Computer graphics
Lesson 4

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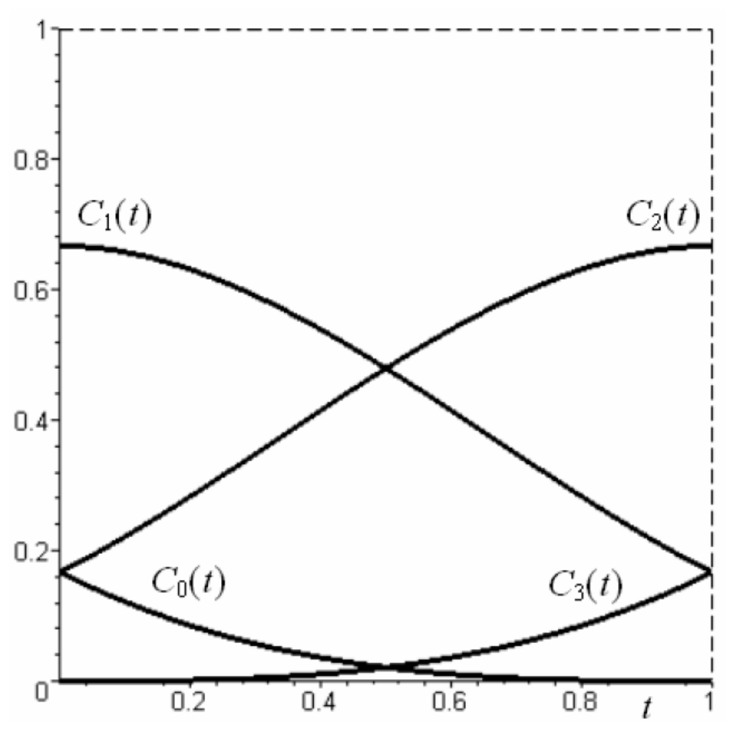
Coons cubic curve

- approximate curve, given points P_0, P_1, P_2 and P_3
- vector equation:

$$\mathbf{P}(t) = C_0(t)\mathbf{P}_0 + C_1(t)\mathbf{P}_1 + C_2(t)\mathbf{P}_2 + C_3(t)\mathbf{P}_3,\ t \in [0,1]$$

basis functions are Coons polynomials:

$$\begin{aligned} C_0(t) &= \frac{1}{6}(1-t)^3, \\ C_1(t) &= \frac{1}{6}(3t^3 - 6t^2 + 4), \\ C_2(t) &= \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1), \\ C_3(t) &= \frac{1}{6}t^3, \end{aligned}$$



