A is strictly diagonally dominant (SDD).

b) What can be said about convergence of the Jacobi method for a system with SDD matrix? What can be said about convergence of this method if the matrix is not SDD?

c) Deduce a matrix formulation of the Jacobi iterative method for a general matrix A decomposed as $A = L + D + P$. What is the sufficient and necessary condition for convergence of this method?

d) Choose $p = 1$ and $X^{(0)} = (-1, 1, 4)^T$ and calculate $X^{(1)}$ using the Jacobi method.

2. Consider the nonlinear system

$$\begin{aligned} x^2 + 4y^2 &= 4 \\ x^2 - y &= 2 \end{aligned}$$

a) Find the solutions graphically.

b) Derive the linear system which is to be solved in every step of the Newton's method for solution of a system $F(x, y) = 0$, where $F(x, y) = (f(x, y), g(x, y))^T$.

c) Choose $X^{(0)} = (2, 1)^T$ and compute $X^{(1)}$ using the Newton's method.

3. Consider Cauchy problem

$$y''' = \frac{x}{y' + 1} + \sqrt{y - 3}\ , \qquad y(0) = 4,\ y'(0) = -2,\ y''(0) = 1$$

a) Find the domain of existence and uniqueness of the solution.

b) Choose the step size $h = 0.1$ and compute approximate value of $y(0.2)$ and $y'(0.2)$ using the Euler method.

4. Consider mixed problem for wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2} \qquad \text{in} \quad \Omega = \{[x, t] : x \in (0, 3),\ t > 0\}\ ,$$

$$u(x, 0) = 0, \qquad \frac{\partial u}{\partial t}(x, 0) = 1, \qquad u(0, t) = 5t^2 + t, \qquad u(3, t) = \sin t\ .$$

a) Derive the explicit finite difference scheme for an equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$.

b) Choose space step $h = 0.5$ and time step $\tau = 0.2$ and find the approximate value of $u(0.5, 0.4)$ (use linear extrapolation for the first time level). Will the explicit scheme be stable for this choice of h and τ?

c) Which difference formula approximates a second derivative with precision $\mathcal{O}(h^2)$? Write down the formula and prove the statement.