Computer graphics
Lesson 2

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Bézier curve of degree *n*

- approximation curve
- given: *n+1* control points ordered in control polygon
- vector equation:

$$\mathbf{P}(t) = \sum_{i=0}^{n} B_{i,n}(t)\mathbf{V}_i = B_{0,n}(t)\mathbf{V}_0 + B_{1,n}(t)\mathbf{V}_1 + \ldots + B_{n,n}(t)\mathbf{V}_n, \; t \in [0,1]$$

coefficients are Bernstein polynomials of degree *n*

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \; t \in [0,1], \; i = 0, \ldots, n$$

i corresponds to the associated control point

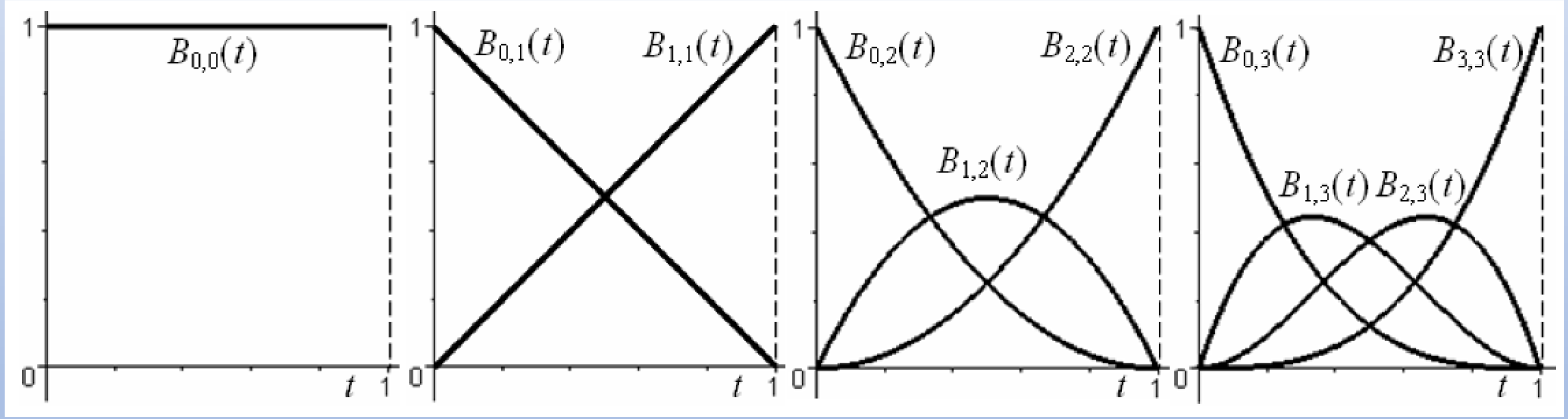
n denotes the curve degree

- curve pass through the first and last control point

Bézier curve

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \; t \in [0,1], \; i = 0, \ldots, n$$

- all Bernstein polynomials are non-negative
- sum of all Bernstein polynomials of the same degree is 1



Bézier curve

- cases:
  1. given by one control point → curve is degenerated to the point

     $$\mathbf{P}(t) = B_{0,0}(t)\mathbf{V}_0 = \mathbf{V}_0,\ \ t \in [0,1]$$

  2. given by two control points → curve is straight line segment

     $$\mathbf{P}(t) = B_{0,1}(t)\mathbf{V}_0 + B_{1,1}(t)\mathbf{V}_1 = (1-t)\mathbf{V}_0 + t\mathbf{V}_1,\ \ t \in [0,1]$$

  3. given by three control points → Bézier quadratic curve

     $$\mathbf{P}(t) = (1-t)^2\mathbf{V}_0 + 2t(1-t)\mathbf{V}_1 + t^2\mathbf{V}_2,\ \ t \in [0,1]$$

  4. given by four control points → Bézier cubic curve

     $$\mathbf{P}(t) = (1-t)^3\mathbf{V}_0 + 3t(1-t)^2\mathbf{V}_1 + 3t^2(1-t)\mathbf{V}_2 + t^3\mathbf{V}_3,\ \ t \in [0,1]$$

Bézier curve

- Example 2.3

- Example 2.4

Properties of Bézier curve

- interpolates endpoints of the control polygon
- tangent vector $P'(0)$ at V_0 equals *n*-multiple of the first control polygon leg → $P'(0) = n\overrightarrow{V_0 V_1} = n(V_1 - V_0)$
- tangent vector $P'(1)$ at V_n equals *n*-multiple of the first control polygon leg → $P'(1) = n\overrightarrow{V_{n-1} V_n} = n(V_n - V_{n-1})$
- for all control points collinear → Bézier curve is straight line segment (*linear accuracy*)

De Casteljau algorithm for Bézier curve

- geometric interpretation of Bernstein polynomials derivation
- for construction of curve point $P(\alpha)$, $\alpha \in \langle 0; 1 \rangle$

- **process**:
  1. choose $\alpha \in \langle 0; 1 \rangle$
  2. divide all legs of control polygon in ratio $\alpha : (1 - \alpha) \rightarrow$ get points A_i
  3. connect points A_i by polyline to a new control polygon with *n-1* legs
  4. divide all legs of control polygon A_i in ratio $\alpha : (1 - \alpha) \rightarrow$ get points B_i
  5. connect points B_i by polyline to a new control polygon with *n-2* legs
  6. repeat these steps until the control polygon is only one line segment
  7. divide the last leg in ratio $\alpha : (1 - \alpha) \rightarrow$ get point $P(\alpha)$, the last leg is tangent line $P'(\alpha)$, tangent vector is *n*-times the last leg
- Example 2.7

Continuity at common point of Bézier curve

- general: Bézier curves are independent to each other
- suppose two Bézier cubic curve:

$$\mathbf{P}(t) = B_{0,3}(t)\mathbf{V}_0 + B_{1,3}(t)\mathbf{V}_1 + B_{2,3}(t)\mathbf{V}_2 + B_{3,3}(t)\mathbf{V}_3,\ t \in [0, 1]$$

$$\mathbf{R}(s) = B_{0,3}(s)\mathbf{W}_0 + B_{1,3}(s)\mathbf{W}_1 + B_{2,3}(s)\mathbf{W}_2 + B_{3,3}(s)\mathbf{W}_3,\ s \in [0, 1]$$

where points V_i are know and points W_i are unknown

$\rightarrow$ express points W_i in terms of points V_i to connect these curves with C^0, C^1 or C^2 continuity

Continuity at common point of Bézier curve

1. **continuity of the zeroth order** = function values of vector functions are equal

2. **continuity of the first order** = function values of the first derivatives of vector functions are equal

3. **continuity of the second order** = function values of the second derivatives of vector functions are equal

* C^0 and C^1 continuity conditions are valid for BC of any degree

Continuity at common point of Bézier curve

- **Example 2.8**

- **Example 2.9**

- **Example 2.10**