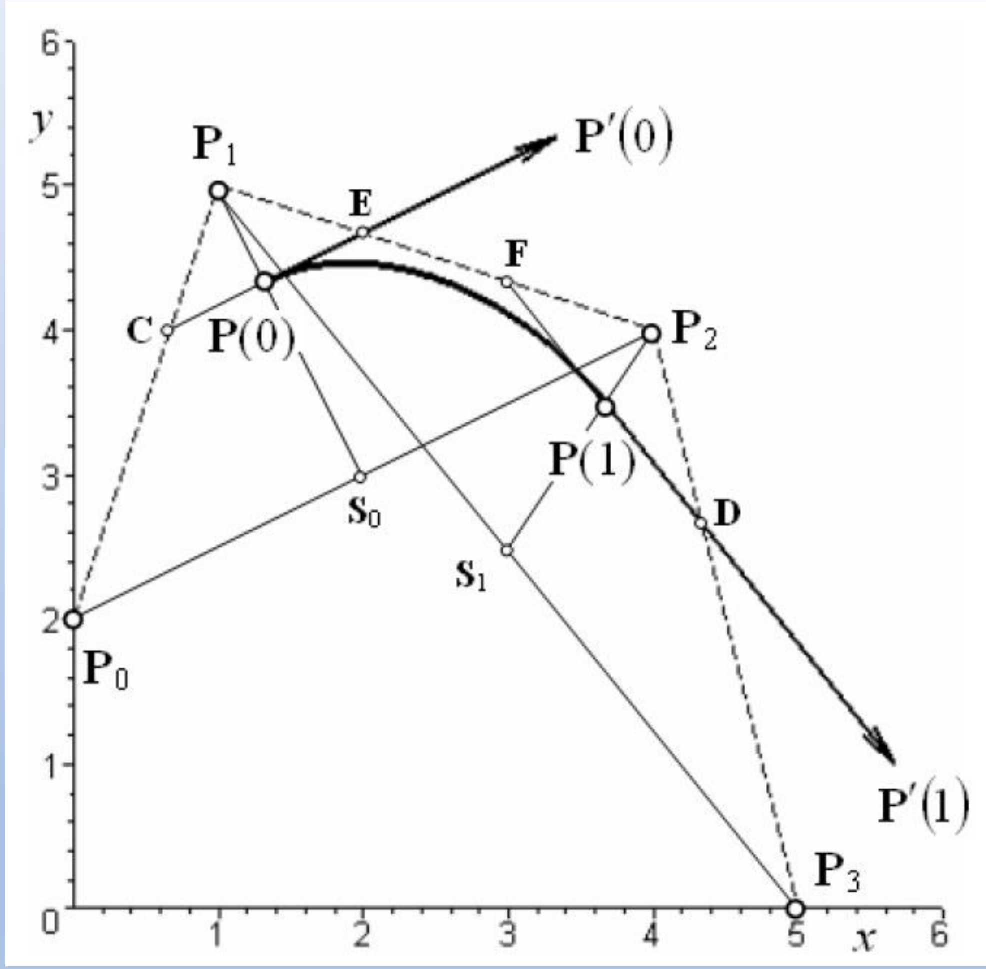
Coons cubic curve

- **properties:**
  1. values of Coons polynomials are <1 for any value of parameter *t* $\rightarrow$ curve does not pass through any given control point
  2. point $\mathrm{P}(0)$ lies at the „*anticentroid*“ of triangle $P_0P_1P_2$ constructed with respect to control point P_1
  3. point $\mathrm{P}(1)$ lies at the „*anticentroid*“ of triangle $P_1P_2P_3$ constructed with respect to control point P_2
  4. tangent vectors in $\mathrm{P}(0)$ and $\mathrm{P}(1)$ are given by equations:

$$\mathbf{P}'(0) = \frac{1}{2}\overrightarrow{\mathbf{P}_0\mathbf{P}_2} = \frac{1}{2}(\mathbf{P}_2 - \mathbf{P}_0)$$

$$\mathbf{P}'(1) = \frac{1}{2}\overrightarrow{\mathbf{P}_1\mathbf{P}_3} = \frac{1}{2}(\mathbf{P}_3 - \mathbf{P}_1)$$

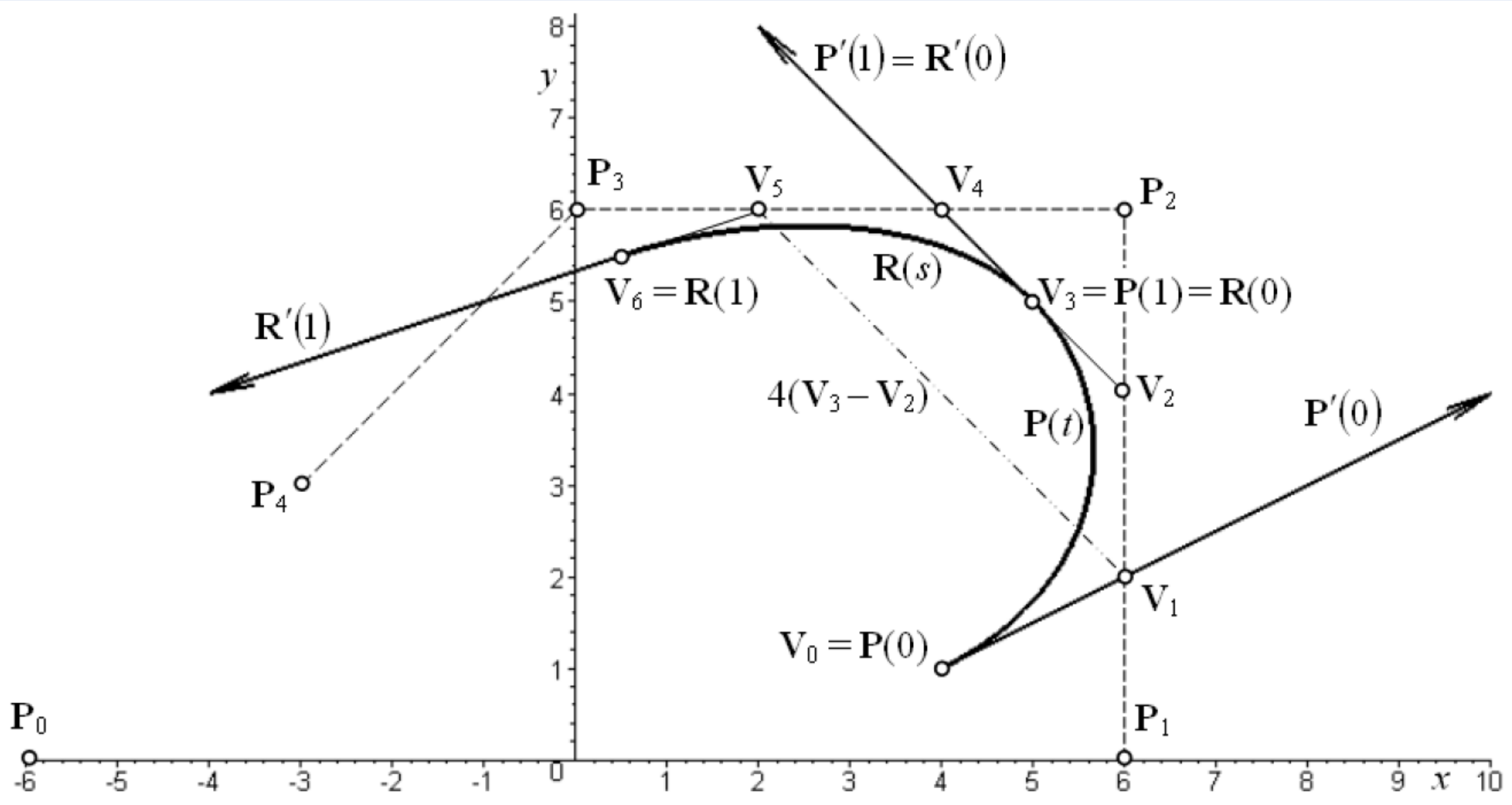
Coons cubic curve



5. tangent line at the initial point intersects legs P_0P_1 and P_1P_2 at one third from control point P_1 (points *C*, *E*) and leg P_1P_2 intersects the tangent vector $P'(0)$ at one third from point $P(0)$ (point *E*)
6. tangent line at the terminal point intersects legs P_1P_2 and P_2P_3 at one third from control point P_2 (points F, D) and leg P_2P_3 intersects the tangent vector $P'(1)$ at one third from point $P(1)$ (point *D*)

Coons cubic B-spline

- piecewise C^2 continuous curve made of segments from Coons cubic curves with control points P_0, P_1, P_2, P_3 and P_1, P_2, P_3, P_4 etc.



Coons cubic B-spline

- given by a sequence of control points $P_0,\ P_1,\ \ldots\ ,\ P_n,\ n \geq 4$ in space, a uniform B-spline curve of third degree R(t) compounded from *n-2* Coons cubic curves is called Coons cubic B-spline

$$
\begin{aligned}
\mathbf{R}_0(t) &= C_0(t)\mathbf{P}_0 + C_1(t)\mathbf{P}_1 + C_2(t)\mathbf{P}_2 + C_3(t)\mathbf{P}_3,\ t \in [0,1],\\
\mathbf{R}_1(t) &= C_0(t)\mathbf{P}_1 + C_1(t)\mathbf{P}_2 + C_2(t)\mathbf{P}_3 + C_3(t)\mathbf{P}_4,\ t \in [0,1],\\
&\vdots\\
\mathbf{R}_{n-3}(t) &= C_0(t)\mathbf{P}_{n-3} + C_1(t)\mathbf{P}_{n-2} + C_2(t)\mathbf{P}_{n-1} + C_3(t)\mathbf{P}_n,\ t \in [0,1]
\end{aligned}
$$

- closed or open

- **Example 2.13**

Coons cubic B-spline

- **properties:**
  - control polygon is created by at least five control points
  - If the last three control points are identical with the first three control points, i.e. $P_n = P_2$, $P_{n-1} = P_1$, $P_{n-2} = P_0$, Coons cubic B-spline is closed, otherwise it is open
  - does not pass through any control point of its control polygon
  - is created by *n-2* C^2 continuously joined Coons cubic curves, endpoints of these curves are called ***knots*** of Coons cubic B-spline
  - knots and tangent vectors at these knots can be constructed according to the properties

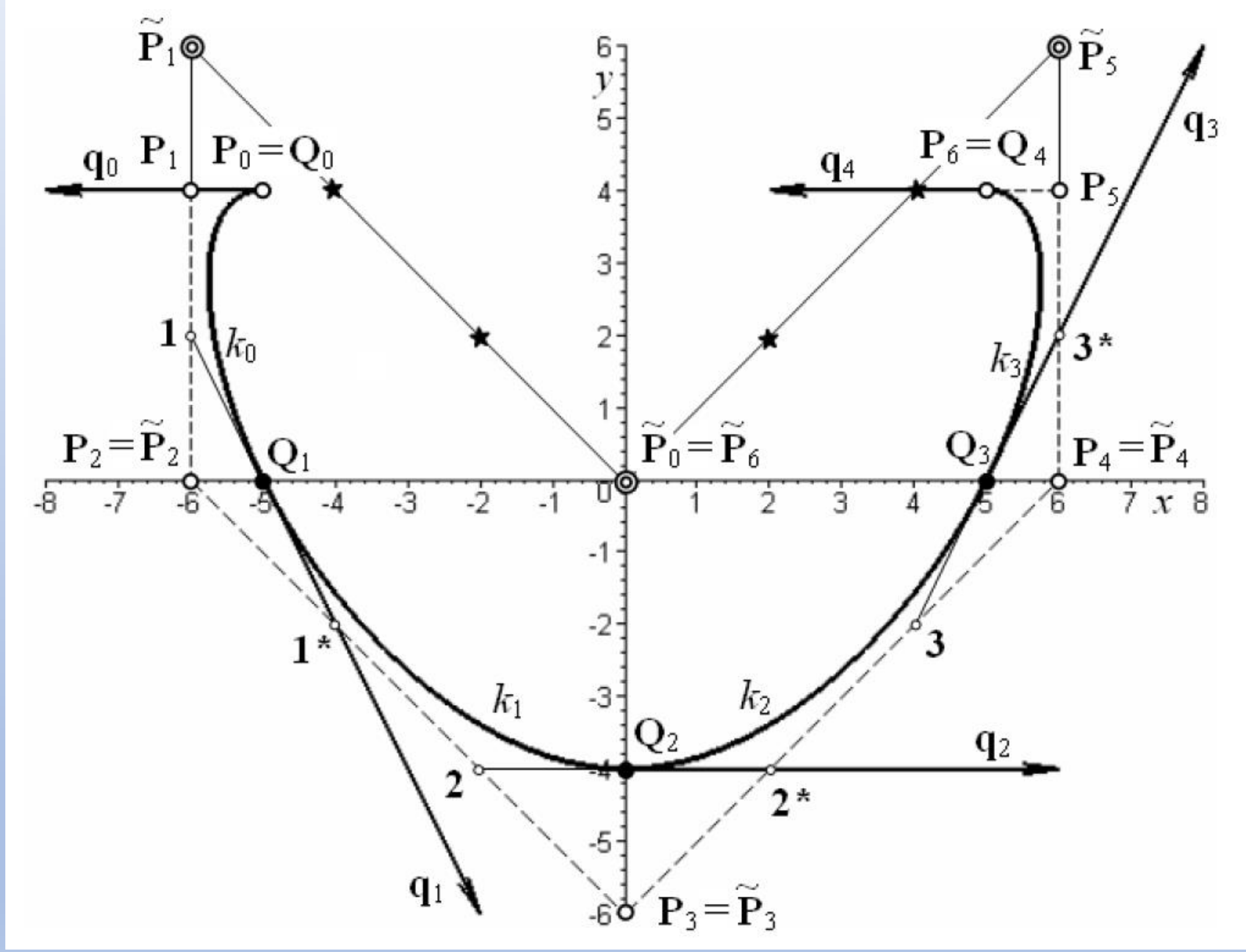
Coons cubic B-spline

- **properties:**
  - is a piecewise dfiened curve by partially overlapping control polygons (a change of position of one control point does not cause the change of whole Coons cubic B-spline, it influences the shape of those individual Coons cubic curves, of which vector equation contains the changing control point)
  - the domain of each individual Coons cubic curve is $t \in [0; 1]$ $\rightarrow$ the curve is called a ***uniform curve*** or ***curve with a uniform parametrization***

- **Exercise 2.19**

Clamped curve

- = Uniform clamped B-spline curve of 3th degree
- segments are created by Bézier cubic curves/Coons cubic curves
- initial point is „anticentroid" Q_0 of triangle $\tilde{P}_0\tilde{P}_1\tilde{P}_2$ constructed with respect to point $\tilde{P}_1$
- terminal point is „anticentroid" Q_4 of triangle $\tilde{P}_4\tilde{P}_5\tilde{P}_6$ constructed with respect to point $\tilde{P}_5$
- P_1 is at the first third of tangent vector q_0, on the leg $\tilde{P}_1\tilde{P}_2$ at one third from point $\tilde{P}_1$
- P_1 is on the leg $\tilde{P}_4\tilde{P}_5$ at one third from point $\tilde{P}_4$



Clamped curve

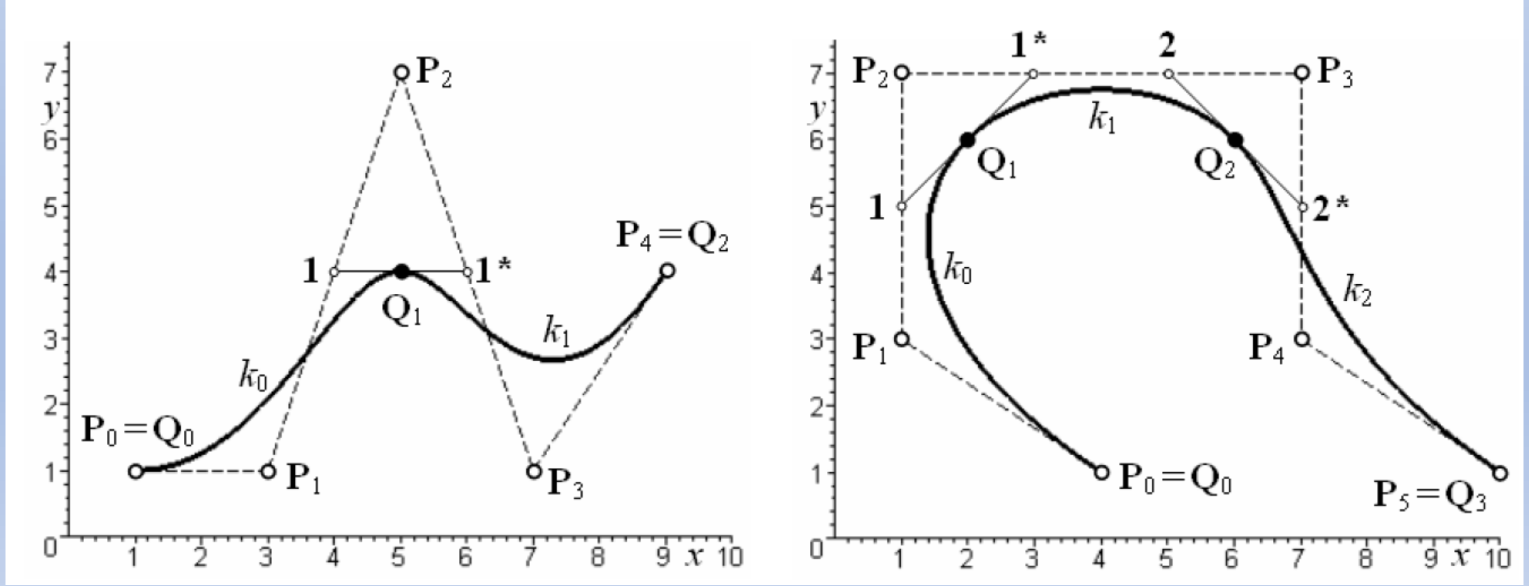
- transformation formulas between control points $P_0, \ldots, P_n$ of clamped curve and control points $\tilde{P}_0, \ldots, \tilde{P}_n$ of open Coons cubic B-spline:

$$
\begin{aligned}
\mathbf{P}_0 &= \mathbf{Q}_0 = \tfrac{1}{6}\widetilde{\mathbf{P}}_0 + \tfrac{2}{3}\widetilde{\mathbf{P}}_1 + \tfrac{1}{6}\widetilde{\mathbf{P}}_2, \\
\mathbf{P}_1 &= \tfrac{2}{3}\widetilde{\mathbf{P}}_1 + \tfrac{1}{3}\widetilde{\mathbf{P}}_2, \\
\mathbf{P}_i &= \widetilde{\mathbf{P}}_i, \; i = 2, \ldots, n-2, \\
\mathbf{P}_{n-1} &= \tfrac{1}{3}\widetilde{\mathbf{P}}_{n-2} + \tfrac{2}{3}\widetilde{\mathbf{P}}_{n-1}, \\
\mathbf{P}_n &= \mathbf{Q}_{n-2} = \tfrac{1}{6}\widetilde{\mathbf{P}}_{n-2} + \tfrac{2}{3}\widetilde{\mathbf{P}}_{n-1} + \tfrac{1}{6}\widetilde{\mathbf{P}}_n
\end{aligned}
$$

- first 2 and the last 2 curve segments are created by Bézier curves and all inner curve segments are created by Coons cubic curves (Coons cubic B-spline)

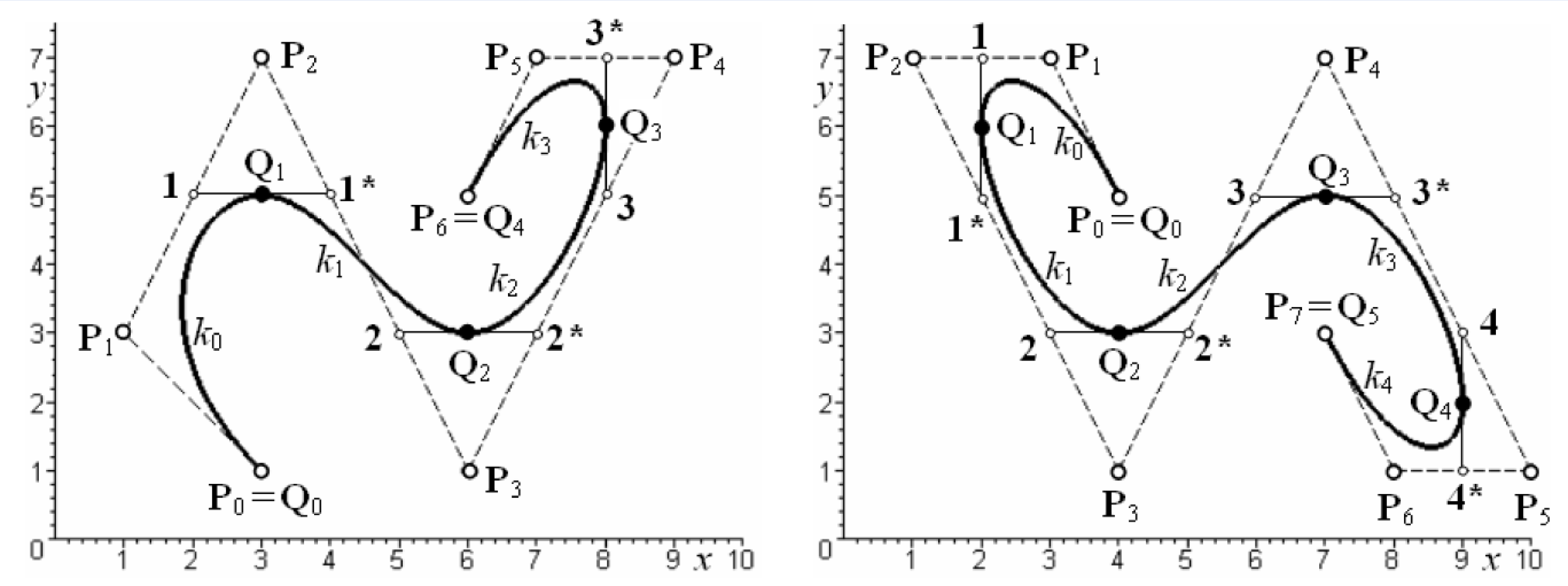
Clamped curve

- **properties:**
  - from properties of Bézier cubic curve, Coons cubic curve and Coons cubic B-spline
  - for n=3: only one curve segment → Bézier cubic curve
  - for n=4: two curve segments → Bézier cubic curves
  - for n=5: three curve segments → Bézier cubic curves



Clamped curve

- **properties:**
    - for n=6: four curve segments → Bézier cubic curves
    - for n=7: five curve segments → Bézier cubic curves, the middle curve segment is simultaneously Coons cubic curve



Clamped curve

- **properties:**
  - for n>7: n-2 curve segments → first two and the last two curve segments are Bézier cubic curves, remaining n-6 internal curve segments are C^2 continuously joined Coons cubic curves (open Coons cubic B-spline)

  - for n≥3 it is possible to create the clamped curve as a set of C^2 continuously joined Bézier cubic curves

Clamped curve

- **construction of knots of clamped curve for n>7:**
    1. initial point Q_0 is equal to the first control point P_0
    2. terminal point Q_{n-2} is equal to the last control point P_n
    3. do **not** divide the first and the last leg of control polygon
    4. divide the second and the next-to-last leg of the control polygon in halves → points 1 and $(n-3)^*$
    5. divide the remaining internal legs in thirds → points 1^*, 2, 2^*,…
    6. construct straight line segments 11^*, 22^*, …
    7. knots Q_1, Q_2, … lie at the centers of straight line segments 11^*, 22^*, …

Clamped curve

- **Exercise 2.26**
- **Exercise 2.27 (***try yourself***)**